HIGH FREQUENCY FORCING OF AN ATTRACTING HETEROCLINIC CYCLE

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Abstract. This article is concerned with the effect of time-periodic forcing on a vector field exhibiting an attracting heteroclinic network. We show that as the forcing frequency tends to infinity, the dynamics reduces to that of a network under constant forcing, the constant being the average value of the forcing term. We also show that under small constant forcing the network breaks up into an attracting periodic solution that persists for periodic forcing of high frequency.

Keywords: periodic forcing, attracting heteroclinic cycle, averaged system, high frequency

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1. Introduction

Heteroclinic cycles organize the dynamics in a wide range of systems: ecological models of competing species [1, 12, 15], thermal convection [17, 19], game theory [6, 10] and climate science [4]. A paradigmatic example of a robust heteroclinic cycle occurs Guckenheimer and Holmes three-dimensional system [12], also studied by May and Leonard [15] and by Busse and Heikes [5]. Although their initial models have periodic forcing terms, all the theory has been developed for the autonomous case. In this case, the equations are symmetric under permutation of coordinates.

With the forcing terms removed, each of the equilibria on the coordinate axes is of saddle type, and the existence of connecting orbits has been proved. Moreover, attracting heteroclinic networks have been found in an open set in the space of parameters [12]. Other examples from the dissipative category include the equations of Lorenz, Duffing and Lorentz gases acted on by external forces [7].

In a series of papers, the authors of [9, 18, 22, 23] considered the effect of small-amplitude time-periodic forcing of an attracting heteroclinic network and describe how to reduce the dynamics to a two-dimensional map. In the limit where the heteroclinic cycle loses asymptotic stability, intervals of frequency locking appear. In the opposite limit, where the heteroclinic cycle becomes strongly stable, no frequency locking is observed. See also [20] where the author proved that rank-one strange attractors are abundant near an attracting cycle.

In the present article, we examine the effect of high frequency periodic forcing on a system of differential equations constructed in Aguiar *et al* [2], that in the absence of forcing exhibits an asymptotically stable network. We start by deriving the first return map near a cycle of the network. We prove that the dynamics of the system is equivalent to the averaged one as the frequency of the forcing

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tends to $+\infty$. Although our computations are done on the example of [2], our results are of far wider interest than the specific problem studied in this paper.

Structure. This article is organized as follows. In Section 2 we describe our object of study and, for the sake of completeness, in Section 3 we recall some of its properties for the organising center ($\mu = \nu = 0$). In Section 4, we obtain the expressions that will be used to compute the first return map to a given cross-section. We also derive auxiliary results that will be helpful to analyse the asymptotic coefficients of the first return map. In Section 5, we state the main contribution of this paper. as well as some dynamical consequences. The existence of an attracting and hyperbolic solution near the cycle under high frequency forcing is explored in Section 6. We finish the article with a short discussion in Section 7, where we fit our main result in the literature.

2. The object of study

Our object of study is the following two-parameter family of ordinary differential equations

$$\dot{X} = F_{(\nu,\mu)}(X,t)$$

defined in $X = (x, y, z) \in \mathbf{R}^3$:

$$\left\{ \begin{array}{l} \dot{x} = x(1-r^2) - \alpha xz + \beta xz^2 + (1-x)[\nu f(2\omega t) + \mu] \\ \dot{y} = y(1-r^2) + \alpha yz + \beta yz^2 \\ \dot{z} = z(1-r^2) - \alpha(y^2 - x^2) - \beta z(x^2 + y^2) \end{array} \right. \tag{2.1}$$

where

- $f(s) = \sin s$,
- $r^2 = x^2 + y^2 + z^2$,
- $\omega \in \mathbf{R}^+$ and
- $\mu, \nu \in \mathbf{R}_0^+$ are two small parameters.

We also assume that:

$$\beta < 0 < \alpha, \qquad |\beta| < \alpha \quad \Rightarrow \quad \beta^2 < 8\alpha^2. \tag{2.2}$$

Concerning the equation (2.1), the amplitude of the autonomous perturbation is governed by the parameter μ whereas ν controls the amplitude of the non-autonomous term. The parameter ω is what we call the *frequency* of the periodic forcing.

Our choice of perturbing term $(1-x)[\nu f(2\omega t) + \mu]$ is made for two reasons: it simplifies the computations and allows comparison with previous work by other authors [2, 9, 13, 14, 18, 23]. Note that we may split the family of perturbations as follows:

$$(1-x)[\nu f(2\omega t) + \mu] = \nu \underbrace{(1-x)f(2\omega t)}_{\text{non-autonomous forcing}} + \mu \underbrace{(1-x)}_{\text{autonomous}}$$

where:

$$\int_0^{\pi/\omega} (1-x) f(2\omega t) dt = 0.$$

Our main result states that in the limit when the forcing frequency $\omega \to +\infty$, the effect of the perturbation governed by (ν, μ) is reduced to the effect of the autonomous term. In other words, we prove that the dynamics associated to $\lim_{\omega \to +\infty} F_{(\nu,\mu)}$ is qualitatively the same as that of the averaged system $F_{(0,\mu)}$. This agrees well with other results in the literature. See Section 7 for a discussion on the topic.

3. The unperturbed system ($\nu = \mu = 0$)

In this section we recall some basic features associated to the system (2.1) when $\nu = \mu = 0$ and we construct suitable cross sections near the equilibria where the local and global maps will be defined.

3.1. **Dynamics.** The vector field $F_{(0,0)}(x, y, z, t)$ is exactly one of the examples proposed by Aguiar *et al* [2, Theorem 7]. For the sake of completeness, we recall some of its properties. The vector field $F_{(0,0)}$ is equivariant[1] under the compact Lie group $\mathcal{G} \subset \mathbb{O}(3)$ generated by the linear maps:

$$\kappa_1(x, y, z) = (-y, x, -z) \qquad \text{and} \qquad \kappa_2(x, y, z) = (x, -y, z).$$

The action of $\mathcal{G}$ on $\mathbf{R}^3$ has the following symmetry planes:

$$\begin{aligned} Fix(\kappa_2 \circ \kappa_1^2) &= \{(x, y, z) \in \mathbf{R}^3 : x = 0\} \\ Fix(\kappa_2) &= \{(x, y, z) \in \mathbf{R}^3 : y = 0\}. \end{aligned}$$

and symmetry axes:

$$\begin{aligned} Fix(\kappa_1^2) &= \{(x, y, z) \in \mathbf{R}^3 : x = 0 \quad \text{and} \quad y = 0\}, \\ Fix(\kappa_2 \circ \kappa_1^3) &= \{(x, y, z) \in \mathbf{R}^3 : x = y \quad \text{and} \quad z = 0\}, \\ Fix(\kappa_2 \circ \kappa_1) &= \{(x, y, z) \in \mathbf{R}^3 : x = -y \quad \text{and } z = 0\}. \end{aligned}$$

The unit sphere $\mathbf{S}^2$ is flow-invariant and attracts all trajectories, except the origin, which is repelling. As illustrated in Figure 1, the intersection of this sphere with $Fix(\kappa_1^2)$ gives rise to two saddle-type equilibria

$$\mathbf{v}_+ = (0, