Generalizations (in the spirit of Koshliakov) of some formulas from Ramanujan's Lost Notebook

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Dedicated to the memory of our Friend and Professor José Carlos Petronilho

Abstract

In his lost notebook, Ramanujan recorded beautiful identities. These include earlier versions of Koshliakov's formula for the divisor function and the transformation formula for the logarithm of Dedekind's $\eta-$function. In this paper we establish some generalizations of these formulas of Ramanujan in a setting that only recently reemerged in the literature and which concerns a beautiful theory due to Koshliakov.

Contents

1 Introduction

Let $\zeta(s)$ denote the Riemann zeta function. On entry 8 of Chapter 15 of his second notebook [3, 5], Ramanujan stated the beautiful formula for $\zeta(1/2)$,

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Entry 8: If $x>0$, then the following identity holds

$$\sum_{n=1}^{\infty}\frac{1}{e^{n^2x}-1}=\frac{1}{4}+\frac{\pi^2}{6x}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta\left(\frac{1}{2}\right)+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(\frac{\cos(2\pi\sqrt{\pi n/x})-\sin(2\pi\sqrt{\pi n/x})-e^{-2\pi\sqrt{\pi n/x}}}{\cosh(2\pi\sqrt{\pi n/x})-\cos(2\pi\sqrt{\pi n/x})}\right),\quad x>0. \tag{1.1}$$

On Entry 8.3.1., page 332, of his lost notebook, Ramanujan restates (1.1) in two different ways, which might be an indication that this representation for $\zeta(1/2)$ was dear to him.

The first proof of (1.1) was given by Berndt and Evans [5], who employed the Poisson summation formula and the transformation formula for Jacobi's $\theta-$function. After this, several generalizations and analogues of (1.1) have been obtained, including character analogues and evaluations of the Riemann zeta function at certain rational points. The reader may find in [[1], pp. 191-193] an exhaustive account of the activity around (1.1).

Of course, (1.1) coexists alongside other beautiful formulas in Ramanujan's lost notebook. For instance, on page 253 [1, 7, 21], Ramanujan states the following formula, quoted from [[1], p. 94],

Entry 3.3.1. (p. 253) Let $\sigma_k(n)=\sum_{d|n}d^k$, and let $\zeta(s)$ denote the Riemann zeta function. If α and β are positive numbers such that $\alpha\beta=\pi^2$ and if s is any complex number, then

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\beta)=$$
$$\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\left\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\right\}+\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\left\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\right\}. \tag{1.2}$$

The appearance of the modified Bessel function and the divisor functions on the infinite series on the left side of (1.2) is somewhat remindful of the Fourier expansion of the non-holomorphic Eisenstein series or of the Epstein zeta function attached to a binary quadratic form. In fact, the first proof of (1.2), due to Guinand [13], used a formula of Watson [26], which states that

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+x^2)^s}=\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)-\frac{x^{-2s}}{2}+\frac{2\,\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi nx). \tag{1.3}$$

for every $x>0$ and $\text{Re}(s)>\frac{1}{2}$. Due to the fact that Guinand's proof seems to be the first appearing in the literature, (1.2) is usually called "Ramanujan-Guinand formula".

As remarked by Berndt, Lee and Sohn [[7], p. 23], Guinand's proof of (1.2) via (1.3) is "completely independent of any considerations of nonanalytic Eisenstein series".

However, proving (1.3) is in fact the first step in almost all the proofs of the aforementioned Fourier expansion of Epstein zeta functions. This expansion was firstly considered by A. Selberg and S. Chowla in 1949, who started with the Epstein zeta function [9]

$$\zeta_Q(s)=\sum_{m,n\neq 0}\frac{1}{(am^2+bmn+cn^2)^s},\quad \text{Re}(s)>1, \tag{1.4}$$

(where $m,\,n\neq 0$ here means that only the term $m=n=0$ is omitted from the sum) and announced the following formula, valid in the entire complex plane,

$$a^s\Gamma(s)\,\zeta_Q(s)=2\Gamma(s)\zeta(2s)+2k^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)$$
$$+\,8k^{1/2-s}\pi^s\sum_{n=1}^{\infty}n^{s-1/2}\sigma_{1-2s}(n)\,\cos\left(n\pi b/a\right)K_{s-1/2}\left(2\pi k\,n\right). \tag{1.5}$$

Here, $d := b^2 - 4ac$ is the discriminant of the quadratic form, $k^2 := |d|/4a^2$ and $\sigma_\nu(n) = \sum_{d|n} d^\nu$. Also, $Q(x,y) = ax^2 + bx\,y + cy^2$ denotes a real and positive definite quadratic form.

It is now clear that the appearance of the divisor function in (1.5) is in some form connected to the identity of Ramanujan (1.2) and if one analyzes the underlying structure of the proofs of (1.5), they tend to rely in one way or another on Watson's formula (1.3).

By using (1.5), Selberg and Chowla [23] established several new results and reproved others by a new method. For example, using their formula, they were able to derive a new proof of the functional equation for $\zeta_Q(s)$. They also obtained from (1.5) a very simple proof of Kronecker's limit formula,

$$\zeta_Q\left(s\right) = \frac{2\pi}{\sqrt{|d|}}\,\frac{1}{s-1} + \frac{2\pi}{\sqrt{|d|}}\,\left(2\gamma - \log\left(\frac{|d|}{a}\right) - 4\log\left(|\eta(\tau)|\right)\right) + O(s-1), \tag{1.6}$$

where γ is the Euler-Mascheroni constant, $\tau := \frac{b+i\sqrt{|d|}}{2a} \in \mathbb{H}$ and $\eta(\tau)$ denotes the Dedekind $\eta-$function,

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}}\,\prod_{m=1}^{\infty}\left(1 - e^{2\pi i m\tau}\right), \quad \text{Im}(\tau) > 0. \tag{1.7}$$

In particular, when $Q(m,n) = m^2 + cn^2$, $c > 0$, the particular case of (1.6) holds

$$\zeta_Q\left(s\right) := \zeta_2(s,c) = \frac{\pi}{\sqrt{c}}\,\frac{1}{s-1} + \frac{\pi}{\sqrt{c}}\,\left(2\gamma - \log\left(4c\right) - 4\log\left(\left|\eta\left(i\sqrt{c}\right)\right|\right)\right) + O(s-1). \tag{1.8}$$

The function defined by (1.7) also played an important role in Ramanujan's lost notebook, right after the statement (1.2). He completes page 253 with two corollaries, the first of them being equivalent to a transformation formula for (1.7).

Entry 3.3.2. (p. 253) Let α and β be two positive numbers such that $\alpha\beta = \pi^2$. Then

$$\sum_{n=1}^{\infty}\sigma_{-1}(n)\,e^{-2n\alpha} - \sum_{n=1}^{\infty}\sigma_{-1}(n)\,e^{-2n\beta} = \frac{\beta-\alpha}{12} + \frac{1}{4}\log\left(\frac{\alpha}{\beta}\right). \tag{1.9}$$

Finally, Ramanujan uses (1.2) to derive the most famous formula attributed to Koshliakov [17]. Preceding Koshliakov's contributions by more than a decade, Ramanujan states:

Entry 3.3.3 (p. 253): Let α and β denote positive numbers such that $\alpha\beta = \pi^2$. Then the following identity takes place

$$\sqrt{\alpha}\left(\frac{\gamma}{4} - \frac{\log(4\beta)}{4} + \sum_{n=1}^{\infty} d(n)\,K_0(2n\alpha)\right) = \sqrt{\beta}\left(\frac{\gamma}{4} - \frac{\log(4\alpha)}{4} + \sum_{n=1}^{\infty} d(n)\,K_0(2n\beta)\right). \tag{1.10}$$

It is the purpose of the present paper to extend Ramanujan's entries (1.1), (1.2), (1.9) and (1.10). In pursuing (1.2) and (1.9) we were led to the study of generalizations of Watson's formula (1.3), as well as extensions of Kronecker's limit formula (1.6) (see sections 4 and 5 below).

We now explain the nature of the generalizations here proposed. These have to do with a manuscript of N. S. Koshliakov which has recently found a renewed interest. In a very long paper [18], Koshliakov introduced the zeta function [[18], p. 6]

$$\zeta_p(s) = \sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\lambda_n^s}, \quad \text{Re}(s) > 1, \tag{1.11}$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence defined by the positive roots of the transcendental equation

$$\tan(\pi y) = -\frac{y}{p}, \quad p > 0. \tag{1.12}$$

From the fact that $n - \frac{1}{2} < \lambda_n < n$, the series (1.11) converges absolutely in the half-plane $\mathrm{Re}(s) > 1$. In particular, $\zeta_p(s)$ generalizes the Riemann zeta function. From the structure of the equation (1.12), one may see that

$$\lim_{p\to\infty} \zeta_p(s) = \zeta(s), \quad \lim_{p\to 0^+} \zeta_p(s) = (2^s - 1)\,\zeta(s).$$

Koshliakov connected the study of the zeta function with a second generalized zeta function, namely, $\eta_p(s)$, being defined by

$$\eta_p(s) := \sum_{k=1}^{\infty} \frac{(s, 2\pi pk)_k}{k^s}, \quad \mathrm{Re}(s) > 1, \tag{1.13}$$

where

$$(s, \nu k)_k := \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-x} \left(\frac{k\nu - x}{k\nu + x}\right)^k dx. \tag{1.14}$$

On page 20 of his paper [18], it is possible to look at an argument concerning the uniform and absolute convergence of the series (1.13) for $\mathrm{Re}(s) > 1$. This property was also rederived on page 26, formula (57), where it was proved that

$$\lim_{k\to\infty} (s, \lambda k)_k = \frac{1}{\left(1 + \frac{2}{\lambda}\right)^s}, \tag{1.15}$$

for every $\lambda > 0$.

Besides studying the analytic continuations of (1.11) and (1.13), Koshliakov proved that $\zeta_p(s)$ and $\eta_p(s)$ are connected through the functional equation

$$\zeta_p(1-s) = \frac{2\,\cos\left(\frac{\pi s}{2}\right)\Gamma(s)}{(2\pi)^s} \eta_p(s). \tag{1.16}$$

After developing an analytic structure for the functions $\zeta_p(s)$ and $\eta_p(s)$, Koshliakov enters the realm of summation formulas, proving new analogues of Poisson's summation formula, introducing new generalizations of the digamma function and Bernoulli numbers (building on a previous work [16]) and even extending an integral formula of Ramanujan (see [[18], p. 150, eq. (19)], where a generalization of the main formula from Ramanujan's paper [20] is derived).

It is not an overstatement to declare that this manuscript of Koshliakov is truly a masterpiece and unfortunately was kept in obscurity since its publication in 1949. Thanks to the recent outstanding work of A. Dixit and R. Gupta [11], Koshliakov's main results were examined and extended in several directions.

Besides making Koshliakov's ideas accessible to a modern mathematical community, Dixit and Gupta also took the setting of Koshliakov in order to obtain new particular formulas. For example (among many others, which include other relations that can be found in the Lost Notebook) they extended a famous formula of Ramanujan for $\zeta(2n+1)$ [[11], p. 13, Theorem 4.1.] [1] and from this extension they could achieve new identities by taking a suitable choice of the parameter p.

The remarkable work of Dixit and Gupta was then followed by a paper [4] (written jointly with Berndt and Zaharescu) where new analogues of the Abel-Plana formula were considered.

[1] see also [6] for a very nice survey about this formula of Ramanujan.

It is in the line of reasoning of the contributions given in [11] and [4] that our main results are here presented. Our paper is organized as follows: in section 2 we establish a generalization (in Koshliakov's setting) of Ramanujan's formula for $\zeta(1/2)$ (1.1). In sections 3 and 4 we generalize Watson's formula (1.3), as well as the Kronecker limit formula for 'diagonal quadratic forms', (1.8), in two directions.

At last, we devote the fifth section of our paper to the generalization of Entries 3.3.1, 3.3.2 and 3.3.3 stated above.

Before going to the main sections of this paper, we remark some additional facts about the zeta function (1.13), which will be useful in the sequel. Since $\eta_p(s)$ is not a Dirichlet series in the classical sense, we need some particular properties that can be extracted from the Mellin representation (1.14). In fact, Koshliakov [[18], p. 25, eq. (48)] could write (1.14) as a combination of fractional integrals of the form

$$\Gamma(s)\,(s,\lambda)_k=\Gamma(s)(-1)^k+\Gamma(s)\,k^s e^{\lambda}\sum_{\ell=1}^{k}\left(\begin{array}{c}k\\ \ell\end{array}\right)\left(\frac{2\lambda}{k}\right)^{\ell}(-1)^{k-\ell}\int_k^{\infty}t^{-s}e^{-\frac{\lambda}{k}t}\,\frac{(t-k)^{\ell-1}}{(\ell-1)!}\,dt, \tag{1.17}$$

which, when used in the definition (1.13), gives the expression for $\eta_p(s)$

$$\eta_p(s):=\sum_{k=1}^{\infty}\frac{(s,2\pi pk)_k}{k^s}=\left(2^{1-s}-1\right)\zeta(s)+\sum_{k=1}^{\infty}e^{2\pi pk}\sum_{\ell=1}^{k}\left(\begin{array}{c}k\\ \ell\end{array}\right)(4\pi p)^{\ell}\,(-1)^{k-\ell}\int_k^{\infty}t^{-s}e^{-2\pi pt}\,\frac{(t-k)^{\ell-1}}{(\ell-1)!}\,dt, \tag{1.18}$$

being valid when $\text{Re}(s)>1$. Representation (1.18) will prove to be very useful in section 3 of this paper, where two generalizations of Watson's formula (1.3) will be given.

Throughout this paper we shall also employ the notation introduced by Koshliakov: if $p,p'\in\mathbb{R}^{+}$, we define the functions $\sigma(t)$ and $\sigma'(t)$ as being [[18], p. 6]

$$\sigma(t):=\frac{p+t}{p-t},\ \ \sigma'(t):=\frac{p'+t}{p'-t}. \tag{1.19}$$

By [[18], p. 17, eq. (18)], we know that the function $\frac{1}{\sigma(t)\,e^{2\pi t}-1}$ is associated with the analytic continuation of $\eta_p(s)$. The analogue of this function attached to $\zeta_p(s)$ is[2] [[18], p. 44, eq. (33)],

$$\sigma_p(z)=\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,e^{-\lambda_n z},\quad \text{Re}(z)>0. \tag{1.20}$$

For additional standard facts about the zeta functions (1.11) and (1.13) we refer to Koshliakov's own manuscript [18] and to Dixit and Gupta's paper [11].

2 Generalization of Entry 8.3.1., p. 332 of Ramanujan's Lost Notebook

We are now ready to establish a general version of Entry 8.3.1.

[2]In this paper we will try to preserve Koshliakov's notation, even when it differs from standard notation (see (4.14) below). So we urge the reader to not misinterpret $\sigma_p(z)$ as the standard divisor function, $\sigma_k(n)$, which plays an important role in formula (1.2). The only section of the paper where we will need the divisor function will be Section 5, but fortunately in this section we will not use Koshliakov's function $\sigma_p(z)$.

Theorem 2.1. *Let $p, p' \in \mathbb{R}_+$ and λ_n, λ'_n be the sequences of numbers satisfying respectively the transcendental equations*

$$\tan(\pi y) = -\frac{y}{p}, \quad \tan(\pi y) = -\frac{y}{p'}. \tag{2.1}$$

Moreover, let $\sigma(t)$, $\sigma'(t)$ be defined by (1.19). Then, for every $x > 0$, the following identity takes place

$$\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2} \cdot \frac{1}{\sigma'\left(\frac{\lambda_n^2 x}{2\pi}\right) e^{\lambda_n^2 x}-1} = \frac{1}{4\left(1+\frac{1}{\pi p}\right)} + \frac{\pi^2}{6x} \frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\zeta_{p'}\left(\frac{1}{2}\right) + $$
$$+ \frac{1}{2}\sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2} \frac{1}{\sqrt{\lambda'_n}} G_p\left(\frac{2\pi\lambda'_n}{x}\right), \tag{2.2}$$

where, for $a > 0$, $G_p(a)$ is explicitly given by

$$G_p(a) = \frac{\left(p^2-a\right)\left\{\cos(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})\right\} - e^{-\pi\sqrt{2a}}\left(p^2-\sqrt{2a}\,p+a\right) - \sqrt{2a}\,p\left\{\cos(\pi\sqrt{2a})+\sin(\pi\sqrt{2a})\right\}}{p^2\left(\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})\right)+\sqrt{2a}\,p\left(\sinh(\pi\sqrt{2a})+\sin(\pi\sqrt{2a})\right)+a\left(\cosh(\pi\sqrt{2a})+\cos(\pi\sqrt{2a})\right)}. \tag{2.3}$$

Proof. By [[18], p. 32, Chapter 1, eq. (74)], for $\mathrm{Re}(s) > 1$ one has the representation

$$\zeta_{p'}(1-s) = 2\cos\left(\frac{\pi s}{2}\right) \int_0^{\infty} \frac{x^{s-1}}{\sigma'(x)\,e^{2\pi x}-1}\, dx. \tag{2.4}$$

Therefore, from Mellin's inversion formula and the absolute convergence of the series (1.11) for $\mathrm{Re}(s) > \mu > 1$,

$$\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2} \cdot \frac{1}{\sigma'\left(\frac{\lambda_n^2 x}{2\pi}\right) e^{\lambda_n^2 x}-1} = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \frac{\zeta_{p'}(1-s)\,\zeta_p(2s)}{2\cos\left(\frac{\pi s}{2}\right)} \left(\frac{x}{2\pi}\right)^{-s} ds. \tag{2.5}$$

Now we shift the line of integration in (2.5) to $\mathrm{Re}(s) = \frac{1}{2}-\mu$. To do this we just need to integrate along a positively oriented rectangular contour $\mathscr{R}_\mu(T)$ with vertices $\mu \pm iT$ and $\frac{1}{2}-\mu\pm iT$: by an application of Cauchy's residue Theorem, we have that

$$\left\{\int_{\mu-iT}^{\mu+iT} + \int_{\mu+iT}^{\frac{1}{2}-\mu+iT} + \int_{\frac{1}{2}-\mu+iT}^{\frac{1}{2}-\mu-iT} + \int_{\frac{1}{2}-\mu-iT}^{\mu-iT}\right\} \frac{\zeta_{p'}(1-s)\,\zeta_p(2s)}{2\cos\left(\frac{\pi s}{2}\right)} \left(\frac{x}{2\pi}\right)^{-s} ds = 2\pi i\, R_{p,p'}(x), \tag{2.6}$$

where $R_{p,p'}(x)$ denotes the sum of the residues of the integrand inside $\mathscr{R}_\mu(T)$. Since $\zeta_p(-2k) = 0$ for every $k \in \mathbb{N}$ [[18], p. 22, eq. (37)], we know that the points $s = 2k+1$, $k \in \mathbb{Z}\setminus\{0\}$ are removable singularities of the integrated function. Thus, the integrand above has only three simple poles, which are located at the points $s = 0, \frac{1}{2}, 1$. It is simple to see that the residual function $R_{p,p'}(x)$ can be explicitly written as

$$R_{p,p'}(x) = \frac{1}{4\left(1+\frac{1}{\pi p}\right)} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta_{p'}\left(\frac{1}{2}\right) + \frac{\pi^2}{6x} \frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2}, \tag{2.7}$$

because $\zeta_p(0) = -\frac{1}{2}\,\frac{1}{1+\frac{1}{\pi p}}$ and $\zeta_p(2) = \frac{\pi^2}{6}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{(1+\frac{1}{\pi p})^2}$ [[18], p. 22, eq. (34), (39)].

We now estimate in an elementary way the integrals along the horizontal segments $\left[\frac{1}{2}-\mu\pm iT,\, \mu\pm iT\right]$ when $T\to\infty$: indeed,

$$\int_{\frac{1}{2}-\mu\pm iT}^{\mu\pm iT} \left|\frac{\zeta_p(1-s)\,\zeta_p(2s)}{2\cos\left(\frac{\pi s}{2}\right)} \left(\frac{x}{2\pi}\right)^{-s}\right| |ds| \le \frac{T^A e^{-\frac{\pi}{2}T}}{\log\left(\frac{x}{2\pi}\right)} \left[\left(\frac{x}{2\pi}\right)^{\mu-\frac{1}{2}} - \left(\frac{x}{2\pi}\right)^{-\mu}\right] \to 0 \tag{2.8}$$

as $T \to \infty$. Here, $A := A(\mu)$ is a positive number depending on μ: we can write it explicitly by using the convex estimates for $\zeta_p(s)$ found by Koshliakov [[18], p. 24, eq. (44)],

$$\zeta_p(\sigma+it) = \begin{cases} O\left(|t|^{1-\sigma}\log|t|\right) & \frac{1}{2}\leq\sigma\leq 1, \\ O\left(|t|^{\frac{1}{2}}\log|t|\right) & 0<\sigma\leq\frac{1}{2}, \quad |t|\to\infty. \\ O\left(|t|^{\frac{1}{2}-\sigma}\log|t|\right) & \sigma\leq 0. \end{cases} \tag{2.9}$$

Combining (2.5) with (2.6) and using the previous bounds (2.8) for the integrals along the horizontal segments, we are able to obtain

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\cdot\frac{1}{\sigma'\left(\frac{\lambda_n^2 x}{2\pi}\right)e^{\lambda_n^2 x}-1} = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\zeta_p(1-2s)\,\zeta_{p'}(s+\frac{1}{2})}{\sqrt{2}\left[\cos\left(\frac{\pi s}{2}\right)+\sin\left(\frac{\pi s}{2}\right)\right]}\left(\frac{x}{2\pi}\right)^{s-\frac{1}{2}}ds+$$
$$+\frac{1}{4\left(1+\frac{1}{\pi p}\right)}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta_{p'}\left(\frac{1}{2}\right)+\frac{\pi^2}{6x}\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2}, \tag{2.10}$$

where in the last step we took the change of variables $s \leftrightarrow \frac{1}{2}-s$ and the explicit expression (2.7). Since we have chosen $\mu>1$, we can use the representation of $\zeta_{p'}\left(s+\frac{1}{2}\right)$ as a Dirichlet series to write the integral on the right side of (2.10) as

$$\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\sqrt{\frac{\pi}{\lambda_n' x}}\,\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\zeta_p(1-2s)}{\cos\left(\frac{\pi s}{2}\right)+\sin\left(\frac{\pi s}{2}\right)}\left(\frac{x}{2\pi\lambda_n'}\right)^{s}ds. \tag{2.11}$$

Finally, we evaluate the remaining integral by using the functional equation for $\zeta_p(s)$ (1.16) and making some elementary reductions, from which we obtain the expression

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\zeta_p(1-2s)}{\cos\left(\frac{\pi s}{2}\right)+\sin\left(\frac{\pi s}{2}\right)}\left(\frac{x}{2\pi\lambda_n'}\right)^{s}ds = \sum_{m=1}^{\infty}\frac{1}{2\pi i}\int_{2\mu-i\infty}^{2\mu+i\infty}\left\{\cos\left(\frac{\pi s}{4}\right)-\sin\left(\frac{\pi s}{4}\right)\right\}\Gamma(s)\,(s,\,2\pi pm)_m\left(\frac{x}{8\pi^3 m^2\lambda_n'}\right)^{s/2}ds \tag{2.12}$$

in its turn being justified by the absolute convergence of $\eta_p(s)$ in the half-plane $\mathrm{Re}(s)>2\mu$. Combining (1.17) with Stirling's formula for $\Gamma(s)$, we have for $s=\sigma+it$, $a\leq\sigma\leq b$ and $z\in\mathbb{C}$,

$$\left|\Gamma(s)\,(s,\lambda)_k\,z^{-s}\right| \leq e^{-\sigma\log|z|}\,C(\sigma,k,\lambda)\,|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|+\arg(z)\,t}, \quad |t|\to\infty,$$

where $C(\sigma,k,\lambda)$ is a positive constant only depending on these fixed parameters.

Since $\Gamma(s)\,(s,\lambda)_k\,z^{-s}$ defines an analytic function for $\mathrm{Re}(s)>0$, we have from (1.14), Mellin's inversion formula and analytic continuation with respect to z in the half-plane $\mathrm{Re}(z)>0$, that the following integral representation is valid

$$f_k(z,\lambda) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\,(s,\lambda)_k\,z^{-s}ds = \left(\frac{\lambda-z}{\lambda+z}\right)^k e^{-z}, \quad \sigma>0,\ \mathrm{Re}(z)>0. \tag{2.13}$$

Applying (2.11) and using (2.12) and (2.13), we immediately obtain

$$\sum_{m=1}^{\infty}\frac{1}{2\pi i}\int_{2\mu-i\infty}^{2\mu+i\infty}\left\{\cos\left(\frac{\pi s}{4}\right)-\sin\left(\frac{\pi s}{4}\right)\right\}\Gamma(s)\ (s,\,2\pi p m)_m\left(\frac{x}{8\pi^3 m^2\lambda_n'}\right)^{s/2}=$$

$$=\sqrt{2}\,\mathrm{Im}\left\{e^{i\frac{\pi}{4}}\sum_{m=1}^{\infty}\exp\left(-\sqrt{\frac{8\pi^3 m^2\lambda_n'}{x}}\,e^{i\frac{\pi}{4}}\right)\left(\frac{p-\sqrt{\frac{2\pi\lambda_n'}{x}}\,e^{i\frac{\pi}{4}}}{p+\sqrt{\frac{2\pi\lambda_n'}{x}}\,e^{i\frac{\pi}{4}}}\right)^m\right\}$$

$$=\sqrt{2}\,\mathrm{Im}\left\{\frac{e^{i\frac{\pi}{4}}}{\sigma\left(\sqrt{\frac{2\pi\lambda_n'}{x}}\,e^{i\frac{\pi}{4}}\right)\,e^{2\pi\sqrt{\frac{2\pi\lambda_n'}{x}}\,e^{i\frac{\pi}{4}}}-1}\right\},\tag{2.14}$$

where in the last step we have used the geometric series (recall once more (1.19))

$$\frac{1}{\sigma(z)\,e^{2\pi z}-1}=\sum_{m=1}^{\infty}\left(\frac{p-z}{p+z}\right)^m e^{-2\pi m z},\quad \mathrm{Re}(z)>0.\tag{2.15}$$

Returning to (2.10) and to (2.11) and collecting (2.14), we are able to derive the identity

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\cdot\frac{1}{\sigma'\left(\frac{\lambda_n^2 x}{2\pi}\right)e^{\lambda_n^2 x}-1}=\frac{1}{4\left(1+\frac{1}{\pi p}\right)}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta_{p'}\left(\frac{1}{2}\right)+\frac{\pi^2}{6x}\,\frac{1+\frac{3}{\pi p}\left(1+\frac{1}{\pi p}\right)}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2}+$$

$$+\sqrt{\frac{2\pi}{x}}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\frac{1}{\sqrt{\lambda_n'}}\,\mathrm{Im}\left\{\frac{e^{i\frac{\pi}{4}}}{\sigma\left(\sqrt{\frac{2\pi\lambda_n'}{x}}\,e^{i\frac{\pi}{4}}\right)\,e^{2\pi\sqrt{\frac{2\pi\lambda_n'}{x}}\,e^{i\frac{\pi}{4}}}-1}\right\}.\tag{2.16}$$

Now our proof is almost finished, with the only step remaining being to simplify the imaginary part of the term in the braces. This comes after straightforward manipulations: it is very simple to show that the denominator inside the braces above can be simplified to ($a:=\sqrt{2\pi\lambda_n'/x}$)

$$\frac{e^{\pi\sqrt{2a}}}{p^2-\sqrt{2a}\,p+a}\left\{\cos\left(\pi\sqrt{2a}\right)(p^2-a)-\sqrt{2a}\,p\,\sin\left(\pi\sqrt{2a}\right)+i\left[\sqrt{2a}\,p\,\cos\left(\pi\sqrt{2a}\right)+(p^2-a)\,\sin\left(\pi\sqrt{2a}\right)\right]\right\}-1,\tag{2.17}$$

and from this it is very simple to deduce the expression for $\left|\sigma\left(\sqrt{a}\,e^{i\frac{\pi}{4}}\right)\,e^{2\pi\sqrt{a}\,e^{i\frac{\pi}{4}}}-1\right|^2$ as

$$\frac{e^{2\pi\sqrt{2a}}}{(p^2-\sqrt{2a}\,p+a)^2}\left(p^4+a^2\right)-2\,\frac{e^{\pi\sqrt{2a}}}{p^2-\sqrt{2a}\,p+a}\left\{\cos(\pi\sqrt{2a})\,(p^2-a)-\sqrt{2a}\,p\,\sin(\pi\sqrt{2a})\right\}+1\tag{2.18}$$

$$=\frac{e^{2\pi\sqrt{2a}}\left(p^2+\sqrt{2a}\,p+a\right)}{p^2-\sqrt{2a}\,p+a}-2\,\frac{e^{\pi\sqrt{2a}}}{p^2-\sqrt{2a}\,p+a}\left\{\cos(\pi\sqrt{2a})\,(p^2-a)-\sqrt{2a}\,p\,\sin(\pi\sqrt{2a})\right\}+1\tag{2.19}$$

Taking the conjugate of (2.17) and using (2.18), we have after simple reductions that $F_p(a)$ can be written as

$$F_p(a):=\mathrm{Im}\left\{\frac{e^{i\frac{\pi}{4}}}{\sigma\left(\sqrt{a}\,e^{i\frac{\pi}{4}}\right)\,e^{2\pi\sqrt{a}\,e^{i\frac{\pi}{4}}}-1}\right\}=\frac{G_p(a)}{2\sqrt{2}},$$

where $G_p(a)$ is given by (2.3). This completes the proof of Ramanujan's formula. □

We now derive particular cases of Ramanujan-type formulas, recovering some results previously known but also gaining new insight. Since the proofs of all the corollaries are nothing but a specialization of (2.2) as p or p' tend to 0^+ or ∞, we will omit them.

We start with the following result, which can be obtained by letting $p'\to\infty$ in the previous Theorem. It provides an infinite number of representations for $\zeta(1/2)$.

Corollary 2.1. *For every $p \in \mathbb{R}^+$ and $x > 0$, we have the infinitely many representations for $\zeta(1/2)$,*

$$\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2} \frac{1}{e^{\lambda_n^2 x}-1} = \frac{1}{4\left(1+\frac{1}{\pi p}\right)} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta\left(\frac{1}{2}\right) + \frac{\pi^2}{6x} \frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p}\right)^2} + \frac{1}{2}\sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \times$$

$$\times \frac{\left(p^2-\frac{2\pi n}{x}\right)\left\{\cos(2\pi\sqrt{\frac{\pi n}{x}})-\sin(2\pi\sqrt{\frac{\pi n}{x}})\right\} - e^{-2\pi\sqrt{\frac{\pi n}{x}}}\left(p^2-2\sqrt{\frac{\pi n}{x}}\,p+\frac{2\pi n}{x}\right) - 2\sqrt{\frac{\pi n}{x}}\,p\left\{\cos(2\pi\sqrt{\frac{\pi n}{x}})+\sin(2\pi\sqrt{\frac{\pi n}{x}})\right\}}{p^2\left(\cosh(2\pi\sqrt{\frac{\pi n}{x}})-\cos(2\pi\sqrt{\frac{\pi n}{x}})\right)+2\sqrt{\frac{\pi n}{x}}\,p\left(\sinh(2\pi\sqrt{\frac{\pi n}{x}})+\sin(2\pi\sqrt{\frac{\pi n}{x}})\right)+\frac{2\pi n}{x}\left(\cosh(2\pi\sqrt{\frac{\pi n}{x}})+\cos(2\pi\sqrt{\frac{\pi n}{x}})\right)}.$$

In particular, Ramanujan's formula (1.1) holds, together with

$$\sum_{n=1}^{\infty} \frac{1}{e^{(2n-1)^2 x}-1} = \frac{1}{4}\sqrt{\frac{\pi}{x}}\,\zeta\left(\frac{1}{2}\right) + \frac{\pi^2}{8x} - \frac{1}{4}\sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \frac{\cos\left(\pi\sqrt{\frac{\pi n}{x}}\right)-\sin\left(\pi\sqrt{\frac{\pi n}{x}}\right)+e^{-\pi\sqrt{\frac{\pi n}{x}}}}{\cosh\left(\pi\sqrt{\frac{\pi n}{x}}\right)+\cos\left(\pi\sqrt{\frac{\pi n}{x}}\right)}, \quad x>0. \quad (2.20)$$

If, instead, we take $p \to \infty$ and fix p', we are able to derive the result.

Corollary 2.2. *For every $p' \in \mathbb{R}^+$ and any $x > 0$, the following identity of Ramanujan type holds*

$$\sum_{n=1}^{\infty} \frac{1}{\sigma'\left(\frac{n^2 x}{2\pi}\right) e^{n^2 x}-1} = \frac{1}{4} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta_{p'}\left(\frac{1}{2}\right) + \frac{\pi^2}{6x}\frac{1}{1+\frac{1}{\pi p'}} +$$

$$+\frac{1}{2}\sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2} \frac{1}{\sqrt{\lambda_n'}} \cdot \left\{\frac{\cos\left(2\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)-\sin\left(2\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)-e^{-2\pi\sqrt{\frac{\pi\lambda_n'}{x}}}}{\cosh\left(2\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)-\cos\left(2\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)}\right\}.$$

In particular,

$$-\sum_{n=1}^{\infty} \frac{1}{e^{n^2 x}+1} = \frac{1}{4}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\left(\sqrt{2}-1\right)\zeta\left(\frac{1}{2}\right)+\sqrt{\frac{\pi}{2x}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} \cdot \left\{\frac{\cos\left(\pi\sqrt{\frac{\pi(4n-2)}{x}}\right)-\sin\left(\pi\sqrt{\frac{\pi(4n-2)}{x}}\right)-e^{-\pi\sqrt{\frac{\pi(4n-2)}{x}}}}{\cosh\left(\pi\sqrt{\frac{\pi(4n-2)}{x}}\right)-\cos\left(\pi\sqrt{\frac{\pi(4n-2)}{x}}\right)}\right\}.$$

If we let $p' \to 0^+$ and fix $p \in \mathbb{R}^+$ in our Theorem 2.1, we are still able to find alternative representations for $\zeta\left(\frac{1}{2}\right)$. This gives the following Corollary.

Corollary 2.3. *For every $p \in \mathbb{R}^+$ and $x > 0$, we have the representations for $\zeta(1/2)$,*

$$-\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2} \cdot \frac{1}{e^{\lambda_n^2 x}+1} = \frac{1}{4\left(1+\frac{1}{\pi p}\right)} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\left(\sqrt{2}-1\right)\zeta\left(\frac{1}{2}\right) + \sqrt{\frac{\pi}{2x}} \sum_{n=1}^{\infty} \frac{G_p\left(\frac{\pi(2n-1)}{x}\right)}{\sqrt{2n-1}},$$

where $G_p(a)$ is given by (2.3). In particular, the following relation takes place

$$\sum_{n=1}^{\infty} \frac{1}{e^{(2n-1)^2 x}+1} = \frac{1}{4}\sqrt{\frac{\pi}{x}}\left(1-\sqrt{2}\right)\zeta\left(\frac{1}{2}\right)+\frac{1}{4}\sqrt{\frac{2\pi}{x}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} \cdot \frac{\cos\left(\pi\sqrt{\frac{\pi(2n-1)}{2x}}\right)-\sin\left(\pi\sqrt{\frac{\pi(2n-1)}{2x}}\right)+e^{-\pi\sqrt{\frac{\pi(2n-1)}{2x}}}}{\cosh\left(\pi\sqrt{\frac{\pi(2n-1)}{2x}}\right)+\cos\left(\pi\sqrt{\frac{\pi(2n-1)}{2x}}\right)}.$$

Finally, we state the corollary that we obtain once we take $p \to 0^+$ in Theorem 2.1 above.

Corollary 2.4. *For every $p' \in \mathbb{R}^+$ and $x > 0$, the following identity takes place*

$$\sum_{n=1}^{\infty} \frac{1}{\sigma'\left(\frac{(2n-1)^2 x}{2\pi}\right) e^{(2n-1)^2 x}-1} = \frac{\pi^2}{8x} \cdot \frac{1}{1+\frac{1}{\pi p'}} + \frac{1}{4}\sqrt{\frac{\pi}{x}}\zeta_{p'}\left(\frac{1}{2}\right) +$$

$$+\frac{1}{4}\sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2} \cdot \frac{1}{\sqrt{\lambda_n'}} \cdot \frac{\sin\left(\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)-\cos\left(\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)-e^{-\pi\sqrt{\frac{\pi\lambda_n'}{x}}}}{\cosh\left(\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)+\cos\left(\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)}.$$

3 A Generalization of Watson's formula

3.1 First Analogue of Watson's Formula

In order to generalize Watson's result (1.3) in Koshliakov's setting, we first need to generalize the series on the left side of (1.3). For $\mathrm{Re}(x)>0$ and $\mathrm{Re}(s)>\frac{1}{2}$, we shall consider

$$\varphi_p(s,x):=\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s},\quad \mathrm{Re}(s)>\frac{1}{2}. \tag{3.1}$$

This constitutes an analogue the infinite series appearing in (1.3), which is obtained in the limiting case $p\to\infty$. We now extend Watson's formula.

Theorem 3.1. *Let $x>0$ and $Re(s)>\frac{1}{2}$. Then the following generalization of Watson's formula (1.3) takes place*

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s}=\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)-\frac{1}{2}\,\frac{x^{-2s}}{1+\frac{1}{\pi p}}+\frac{2\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{m=1}^{\infty}(-1)^m m^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi m x)+$$
$$+\frac{2\,\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{m=1}^{\infty}e^{2\pi p m}m^{s-\frac{1}{2}}\sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{m-\ell}\left(4\pi m p\right)^{\ell}\int_1^{\infty}t^{s-\frac{1}{2}}\,e^{-2\pi m p t}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,K_{s-\frac{1}{2}}\left(2\pi x m t\right)\,dt. \tag{3.2}$$

In particular, for $\frac{1}{2}<Re(s)<1$, one has the integral representation

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s}=\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)-\frac{1}{2}\,\frac{x^{-2s}}{1+\frac{1}{\pi p}}+$$
$$+2^{2-2s}\,x^{1-2s}\sin(\pi s)\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left((2y+1)\,x\right)e^{2\pi(2y+1)x}-1}dy, \tag{3.3}$$

with the right-hand side of the previous expression being the analytic continuation of the series (3.1) to the half-plane $Re(s)<\frac{1}{2}$

Proof. Let $\mu>\frac{1}{2}$ and consider a fixed $s\in\mathbb{C}$ such that $\mathrm{Re}(s)>\mu$. Under this condition, the following integral representation holds [[12], p. 348, eq. 7.3.15]

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,x^{-2z}dz=\frac{\Gamma(s)}{(1+x^2)^s},\quad x>0. \tag{3.4}$$

Using this integral representation in the series (3.1), we have that

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s}=\frac{x^{-2s}}{\Gamma(s)}\,\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,\zeta_p(2z)\,x^{2z}dz \tag{3.5}$$

where the interchange of the infinite series with the contour integral (3.4) is justified by absolute convergence of $\zeta_p(2z)$ in the half-plane $\mathrm{Re}(z)>\mu>\frac{1}{2}$ and Stirling's formula $|\,|\Gamma\left(\mu+it\right)\Gamma\left(s-\mu-it\right)|=O\left(|t|^{\mathrm{Re}(s)-1}e^{-\pi|t|}\right)$, as $|t|\to\infty$.

As in the proof of Theorem 2.1., let us now move the line of integration to $\mathrm{Re}(z)=\frac{1}{2}-\mu$ and integrate along a positively oriented rectangular contour $\mathscr{R}_{\mu}(T)$ containing the vertices $\mu\pm iT$ and $\frac{1}{2}-\mu\pm iT$, $T>0$. Due to the trivial zeros of $\zeta_p(2z)$ and the choice $\mathrm{Re}(s)>\mu>\frac{1}{2}$, the only singularities of the integrand inside $\mathscr{R}_{\mu}(T)$ are located at $z=\frac{1}{2}$ and at $z=0$. Like in (2.8), we can estimate trivially the integrals along the horizontal segments $\left[\frac{1}{2}-\mu\pm iT,\,\mu\pm iT\right]$ when $T\to\infty$ and we can show that they vanish. An application of Cauchy's residue theorem gives

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,\zeta_p(2z)\,x^{2z}dz=\frac{1}{2\pi i}\int_{\frac{1}{2}-\mu-i\infty}^{\frac{1}{2}-\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,\zeta_p(2z)\,x^{2z}dz-\frac{1}{2}\,\frac{\Gamma(s)}{1+\frac{1}{\pi p}}+\frac{\sqrt{\pi}\,x}{2}\Gamma\left(s-\frac{1}{2}\right).\tag{3.6}$$

Invoking the functional equation for $\zeta_p(z)$ (1.16), we can simplify the integral on the right-hand side of (3.6) in the form

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-\mu-i\infty}^{\frac{1}{2}-\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,\zeta_p(2z)\,x^{2z}dz=\sqrt{\pi}\,x\sum_{m=1}^{\infty}\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(z\right)\,\Gamma\left(s+z-\frac{1}{2}\right)\,(2z,2\pi pm)_m\,(\pi xm)^{-2z}\,dz,$$

because the representation of $\eta_p(2z)$, (1.13), converges absolutely in the half-plane $\mathrm{Re}(z)>\frac{1}{2}$.

For each fixed $m\in\mathbb{N}$, we now evaluate the integral

$$I_{m,p}(s,x):=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}(2z,2\pi pm)_m\,\Gamma\left(z\right)\,\Gamma\left(s+z-\frac{1}{2}\right)\,(\pi\,x\,m)^{-2z}\,dz.$$

This is possible by using representation (1.17), which gives

$$I_{m,p}(s,x)=I_{m,p}^{(1)}(s,x)+I_{m,p}^{(2)}(s,x),$$

with

$$I_{m,p}^{(1)}(s,x)=\frac{(-1)^m}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(z\right)\,\Gamma\left(s+z-\frac{1}{2}\right)\,(\pi\,x\,m)^{-2z}\,dz=2\,(-1)^m\,(\pi x\,m)^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}\left(2\pi x\,m\right)\tag{3.7}$$

and

$$\begin{aligned}I_{m,p}^{(2)}(s,x)&=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}e^{2\pi pm}\sum_{\ell=1}^{m}\binom{m}{\ell}(4\pi p)^{\ell}\,(-1)^{m-\ell}\int_{m}^{\infty}t^{-2z}e^{-2\pi p\,t}\,\frac{(t-m)^{\ell-1}}{(\ell-1)!}\,\Gamma\left(z\right)\,\Gamma\left(s+z-\frac{1}{2}\right)\,(\pi x)^{-2z}\,dt\,dz\\&=e^{2\pi pm}\sum_{\ell=1}^{m}\binom{m}{\ell}(4\pi p)^{\ell}\,(-1)^{m-\ell}\int_{m}^{\infty}e^{-2\pi pt}\,\frac{(t-m)^{\ell-1}}{(\ell-1)!}\,\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(z\right)\,\Gamma\left(s+z-\frac{1}{2}\right)\,(\pi xt)^{-2z}\,dz\,dt\\&=2\,(\pi x)^{s-\frac{1}{2}}e^{2\pi pm}m^{s-\frac{1}{2}}\sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{m-\ell}\,(4\pi pm)^{\ell}\int_{1}^{\infty}t^{s-\frac{1}{2}}\,e^{-2\pi mpt}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,K_{s-\frac{1}{2}}\left(2\pi xmt\right)\,dt,\end{aligned}$$

where the interchange of the operations is possible due to absolute convergence, which in its turn is justified by Stirling's formula for the product $\Gamma(z)\,\Gamma(s+z-\frac{1}{2})$. On the third equality above, as well as in (3.7), we have used the well-known integral representation for the Macdonald function [[12], p. 349, 7.3 (17)]

$$K_{\nu}(x)=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}2^{s-2}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right)\,x^{-s}ds,\quad x>0,\ \ \mu:=\mathrm{Re}(s)>\max\{0,\mathrm{Re}(\nu)\}.\tag{3.8}$$

Combining all the expressions given above yields the formula

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s}&=\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)-\frac{1}{2}\,\frac{x^{-2s}}{1+\frac{1}{\pi p}}+\frac{2\pi^s\,x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{m=1}^{\infty}(-1)^m\,m^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}\left(2\pi xm\right)\\&+\frac{2\,\pi^s\,x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{m=1}^{\infty}e^{2\pi pm}m^{s-\frac{1}{2}}\sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{m-\ell}\,(4\pi mp)^{\ell}\int_{1}^{\infty}t^{s-\frac{1}{2}}\,e^{-2\pi mpt}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,K_{s-\frac{1}{2}}\left(2\pi xmt\right)\,dt.\end{aligned}\tag{3.9}$$

Although (3.9) constitutes already a generalization of the classical case (1.3), we will now write the latter series in a more elegant form, which is well-defined in the half-plane $\text{Re}(s) \leq \frac{1}{2}$. We now claim that the right-hand side of (3.9) constitutes the analytic continuation of the series $\varphi_p(s,x)$, (3.1), to the entire complex plane. We prove this claim by showing that the series on the right defines an entire function of $s \in \mathbb{C}$. Indeed, it is enough to check this for any s in the half-plane $\text{Re}(s) \leq \frac{1}{2}$ because for $\text{Re}(s) > \frac{1}{2}$, the absolute convergence of the left-hand side of (3.9) already assures this.

From now on, we analyze the infinite series on the left of (3.9) under the hypothesis that $\text{Re}(s) < 1$, which obviously includes the remaining case $\text{Re}(s) \leq \frac{1}{2}$. To that end, recall the well-known integral formula [[19], p. 252, relation 10.32.8]

$$K_{\nu}(z) = \frac{\sqrt{\pi} z^{\nu}}{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)} \int_1^{\infty} e^{-zt}\left(t^2-1\right)^{\nu-\frac{1}{2}} dt, \quad \text{Re}(\nu) > -\frac{1}{2}, \quad |\arg(z)| < \frac{\pi}{2}. \tag{3.10}$$

From the fact that $K_{-\nu}(z) = K_{\nu}(z)$, (3.10) can be rewritten in the equivalent form

$$\left(\frac{z}{2}\right)^{\nu} K_{\nu}(z) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\nu\right)} \int_1^{\infty} e^{-zt}\left(t^2-1\right)^{-\nu-\frac{1}{2}} dt, \ \text{Re}(\nu) < \frac{1}{2}, \ |\arg(z)| < \frac{\pi}{2}, \tag{3.11}$$

that can be simplified to

$$\left(\frac{z}{2}\right)^{\nu} K_{\nu}(z) = \frac{2^{-2\nu}\sqrt{\pi}\, e^{-z}}{\Gamma\left(\frac{1}{2}-\nu\right)} \int_0^{\infty} e^{-2zt} t^{-\nu-\frac{1}{2}}(t+1)^{-\nu-\frac{1}{2}} dt, \ \text{Re}(\nu) < \frac{1}{2}, \ |\arg(z)| < \frac{\pi}{2}. \tag{3.12}$$

Assuming that s lies in the region $\text{Re}(s) \leq \frac{1}{2}$ and using (3.12), we see that the fractional integral appearing on the right-hand side of (3.9) can be simplified in the following manner,

$$J_{p,\ell}(m,x) := \int_1^{\infty} t^{s-\frac{1}{2}}\, e^{-2\pi mpt}\, \frac{(t-1)^{\ell-1}}{(\ell-1)!}\, K_{s-\frac{1}{2}}(2\pi xmt)\, dt$$

$$= \frac{2^{1-2s}\sqrt{\pi}}{\Gamma(1-s)} \,(\pi xm)^{\frac{1}{2}-s} \int_0^{\infty} \exp\left(-2\pi m\left(p+(2y+1)\,x\right)\right)\left(2\pi m\left(p+(2y+1)\,x\right)\right)^{-\ell} \frac{dy}{y^s\,(y+1)^s}$$

where the interchange of the orders of integration comes from the absolute convergence of both integrals under the hypothesis $\text{Re}(s) \leq \frac{1}{2} < 1$. Note now that the integral in the last expression converges absolutely, which allows us to take the finite sum over $\ell \in \{0, ..., m\}$, resulting in

$$\sum_{m=1}^{\infty} e^{2\pi pm} m^{s-\frac{1}{2}} \sum_{\ell=1}^{m} \binom{m}{\ell} (-1)^{m-\ell}\,(4\pi mp)^{\ell}\, J_{p,\ell}(m,x) =$$

$$= \frac{2^{1-2s}\,\pi^{1-s} x^{\frac{1}{2}-s}}{\Gamma(1-s)} \sum_{m=1}^{\infty} e^{2\pi pm} \int_0^{\infty} \left\{\left(\frac{p-(2y+1)x}{p+(2y+1)x}\right)^m + (-1)^{m-1}\right\} \frac{e^{-2\pi m(p+(2y+1)x)}}{y^s\,(y+1)^s}\, dy$$

$$= \frac{2^{1-2s}\,\pi^{1-s} x^{\frac{1}{2}-s}}{\Gamma(1-s)} \int_0^{\infty} \frac{y^{-s}(y+1)^{-s}}{\sigma\left((2y+1)\,x\right) e^{2\pi(2y+1)x}-1} dy + \frac{2^{1-2s}\,\pi^{1-s} x^{\frac{1}{2}-s}}{\Gamma(1-s)} \sum_{m=1}^{\infty} (-1)^{m-1} \int_0^{\infty} \frac{e^{-2\pi m(2y+1)x}}{y^s(y+1)^s}\, dy \tag{3.13}$$

where in the last step we have used once more absolute convergence and the geometric series (2.15). Clearly, the former integral converges absolutely and uniformly for every s in the half-plane $\text{Re}(s) < 1$, as the elementary bound shows

$$\left|\int_0^{\infty} \frac{y^{-s}(y+1)^{-s}}{\sigma\left((2y+1)\,x\right) e^{2\pi(2y+1)x}-1}\, dy\right| \leq \int_0^{\infty} \frac{y^{-\sigma}(y+1)^{-\sigma}}{e^{2\pi(2y+1)x}-1} dy. \tag{3.14}$$

This proves that the infinite series on the right of (3.9) defines an entire function of $s \in \mathbb{C}$ [[28], p. 92] and then (3.2) constitutes the analytic continuation of the series to the half-plane $\text{Re}(s) < \frac{1}{2}$. Finally, recalling once more (3.12), we note that the second series on the last equation of (3.13) is precisely the symmetric of the infinite series appearing on the top of the right-hand side of (3.9). Joining (3.13) with (3.9) leads to (3.3) as an equivalent continuation to $\text{Re}(s) < \frac{1}{2}$. □

Remark 3.1. It is also possible to write the series on the right-hand side (3.2) as involving integrals of the Whittaker function. Expressions like this are also helpful to have a grasp on the analytic continuation of the series (3.1). We could also replace the integral in (3.3) by an integral with respect to a Hankel contour. This would allow a representation valid for every $s \in \mathbb{C}$. For our purposes, however, it suffices to use the above representation (3.3).

By letting $p \to \infty$ in (3.2) or in (3.3), we can recover Watson's formula (1.3) and by letting $p \to 0^{+}$, we can derive an analogue of it. We remark that our next formula (3.15) could be also achieved by the general methods given in [[22], Example 5.1, eq. (5.2)].

Corollary 3.1. *Let s be any complex number such that $Re(s) > \frac{1}{2}$ and $x > 0$. Then Watson's formula (1.3) holds. Moreover, one has the companion identity*

$$\sum_{n=1}^{\infty} \frac{1}{\left((2n-1)^2 + x^2\right)^s} = \frac{\sqrt{\pi}\, x^{1-2s}}{4\Gamma(s)}\Gamma\left(s - \frac{1}{2}\right) + \frac{2^{\frac{1}{2}-s}\, x^{\frac{1}{2}-s}\, \pi^s}{\Gamma(s)} \sum_{m=1}^{\infty} (-1)^m m^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(\pi x m). \tag{3.15}$$

Proof. Since the proofs of (3.15) and (1.3) are analogous, we will only see that (3.3) reduces to (1.3) as $p \to \infty$. For this, it suffices to analyze the continuation of the integral in (3.3) to the region $\text{Re}(s) > 1$ in this limit. Assume that $\text{Re}(s) < 1$: by an application of the dominated convergence Theorem, we can take $p \to \infty$ on the right-hand side of (3.3), giving

$$\begin{aligned} &\frac{\sqrt{\pi}\, x^{1-2s}}{2\Gamma(s)}\Gamma\left(s - \frac{1}{2}\right) - \frac{x^{-2s}}{2} + 2^{1-2s}\, x^{1-2s} \sin(\pi s) \int_0^{\infty} \frac{y^{-s}(y+1)^{-s}}{e^{2\pi(2y+1)x} - 1} dy \\ &= \frac{\sqrt{\pi}\, x^{1-2s}}{2\Gamma(s)}\Gamma\left(s - \frac{1}{2}\right) - \frac{x^{-2s}}{2} + 2^{1-2s}\, x^{1-2s} \sin(\pi s) \sum_{n=1}^{\infty} e^{-2\pi n x} \int_0^{\infty} y^{-s}(y+1)^{-s} e^{-4\pi n x\, y} dy \\ &= \frac{\sqrt{\pi}\, x^{1-2s}}{2\Gamma(s)}\Gamma\left(s - \frac{1}{2}\right) - \frac{x^{-2s}}{2} + \frac{\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\, K_{s-\frac{1}{2}}(2\pi n x), \quad \text{Re}(s) < \frac{1}{2}, \end{aligned} \tag{3.16}$$

where the second step is justified by invoking the geometric series and arguing by absolute convergence (recall that $K_\nu(x) = O(e^{-x}/\sqrt{x}),\ x \to \infty$) and the last equality comes from the integral representation (3.12). Since the series on the right-hand side of (3.16) converges absolutely and uniformly for every $s \in \mathbb{C}$ and its summands are entire functions of s, we see that the expression (3.16) is valid for any $s \in \mathbb{C} \setminus \{\frac{1}{2} - k,\, k \in \mathbb{N}_0\}$. Therefore, (3.16) and the series (3.1) must coincide for $\text{Re}(s) > \frac{1}{2}$. □

We now obtain a result in which the restriction $\frac{1}{2} < \text{Re}(s) < 1$ in our Theorem 3.1. is relaxed. Our proof is completely analogous to Watson's [[26], p. 300].

Corollary 3.2. *Let $N > 0$ be an integer. For every $x > 0$ and every s satisfying the condition $-N + \frac{1}{2} <$*

Re$(s) < 1$, *we have the identity*

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\left\{\frac{1}{\left(\lambda_n^2+x^2\right)^s}-\sum_{m=0}^{N-1}\binom{-s}{m}\frac{x^{2m}}{\lambda_n^{2s+2m}}\right\}+\sum_{m=0}^{N-1}\binom{-s}{m}x^{2m}\zeta_p\left(2s+2m\right)=$$

$$=\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)-\frac{1}{2}\frac{x^{-2s}}{1+\frac{1}{\pi p}}+2^{2-2s}\,x^{1-2s}\sin(\pi s)\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left((2y+1)\,x\right)e^{2\pi(2y+1)x}-1}dy. \qquad (3.17)$$

Proof. Note that, for $\text{Re}(s) > \frac{1}{2}$, we can write the left-hand side of (3.3) in the following form

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\left\{\frac{1}{\left(\lambda_n^2+x^2\right)^s}-\sum_{m=0}^{N-1}\binom{-s}{m}\frac{x^{2m}}{\lambda_n^{2s+2m}}\right\}+\sum_{m=0}^{N-1}\binom{-s}{m}x^{2m}\zeta_p\left(2s+2m\right). \qquad (3.18)$$

Let N be a positive such that, for $n > N$ the $x/\lambda_n < 1$. From the generalized binomial theorem and $n > N$, we have

$$\frac{1}{\left(\lambda_n^2+x^2\right)^s}=\sum_{m=0}^{\infty}\binom{-s}{m}\frac{x^{2m}}{\lambda_n^{2m+2s}}.$$

Hence, (3.18) can be actually expressed as

$$\sum_{n=1}^{N}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\left\{\frac{1}{\left(\lambda_n^2+x^2\right)^s}-\sum_{m=0}^{N-1}\binom{-s}{m}\frac{x^{2m}}{\lambda_n^{2s+2m}}\right\}+$$

$$\sum_{n=N+1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\left\{\sum_{m=N}^{\infty}\binom{-s}{m}\frac{x^{2m}}{\lambda_n^{2s+2m}}\right\}+\sum_{m=0}^{N-1}\binom{-s}{m}x^{2m}\zeta_p\left(2s+2m\right). \qquad (3.19)$$

Note that the general term of the infinite series $\sum_{m\geq N+1}$ is $O\left(\lambda_n^{-2\sigma-2N}\right)$ for large n. This means that this series converges absolutely and uniformly with respect to s in the region $\text{Re}(s) > -N+\frac{1}{2}$ (because $\lambda_n \geq n-\frac{1}{2}$ for every $p\in\mathbb{R}_+$). Since any of its terms are analytic functions of s, an application of Weierstrass' M-test shows that (3.18) represents an analytic function in the region $\text{Re}(s) > -N+\frac{1}{2}$. By our argument above, we know that the right-hand side of (3.17) represents also an analytic function in the half-plane $\text{Re}(s) < 1$. Thus, by the principle of analytic continuation, we have that (3.17) must hold. □

From the previous result, we can obtain the following identity.

Corollary 3.3. *For every* $p\in\mathbb{R}_+$ *and* $x > 0$, *the following formula holds*

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\left\{\frac{1}{\sqrt{\lambda_n^2+x^2}}-\frac{1}{\lambda_n}\right\}+C_p^{(1)}+\log\left(\frac{x}{2}\right)+\frac{1}{2x}\cdot\frac{1}{1+\frac{1}{\pi p}}=2\int_0^{\infty}\frac{1}{\sqrt{y^2+y}}\cdot\frac{dy}{\sigma\left((2y+1)\,x\right)e^{2\pi(2y+1)x}-1},$$

where $C_p^{(1)}$ *denotes Koshliakov's generalization of Euler's constant* γ *[[18], p. 46, eq. (46)],*

$$C_p^{(1)}:=\lim_{n\to\infty}\left\{\sum_{j=1}^{n-1}\frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2}\cdot\frac{1}{\lambda_j}-\log\left(\lambda_n\right)\right\}. \qquad (3.20)$$

Proof. Let us take $N=1$ in the previous Corollary and take the limit $s\to\frac{1}{2}$ on both sides of (3.17): this yields

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\left\{\frac{1}{\sqrt{\lambda_n^2+x^2}}-\frac{1}{\lambda_n}\right\}+\lim_{s\to\frac{1}{2}}\left[\zeta_p(2s)-\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)\right]$$

$$=-\frac{1}{2x}\cdot\frac{1}{1+\frac{1}{\pi p}}+2\int_0^{\infty}\frac{1}{\sqrt{y^2+y}}\cdot\frac{dy}{\sigma\left((2y+1)\,x\right)e^{2\pi(2y+1)x}-1}.$$

To find the limit as $s \to \frac{1}{2}$, we use the well-known Laurent expansions around this point,

$$\frac{1}{\Gamma(s)} = \frac{1}{\sqrt{\pi}} + \frac{\gamma + 2\log(2)}{\sqrt{\pi}} \left(s - \frac{1}{2}\right) + O\left(\left(s - \frac{1}{2}\right)^2\right), \tag{3.21}$$

$$x^{-2s} = x^{-1} \left(1 - 2\log(x) \left(s - \frac{1}{2}\right) + O\left(\left(s - \frac{1}{2}\right)^2\right)\right), \tag{3.22}$$

$$\Gamma\left(s - \frac{1}{2}\right) = \frac{1}{s - \frac{1}{2}} - \gamma + O\left(s - \frac{1}{2}\right) \tag{3.23}$$

and finally,

$$\zeta_p(2s) = \frac{1}{2s-1} + C_p^{(1)} + O\left(s - \frac{1}{2}\right), \tag{3.24}$$

with the last expression coming from [[18], p. 48, eq. (51)]. Deducing our corollary from (3.21)-(3.24) is now immediate. □

By letting $p \to \infty$ and $p \to 0^+$ we obtain a formula also obtained by Watson and an analogue of it, which seems to be new.

Corollary 3.4. *For $x > 0$, the following formulas are valid*

$$\sum_{n=1}^{\infty} \left\{\frac{1}{\sqrt{n^2 + x^2}} - \frac{1}{n}\right\} + \frac{1}{2x} + \gamma + \log\left(\frac{x}{2}\right) = 2\sum_{n=1}^{\infty} K_0(2\pi n x), \tag{3.25}$$

$$\sum_{n=1}^{\infty} \left\{\frac{1}{\sqrt{(2n-1)^2 + x^2}} - \frac{1}{2n-1}\right\} + \frac{\gamma}{2} + \frac{\log(x)}{2} = \sum_{n=1}^{\infty} (-1)^n K_0(\pi n x). \tag{3.26}$$

As a particular case of our generalization of Watson's result, we can evaluate the series (3.1) for integer argument s. We will do this for $s = 1, 2$. We remark that there are other ways of proving both formulas of our next corollary, for instance involving residue theory. In fact, Koshliakov gave a direct proof of this result on page 34 of his paper [18]. We only give a proof of the first formula, the second one being analogous.

Corollary 3.5. *Let $p \in \mathbb{R}_+$ and $x > 0$. Then the following identities take place*

$$\sum_{n=1}^{\infty} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \frac{1}{x^2 + \lambda_n^2} = \frac{\pi}{2x} - \frac{1}{1 + \frac{1}{\pi p}} \cdot \frac{1}{2x^2} + \frac{\pi}{x} \frac{1}{\sigma(x)\, e^{2\pi x} - 1}, \tag{3.27}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \frac{1}{\left(\lambda_n^2 + x^2\right)^2} &= \frac{\pi}{4x^3} - \frac{1}{2x^4} \cdot \frac{1}{1 + \frac{1}{\pi p}} + \\ &+ \frac{\pi^2}{x^2\left(\sigma(x)\, e^{2\pi x} - 1\right)} \cdot \left[\frac{1}{2\pi x} + \left(1 + \frac{p}{\pi\left(p^2 - x^2\right)}\right) \frac{\sigma(x)\, e^{2\pi x}}{\sigma(x)\, e^{2\pi x} - 1}\right]. \end{aligned} \tag{3.28}$$

Proof. Taking $s = 1$ in formula (3.2) and use the special value for the Macdonald function,

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}, \quad x > 0, \tag{3.29}$$

we deduce

$$\sum_{n=1}^{\infty}\frac{p^{2}+\lambda_{n}^{2}}{p\left(p+\frac{1}{\pi}\right)+\lambda_{n}^{2}}\,\frac{1}{\lambda_{n}^{2}+x^{2}}=\frac{\pi}{2x}-\frac{1}{1+\frac{1}{\pi p}}\cdot\frac{1}{2x^{2}}+\frac{\pi}{x}\sum_{m=1}^{\infty}(-1)^{m}\,e^{-2\pi mx}+$$

$$+\frac{\pi}{x}\sum_{m=1}^{\infty}e^{2\pi pm}\sum_{\ell=1}^{m}\left(\begin{array}{c}m\\ \ell\end{array}\right)(-1)^{m-\ell}\left(4\pi mp\right)^{\ell}\int_{1}^{\infty}e^{-2\pi mt(x+p)}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,dt=$$

$$=\frac{\pi}{2x}-\frac{1}{1+\frac{1}{\pi p}}\cdot\frac{1}{2x^{2}}+\frac{\pi}{x}\sum_{m=1}^{\infty}e^{-2\pi mx}\sum_{\ell=0}^{m}\left(\begin{array}{c}m\\ \ell\end{array}\right)(-1)^{m-\ell}\left(4\pi mp\right)^{\ell}\left(2\pi m(x+p)\right)^{-\ell} \tag{3.30}$$

$$=\frac{\pi}{2x}-\frac{1}{1+\frac{1}{\pi p}}\cdot\frac{1}{2x^{2}}+\frac{\pi}{x}\sum_{m=1}^{\infty}\left(\frac{p-x}{p+x}\right)^{m}e^{-2\pi mx}=\frac{\pi}{2x}-\frac{1}{1+\frac{1}{\pi p}}\cdot\frac{1}{2x^{2}}+\frac{\pi}{x}\,\frac{1}{\sigma(x)\,e^{2\pi x}-1}. \tag{3.31}$$

□

Remark 3.2. By invoking (3.2) and to the representation via the Bessel polynomials

$$K_{n-\frac{1}{2}}(x)=\sqrt{\frac{\pi}{2x}}\,e^{-x}\sum_{k=0}^{n-1}\frac{(n-1+k)!}{k!\,(n-1-k)!\,(2x)^{k}},\quad n\in\mathbb{N},$$

it is possible to compute the values of (3.1) for every integer s. Moreover, we may connect the resulting values with identities involving $\zeta_p(2n-1)$, reminiscent of Terras' representations [25] representations (c.f. [15]).

3.2 Second Analogue of Watson's formula

We now develop a second analogue of Watson's formula which will be useful to derive the functional equation for a generalized Epstein zeta function. Another way of considering an analogue of (1.3) in Koshliakov's setting is by starting with its right-hand side, this is, to consider

$$\tilde{\varphi}_p(x):=\sum_{n=1}^{\infty}\frac{p^{2}+\lambda_{n}^{2}}{p\left(p+\frac{1}{\pi}\right)+\lambda_{n}^{2}}\,\lambda_{n}^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi\lambda_{n}x),\quad x>0.$$

We will now establish the following result, which acts as a complementary case to Theorem 3.1.

Theorem 3.2. *Let $Re(s)>1$ and $x>0$. Then the following generalization of Watson's formula takes place*

$$\sum_{n=1}^{\infty}\frac{p^{2}+\lambda_{n}^{2}}{p\left(p+\frac{1}{\pi}\right)+\lambda_{n}^{2}}\,\lambda_{n}^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi\lambda_{n}x)$$

$$=-\frac{\pi^{\frac{1}{2}-s}x^{\frac{1}{2}-s}}{4}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p}}+\frac{\Gamma(s)\pi^{-s}x^{-s-\frac{1}{2}}}{4}+\pi\int_{0}^{\infty}\frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(2\pi xy)}{\sigma\left(y\right)e^{2\pi y}-1}\,dy, \tag{3.32}$$

where $J_\nu(x)$ denotes the Bessel function of the first kind.

Proof. Let $s\in\mathbb{C}$ and choose $\mu>\mathrm{Re}(s)+\frac{1}{2}$. Using the integral representation (3.8), arguing by absolute convergence of the Dirichlet series $\zeta_p(s)$ for $\mathrm{Re}(s)>1$ and taking into account our choice of μ, we have

$$\sum_{n=1}^{\infty}\frac{p^{2}+\lambda_{n}^{2}}{p\left(p+\frac{1}{\pi}\right)+\lambda_{n}^{2}}\,\lambda_{n}^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi\lambda_{n}x)=\frac{1}{8\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(\frac{w+\frac{1}{2}-s}{2}\right)\Gamma\left(\frac{w+s-\frac{1}{2}}{2}\right)\zeta_p\left(w+\frac{1}{2}-s\right)(\pi x)^{-w}\,dw$$

$$=\frac{(\pi x)^{\frac{1}{2}-s}}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(z\right)\Gamma\left(s+z-\frac{1}{2}\right)\zeta_p\left(2z\right)\,(\pi x)^{-2z}\,dz$$

$$=\frac{(\pi x)^{\frac{1}{2}-s}}{2}\,\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(z\right)\Gamma\left(s+z-\frac{1}{2}\right)\zeta_p\left(2z\right)\,(\pi x)^{-2z}\,dz. \tag{3.33}$$

where $\sigma > \frac{1}{2}$. We will now assume that $\mathrm{Re}(s) > \sigma > \frac{1}{2}$ and change the line of integration in (3.33) to $\mathrm{Re}(z) = \frac{1}{2} - \sigma$: doing so we pass by two poles located at $z = \frac{1}{2}$ and $z = 0$, since $\mathrm{Re}(s) > \sigma$ by hypothesis. An application of Cauchy's residue Theorem, combined with the convex estimates (2.9), gives

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(z)\,\Gamma\left(s+z-\frac{1}{2}\right)\zeta_p(2z)\,(\pi x)^{-2z}\,dz = \frac{1}{2\pi i}\int_{\frac{1}{2}-\sigma-i\infty}^{\frac{1}{2}-\sigma+i\infty}\Gamma(z)\,\Gamma\left(s+z-\frac{1}{2}\right)\zeta_p(2z)\,(\pi x)^{-2z}\,dz - \frac{1}{2}\cdot\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p}} + \frac{\Gamma(s)}{2x\sqrt{\pi}}. \tag{3.34}$$

Finally, we may use the functional equation for the Koshliakov zeta function (1.16) to deduce that

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-\sigma-i\infty}^{\frac{1}{2}-\sigma+i\infty}\Gamma(z)\,\Gamma\left(s+z-\frac{1}{2}\right)\zeta_p(2z)\,(\pi x)^{-2z}\,dz = \frac{1}{2\pi i}\frac{x^{-1}}{\sqrt{\pi}}\int_{\sigma-i\infty}^{\sigma+i\infty}x^{2z}\,\Gamma(z)\,\eta_p(2z)\,\Gamma(s-z)\,dz$$
$$= \frac{x^{-1}}{\sqrt{\pi}}\sum_{m=1}^{\infty}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(z)\,\Gamma(s-z)\,(2z,2\pi pm)_m\left(\frac{x}{m}\right)^{2z}dz = \frac{x^{-1}}{\sqrt{\pi}}\sum_{m=1}^{\infty}I_{m,p}^{\star}(s,x). \tag{3.35}$$

As in the proof of the first analogue of Watson's formula (see Theorem 3.1. above), if we use the representation (1.17), we see that

$$I_{m,p}^{\star}(s,x) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(z)\,\Gamma(s-z)\,(2z,2\pi pm)_m\left(\frac{x}{m}\right)^{2z}dz = I_{m,p}^{\star(1)}(s,x) + I_{m,p}^{\star(2)}(s,x), \tag{3.36}$$

where, due to the hypothesis $\mathrm{Re}(s) > \sigma > \frac{1}{2}$ and the integral representation (3.4),

$$I_{m,p}^{\star(1)}(s,x) = \frac{(-1)^m}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(z)\,\Gamma(s-z)\left(\frac{x}{m}\right)^{2z}dz = \frac{(-1)^m\Gamma(s)x^{2s}}{(x^2+m^2)^s}. \tag{3.37}$$

The second integral, $I_{m,p}^{\star(2)}(s,x)$, has a similar representation, given as

$$I_{m,p}^{\star(2)}(s,x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{2\pi pm}\sum_{\ell=1}^{m}\binom{m}{\ell}(4\pi p)^{\ell}(-1)^{m-\ell}\int_m^{\infty}t^{-2z}e^{-2\pi p\,t}\frac{(t-m)^{\ell-1}}{(\ell-1)!}\Gamma(z)\,\Gamma(s-z)\,x^{2z}\,dt\,dz$$
$$= e^{2\pi pm}\sum_{\ell=1}^{m}\binom{m}{\ell}(4\pi p)^{\ell}(-1)^{m-\ell}\int_m^{\infty}e^{-2\pi p\,t}\frac{(t-m)^{\ell-1}}{(\ell-1)!}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(z)\,\Gamma(s-z)\,t^{-2z}\,x^{2z}dz\,dt$$
$$= e^{2\pi pm}\sum_{\ell=1}^{m}\binom{m}{\ell}(4\pi p)^{\ell}(-1)^{m-\ell}\int_m^{\infty}e^{-2\pi p\,t}\frac{(t-m)^{\ell-1}}{(\ell-1)!}\frac{\Gamma(s)x^{2s}}{(x^2+t^2)^s}\,dt, \tag{3.38}$$

where all the steps are justified via absolute convergence. Now, we use the integral representation [[14], p. 702, eq. (6.623.1)]

$$\frac{1}{(x^2+t^2)^s} = \frac{\sqrt{\pi}\,2^{\frac{1}{2}-s}}{\Gamma(s)x^{s-\frac{1}{2}}}\int_0^{\infty}y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(xy)\,e^{-ty}\,dy, \tag{3.39}$$

valid for every $x, t > 0$ and $\mathrm{Re}(s) > 0$. We find an alternative representation for $I_{m,p}^{\star(1)}(s,x)$ in the following form

$$I_{m,p}^{\star(1)}(s,x) = (-1)^m\sqrt{\pi}2^{\frac{1}{2}-s}x^{s+\frac{1}{2}}\int_0^{\infty}y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(xy)\,e^{-my}dy \tag{3.40}$$

It is simple to argue via absolute convergence and the hypothesis $\mathrm{Re}(s) > \frac{1}{2}$ that the orders of integration with respect to t and y can be reversed and this gives the representation for $I_{m,p}^{\star(2)}(s,x)$,

$$\begin{aligned}
I_{m,p}^{\star(2)}(s,x) &= \sqrt{\pi}2^{\frac{1}{2}-s}x^{s+\frac{1}{2}}\,e^{2\pi pm}\sum_{\ell=1}^{m}\begin{pmatrix} m \\ \ell \end{pmatrix}(4\pi p)^{\ell}\,(-1)^{m-\ell}\int_0^\infty y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(xy)\int_m^\infty e^{-2\pi pt}\,\frac{(t-m)^{\ell-1}}{(\ell-1)!}\,e^{-ty}\,dt\,dy \\
&= \sqrt{\pi}2^{\frac{1}{2}-s}x^{s+\frac{1}{2}}\int_0^\infty y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(xy)\sum_{\ell=1}^{m}\begin{pmatrix} m \\ \ell \end{pmatrix}(4\pi p)^{\ell}\,(-1)^{m-\ell}\,(2\pi p+y)^{-\ell}\,e^{-m\,y}\,dy \\
&= \sqrt{\pi}2^{\frac{1}{2}-s}x^{s+\frac{1}{2}}\int_0^\infty y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(xy)\left\{\left(\frac{4\pi p}{2\pi p+y}-1\right)^{m}-(-1)^{m}\right\}e^{-my}\,dy \\
&= \sqrt{\pi}2^{\frac{1}{2}-s}x^{s+\frac{1}{2}}\int_0^\infty y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(xy)\left\{\left(\frac{2\pi p-y}{2\pi p+y}\right)^{m}-(-1)^{m}\right\}e^{-my}\,dy. \qquad (3.41)
\end{aligned}$$

Finally, a combination of (3.37) and (3.41) yields

$$I_{m,p}^{\star}(s,x) = \sqrt{\pi}2^{\frac{1}{2}-s}x^{s+\frac{1}{2}}\int_0^\infty y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(xy)\left(\frac{2\pi p-y}{2\pi p+y}\right)^{m}e^{-my}dy \qquad (3.42)$$

Summing over m, using the well-known uniform bound $|J_\nu(x)| \leq C_\nu/\sqrt{x}$, $x > 0$, and the hypothesis that $\mathrm{Re}(s) > 1$, we find from (3.42) and (2.15) that

$$\sum_{m=1}^{\infty} I_{m,p}^{\star}(s,x) = \sqrt{\pi}2^{\frac{1}{2}-s}x^{s+\frac{1}{2}}\int_0^\infty \frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(xy)}{\sigma\left(\frac{y}{2\pi}\right)e^{y}-1}\,dy. \qquad (3.43)$$

Returning to (3.35) and (3.33) we obtain immediately the second analogue of Watson's formula (3.32).

□

As before, by letting $p \to \infty$ or $p \to 0^{+}$ in (3.32), we shall obtain Watson's formula (1.3) and an analogue of it.

Corollary 3.6. *For every $x > 0$ and Re(s) $> \frac{1}{2}$, the classical Watson's formula (1.3) holds. Moreover, an analogue of (3.15) takes place*

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(x^2+n^2)^s} = -\frac{x^{-2s}}{2}+\frac{2^{\frac{3}{2}-s}\pi^{s}x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{n=1}^{\infty}(2n-1)^{s-\frac{1}{2}}\;K_{s-\frac{1}{2}}\left(\pi(2n-1)x\right). \qquad (3.44)$$

Proof. Assume first that $\mathrm{Re}(s) > 1$: letting $p \to \infty$ in (3.32), we find that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi nx) &= -\frac{\pi^{\frac{1}{2}-s}x^{\frac{1}{2}-s}}{4}\,\Gamma\left(s-\frac{1}{2}\right)+\frac{\Gamma(s)\pi^{-s}x^{-s-\frac{1}{2}}}{4}+\pi\int_0^\infty \frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(2\pi xy)}{e^{2\pi y}-1}\,dy \\
&= -\frac{\pi^{\frac{1}{2}-s}x^{\frac{1}{2}-s}}{4}\,\Gamma\left(s-\frac{1}{2}\right)+\frac{\Gamma(s)\pi^{-s}x^{-s-\frac{1}{2}}}{4}+\pi\sum_{n=1}^{\infty}\int_0^\infty y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(2\pi xy)\,e^{-2\pi ny}\,dy,
\end{aligned}$$

where the interchange of the orders of the integral and the series comes from the well-known bound for $J_\nu(x)$, $|J_\nu(x)| \leq C_\nu/\sqrt{x}$, $x > 0$, which gives

$$\sum_{n=1}^{\infty}\int_0^\infty y^{\mathrm{Re}(s)-\frac{1}{2}}|J_{s-\frac{1}{2}}(2\pi xy)|\,e^{-2\pi ny}\,dy \leq C_s\sum_{n=1}^{\infty}\int_0^\infty y^{\mathrm{Re}(s)-1}\,e^{-2\pi ny}\,dy \leq D_s\sum_{n=1}^{\infty}\frac{1}{n^{\mathrm{Re}(s)}} < \infty, \qquad (3.45)$$

by the hypothesis $\mathrm{Re}(s) > 1$.

Therefore, using (3.39) with x replaced by $2\pi x$ and t by $2\pi n$, we deduce immediately Watson's formula (1.3) when $\text{Re}(s) > 1$. Since both sides of (1.3) represent analytic functions on the half-plane $\text{Re}(s) > \frac{1}{2}$ (note the uniform and absolute convergence of the series on the left in this region), Watson's formula can be proved for every $\text{Re}(s) > \frac{1}{2}$ by the principle of analytic continuation.

Letting $p \to 0^+$ and using the same justification as in (3.45), we find that (3.32) implies

$$2^{\frac{1}{2}-s}\sum_{n=1}^{\infty}(2n-1)^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}\left(\pi(2n-1)x\right)=\frac{\Gamma(s)\pi^{-s}x^{-s-\frac{1}{2}}}{4}+\pi\sum_{n=1}^{\infty}(-1)^n\int_0^{\infty}y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(2\pi xy)\,e^{-2\pi ny}\,dy,\quad \text{Re}(s)>1.$$

Thus, the use of the integral representation (3.39) also proves the desired formula (3.44) for $\text{Re}(s) > 1$. Finally, the extension of (3.44) to the region $\text{Re}(s) > \frac{1}{2}$ follows from analytic continuation.

□

By using an integration by parts on the integral (3.42), it can be actually proved that Watson's formula (3.32) holds for $\text{Re}(s) > \frac{1}{2}$ (see (3.53) below for a justification of a particular case of this). Like in Corollary 3.2., we now establish the analytic continuation of the second analogue of Watson's formula (3.32).

Corollary 3.7. *Let $N > 0$ be an integer. For every $x > 0$ and every s satisfying the condition, Re$(s) > -N - \frac{1}{2}$, one has the identity*

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi\lambda_n x)=-\frac{\pi^{\frac{1}{2}-s}x^{\frac{1}{2}-s}}{4}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p}}+\frac{\Gamma(s)\pi^{-s}x^{-s-\frac{1}{2}}}{4}$$
$$+\frac{x^{s-\frac{1}{2}}}{2\pi^s}\sum_{k=0}^{N}\frac{(-1)^k\,x^{2k}}{k!}\,\Gamma(s+k)\,\eta_p(2s+2k)+\pi\int_0^{\infty}\frac{y^{s-\frac{1}{2}}}{\sigma(y)\,e^{2\pi y}-1}\left\{J_{s-\frac{1}{2}}(2\pi xy)-\sum_{k=0}^{N}\frac{(-1)^k\,(\pi xy)^{s+2k-\frac{1}{2}}}{k!\Gamma\left(s+k+\frac{1}{2}\right)}\right\}dy. \tag{3.46}$$

Proof. For every $x > 0$ and $N \in \mathbb{N}$, let us consider the integral

$$\mathscr{J}_N(x,s):=\int_0^{\infty}\frac{y^{s-\frac{1}{2}}\left\{J_{s-\frac{1}{2}}(2\pi xy)-\sum_{k=0}^{N}\frac{(-1)^k(\pi xy)^{s+2k-\frac{1}{2}}}{k!\Gamma\left(s+k+\frac{1}{2}\right)}\right\}}{\sigma(y)\,e^{2\pi y}-1}\,dy$$

which converges absolutely and uniformly for every $\text{Re}(s) > -N - \frac{1}{2}$. By the power series of the Bessel function of the first kind $J_\nu(z)$ [[19], p. 217, eq. 10.2.2.], for every $\text{Re}(s) > \frac{1}{2}$ we know that the following equality must take place

$$\int_0^{\infty}\frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(2\pi xy)}{\sigma(y)\,e^{2\pi y}-1}\,dy=\mathscr{J}_N(x,s)+\sum_{k=0}^{N}\frac{(-1)^k\,(\pi x)^{s+2k-\frac{1}{2}}}{k!\Gamma\left(s+k+\frac{1}{2}\right)}\int_0^{\infty}\frac{y^{2s+2k-1}}{\sigma(y)\,e^{2\pi y}-1}dy$$
$$=\mathscr{J}_N(x,s)+\frac{x^{s-\frac{1}{2}}}{2\pi^{s+1}}\sum_{k=0}^{N}\frac{(-1)^k\,x^{2k}}{k!}\,\Gamma(s+k)\,\eta_p(2s+2k) \tag{3.47}$$

where we have used the integral representation (2.4), together with the functional equation for $\zeta_p(s)$ as well as Legendre's duplication formula for the Gamma function. Taking now $\text{Re}(s) > \frac{1}{2}$ in (3.47) and combining it with (3.32) (which is also valid for $\text{Re}(s) > \frac{1}{2}$), we may now invoke the principle of analytic continuation to deduce (3.46), completing the proof.

□

From the previous result, we can obtain a nice generalization of Watson's identity (3.25), which is analogous to (3.3).

Corollary 3.8. *For every $p \in \mathbb{R}_{+}$ and $x > 0$, the following identity takes place*

$$\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\, K_0(2\pi\lambda_n x) = \frac{1}{4x} + \frac{C_p^{(2)}}{2} - \frac{e^{2\pi p} Q_{2\pi p}(0)}{1+\frac{1}{\pi p}} + \frac{\log\left(\frac{x}{2}\right)}{2\left(1+\frac{1}{\pi p}\right)} + \pi \int_0^{\infty} \frac{J_0(2\pi x y)-1}{\sigma(y)\, e^{2\pi y}-1}\, dy. \tag{3.48}$$

Proof. We prove (3.48) by taking $N=1$ and letting $s \to \frac{1}{2}$ in (3.46). The conclusion of the proof follows after invoking the Laurent expansion [[18], p. 49, eq. (53)],

$$\eta_p(s) = \frac{1}{1+\frac{1}{\pi p}}\, \frac{1}{s-1} + C_p^{(2)} - \frac{2e^{2\pi p} Q_{2\pi p}(0)}{1+\frac{1}{\pi p}} + O\left(s-1\right), \tag{3.49}$$

together with (3.22), (3.23), as well as

$$\Gamma\left(s\right) = \sqrt{\pi} - \sqrt{\pi}\left(2\log(2)+\gamma\right)\left(s-\frac{1}{2}\right) + O\left(s-\frac{1}{2}\right)^2. \tag{3.50}$$

□

Corollary 3.9. *Watson's formula (3.25) holds. Furthermore, we have the identity*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{x^2+n^2}} = -\frac{1}{2x} + 2\sum_{n=1}^{\infty} K_0\left(\pi(2n-1)x\right), \quad x>0, \tag{3.51}$$

which is analogous to (3.26).

Proof. Taking $p \to \infty$ on (3.48), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} K_0(2\pi n x) &= \frac{1}{4x} + \frac{\gamma}{2} + \frac{\log\left(\frac{x}{2}\right)}{2} + \pi \int_0^{\infty} \frac{J_0(2\pi x y)-1}{e^{2\pi y}-1}\, dy = \frac{1}{4x} + \frac{\gamma}{2} + \frac{\log\left(\frac{x}{2}\right)}{2} + \pi \sum_{n=1}^{\infty} \int_0^{\infty} \left(J_0(2\pi x y)-1\right) e^{-2\pi n y}\, dy \\ &= \frac{1}{4x} + \frac{\gamma}{2} + \frac{\log\left(\frac{x}{2}\right)}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{n^2+x^2}} - \frac{1}{n} \right\},\end{aligned}$$

where the last step comes from (3.39). In the second step, the interchange of the orders of the integral and the geometric series can be justified by absolute convergence: indeed, it follows from an integration by parts that

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left(J_0(2\pi x y)-1\right) e^{-2\pi n y}\, dy = -\sum_{n=1}^{\infty} \frac{x}{n} \int_0^{\infty} e^{-2\pi n y}\, J_1(2\pi x y)\, dy, \tag{3.52}$$

so that, by using once more the uniform bound $|J_\nu(x)| \leq C_\nu/\sqrt{x}$, $x>0$, we have for absolute constants D and D',

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} e^{-2\pi n y} \left|J_1(2\pi x y)\right| dy \leq \frac{D}{\sqrt{2\pi x}} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \frac{e^{-2\pi n y}}{\sqrt{y}}\, dy \leq \frac{D'}{\sqrt{x}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}, \tag{3.53}$$

justifying the passage. To prove (3.51), we let $p \to 0^{+}$ on (3.48) and use the same kind of justifications as in (3.53) in order to get

$$\begin{aligned}\sum_{n=1}^{\infty} K_0\left(\pi\left(2n-1\right)x\right) &= \frac{1}{4x} - \frac{\log(2)}{2} + \pi \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} \left(J_0(2\pi x y)-1\right) e^{-2\pi n y} dy \\ &= \frac{1}{4x} - \frac{\log(2)}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n}{\sqrt{n^2+x^2}} - \frac{(-1)^n}{n} \right\} = \frac{1}{4x} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+x^2}}.\end{aligned}$$

□

Like in Corollary 3.5., it is now possible to derive as a particular case from our analogue of Watson's formula (3.32) a formula obtained by Koshliakov himself [[18], p. 44, eq. (36)], which truly complements (3.27).

Corollary 3.10. *Let $\sigma_p(z)$ be defined by*

$$\sigma_p(z) = \sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\, e^{-\lambda_n z}, \quad Re(z) > 0, \tag{3.54}$$

(c.f. (1.20) above). Then the following formula holds

$$\sigma_p(2\pi x) = -\frac{1}{2}\,\frac{1}{1+\frac{1}{\pi p}} + \frac{1}{2\pi x} + 2\int_0^{\infty} \frac{\sin(xy)}{\sigma(y)\, e^{2\pi y}-1}\, dy. \tag{3.55}$$

Proof. Take $s = 1$ in our second analogue of Watson's formula (3.32) [3]. By appealing to the well-known formula (3.29) and to $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin(x)$, we see that

$$\frac{1}{2\sqrt{x}}\sigma_p(2\pi x) = \frac{1}{2\sqrt{x}} \sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\, e^{-2\pi\lambda_n x} = -\frac{1}{4\sqrt{x}}\,\frac{1}{1+\frac{1}{\pi p}} + \frac{1}{4\pi x^{3/2}} + \pi\int_0^{\infty} \frac{y^{1/2} J_{1/2}(2\pi x y)}{\sigma\,(y)\, e^{2\pi y}-1}\, dy$$

$$= -\frac{1}{4\sqrt{x}}\,\frac{1}{1+\frac{1}{\pi p}} + \frac{1}{4\pi x^{3/2}} + \frac{1}{\sqrt{x}}\int_0^{\infty} \frac{\sin(2\pi x y)}{\sigma(y)\, e^{2\pi y}-1}\, dy,$$

which is equivalent to (3.55). □

3.3 Another proof and a generalization of Watson's formula

In this subsection we give an alternative proof and a new expression respectively for (3.3) and (3.32). We start with a new proof of (3.3).

2nd Proof of (3.3). We evaluate

$$\varphi_p(s,x) := \sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s}, \quad \mathrm{Re}(s) > \frac{1}{2}, \quad x>0, \tag{3.56}$$

by appealing to the representation (3.39). By absolute convergence, we have

$$\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s} = \frac{\sqrt{\pi}\,2^{\frac{1}{2}-s}}{\Gamma(s)x^{s-\frac{1}{2}}}\int_0^{\infty} y^{s-\frac{1}{2}} J_{s-\frac{1}{2}}(xy)\,\sigma_p(y)\, dy, \quad \mathrm{Re}(s) > \frac{1}{2}. \tag{3.57}$$

The justification of the previous formula is similar to (3.53) and we may use an integration by parts and Fubini's Theorem to complete it. Invoking the Basset type representation for $J_\nu(z)$ [[19], p. 224, eq. (10.9.12)],

$$J_\nu(x) = \frac{2^{\nu+1}x^{-\nu}}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right)}\int_1^{\infty} \frac{\sin(xt)}{(t^2-1)^{\nu+\frac{1}{2}}}\, dt, \quad |\mathrm{Re}(\nu)| < \frac{1}{2},\ x>0, \tag{3.58}$$

we find that (3.57) implies

$$\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s} = \frac{2\sin(\pi s)}{\pi x^{2s-1}}\int_0^{\infty}\int_1^{\infty} \sigma_p(y)\,\sin(xyt)\,\frac{dt\, dy}{(t^2-1)^s}$$

$$= \frac{2\sin(\pi s)}{\pi x^{2s-1}}\int_1^{\infty}\int_0^{\infty} \sigma_p(y)\,\sin(xyt)\, dy\,\frac{dt}{(t^2-1)^s}, \quad \frac{1}{2} < \mathrm{Re}(s) < 1 \tag{3.59}$$

[3] recall by the justification given in (3.53) that it is possible to extend (3.32) to $\mathrm{Re}(s) > \frac{1}{2}$.

with the last step being justified by Fubini's theorem and the hypothesis that $\frac{1}{2} < \mathrm{Re}(s) < 1$.

Now, by [[18], p. 45, eq. (40)][4], we know that

$$\frac{1}{\sigma(x)\,e^{2\pi x}-1} = -\frac{1}{2} + \frac{1}{2\pi x}\cdot\frac{1}{1+\frac{1}{\pi p}} + \frac{1}{\pi}\int_0^{\infty}\sin(xt)\,\sigma_p(t)\,dt. \tag{3.60}$$

Using this formula, we are able to evaluate the first integral on (3.59) and we can derive that

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s} = \frac{2\sin(\pi s)}{x^{2s-1}}\int_1^{\infty}\left\{\frac{1}{\sigma(xt)e^{2\pi xt}-1}+\frac{1}{2}-\frac{1}{2\pi xt}\cdot\frac{1}{1+\frac{1}{\pi p}}\right\}\frac{dt}{(t^2-1)^s}.$$

By using some elementary relations for Euler's beta function,

$$\int_1^{\infty}\frac{1}{(t^2-1)^s}\,\frac{dt}{t} = \frac{\pi}{2\sin(\pi s)}, \quad 0<\mathrm{Re}(s)<1, \tag{3.61}$$

$$\int_1^{\infty}\frac{dt}{(t^2-1)^s} = \frac{\Gamma(1-s)\,\Gamma\left(s-\frac{1}{2}\right)}{2\sqrt{\pi}}, \quad \frac{1}{2}<\mathrm{Re}(s)<1, \tag{3.62}$$

we find, for $\frac{1}{2} < \mathrm{Re}(s) < 1$ and $x > 0$,

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s} = \frac{\sqrt{\pi}x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right) - \frac{1}{2}\,\frac{x^{-2s}}{1+\frac{1}{\pi p}} + 2\sin(\pi s)x^{1-2s}\int_1^{\infty}\frac{(t^2-1)^{-s}}{\sigma(xt)e^{2\pi xt}-1}dt, \tag{3.63}$$

from which (3.3) can now be easily derived, after taking a simple change of variable in the integral on the right side of (3.63). □

There is still a plethora of different scenarios where we may generalize Watson's formula (1.3). Here we give yet another generalization that is connected to Koshliakov's first analogue of Poisson's summation formula [[18], p. 58, eq. (V)].

Theorem 3.3. *Let Re$(s) > \frac{1}{2}$ and $x > 0$. Then the following generalization of Watson's formula (1.3) holds*

$$\begin{aligned}&\frac{2^{1-2s}x^{s-\frac{1}{2}}\pi^{-s}}{\Gamma(s)}\sum_{m=1}^{\infty}\int_0^{\infty}\int_0^{1}y^{2s-1}e^{-xy}\left(1-u^2\right)^{s-1}\cos\left(myu+2m\,\arctan\left(\frac{yu}{2\pi p}\right)\right)du\,dy\\ &=\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi\lambda_n x)+\frac{(\pi x)^{\frac{1}{2}-s}}{4}\cdot\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p}}-\frac{\pi^{-s}x^{-s-\frac{1}{2}}\Gamma(s)}{4}.\end{aligned} \tag{3.64}$$

Proof. In the proof Theorem 3.2., we have used the representation (3.39) in order to evaluate the integrals $I_{m,p}^{\star(1)}(s,x)$ and $I_{m,p}^{\star(2)}(s,x)$. Since we can reverse the roles of the variables x and t in the representation (3.39), we may write an alternative version

$$\frac{1}{\left(x^2+t^2\right)^s} = \frac{\sqrt{\pi}\,2^{\frac{1}{2}-s}}{\Gamma(s)t^{s-\frac{1}{2}}}\int_0^{\infty}y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(ty)\,e^{-xy}\,dy,$$

[4] this formula is a corollary of Koshliakov's generalizations of the Abel-Plana formula. See [4] for even more generalizations. This formula is in fact equivalent to (3.27), also due to Koshliakov [[18], p. 34, eq. (76)].

and so, returning to the expression (3.38), we obtain for $I_{m,p}^{\star(2)}(s,x)$,

$$I_{m,p}^{\star(2)}(s,x)=e^{2\pi pm}\sum_{\ell=1}^{m}\left(\begin{array}{c}m\\ \ell\end{array}\right)(4\pi p)^{\ell}\,(-1)^{m-\ell}\int_{m}^{\infty}e^{-2\pi p\,t}\,\frac{(t-m)^{\ell-1}}{(\ell-1)!}\,\frac{\Gamma(s)x^{2s}}{(x^{2}+t^{2})^{s}}\,dt$$

$$=e^{2\pi pm}\sum_{\ell=1}^{m}\left(\begin{array}{c}m\\ \ell\end{array}\right)(4\pi p)^{\ell}\,(-1)^{m-\ell}\int_{m}^{\infty}e^{-2\pi p\,t}\,\frac{(t-m)^{\ell-1}}{(\ell-1)!}\,\frac{\sqrt{\pi}2^{\frac{1}{2}-s}x^{2s}}{t^{s-\frac{1}{2}}}\int_{0}^{\infty}y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}(ty)\,e^{-xy}\,dy\,dt$$

$$=e^{2\pi pm}\sqrt{\pi}2^{\frac{1}{2}-s}x^{2s}\sum_{\ell=1}^{m}\left(\begin{array}{c}m\\ \ell\end{array}\right)(4\pi p)^{\ell}\,(-1)^{m-\ell}\int_{0}^{\infty}y^{s-\frac{1}{2}}\,e^{-xy}\int_{m}^{\infty}e^{-2\pi p\,t}\,\frac{(t-m)^{\ell-1}}{(\ell-1)!}\,\frac{J_{s-\frac{1}{2}}(ty)}{t^{s-\frac{1}{2}}}\,dt\,dy.$$

Using the Poisson representation for the Bessel function of the first kind [[19], p. 224, eq. (10.9.4)]

$$z^{-\nu}J_{\nu}(z)=\frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_{0}^{1}\left(1-u^{2}\right)^{\nu-\frac{1}{2}}\cos(zu)\;du,\quad \mathrm{Re}(\nu)>-\frac{1}{2},\;\; z\in\mathbb{C}, \tag{3.65}$$

we have from absolute convergence that

$$\int_{m}^{\infty}e^{-2\pi p\,t}\,\frac{(t-m)^{\ell-1}}{(\ell-1)!}\,\frac{J_{s-\frac{1}{2}}(ty)}{t^{s-\frac{1}{2}}}\,dt=\frac{y^{s-\frac{1}{2}}2^{\frac{3}{2}-s}}{\sqrt{\pi}\Gamma(s)}\int_{0}^{1}\left(1-u^{2}\right)^{s-1}\int_{m}^{\infty}e^{-2\pi p\,t}\,\frac{(t-m)^{\ell-1}}{(\ell-1)!}\,\cos(yu\,t)\,dt\,du$$

$$=\frac{2^{\frac{3}{2}-s}y^{s-\frac{1}{2}}}{\sqrt{\pi}\Gamma\left(s\right)}\int_{0}^{1}\left(1-u^{2}\right)^{s-1}\mathrm{Re}\left[(2\pi p+iyu)^{-\ell}\,e^{-2\pi mp-imyu}\right]\,du.$$

Therefore, the summation over ℓ yields the result

$$I_{m,p}^{\star(2)}(s,x)=\sqrt{\pi}2^{\frac{1}{2}-s}\,x^{2s}\int_{0}^{\infty}y^{s-\frac{1}{2}}\,e^{-xy}\,\frac{2^{\frac{3}{2}-s}y^{s-\frac{1}{2}}}{\sqrt{\pi}\Gamma\left(s\right)}\int_{0}^{1}\left(1-u^{2}\right)^{s-1}\mathrm{Re}\left[\sum_{\ell=1}^{m}\left(\begin{array}{c}m\\ \ell\end{array}\right)\left(\frac{4\pi p}{2\pi p+iyu}\right)^{\ell}(-1)^{m-\ell}e^{-imyu}\right]du\,dy$$

$$=\sqrt{\pi}2^{\frac{1}{2}-s}\,x^{2s}\int_{0}^{\infty}y^{s-\frac{1}{2}}\,e^{-xy}\,\frac{2^{\frac{3}{2}-s}y^{s-\frac{1}{2}}}{\sqrt{\pi}\Gamma\left(s\right)}\int_{0}^{1}\left(1-u^{2}\right)^{s-1}\mathrm{Re}\left[\left(\frac{p-iy\frac{u}{2\pi}}{p+iy\frac{u}{2\pi}}\right)^{m}e^{-imyu}-(-1)^{m}e^{-imyu}\right]du\,dy$$

$$=\frac{2^{2-2s}}{\Gamma(s)}\,x^{2s}\int_{0}^{\infty}y^{2s-1}\,e^{-xy}\int_{0}^{1}\left(1-u^{2}\right)^{s-1}\,\mathrm{Re}\left[\sigma^{m}\left(-\frac{iyu}{2\pi}\right)e^{-imyu}-(-1)^{m}e^{-imyu}\right]du\,dy,$$

which proves that

$$I_{m,p}^{\star}(s,x)=\frac{2^{2-2s}}{\Gamma(s)}\,x^{2s}\int_{0}^{\infty}\int_{0}^{1}y^{2s-1}\,e^{-xy}\left(1-u^{2}\right)^{s-1}\,\mathrm{Re}\left[\sigma^{m}\left(-\frac{iyu}{2\pi}\right)e^{-imyu}\right]du\,dy.$$

Finally, returning to (3.35) and to (3.34), we obtain

$$\frac{x^{-1}}{\sqrt{\pi}}\sum_{m=1}^{\infty}I_{m,p}^{\star}(s,x)=\frac{2^{2-2s}x^{2s-1}}{\sqrt{\pi}\Gamma(s)}\sum_{m=1}^{\infty}\int_{0}^{\infty}\int_{0}^{1}y^{2s-1}e^{-xy}\left(1-u^{2}\right)^{s-1}\mathrm{Re}\left[\sigma^{m}\left(-\frac{iyu}{2\pi}\right)e^{-imyu}\right]du\,dy$$

$$=\frac{2}{(\pi x)^{\frac{1}{2}-s}}\sum_{n=1}^{\infty}\frac{p^{2}+\lambda_{n}^{2}}{p\left(p+\frac{1}{\pi}\right)+\lambda_{n}^{2}}\,\lambda_{n}^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi\lambda_{n}x)+\frac{1}{2}\cdot\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p}}-\frac{\Gamma\left(s\right)}{2x\sqrt{\pi}},$$

which implies that

$$\frac{2^{1-2s}x^{s-\frac{1}{2}}\pi^{-s}}{\Gamma(s)}\sum_{m=1}^{\infty}\int_{0}^{\infty}\int_{0}^{1}y^{2s-1}e^{-xy}\left(1-u^{2}\right)^{s-1}\mathrm{Re}\left[\sigma^{m}\left(-\frac{iyu}{2\pi}\right)e^{-imyu}\right]du\,dy$$

$$=\sum_{n=1}^{\infty}\frac{p^{2}+\lambda_{n}^{2}}{p\left(p+\frac{1}{\pi}\right)+\lambda_{n}^{2}}\,\lambda_{n}^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi\lambda_{n}x)+\frac{(\pi x)^{\frac{1}{2}-s}}{4}\cdot\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p}}-\frac{\Gamma\left(s\right)(\pi x)^{\frac{1}{2}-s}}{4x\sqrt{\pi}}.$$

Watson's formula (3.64) is now achieved after using the elementary identity

$$\operatorname{Re}\left[\sigma^m\left(-ix\right)e^{-2\pi imx}\right]=\cos\left(2\pi mx+2m\arctan\left(\frac{x}{p}\right)\right). \tag{3.66}$$

□

Corollary 3.11. *Watson's formulas (1.3) and (3.44) hold.*

Proof. We only give details when $p\to\infty$. In fact, for $\operatorname{Re}(s)>\frac{1}{2}$ and $x>0$, the right-hand side of (3.64) reduces to

$$\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi n\,x)+\frac{(\pi x)^{\frac{1}{2}-s}}{4}\Gamma\left(s-\frac{1}{2}\right)-\frac{\pi^{-s}x^{-s-\frac{1}{2}}\Gamma(s)}{4}.$$

At the same time, if we use the Poisson representation (3.65) and (3.39), we see that the left-hand side can be further simplified to

$$\begin{aligned}\frac{2^{1-2s}x^{s-\frac{1}{2}}\pi^{-s}}{\Gamma(s)}\sum_{m=1}^{\infty}\int_0^{\infty}\int_0^{1}y^{2s-1}e^{-xy}\left(1-u^2\right)^{s-1}\cos\left(myu\right)\,du\,dy&=2^{-s-\frac{1}{2}}x^{s-\frac{1}{2}}\pi^{\frac{1}{2}-s}\sum_{m=1}^{\infty}\frac{1}{m^{s-\frac{1}{2}}}\int_0^{\infty}y^{s-\frac{1}{2}}e^{-xy}\,J_{s-\frac{1}{2}}(my)\,dy\\&=\frac{\Gamma(s)x^{s-\frac{1}{2}}\pi^{-s}}{2}\sum_{m=1}^{\infty}\frac{1}{\left(m^2+x^2\right)^s},\quad\operatorname{Re}(s)>\frac{1}{2}.\end{aligned}$$

□

4 Generalizations of the Epstein zeta function

4.1 First Analogue of Epstein's zeta function

Here we will consider a generalization of the Epstein zeta function (1.4) when the summation indices are replaced by two Koshliakov sequences $(\lambda_m,\lambda_n)_{m,n\in\mathbb{Z}}$. Note that if y is a root of the equation (1.12) then $-y$ also is. Therefore, we extend Koshliakov's sequence λ_n to the nonpositive integers by simply setting $\lambda_{-n}:=-\lambda_n$ and $\lambda_0:=0$.

Although we may proceed with any positive quadratic form, say $Q(x,y)=ax^2+bxy+cy^2$, it is enough for the purposes of our paper to consider the following analogue of (1.4),

$$\zeta_{p,p'}(s,c):=\sum_{m,n\neq0}\frac{\left(p^2+\lambda_n^2\right)\cdot\left(p'^2+\lambda_n'^2\right)}{\left(p(p+\frac{1}{\pi})+\lambda_n^2\right)\cdot\left(p'(p'+\frac{1}{\pi})+\lambda_n'^2\right)}\frac{1}{\left(\lambda_m^2+c\lambda_n'^2\right)^s},\quad\operatorname{Re}(s)>1,\;c>0, \tag{4.1}$$

where $m,\,n\neq0$ here means that only the term $m=n=0$ is omitted from the sum above.

It is simple to see that the Dirichlet series defining $\zeta_{p,p'}(s,c)$ converges absolutely in the half-plane $\operatorname{Re}(s)>1$. In the next result we give a formula which provides the continuation of (4.1) to $\operatorname{Re}(s)<1$.

Theorem 4.1. *For any $c>0$ and $p,p'\in\mathbb{R}_+$, consider the analogue of Epstein's zeta function given by (4.1). Then $\zeta_{p,p'}(s,c)$ can be continued to the half-plane Re$(s)<1$ by the following integral formula,*

$$\begin{aligned}\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\zeta_{p,p'}(s,c)&=\frac{2c^{s/2}\,\pi^{-s}\Gamma(s)}{1+\frac{1}{\pi p'}}\,\zeta_p(2s)+2\,c^{\frac{1-s}{2}}\,\pi^{-(s-\frac{1}{2})}\Gamma\left(s-\frac{1}{2}\right)\zeta_{p'}(2s-1)+\\&+\frac{2^{4-2s}\,\pi^{1-s}\,c^{\frac{1-s}{2}}}{\Gamma(1-s)}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\lambda_n'^{1-2s}\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left(\lambda_n'\left(2y+1\right)\sqrt{c}\right)e^{2\pi\lambda_n'(2y+1)\sqrt{c}}-1}dy.\end{aligned} \tag{4.2}$$

Furthermore, (4.2) provides the continuation of $\zeta_{p,p'}(s,c)$ to the entire complex plane as an analytic function everywhere except at a simple pole located at $s=1$, whose residue is

$$Res_{s=1}\zeta_{p,p'}(s,c)=\frac{\pi}{\sqrt{c}}. \tag{4.3}$$

Proof. The proof comes immediately from our first analogue of Watson's formula (3.3). We start by choosing $\mathrm{Re}(s)>\mu>1$ and by writing the generalized Epstein zeta function (4.1) in the following form

$$\zeta_{p,p'}(s,c)=\frac{2}{1+\frac{1}{\pi p'}}\,\zeta_p(2s)+\frac{2c^{-s}}{1+\frac{1}{\pi p}}\,\zeta_{p'}(2s)+4\sum_{m,n=1}^{\infty}\frac{\left(p^2+\lambda_n^2\right)\left(p'^2+\lambda_n'^2\right)}{\left(p(p+\frac{1}{\pi})+\lambda_n^2\right)\left(p'(p'+\frac{1}{\pi})+\lambda_n'^2\right)}\frac{1}{\left(\lambda_m^2+c\lambda_n'^2\right)^s}. \tag{4.4}$$

If, on the double series in (4.4), we fix the variable of summation n and sum over m by using (3.2) with x being replaced by $\sqrt{c}\lambda_n'$, we are able to deduce that

$$\zeta_{p,p'}(s,c)=\frac{2}{1+\frac{1}{\pi p'}}\,\zeta_p(2s)+\frac{2\sqrt{\pi}\,c^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta_{p'}(2s-1)+H_{p,p'}(s,c),\quad \mathrm{Re}(s)>1, \tag{4.5}$$

where

$$\frac{\Gamma(s)\,H_{p,p'}(s,c)}{8\pi^s c^{\frac{1}{4}-\frac{s}{2}}}=\sum_{m,n=1}^{\infty}\frac{\left(p'^2+\lambda_n'^2\right)\,(-1)^m}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\left(\frac{m}{\lambda_n'}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi m\,\lambda_n'\sqrt{c}\right)+$$

$$+\sum_{m,n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\left(\frac{m}{\lambda_n'}\right)^{s-\frac{1}{2}}e^{2\pi pm}\sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{m-\ell}\left(4\pi mp\right)^{\ell}\int_1^{\infty}t^{s-\frac{1}{2}}\,e^{-2\pi mpt}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,K_{s-\frac{1}{2}}\left(2\pi m\sqrt{c}\lambda_n' t\right)\,dt. \tag{4.6}$$

We now claim that the right-hand side of (4.5) constitutes the analytic continuation of $\zeta_{p,p'}(s,\,c)$ to the entire complex plane and when $\mathrm{Re}(s)<1$, it is reduced to (4.2). Since the continuations of $\zeta_p(s)$ and $\zeta_{p'}(s)$ are assured by Koshliakov's paper [18], we only need to focus on the continuation of $H_{p,p'}(s,c)$. This is now analogous to the case where we have treated (3.9) and it is possible to show that it defines an entire function of $s\in\mathbb{C}$.

Take now the expression defining $H_{p,p'}(s,c)$ (4.6) and assume that $\mathrm{Re}(s)<1$. We see from the computations leading to (3.13) that it can be written in the form

$$2^{4-2s}c^{\frac{1}{2}-s}\sin(\pi s)\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\lambda_n'^{1-2s}\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left(\lambda_n'\,(2y+1)\sqrt{c}\right)e^{2\pi\lambda_n'(2y+1)\sqrt{c}}-1}dy,$$

which gives (4.2).

By a standard verification of the right-hand side of (4.5), it is easily seen that $\zeta_{p,p'}(s,c)$ has removable singularities located at $s=\frac{1}{2}-k$, $k\in\mathbb{N}_0$. Since $\zeta_p(2s)$ and $\Gamma\left(s-\frac{1}{2}\right)$ are analytic in a neighborhood of the point $s=1$, we can conclude that $\zeta_{p,p'}(s,c)$ must have a pole located at $s=1$ (coming from the function $\zeta_{p'}(2s-1)$) with residue $\pi/\sqrt{c}$. □

Remark 4.1. It is possible to deform the path of integration on the right-hand side of (4.2) in order to make it valid for every $s\in\mathbb{C}$. See Remark 3.1. above.

Remark 4.2. It is clear from the proof of Corollary 3.1. (see the steps leading to (3.16)) that (4.2) and (4.6) are generalizations of the Selberg-Chowla formula (1.5) when $Q(m,n)=m^2+cn^2$, $c>0$.

Having proved that $\zeta_{p,p'}(s,c)$ possesses a pole at $s=1$, we now study how it behaves around this singularity, generalizing a classical formula due to Kronecker [24] (see (1.8) above).

Corollary 4.1. *Let $p,p'\in\mathbb{R}_+$ and let $\zeta_{p,p'}(s,c)$ be the generalized Epstein $\zeta-$function (4.1). Moreover, let $\sigma(t)$ be defined by (1.19). Then $\zeta_{p,p'}(s,c)$ admits the meromorphic expansion around $s=1$,*

$$\zeta_{p,p'}(s,\,c)=\frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi^2}{3}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2}+$$
$$+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log(4c)+4\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\cdot\frac{\lambda_n'^{-1}}{\sigma\left(\sqrt{c}\lambda_n'\right)e^{2\pi\sqrt{c}\lambda_n'}-1}\right)+O(s-1),\tag{4.7}$$

where $C_p^{(1)}$ is Koshliakov's analogue of the Euler-Mascheroni constant (3.20).

Proof. The proof consists in evaluating the right-hand side of (4.5) when $s\to1$. Let us recall the following expansions around $s=1$,

$$\frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)}=\pi-2\pi\log(2)\,(s-1)+O\,(s-1)^2\tag{4.8}$$

and

$$\zeta_{p'}(2s-1)=\frac{1}{2(s-1)}+C_{p'}^{(1)}+O\,(s-1)\,,\tag{4.9}$$

where $C_{p'}^{(1)}$ is given by (3.20). Combining (4.8) and (4.9), we see that the second term in (4.5) can be written as

$$\frac{2\sqrt{\pi}\,c^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta_{p'}(2s-1)=\frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log(4c)\right)+O\,(s-1)\,.$$

which gives, after the use of (4.5),

$$\zeta_{p,p'}(s,\,c)=\frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log(4c)\right)+\frac{\pi^2}{3}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2}+H_{p,p'}(1,c)+O\,(s-1)\tag{4.10}$$

where we have employed the identity $\zeta_p(2)=\frac{\pi^2}{6}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{(1+\frac{1}{\pi p})^2}$ [[18], p. 22, eq. (39)]. Using once more formula (3.29), and doing no more than the calculations made in (3.30), we can derive that

$$H_{p,p'}(1,c)=\frac{4\pi}{\sqrt{c}}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\cdot\frac{\lambda_n'^{-1}}{\sigma\left(\sqrt{c}\lambda_n'\right)e^{2\pi\sqrt{c}\lambda_n'}-1}\tag{4.11}$$

which proves (4.7). □

Remark 4.3. The infinite series appearing in (4.7) can be regarded as a generalization of the logarithm of Dedekind's $\eta-$function (1.7). Taking $p,p'\to\infty$ in our formula above, the classical limit formula of Kronecker (1.6) can be derived. Analogously to section 2 above, we can obtain several new analogues of Kronecker's limit formula by fixing one of the parameters and varying the other. Of course, when p,p' are both tending to zero or infinity (not necessary the same limit), the formulas resulting from (4.7) may also be achieved through Kronecker's second limit formula [[24], p. 22].

Our first corollary is given when we take $p'\to0^+$ while p is kept fixed.

Corollary 4.2. *For every $p\in\mathbb{R}_+$ and $c>0$, consider the Epstein zeta function defined by*

$$\zeta_{p,0}(s,c):=\sum_{m,n\neq0}\frac{p^2+\lambda_m^2}{p(p+\frac{1}{\pi})+\lambda_m^2}\,\frac{1}{\left(\lambda_m^2+c\,\left(n-\frac{1}{2}\right)^2\right)^s},\quad Re(s)>1.$$

Then $\zeta_{p,0}(s,c)$ admits the meromorphic expansion

$$\zeta_{p,0}(s,c)=\frac{\pi}{\sqrt{c}}\frac{1}{s-1}+\frac{\pi}{\sqrt{c}}\left(2\gamma-\log\left(\frac{c}{4}\right)+8\sum_{n=1}^{\infty}\frac{1}{2n-1}\cdot\frac{1}{\sigma\left(\sqrt{c}\lambda_n'\right)e^{(2n-1)\pi\sqrt{c}}-1}\right)+O(s-1).$$

In particular, the standard expansions holds

$$\zeta_{0,0}(s,c)=\frac{\pi}{\sqrt{c}}\frac{1}{s-1}+\frac{\pi}{\sqrt{c}}\left(2\gamma-\log\left(\frac{c}{4}\right)-8\sum_{n=1}^{\infty}\frac{1}{2n-1}\cdot\frac{1}{e^{(2n-1)\pi\sqrt{c}}+1}\right)+O(s-1).$$

Now, letting $p'\to\infty$ and fixing $p\in\mathbb{R}_+$, we deduce the following result.

Corollary 4.3. *For every $p\in\mathbb{R}_+$ and $c>0$, consider the Epstein zeta function defined by*

$$\zeta_{p,\infty}(s,c)=\sum_{m,n\neq 0}\frac{p^2+\lambda_n^2}{p(p+\frac{1}{\pi})+\lambda_n^2}\frac{1}{(\lambda_m^2+cn^2)^s},\quad Re(s)>1.$$

Then $\zeta_{p,\infty}(s,c)$ has the following meromorphic expansion around $s=1$,

$$\zeta_{p,\infty}(s,\,c)=\frac{\pi}{\sqrt{c}}\frac{1}{s-1}+\frac{\pi^2}{3}\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p}\right)^2}+\frac{\pi}{\sqrt{c}}\left(2\gamma-\log(4c)+4\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{\sigma\left(\sqrt{c}n\right)e^{2\pi\sqrt{c}n}-1}\right)+O(s-1).$$

In particular, the usual Kronecker limit formula (1.8) takes place. Moreover,

$$\zeta_{0,\infty}(s,c)=\frac{\pi}{\sqrt{c}}\frac{1}{s-1}+\pi^2+\frac{\pi}{\sqrt{c}}\left(2\gamma-\log(4c)-4\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{e^{2\pi\sqrt{c}n}+1}\right)+O(s-1)$$

If we now take $p\to\infty$ and fix $p'\in\mathbb{R}_+$, we can derive the following formula.

Corollary 4.4. *For every $p'\in\mathbb{R}_+$ and $c>0$, consider the Epstein zeta function defined by*

$$\zeta_{\infty,p'}(s,c):=\sum_{m,n\neq 0}\frac{p'^2+\lambda_n'^2}{p'(p'+\frac{1}{\pi})+\lambda_n'^2}\frac{1}{(m^2+c\lambda_n'^2)^s},\quad Re(s)>1.$$

Then $\zeta_{\infty,p'}(s,c)$ has a Laurent expansion around $s=1$ given by

$$\zeta_{\infty,p'}(s,\,c)=\frac{\pi}{\sqrt{c}}\frac{1}{s-1}+\frac{\pi^2}{3}\cdot\frac{1}{1+\frac{1}{\pi p'}}+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log(4c)+4\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\cdot\frac{\lambda_n'^{-1}}{e^{2\pi\sqrt{c}\lambda_n'}-1}\right)+O\left(s-1\right)$$

In particular, we have the expansion

$$\zeta_{\infty,0}(s,\,c)=\frac{\pi}{\sqrt{c}}\frac{1}{s-1}+\frac{\pi}{\sqrt{c}}\left(2\gamma-\log\left(\frac{c}{4}\right)+8\sum_{n=1}^{\infty}\frac{1}{2n-1}\cdot\frac{1}{e^{\pi\sqrt{c}(2n-1)}-1}\right)+O\left(s-1\right).$$

Our final corollary of the Kronecker limit formula now comes from taking $p\to 0^+$ on our generalized Kronecker's limit formula.

Corollary 4.5. *For every $p'\in\mathbb{R}_+$ and $c>0$, consider the Epstein zeta function defined by*

$$\zeta_{0,p'}(s,c):=\sum_{m,n\neq 0}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\frac{1}{\left(\left(m-\frac{1}{2}\right)^2+c\,\lambda_n'^2\right)^s},\quad Re(s)>1.$$

Then $\zeta_{0,p'}(s,c)$ admits the meromorphic expansion around $s=1$,

$$\zeta_{0,p'}(s,\,c)=\frac{\pi}{\sqrt{c}}\frac{1}{s-1}+\frac{\pi^2}{1+\frac{1}{\pi p'}}+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log(4c)-4\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\cdot\frac{\lambda_n'^{-1}}{e^{2\pi\sqrt{c}\lambda_n'}+1}\right)+O(s-1).$$

Although it is possible to write a functional equation for (4.1) using the general Selberg-Chowla formula (4.2), we shall omit it for this case because there is an asymmetry between the zeta functions on both sides of it. This asymmetry is due to the drastic differences between Koshliakov's zeta functions $\zeta_p(s)$ and $\eta_p(s)$.

Nevertheless, we end this section with an identity that $\zeta_{p,p'}(s,c)$ shares with the classical Epstein zeta function (1.4). This identity concerns its central value, $\zeta_{p,p'}\left(\frac{1}{2},c\right)$ and may be seen as an analogue of our extension of Ramanujan's formula (2.2).

By using an idea of Selberg and Chowla [10], but closely following the proof by Bateman and Grosswald [2], we shall use this formula to prove that, under the condition of large c, $\zeta_{p,p'}(s,c)$ will have a real zero lying on the interval $\left(\frac{1}{2},1\right)$.

Corollary 4.6. *Let $\sigma(t)$ be defined by (1.19). Then the following identity takes place*

$$\zeta_{p,p'}\left(\frac{1}{2},c\right)=\frac{2C_p^{(1)}+\log\left(\frac{c}{4}\right)-4e^{2\pi p'}Q_{2\pi p'}(0)}{1+\frac{1}{\pi p'}}+2C_{p'}^{(2)}-2\log(2\pi)-2\gamma+$$

$$+8\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\int_0^{\infty}\frac{1}{\sigma\left(\lambda_n'\left(2y+1\right)\sqrt{c}\right)e^{2\pi\lambda_n'(2y+1)\sqrt{c}}-1}\frac{dy}{\sqrt{y^2+y}},\tag{4.12}$$

where $C_p^{(2)}$ denotes Koshliakov's second analogue of the Euler-Mascheroni constant [[18], p. 46, eq. (47)],

$$C_p^{(2)}:=\lim_{n\to\infty}\left\{\sum_{k=1}^{n-1}\frac{(1,2\pi pk)_k}{k}-\frac{\log(n)}{1+\frac{1}{\pi p}}\right\}\tag{4.13}$$

and (in Koshliakov's notation),

$$Q_{\mu}(s)=\int_{\mu}^{\infty}t^{s-1}e^{-t}dt\tag{4.14}$$

denotes the incomplete gamma function $\Gamma(s,\mu)$ [[18], p. 25, eq. (49)].

Proof. We use our generalization of the Selberg-Chowla formula (4.2) and let $s=\frac{1}{2}$. We obtain,

$$\zeta_{p,p'}\left(\frac{1}{2},c\right)=c^{-1/4}\lim_{s\to\frac{1}{2}}\left\{\frac{2c^{s/2}\pi^{-s}\Gamma(s)}{1+\frac{1}{\pi p'}}\zeta_p(2s)+2\,c^{\frac{1-s}{2}}\,\pi^{-\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\zeta_{p'}(2s-1)\right\}+$$

$$+\,8\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\int_0^{\infty}\frac{1}{\sigma\left(\lambda_n'\left(2y+1\right)\sqrt{c}\right)e^{2\pi\lambda_n'(2y+1)\sqrt{c}}-1}\frac{dy}{\sqrt{y^2+y}}.\tag{4.15}$$

In order to compute the Laurent expansion of the first term in the braces, we invoke the well-known expansions around $s=\frac{1}{2}$, (3.22) and (3.23), as well as

$$\Gamma\left(s\right)=\sqrt{\pi}-\sqrt{\pi}\left(2\log(2)+\gamma\right)\left(s-\frac{1}{2}\right)+O\left(s-\frac{1}{2}\right)^2.\tag{4.16}$$

Furthermore, by [[18], pp. 48 and 49, eq. (51) and (54)], we have

$$\zeta_p(2s)=\frac{1}{2s-1}+C_p^{(1)}+O\left(s-\frac{1}{2}\right)\tag{4.17}$$

and

$$\zeta_p(2s-1)=\zeta_p(0)+2\zeta_p'\left(0\right)\left(s-\frac{1}{2}\right)+O\left(s-\frac{1}{2}\right)^2=$$

$$=-\frac{1}{2}\frac{1}{1+\frac{1}{\pi p}}+2\left\{-\frac{\log(2\pi)}{2}+\frac{1}{2}C_p^{(2)}-\frac{\gamma}{2}-\frac{e^{2\pi p}}{1+\frac{1}{\pi p}}Q_{2\pi p}(0)\right\}\left(s-\frac{1}{2}\right)+O\left(s-\frac{1}{2}\right)^2\tag{4.18}$$

where $Q_\mu(s)$ is given by (4.14).

We start by simplifying the first term in the braces: as one easily checks,

$$\frac{2}{1+\frac{1}{\pi p'}}\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\zeta_p(2s)=\frac{2c^{1/4}}{1+\frac{1}{\pi p'}}\left[1-\left(\gamma+\log\left(\frac{4\pi}{\sqrt{c}}\right)\right)\left(s-\frac{1}{2}\right)+O\left(s-\frac{1}{2}\right)^2\right]\times$$
$$\times\left[\frac{1}{2s-1}+C_p^{(1)}+O\left(s-\frac{1}{2}\right)\right]=\frac{2c^{1/4}}{1+\frac{1}{\pi p'}}\left[\frac{1}{2s-1}+C_p^{(1)}-\frac{1}{2}\left(\gamma+\log\left(\frac{4\pi}{\sqrt{c}}\right)\right)+O\left(s-\frac{1}{2}\right)\right]. \qquad (4.19)$$

Analogously, we find that the second term on the braces can be written as

$$2\,c^{\frac{1-s}{2}}\,\pi^{-(s-\frac{1}{2})}\Gamma\left(s-\frac{1}{2}\right)\zeta_{p'}(2s-1)=2c^{1/4}\left[\frac{1}{s-\frac{1}{2}}-\gamma-\log\left(\pi\sqrt{c}\right)+O\left(s-\frac{1}{2}\right)\right]\times$$
$$\times\left[-\frac{1}{2}\frac{1}{1+\frac{1}{\pi p'}}+2\zeta_{p'}'(0)\left(s-\frac{1}{2}\right)+O\left(s-\frac{1}{2}\right)^2\right]=2c^{1/4}\left[\frac{\zeta_{p'}(0)}{s-\frac{1}{2}}+2\zeta_{p'}'(0)-\gamma\zeta_{p'}(0)-\log(\pi\sqrt{c})\,\zeta_{p'}(0)+O\left(s-\frac{1}{2}\right)\right]. \qquad (4.20)$$

Finally, using the evaluation of $\zeta_{p'}'(0)$ given in (4.18) and bringing (4.19) and (4.20) together, we find (4.12) after some straightforward manipulations.

□

Corollary 4.7. *For $p'\to\infty$ and arbitrary $p\in\mathbb{R}_+$, consider the particular case of the Epstein zeta function (4.1)*

$$\zeta_{p,\infty}(s,c):=\sum_{m,n\neq 0}\frac{p^2+\lambda_n^2}{p(p+\frac{1}{\pi})+\lambda_n^2}\cdot\frac{1}{(\lambda_m^2+c\,n^2)^s},\quad Re(s)>1.$$

Then there exists some $c_0>16\pi^2e^{-2C_p^{(1)}}$ such that, for every $c\geq c_0$, $\zeta_{p,\infty}(s,c)$ has a real zero on the interval $\left(\frac{1}{2},1\right)$.

Proof. Like Bateman and Grosswald [2], we bound in a trivial manner the "double series" appearing as the last term on the expression (4.15) defining $\zeta_{p,p'}\left(\frac{1}{2},c\right)$. Indeed, since $\frac{p-t}{p+t}<1$ and $\lambda_n'>n-\frac{1}{2}$ for every $n\in\mathbb{N}$, we have

$$8\sum_{n=1}^{\infty}\left|\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\int_0^{\infty}\frac{1}{\sigma\left(\lambda_n'\,(2y+1)\sqrt{c}\right)e^{2\pi\lambda_n'(2y+1)\sqrt{c}}-1}\,\frac{dy}{\sqrt{y^2+y}}\right|\leq$$
$$\leq 8\sum_{n=1}^{\infty}\int_0^{\infty}\frac{1}{e^{2\pi\lambda_n'(2y+1)\sqrt{c}}-1}\,\frac{dy}{\sqrt{y^2+y}}=8\sum_{m,n=1}^{\infty}\int_0^{\infty}\frac{e^{-2\pi m\lambda_n'(2y+1)\sqrt{c}}}{\sqrt{y^2+y}}\,dy\leq$$
$$\leq 8\sum_{m,n=1}^{\infty}K_0\left(\pi m\,(2n-1)\sqrt{c}\right)\leq 8\sum_{n=1}^{\infty}d(n)\,K_0\left(\pi n\sqrt{c}\right). \qquad (4.21)$$

Using now a similar argument as in [[2], p. 371, eq. (20)], we get the simple bound

$$8\sum_{n=1}^{\infty}d(n)\,K_0\left(\pi n\sqrt{c}\right)<\frac{2^{5/2}\,e^{-\pi\sqrt{c}}}{c^{1/4}}. \qquad (4.22)$$

Therefore, for some $-1<\theta<1$, it follows from (4.12) and (4.22) that

$$\zeta_{p,p'}\left(\frac{1}{2},c\right)=\frac{2C_p^{(1)}+\log\left(\frac{c}{4}\right)-4e^{2\pi p'}Q_{2\pi p'}(0)}{1+\frac{1}{\pi p'}}+2C_{p'}^{(2)}-2\log(2\pi)-2\gamma+\frac{2^{5/2}\theta e^{-\pi\sqrt{c}}}{c^{1/4}}.$$

Thus, if $p' \to \infty$ we see from the previous expression that

$$\zeta_{p,\infty}\left(\frac{1}{2},c\right)=2C_p^{(1)}+\log\left(\frac{c}{4}\right)-2\log(2\pi)+\theta\,\frac{2^{5/2}e^{-\pi\sqrt{c}}}{c^{1/4}}>0$$

provided $c \geq c_0 > 16\pi^2 e^{-2C_p^{(1)}}$ and c_0 large enough. By (4.3), we know that $\lim_{s\to 1^-}\zeta_{p,p'}(s,c)=-\infty$ and so an immediate application of the intermediate value theorem shows that there exists some $\sigma_0 \in \left(\frac{1}{2},1\right)$ for which $\zeta_{p,\infty}(\sigma_0,c)=0$. □

4.2 A second Analogue of Epstein's zeta function

Let $c>0$, $p,\,p'>0$ and let $\lambda_n,\,\lambda_n'$ be the Koshliakov sequences satisfying the transcendental equations (2.1). For $\text{Re}(s)>1$, we define the second analogue of Epstein's zeta function as

$$\tilde{\zeta}_{p,p'}(s,c):=\frac{2c^{-s}}{1+\frac{1}{\pi p}}\,\zeta_{p'}(2s)+\frac{2}{1+\frac{1}{\pi p'}}\,\eta_p(2s)+\frac{8\,\pi^{s+1}c^{\frac{1}{4}-\frac{s}{2}}}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\lambda_n'^{\frac{1}{2}-s}\int_0^{\infty}\frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}\left(2\pi\lambda_n'\sqrt{c}\,y\right)}{\sigma\left(y\right)e^{2\pi y}-1}\,dy. \tag{4.23}$$

Although this definition seems to have a drastically different aspect to the classical Epstein zeta function (1.4), in view of the second analogue of Watson's formula (3.32), we may expect to recover the classical cases when p,p' tend to zero of infinity. In the next lemma we check that this is the case.

Lemma 4.1. *Let Re$(s)>1$ and $c>0$. Then we have the limiting cases*

$$\lim_{p,p'\to\infty}\tilde{\zeta}_{p,p'}(s,c)=\sum_{m,n\neq 0}\frac{1}{\left(m^2+cn^2\right)^s} \tag{4.24}$$

and

$$\lim_{p,p'\to 0^+}\tilde{\zeta}_{p,p'}(s,c)=\sum_{m\neq 0,n\neq 0}\frac{(-1)^m}{\left(m^2+c\left(n-\frac{1}{2}\right)^2\right)^s}. \tag{4.25}$$

Analogously,

$$\lim_{p\to 0^+,p'\to\infty}\tilde{\zeta}_{p,p'}(s,c)=\sum_{m\neq 0,\,n\in\mathbb{Z}}\frac{(-1)^m}{\left(m^2+c\,n^2\right)^s}, \tag{4.26}$$

$$\lim_{p\to\infty,p'\to 0^+}\tilde{\zeta}_{p,p'}(s,c)=\sum_{m\in\mathbb{Z},n\neq 0}\frac{1}{\left(m^2+c\,\left(n-\frac{1}{2}\right)^2\right)^s}. \tag{4.27}$$

Proof. It suffices to prove the case (4.24), the remaining ones being analogous. By definition (4.23) and by appealing to the steps given in (3.45) and using once more the Laplace representation (3.39), we obtain after some straightforward simplifications,

$$\begin{aligned}\lim_{p,p'\to\infty}\tilde{\zeta}_{p,p'}(s,c)&:=2c^{-s}\,\zeta(2s)+2\,\zeta(2s)+\frac{8\,\pi^{s+1}c^{\frac{1}{4}-\frac{s}{2}}}{\Gamma(s)}\sum_{n=1}^{\infty}n^{\frac{1}{2}-s}\int_0^{\infty}\frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}\left(2\pi n\sqrt{c}\,y\right)}{e^{2\pi y}-1}\,dy\\&=2c^{-s}\,\zeta(2s)+2\,\zeta(2s)+\frac{8\,\pi^{s+1}c^{\frac{1}{4}-\frac{s}{2}}}{\Gamma(s)}\sum_{m,n=1}^{\infty}n^{\frac{1}{2}-s}\int_0^{\infty}y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}\left(2\pi n\sqrt{c}\,y\right)\,e^{-2\pi my}dy\\&=2c^{-s}\,\zeta(2s)+2\,\zeta(2s)+4\sum_{m,n=1}^{\infty}\frac{1}{\left(m^2+cn^2\right)^s}=\sum_{m,n\neq 0}\frac{1}{\left(m^2+cn^2\right)^s},\quad\text{Re}(s)>1.\end{aligned} \tag{4.28}$$

□

Like in the previous section, we study the analytic continuation of the Epstein zeta function (4.23). It is also apparent from the limiting cases (4.24)-(4.27) that the analytic continuation of (4.23) should be similar to the analytic continuation of the Classical Epstein zeta function (1.4).

Our next result gives another generalization of the Selberg-Chowla formula (1.5) and the analytic continuation of $\tilde{\zeta}_{p,p'}(s,c)$.

Theorem 4.2. *For every $Re(s)>1$, the following generalization of the Selberg-Chowla formula takes place*

$$\tilde{\zeta}_{p,p'}(s,c)=\frac{2}{1+\frac{1}{\pi p'}}\,\eta_p(2s)+\frac{2\sqrt{\pi}\,c^{\frac{1}{2}-s}}{\Gamma(s)}\cdot\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p}}\,\zeta_{p'}(2s-1)+$$
$$+\frac{8\,\pi^s c^{\frac{1}{4}-\frac{s}{2}}}{\Gamma(s)}\sum_{m,n=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\,\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\left(\frac{\lambda_m}{\lambda_n'}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi\sqrt{c}\lambda_m\lambda_n'\right). \tag{4.29}$$

Furthermore, (4.29) provides the continuation of $\tilde{\zeta}_{p,p'}(s,c)$ as an analytic function for every $s\in\mathbb{C}$ except at a simple pole located at $s=1$, whose residue is

$$Res_{s=1}\tilde{\zeta}_{p,p'}(s,c)=\frac{\pi}{\sqrt{c}\left(1+\frac{1}{\pi p}\right)}. \tag{4.30}$$

Proof. The proof of (4.29) comes immediately from our second analogue of Watson's formula (3.32). Indeed, if we assume that $\text{Re}(s)>1$ and apply (3.32) to the third term of (4.23), we obtain

$$\tilde{\zeta}_{p,p'}(s,c):=\frac{2c^{-s}}{1+\frac{1}{\pi p}}\,\zeta_{p'}(2s)+2\eta_p(2s)+\frac{8\,\pi^s c^{\frac{1}{4}-\frac{s}{2}}}{\Gamma(s)}\sum_{m,n=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\cdot\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\left(\frac{\lambda_m}{\lambda_n'}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi\sqrt{c}\lambda_m\lambda_n'\right)$$
$$+\frac{8\pi^s c^{1/4-\frac{s}{2}}}{\Gamma(s)}\left\{\frac{\pi^{\frac{1}{2}-s}\left(\sqrt{c}\right)^{\frac{1}{2}-s}}{4}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p}}\,\zeta_{p'}(2s-1)-\frac{\Gamma(s)\pi^{-s}c^{-\frac{s}{2}-\frac{1}{4}}}{4}\zeta_{p'}(2s)\right\}=2\eta_p(2s)+\frac{2\sqrt{\pi}\,c^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)\zeta_{p'}(2s-1)}{\Gamma(s)(1+\frac{1}{\pi p})}+$$
$$+\frac{8\,\pi^s c^{\frac{1}{4}-\frac{s}{2}}}{\Gamma(s)}\sum_{m,n=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\,\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\left(\frac{\lambda_m}{\lambda_n'}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi\sqrt{c}\lambda_m\lambda_n'\right),$$

which proves (4.29). To deduce the analytic continuation of $\tilde{\zeta}_{p,p'}(s,c)$ from this, we recall that proving that

$$\tilde{H}_{p,p'}(s,c):=\sum_{m,n=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\,\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\left(\frac{\lambda_m}{\lambda_n'}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi\sqrt{c}\lambda_m\lambda_n'\right) \tag{4.31}$$

is entire requires no more than repeating the argument given in [[2], p. 368, Lemma 2]. Therefore, any potential meromorphic behavior of $\tilde{\zeta}_{p,p'}(s,c)$ is due to the function

$$\tilde{G}_{p,p'}(s,c):=\frac{2}{1+\frac{1}{\pi p'}}\,\eta_p(2s)+\frac{2\sqrt{\pi}\,c^{\frac{1}{2}-s}}{\Gamma(s)}\cdot\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p}}\,\zeta_{p'}(2s-1).$$

It is straightforward to check that $\tilde{G}_{p,p'}(s,c)$ has removable singularities at $s=\frac{1}{2}-k$, $k\in\mathbb{N}_0$. Since $\eta_p(2s)$ and $\Gamma\left(s-\frac{1}{2}\right)$ are analytic in a neighborhood of $s=1$, we conclude that $\tilde{\zeta}_{p,p'}(s,c)$ must have a simple pole at $s=1$ (coming from the contribution of $\zeta_{p'}(2s-1)$) with residue explicitly given by (4.30). □

Like in the previous section, we may get a generalization of Kronecker's limit formula (1.8) in another direction. The following corollary provides such generalization.

Corollary 4.8. *Let $p,\ p'\in\mathbb{R}_+$, $c>0$ and let $\tilde{\zeta}_{p,p'}(s,c)$ be the generalized Epstein zeta function (4.1). Then $\tilde{\zeta}_{p,p'}(s,c)$ admits the following meromorphic expansion around $s=1$,*

$$\zeta_{p,p'}(s,\,c)=\frac{\pi}{\sqrt{c}\left(1+\frac{1}{\pi p}\right)}\,\frac{1}{s-1}+\frac{2}{1+\frac{1}{\pi p'}}\,\eta_p(2)+$$
$$+\frac{\pi}{\sqrt{c}}\left(\frac{2C_{p'}^{(1)}-\log\left(4c\right)}{1+\frac{1}{\pi p}}+4\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\lambda_n'^{-1}\,\sigma_p\left(2\pi\sqrt{c}\lambda_n'\right)\right)+O(s-1), \tag{4.32}$$

where $\sigma_p(z)$ is defined by (1.20).

Proof. Like in the proof of Corollary 4.1., we appeal to the Laurent expansions (4.8) and (4.9), which show that

$$\frac{2\sqrt{\pi}\,c^{\frac{1}{2}-s}}{\Gamma(s)}\cdot\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p}}\,\zeta_{p'}(2s-1)=\frac{\pi}{\sqrt{c}\left(1+\frac{1}{\pi p}\right)}\,\frac{1}{s-1}+\frac{\pi}{\sqrt{c}\left(1+\frac{1}{\pi p}\right)}\left(2C_{p'}^{(1)}-\log\left(4c\right)\right)+O\left(s-1\right). \quad (4.33)$$

Therefore, using the generalized Selberg-Chowla formula (4.29) and (3.29) and using the limit $s\rightarrow 1$, we conclude

$$\tilde{\zeta}_{p,p'}(s,c)=\frac{2}{1+\frac{1}{\pi p'}}\,\eta_p(2)+\frac{1}{1+\frac{1}{\pi p}}\left\{\frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log\left(4c\right)\right)\right\}$$
$$+\frac{4\pi}{\sqrt{c}}\sum_{m,n=1}^{\infty}\frac{\left(p^2+\lambda_m^2\right)\left(p'^2+\lambda_n'^2\right)\,\lambda_n'^{-1}}{\left(p\left(p+\frac{1}{\pi}\right)+\lambda_m^2\right)\left(p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2\right)}\cdot e^{-2\pi\sqrt{c}\lambda_m\lambda_n'}+O\left(s-1\right).$$

One finally obtains (4.32) after using the definition of $\sigma_p(z)$ (1.20). □

We may use the previous result to derive more analogues of Corollaries 4.2, 4.3., 4.4. and 4.5. Unlike the previous analogue of the Epstein zeta function, the symmetries provided by the Selberg-Chowla formula (4.29) can be used to derive (as an easy consequence) a symmetric functional equation for $\tilde{\zeta}_{p,p'}(s,c)$.

Corollary 4.9. *For every $s\in\mathbb{C}$, the following functional equation holds*

$$\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\tilde{\zeta}_{p,p'}\left(s,c\right)=\left(\frac{\pi}{\sqrt{c}}\right)^{-(1-s)}\Gamma\left(1-s\right)\,\tilde{\zeta}_{p',p}\left(1-s,c\right). \quad (4.34)$$

Proof. Indeed, using the Selberg-Chowla formula (4.29) with s being replaced by $1-s$, we are able to see that

$$\tilde{\zeta}_{p,p'}\left(1-s,c\right)=\frac{2}{1+\frac{1}{\pi p'}}\,\eta_p(2-2s)+\frac{2\sqrt{\pi}\,c^{s-\frac{1}{2}}}{\Gamma(1-s)}\cdot\frac{\Gamma\left(\frac{1}{2}-s\right)}{1+\frac{1}{\pi p}}\,\zeta_{p'}(1-2s)+$$
$$+\frac{8\,\pi^{1-s}c^{\frac{s}{2}-\frac{1}{4}}}{\Gamma(1-s)}\sum_{m,n=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\,\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\left(\frac{\lambda_n'}{\lambda_m}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi\sqrt{c}\,\lambda_m\lambda_n'\right). \quad (4.35)$$

Assume now that $\mathrm{Re}(s)>1$: if we start by evaluating the Bessel series on the right-hand side of (4.35) by summing first with respect to the variable of summation n, we get (according to the second analogue of Watson's formula (3.32)),

$$\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\lambda_n'^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}\left(2\pi\lambda_m\sqrt{c}\lambda_n'\right)=$$
$$=-\frac{\pi^{\frac{1}{2}-s}(\lambda_m\sqrt{c})^{\frac{1}{2}-s}}{4}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{1+\frac{1}{\pi p'}}+\frac{\Gamma(s)\pi^{-s}\left(\lambda_m\sqrt{c}\right)^{-s-\frac{1}{2}}}{4}+\pi\int_0^{\infty}\frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}\left(2\pi\lambda_m\sqrt{c}y\right)}{\sigma'\left(y\right)e^{2\pi y}-1}\,dy. \quad (4.36)$$

Sum now the right-hand side of the previous equality with respect to m and use the hypothesis $\mathrm{Re}(s)>1$ to obtain (here $\mathcal{A}:=\tilde{\zeta}_{p,p'}\left(1-s,c\right)$)

$$\mathcal{A}=\frac{2\eta_p(2-2s)}{1+\frac{1}{\pi p'}}+\frac{2\sqrt{\pi}\,c^{s-\frac{1}{2}}}{\Gamma(1-s)}\cdot\frac{\Gamma\left(\frac{1}{2}-s\right)\zeta_{p'}(1-2s)}{1+\frac{1}{\pi p}}+\frac{8\,\pi^{1-s}c^{\frac{s}{2}-\frac{1}{4}}}{\Gamma(1-s)}\left[\frac{\Gamma(s)\pi^{-s}c^{-\frac{s}{2}-\frac{1}{4}}\zeta_p(2s)}{4}-\frac{\pi^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)c^{\frac{1}{4}-\frac{s}{2}}\zeta_p(2s-1)}{4\left(1+\frac{1}{\pi p'}\right)}\right]$$
$$+\frac{8\,\pi^{2-s}c^{\frac{s}{2}-\frac{1}{4}}}{\Gamma(1-s)}\sum_{m=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\lambda_m^{\frac{1}{2}-s}\int_0^{\infty}\frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}\left(2\pi\lambda_m\sqrt{c}y\right)}{\sigma'\left(y\right)e^{2\pi y}-1}\,dy.$$

Appealing to the functional equation for $\zeta_p(2s-1)$, (1.16), we are able to simplify the previous expression to:

$$\tilde{\zeta}_{p,p'}\left(1-s,c\right)=\frac{2\sqrt{\pi}\,c^{s-\frac{1}{2}}}{\Gamma(1-s)}\cdot\frac{\Gamma\left(\frac{1}{2}-s\right)}{1+\frac{1}{\pi p}}\,\zeta_{p'}(1-2s)+\frac{8\,\pi^{1-s}c^{\frac{s}{2}-\frac{1}{4}}}{\Gamma(1-s)}\times$$
$$\times\left[\frac{\Gamma(s)\pi^{-s}c^{-\frac{s}{2}-\frac{1}{4}}}{4}\zeta_p(2s)+\pi\sum_{m=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\lambda_m^{\frac{1}{2}-s}\int_0^{\infty}\frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}\left(2\pi\lambda_m\sqrt{c}y\right)}{\sigma'\left(y\right)e^{2\pi y}-1}\,dy\right]. \quad (4.37)$$

Henceforth, we see that (4.37) gives

$$\left(\frac{\pi}{\sqrt{c}}\right)^{-(1-s)}\Gamma(1-s)\,\tilde{\zeta}_{p,p'}(1-s,c)=2\pi^{s-\frac{1}{2}}c^{\frac{s}{2}}\cdot\frac{\Gamma\left(\frac{1}{2}-s\right)}{1+\frac{1}{\pi p}}\zeta_{p'}(1-2s)+2\Gamma(s)\pi^{-s}c^{-\frac{s}{2}}\zeta_p(2s)$$
$$+8\pi c^{\frac{1}{4}}\sum_{m=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\lambda_m^{\frac{1}{2}-s}\int_0^{\infty}\frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}\left(2\pi\lambda_m\sqrt{c}y\right)}{\sigma'(y)\,e^{2\pi y}-1}\,dy. \tag{4.38}$$

Invoking now the definition of the second analogue of Epstein's zeta function, (4.23) (with p and p' being switched), we are also able to check that

$$\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\tilde{\zeta}_{p',p}(s,c):=\frac{2c^{-s/2}\,\pi^{-s}\Gamma(s)\,\zeta_p(2s)}{1+\frac{1}{\pi p'}}+\frac{2c^{s/2}\,\pi^{-s}\Gamma(s)\eta_{p'}(2s)}{1+\frac{1}{\pi p}}+$$
$$+\,8\pi c^{\frac{1}{4}}\sum_{m=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\lambda_m^{\frac{1}{2}-s}\int_0^{\infty}\frac{y^{s-\frac{1}{2}}J_{s-\frac{1}{2}}\left(2\pi\lambda_m\sqrt{c}y\right)}{\sigma'(y)\,e^{2\pi y}-1} \tag{4.39}$$

for every $\text{Re}(s)>1$. But the right-hand sides of (4.37) and (4.39) coincide, due to the functional equation for $\zeta_{p'}(1-2s)$. Therefore, equating both sides of (4.37) and (4.39) (and replacing p by p') gives (4.34). □

As a particular case of the previous theorem, we can write a simple functional equation for the particular case of (4.1) studied in Corollary 4.4.

Corollary 4.10. *For $c>0$, let $\zeta_{p,p'}(s,c)$ be the first analogue of Epstein's zeta function (4.1). Furthermore, for every $p'\in\mathbb{R}_+$, let*

$$\zeta_{\infty,p'}(s,c):=\sum_{m,n\neq 0}\frac{p'^2+\lambda_n'^2}{p'(p'+\frac{1}{\pi})+\lambda_n'^2}\frac{1}{(m^2+c\lambda_n'^2)^s},\quad Re(s)>1.$$

Then the analytic continuation of $\zeta_{\infty,p'}(s,c)$ satisfies the functional equation

$$\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\zeta_{\infty,p'}(s,c)=\left(\frac{\pi}{\sqrt{c}}\right)^{-(1-s)}\Gamma(1-s)\,\tilde{\zeta}_{p',\infty}(1-s,c)\,. \tag{4.40}$$

Proof. Indeed, by letting $p\to\infty$, we see from a simple adaptation of (4.28) that

$$\tilde{\zeta}_{\infty,p'}(s,c):=\lim_{p\to\infty}\tilde{\zeta}_{p,p'}(s,c)=\sum_{m,n\neq 0}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\frac{1}{(m^2+c\lambda_n'^2)^s}=\zeta_{\infty,p'}(s,c),\quad \text{Re}(s)>1.$$

Therefore, (4.40) follows from (4.34). □

We end this section by establishing an analogue of Corollary 4.6. for $\tilde{\zeta}_{p,p'}(s,c)$.

Corollary 4.11. *Let $\sigma(t)$ be defined by (1.19). Then the following identity takes place,*

$$\tilde{\zeta}_{p,p'}\left(\frac{1}{2},c\right)=\frac{2C_p^{(2)}-\gamma-\log\left(\frac{4\pi}{\sqrt{c}}\right)}{1+\frac{1}{\pi p'}}+\frac{2C_{p'}^{(2)}-2\log(2\pi)-2\gamma}{1+\frac{1}{\pi p}}+\frac{\log(\pi\sqrt{c})+\gamma-4e^{2\pi p}\,Q_{2\pi p}(0)-4e^{2\pi p'}Q_{2\pi p'}(0)}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)}$$
$$+\,8\sum_{m,n=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\cdot\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}K_0\left(2\pi\sqrt{c}\lambda_m\lambda_n'\right), \tag{4.41}$$

where $C_p^{(2)}$ and $Q_\mu(s)$ are respectively defined by (4.13) and (4.14).

Proof. Using the second Selberg-Chowla formula (4.29), we find that, in an analogous way to (4.15),

$$\tilde{\zeta}_{p,p'}\left(\frac{1}{2},c\right)=c^{-1/4}\lim_{s\to\frac{1}{2}}\left\{\frac{2c^{s/2}}{1+\frac{1}{\pi p'}}\pi^{-s}\,\Gamma(s)\,\eta_p(2s)+\frac{2c^{\frac{1-s}{2}}}{1+\frac{1}{\pi p}}\pi^{-\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\zeta_{p'}(2s-1)\right\}$$
$$+\,8\sum_{m,n=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\cdot\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}K_0\left(2\pi\sqrt{c}\lambda_m\lambda_n'\right).$$

We have seen already (c.f. (4.20) above) that the second term on the braces has the Laurent expansion

$$\frac{2c^{\frac{1-s}{2}}}{1+\frac{1}{\pi p}}\,\pi^{-\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\,\zeta_{p'}(2s-1)=$$

$$=\frac{2c^{1/4}}{1+\frac{1}{\pi p}}\left\{\frac{\zeta_{p'}(0)}{s-\frac{1}{2}}+2\zeta_{p'}'(0)-\gamma\zeta_{p'}(0)-\log(\pi\sqrt{c})\,\zeta_{p'}(0)+O\left(s-\frac{1}{2}\right)\right\}, \tag{4.42}$$

with $\zeta_{p'}'(0)$ being given in the expression (4.18). Therefore, we just need to find the Laurent expansion of the first term in the braces. Invoking (4.16) and [[18], p. 49, eq. (53)],

$$\eta_p(2s)=\frac{1}{1+\frac{1}{\pi p}}\cdot\frac{1}{2s-1}+C_p^{(2)}-\frac{2e^{2\pi p}}{1+\frac{1}{\pi p}}Q_{2\pi p}(0)+O\left(s-\frac{1}{2}\right), \tag{4.43}$$

we find, just like in (4.19),

$$\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\eta_p(2s)=c^{1/4}\left[\frac{1}{1+\frac{1}{\pi p}}\cdot\frac{1}{2s-1}+C_p^{(2)}-\frac{2e^{2\pi p}}{1+\frac{1}{\pi p}}Q_{2\pi p}(0)-\frac{1}{2}\left(\gamma+\log\left(\frac{4\pi}{\sqrt{c}}\right)\right)+O\left(s-\frac{1}{2}\right)\right]. \tag{4.44}$$

By combining (4.42) and (4.44), (4.41) follows. □

5 Generalizations of Entries 3.3.1 - 3.3.3., pp. 253-254 of Ramanujan's Lost Notebook

Building on the work of the previous two sections, we now establish a generalization of the Ramanujan-Guinand formula[5] (1.2), stated at the introduction as "Entry 3.3.1.". However, before stating it, we need a Lemma that motivates the general aspect of the Modified Bessel functions showing up in (1.2).

Lemma 5.1. *For $x>0$ and $Re(\nu)<\frac{1}{2}$, let*

$$\mathcal{K}_{\nu,p}(x)=\int_0^{\infty}\frac{y^{-\nu-\frac{1}{2}}(y+1)^{-\nu-\frac{1}{2}}}{\sigma\left(\frac{x}{2\pi}\,(2y+1)\,\right)e^{(2y+1)x}-1}dy,\quad x>0,\ Re(\nu)<\frac{1}{2}, \tag{5.1}$$

where $\sigma(t)$ is defined by (1.19).

Then one has the limiting cases:

$$\lim_{p\to\infty}\mathcal{K}_{\nu,p}(x)=\frac{\Gamma\left(\frac{1}{2}-\nu\right)(2x)^{\nu}}{\sqrt{\pi}}\sum_{n=1}^{\infty}n^{\nu}\,K_{\nu}(xn), \tag{5.2}$$

while

$$\lim_{p\to 0^+}\mathcal{K}_{\nu,p}(x)=\frac{\Gamma\left(\frac{1}{2}-\nu\right)(2x)^{\nu}}{\sqrt{\pi}}\sum_{n=1}^{\infty}(-1)^n\,n^{\nu}\,K_{\nu}(xn). \tag{5.3}$$

Proof. Indeed, by a simple use of the dominated convergence theorem, the power series (2.15), the absolute convergence of the iteration of the series and the integral for $\mathrm{Re}(\nu)<\frac{1}{2}$ and finally the integral representation (3.12), we obtain

$$\lim_{p\to\infty}\mathcal{K}_{\nu,p}(x)=\int_0^{\infty}\frac{y^{-\nu-\frac{1}{2}}(y+1)^{-\nu-\frac{1}{2}}}{e^{(2y+1)x}-1}dy=\sum_{n=1}^{\infty}\int_0^{\infty}y^{-\nu-\frac{1}{2}}(y+1)^{-\nu-\frac{1}{2}}\,e^{-(2y+1)xn}dy$$

$$=\frac{\Gamma\left(\frac{1}{2}-\nu\right)(2x)^{\nu}}{\sqrt{\pi}}\sum_{n=1}^{\infty}n^{\nu}\,K_{\nu}(xn),$$

[5] As stated in footnote 1, page 6, of this paper, in this section $\sigma_k(n)$ will always denote the usual divisor function and not Koshliakov's function (1.20).

while

$$\lim_{p\to 0^+}\mathcal{K}_{\nu,p}(x) = -\int_0^\infty \frac{y^{-\nu-\frac{1}{2}}(y+1)^{-\nu-\frac{1}{2}}}{e^{(2y+1)x}+1}dy = \sum_{n=1}^\infty (-1)^n \int_0^\infty y^{-\nu-\frac{1}{2}}(y+1)^{-\nu-\frac{1}{2}}\, e^{-(2y+1)xn}dy$$

$$= \frac{\Gamma\left(\frac{1}{2}-\nu\right)(2x)^\nu}{\sqrt{\pi}}\sum_{n=1}^\infty (-1)^n\, n^\nu\, K_\nu(xn).$$

□

Theorem 5.1. *If α and β are positive numbers such that $\alpha\beta=\pi^2$ and if s is a complex number such that $Re(s)<1$, then the following identity takes place*

$$\frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\alpha^{\frac{1-s}{2}}\sum_{n=1}^\infty \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{-s}\,\mathcal{K}_{\frac{s}{2},p}\left(2\lambda_n\alpha\right) - \frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\beta^{\frac{1-s}{2}}\sum_{n=1}^\infty \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{-s}\,\mathcal{K}_{\frac{s}{2},p}\left(2\lambda_n\beta\right)$$
$$= \frac{\Gamma\left(-\frac{s}{2}\right)\eta_p(-s)}{4\left(1+\frac{1}{\pi p}\right)}\left\{\beta^{\frac{1+s}{2}}-\alpha^{\frac{1+s}{2}}\right\} + \frac{\Gamma\left(\frac{s}{2}\right)\zeta_p(s)}{4}\left\{\beta^{\frac{1-s}{2}}-\alpha^{\frac{1-s}{2}}\right\}, \quad (5.4)$$

where $\mathcal{K}_{\nu,p}(x)$ denotes the integral defined by (5.1).

Proof. In Theorem 4.1. we have seen a representation (namely (4.2)) which extends the Epstein zeta function, $\zeta_{p,p'}(s,c)$, to the half-plane $\mathrm{Re}(s)<1$. It was deduced by applying the first analogue of Watson's formula to the sum with respect to the variable of summation m. But in the region of absolute convergence of $\zeta_{p,p'}(s,c)$ it is possible to reverse the orders of summation: indeed, for $\mathrm{Re}(s)>1$,

$$\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\zeta_{p,p'}(s,c) = \left(\frac{\pi}{\sqrt{1/c}}\right)^{-s}\Gamma(s)\sum_{m,n\neq 0}\frac{\left(p^2+\lambda_n^2\right)\cdot\left(p'^2+\lambda_n'^2\right)}{\left(p(p+\frac{1}{\pi})+\lambda_n^2\right)\cdot\left(p'(p'+\frac{1}{\pi})+\lambda_n'^2\right)}\frac{1}{\left(\lambda_n'^2+\lambda_m^2c^{-1}\right)^s}$$
$$= \left(\frac{\pi}{\sqrt{1/c}}\right)^{-s}\Gamma(s)\,\zeta_{p',p}\left(s,\frac{1}{c}\right). \quad (5.5)$$

Therefore, an alternative representation of $\zeta_{p,p'}(s,c)$ can be obtained once we apply Theorem 4.1. to $\zeta_{p',p}\left(s,\frac{1}{c}\right)$. In fact, using (4.2) with c being replaced by $1/c$ and with p being replaced by p', we find that an equivalent continuation of $\zeta_{p,p'}(s,c)$ to the half-plane $\mathrm{Re}(s)<1$ is

$$\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\zeta_{p,p'}(s,\,c) = \frac{2c^{-s/2}\,\pi^{-s}\Gamma(s)}{1+\frac{1}{\pi p}}\zeta_{p'}(2s) + 2\,c^{\frac{s-1}{2}}\,\pi^{-\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\zeta_p(2s-1)+$$
$$+\frac{2^{4-2s}\,\pi^{1-s}\,c^{\frac{s-1}{2}}}{\Gamma(1-s)}\sum_{n=1}^\infty\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{1-2s}\int_0^\infty \frac{y^{-s}(y+1)^{-s}}{\sigma'\left(\frac{\lambda_n}{\sqrt{c}}(2y+1)\right)e^{\frac{2\pi\lambda_n}{\sqrt{c}}(2y+1)}-1}\,dy. \quad (5.6)$$

Since (4.2) and (5.6) both represent the analytic continuation of $\zeta_{p,p'}(s,c)$ to the region $\mathrm{Re}(s)<1$, by uniqueness of the continuation we must have

$$\frac{2c^{s/2}\,\pi^{-s}\Gamma(s)}{1+\frac{1}{\pi p'}}\zeta_p(2s) + 2\,c^{\frac{1-s}{2}}\,\pi^{-\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\zeta_{p'}(2s-1)+$$
$$+\frac{2^{4-2s}\,\pi^{1-s}\,c^{\frac{1-s}{2}}}{\Gamma(1-s)}\sum_{n=1}^\infty\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\lambda_n'^{1-2s}\int_0^\infty \frac{y^{-s}(y+1)^{-s}}{\sigma\left(\lambda_n'(2y+1)\sqrt{c}\right)e^{2\pi\lambda_n'(2y+1)\sqrt{c}}-1}dy$$
$$= \frac{2c^{-s/2}\,\pi^{-s}\Gamma(s)}{1+\frac{1}{\pi p}}\zeta_{p'}(2s) + 2\,c^{\frac{s-1}{2}}\,\pi^{-\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\zeta_p(2s-1)+$$
$$+\frac{2^{4-2s}\,\pi^{1-s}\,c^{\frac{s-1}{2}}}{\Gamma(1-s)}\sum_{n=1}^\infty\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{1-2s}\int_0^\infty \frac{y^{-s}(y+1)^{-s}}{\sigma'\left(\frac{\lambda_n}{\sqrt{c}}(2y+1)\right)e^{\frac{2\pi\lambda_n'}{\sqrt{c}}(2y+1)}-1}\,dy,\ \mathrm{Re}(s)<1. \quad (5.7)$$

Take the substitution $c=\alpha^2/\pi^2$, replace s by $\frac{1+s}{2}$ and let $p=p'$ in (5.7): recalling the condition $\alpha\beta=\pi^2$, we may reduce (5.7) to

$$\frac{\pi^{-(s+1)}\,\Gamma\left(\frac{s+1}{2}\right)}{4\left(1+\frac{1}{\pi p}\right)}\,\zeta_p(s+1)\left\{\alpha^{\frac{1+s}{2}}-\beta^{\frac{1+s}{2}}\right\}+\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\zeta_p(s)\left\{\alpha^{\frac{1-s}{2}}-\beta^{\frac{1-s}{2}}\right\}$$

$$=\frac{2^{-s}}{\Gamma\left(\frac{1-s}{2}\right)}\beta^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{-s}\int_0^{\infty}\frac{y^{-\frac{s+1}{2}}(y+1)^{-\frac{s+1}{2}}}{\sigma\left(\frac{\lambda_n\beta}{\pi}\,(2y+1)\right)e^{2\beta\lambda_n(2y+1)}-1}\,dy-$$

$$-\frac{2^{-s}}{\Gamma\left(\frac{1-s}{2}\right)}\alpha^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{-s}\int_0^{\infty}\frac{y^{-\frac{s+1}{2}}(y+1)^{-\frac{s+1}{2}}}{\sigma\left(\frac{\lambda_n\alpha}{\pi}\,(2y+1)\right)e^{2\alpha\lambda_n(2y+1)}-1}\,dy \tag{5.8}$$

Now, (5.4) is finally obtained by invoking the functional equation to $\zeta_p(s+1)$ on the first term lying on the left of (5.8) and invoking the definition of $\mathcal{K}_{\nu,p}(x)$ (5.1). □

Remark 5.1. By taking a proper deformation of the contour defining the integral (5.1), it is possible to have a formula similar to (5.4) and valid for every $s\in\mathbb{C}$. See Remark 3.2. above.

With a slight change in the sequence of Ramanujan's statements in [21], we now establish a generalized version of Entry 3.3.3., also known as Koshliakov's formula (1.10).

Corollary 5.1. *Let α and β denote positive numbers such that $\alpha\beta=\pi^2$. Define*

$$\gamma_p:=C_p^{(2)}+\frac{C_p^{(1)}}{1+\frac{1}{\pi p}}-\gamma-\frac{2e^{2\pi p}\,Q_{2\pi p}(0)}{1+\frac{1}{\pi p}}-\log(2\pi)\left(1-\frac{1}{1+\frac{1}{\pi p}}\right), \tag{5.9}$$

where $C_p^{(1)}$, $C_p^{(2)}$ and $Q_\mu(s)$ are respectively given by (3.20), (4.13) and (4.14).

Then the following generalization of Entry 3.3.3, (1.10), holds

$$\sqrt{\alpha}\left(\frac{1}{4}\gamma_p-\frac{\log(4\beta)}{4\left(1+\frac{1}{\pi p}\right)}+\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\mathcal{K}_{0,p}\left(2\alpha\lambda_n\right)\right)=\sqrt{\beta}\left(\frac{1}{4}\gamma_p-\frac{\log(4\alpha)}{4\left(1+\frac{1}{\pi p}\right)}+\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\mathcal{K}_{0,p}\left(2\beta\lambda_n\right)\right), \tag{5.10}$$

where $\mathcal{K}_{\nu,p}(x)$ denotes the integral defined by (5.1).

Proof. The proof comes from letting $s\to0$ in the generalized Ramanujan-Guinand's formula (5.4) and from invoking the Laurent expansion for $\Gamma(s)$ around $s=0$, (3.23), together with [[18], p. 22, pp. 48 and 49, eqs. (52), (54)]

$$\zeta_p(s)=-\frac{1}{2}\frac{1}{1+\frac{1}{\pi p}}+\left(\frac{1}{2}C_p^{(2)}-\frac{\gamma}{2}-\frac{e^{2\pi p}}{1+\frac{1}{\pi p}}Q_{2\pi p}(0)-\frac{\log(2\pi)}{2}\right)s+O\left(s^2\right) \tag{5.11}$$

and

$$\eta_p(s)=-\frac{1}{2}+\left(\frac{1}{2}C_p^{(1)}-\frac{\gamma}{2}-\frac{\log(2\pi)}{2}\right)s+O\left(s^2\right). \tag{5.12}$$

Using these expansions under the limit $s\to0$ and rearranging the terms gives (5.10). □

By letting $p\to0^+$ or $p\to\infty$, we are able to deduce formulas akin to the classical ones, because in these cases the integral (5.1) can be written as a series of Bessel functions (see Lemma 5.1. above).

Corollary 5.2. *If α and β are positive numbers such that $\alpha\beta=\pi^2$ and if s is any complex number, then the following identities take place*

$$\sqrt{\alpha}\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{s}{2}}K_{\frac{s}{2}}\left(2m\,n\alpha\right)-\sqrt{\beta}\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{s}{2}}K_{\frac{s}{2}}\left(2m\,n\beta\right)=$$

$$\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\left\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\right\}+\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\left\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\right\}, \tag{5.13}$$

$$\sqrt{\alpha}\sum_{m,n=1}^{\infty}\frac{(-1)^m m^{s/2}}{(2n-1)^{s/2}}K_{\frac{s}{2}}\left((2n-1)\,m\,\alpha\right)-\sqrt{\beta}\sum_{m,n=1}^{\infty}\frac{(-1)^m m^{s/2}}{(2n-1)^{s/2}}K_{\frac{s}{2}}\left((2n-1)\,m\,\beta\right)$$
$$=\frac{\Gamma\left(\frac{s}{2}\right)\left(2^{s/2}-2^{-s/2}\right)\zeta(s)}{4}\left\{\beta^{\frac{1-s}{2}}-\alpha^{\frac{1-s}{2}}\right\}. \tag{5.14}$$

Moreover, we have

$$\sqrt{\alpha}\left(\frac{1}{4}\gamma-\frac{1}{4}\log(4\beta)+\sum_{m,n=1}^{\infty}K_0(2mn\,\alpha)\right)=\sqrt{\beta}\left(\frac{1}{4}\gamma-\frac{1}{4}\log(4\alpha)+\sum_{m,n=1}^{\infty}K_0(2mn\,\beta)\right), \tag{5.15}$$

$$\sqrt{\alpha}\left(\sum_{m,n=1}^{\infty}(-1)^m\,K_0\left((2n-1)\,m\,\alpha\right)-\frac{\log(2)}{4}\right)=\sqrt{\beta}\left(\sum_{m,n=1}^{\infty}(-1)^m K_0\left((2n-1)\,m\,\beta\right)-\frac{\log(2)}{4}\right). \tag{5.16}$$

Proof. We only prove the classical Ramanujan-Guinand formula (5.13) from our generalized version of Entry 3.3.1. The proofs of the remaining formulas (5.14), (5.15) and (5.16) follow the same principle. The only part that it is not immediate is the recovery of the Modified Bessel function appearing in (5.13) from the left-hand side of (5.4). However, this was done in Lemma 5.1. above. Still, assuming that $\mathrm{Re}(s)<1$ and justifying once more all the intermediate steps by absolute convergence, we find that the first term on the left of (5.4) is

$$\frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\alpha^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^{\infty}\frac{y^{-\frac{s+1}{2}}(y+1)^{-\frac{s+1}{2}}}{e^{2\alpha n(2y+1)}-1}dy=\frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\alpha^{\frac{1-s}{2}}\sum_{m,n=1}^{\infty}\frac{1}{n^s}\int_0^{\infty}e^{-2\alpha m\,n(2y+1)}y^{-\frac{s+1}{2}}(y+1)^{-\frac{s+1}{2}}dy$$
$$=\sqrt{\alpha}\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{s}{2}}K_{\frac{s}{2}}\left(2m\,n\alpha\right)=\sqrt{\alpha}\sum_{n=1}^{\infty}\sum_{d|n}d^{-s}\,n^{\frac{s}{2}}\,K_{\frac{s}{2}}\left(2n\alpha\right)=\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}\left(2n\alpha\right). \tag{5.17}$$

Note now that the expression on the extreme right-hand side of (5.17) defines a series that converges uniformly and absolutely for every $s\in\mathbb{C}$. Since its summands are analytic functions of s, this series defines an analytic function of s. By the principle of analytic continuation, we now see that (5.13) must hold for every $s\in\mathbb{C}$. ☐

We now note that our proof of Theorem 5.1. can be actually used to prove a more general result than (5.4). Note that we can use the identity (5.7), take the substitutions $c=\alpha^2/\pi^2$, $s\leftrightarrow\frac{1+s}{2}$ but not imposing that $p=p'$. In this case we get the following result.

Theorem 5.2. *If α and β are positive numbers such that $\alpha\beta=\pi^2$ and if s is a complex number such that $Re(s)<1$, then the following identity takes place*

$$\frac{\Gamma\left(-\frac{s}{2}\right)}{4}\left\{\frac{\beta^{\frac{s+1}{2}}}{1+\frac{1}{\pi p}}\eta_{p'}(-s)-\frac{\alpha^{\frac{s+1}{2}}}{1+\frac{1}{\pi p'}}\eta_p(-s)\right\}+\frac{\Gamma\left(\frac{s}{2}\right)}{4}\left\{\beta^{\frac{1-s}{2}}\zeta_p(s)-\alpha^{\frac{1-s}{2}}\zeta_{p'}(s)\right\}=$$
$$=\frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\,\alpha^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\lambda_n'^{-s}\,\mathcal{K}_{\frac{s}{2},p}\left(2\lambda_n'\alpha\right)-\frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\,\beta^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{-s}\mathcal{K}_{\frac{s}{2},p'}\left(2\lambda_n\beta\right), \tag{5.18}$$

where $\mathcal{K}_{\nu,p}(x)$ denotes the integral defined by (5.1).

Using this more general version of the Ramanujan-Guinand formula, we can establish a wide class of Corollaries. The following result comes easily from taking $p'\to\infty$ in (5.18) and appealing to Lemma 5.1.

Corollary 5.3. *If α and β are positive numbers such that $\alpha\beta=\pi^2$ and if s is a complex number satisfying the* *condition $Re(s) < 1$, then the following identity takes place*

$$\frac{\Gamma\left(-\frac{s}{2}\right)}{4}\left\{\frac{\beta^{\frac{s+1}{2}}}{1+\frac{1}{\pi p}}\zeta(-s)-\alpha^{\frac{s+1}{2}}\eta_p(-s)\right\}+\frac{\Gamma\left(\frac{s}{2}\right)}{4}\left\{\beta^{\frac{1-s}{2}}\zeta_p(s)-\alpha^{\frac{1-s}{2}}\zeta(s)\right\}=$$

$$=\frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\alpha^{\frac{1-s}{2}}\sum_{n=1}^{\infty}n^{-s}\mathcal{K}_{\frac{s}{2},p}\left(2n\alpha\right)-\sqrt{\beta}\sum_{m,n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\left(\frac{m}{\lambda_n}\right)^{\frac{s}{2}}K_{\frac{s}{2}}\left(2m\lambda_n\beta\right). \tag{5.19}$$

In particular, (1.2) holds. Moreover,

$$\frac{\Gamma\left(-\frac{s}{2}\right)\zeta(-s)}{4}\alpha^{\frac{s+1}{2}}(1-2^{1+s})+\frac{\Gamma\left(\frac{s}{2}\right)\zeta(s)}{4}\left\{\beta^{\frac{1-s}{2}}(2^s-1)-\alpha^{\frac{1-s}{2}})\right\}=$$

$$=\sqrt{\alpha}\sum_{m,n=1}^{\infty}(-1)^n\left(\frac{n}{m}\right)^{s/2}K_{\frac{s}{2}}\left(2mn\,\alpha\right)-2^{s/2}\sqrt{\beta}\sum_{m,n=1}^{\infty}\left(\frac{m}{2n-1}\right)^{\frac{s}{2}}K_{\frac{s}{2}}\left((2n-1)\,m\beta\right),$$

for every $s\in\mathbb{C}$.

Our next corollary is obtained when we take $p'\to 0^+$ in (5.18) and use Lemma 5.1.

Corollary 5.4. *If α and β are positive numbers such that $\alpha\beta=\pi^2$ and if s is a complex number satisfying the condition $Re(s) < 1$, then the following identity takes place*

$$\frac{\Gamma\left(-\frac{s}{2}\right)\zeta(-s)}{4}\frac{\beta^{\frac{s+1}{2}}}{1+\frac{1}{\pi p}}\left(2^{1+s}-1\right)+\frac{\Gamma\left(\frac{s}{2}\right)}{4}\left\{\beta^{\frac{1-s}{2}}\zeta_p(s)-\alpha^{\frac{1-s}{2}}\left(2^s-1\right)\zeta(s)\right\}=$$

$$=\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\alpha^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{\mathcal{K}_{\frac{s}{2},p}\left((2n-1)\alpha\right)}{(2n-1)^s}-\sqrt{\beta}\sum_{m,n=1}^{\infty}\frac{(p^2+\lambda_n^2)\,(-1)^m}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\left(\frac{m}{\lambda_n}\right)^{s/2}K_{\frac{s}{2}}(2m\,\lambda_n\beta).$$

In particular, (5.14) holds.

Remark 5.2. Another proof of Corollary 5.3. can be obtained if we reverse the roles of p and p' and use the functional equation (4.40). Writing each side of (4.40) by using their respective analogue of the Selberg-Chowla formula, (4.2) or (4.29), we can easily derive (5.19).

Finally, we derive a generalized transformation formula for the logarithm of the Dedekind $\eta-$function. We remark that a particular case of our formula (when $p=p'$) was derived by Dixit and Gupta [[11], Theorem 4.5., p. 15]. Our proof is drastically different because we use the first analogue of Kronecker's limit formula (4.7) given in the previous section.

Corollary 5.5. *Let $p,p'\in\mathbb{R}_+$ and $\sigma(t)$, $\sigma'(t)$ be defined by (1.19). Then, if $\alpha,\beta>0$ are such that $\alpha\beta=\pi^2$, the following generalization of Entry 3.3.2. takes place*

$$\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\cdot\frac{\lambda_n'^{-1}}{\sigma\left(\frac{\lambda_n'\alpha}{\pi}\right)e^{2\alpha\lambda_n'}-1}-\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\cdot\frac{\lambda_n^{-1}}{\sigma'\left(\frac{\lambda_n\beta}{\pi}\right)e^{2\beta\lambda_n}-1}$$

$$=\frac{1}{12\left(1+\frac{1}{\pi p}\right)\left(1+\frac{1}{\pi p'}\right)}\left\{\beta\,\frac{1+\frac{3}{\pi p'}(1+\frac{1}{\pi p'})}{1+\frac{1}{\pi p'}}-\alpha\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{1+\frac{1}{\pi p}}\right\}+\frac{C_p^{(1)}-C_{p'}^{(1)}}{2}+\frac{1}{4}\log\left(\frac{\alpha}{\beta}\right). \tag{5.20}$$

Proof. We have seen in the proof of Theorem 5.1. that the relation holds

$$\zeta_{p,p'}(s,\,c)=c^{-s}\,\zeta_{p',p}\left(s,\,\frac{1}{c}\right). \tag{5.21}$$

Employing Kronecker's limit formula to $\zeta_{p',p}\left(s,\,\frac{1}{c}\right)$ (i.e., using (4.7) with c replaced by c^{-1} and p by p'), we deduce that $c^{-s}\zeta_{p',p}\left(s,\,\frac{1}{c}\right)$ must have the meromorphic expansion

$$c^{-s}\zeta_{p,p'}\left(s,\frac{1}{c}\right) = \frac{\pi}{\sqrt{c}}\,\frac{1}{s-1} - \frac{\pi\log(c)}{\sqrt{c}} + \frac{\pi^2}{3c}\,\frac{1+\frac{3}{\pi p'}(1+\frac{1}{\pi p'})}{\left(1+\frac{1}{\pi p}\right)\left(1+\frac{1}{\pi p'}\right)^2} +$$

$$+\pi\sqrt{c}\left(2C_p^{(1)} - \log\left(\frac{4}{c}\right) + 4\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\cdot\frac{\lambda_n^{-1}}{\sigma'\left(\sqrt{\frac{1}{c}}\lambda_n\right)e^{2\pi\sqrt{\frac{1}{c}}\lambda_n}-1}\right) + O\left(s-1\right) \qquad (5.22)$$

Since relation (5.21) holds for every $s\in\mathbb{C}$ by analytic continuation, the constant terms of the meromorphic expansions (5.22) and (4.7) must be the same. Henceforth, equating the constant terms of (4.7) and (5.22) and replacing $c=\alpha^2/\pi^2$ gives (5.20). □

Our next corollary presents a particular case of the previous result when we take $p\to 0^+$ and $p'\to\infty$. It is not explicitly given in [11], so we shall state it here.

Corollary 5.6. *Let $\alpha,\beta>0$ be such that $\alpha\beta=\pi^2$. Then the following identity takes place*

$$\sum_{n=1}^{\infty}\frac{1}{n\left(e^{2\alpha n}+1\right)} + 2\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\,\frac{1}{\left(e^{\beta(2n-1)}-1\right)} = \frac{\alpha}{4} - \log(2) - \frac{1}{4}\log\left(\frac{\alpha}{\beta}\right).$$

Finally, using the same method as in the previous Corollary 5.5. (proof of a generalized Entry 3.3.2.), we state the final result of this paper, which is also an extension of a formula found in one of Ramanujan's notebooks [[21], p. 318], although discovered earlier by Schlömlich (see [[3], p. 256] for a historical account of this particular formula). We will omit its proof, because it consists in using (5.20) with $p=p'$, dividing by $\beta-\alpha$ and then let $\alpha\to\beta$.

Corollary 5.7. *Let $p\in\mathbb{R}_+$ and let $\sigma(t)$ be defined by (1.19). Then the following identity holds*

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{e^{2\pi\lambda_n}}{\left(\sigma(\lambda_n)e^{2\pi\lambda_n}-1\right)^2}\cdot\left\{\pi\,\sigma\left(\lambda_n\right)+\frac{p}{\left(p-\lambda_n\right)^2}\right\} = \frac{\pi}{24}\cdot\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p}\right)^3} - \frac{1}{8}.$$

Taking $p\to\infty$ gives Schlömlich's formula. By taking the limit $p\to 0^+$ we obtain a companion of it, which seems to be new. We state them in the following Corollary.

Corollary 5.8. *The following identities hold:*

$$\sum_{n=1}^{\infty}\frac{e^{2\pi n}}{\left(e^{2\pi n}-1\right)^2} = \frac{1}{24} - \frac{1}{8\pi}, \qquad \sum_{n=1}^{\infty}\frac{e^{\pi(2n-1)}}{\left(e^{\pi(2n-1)}+1\right)^2} = \frac{1}{8\pi}. \qquad (5.23)$$

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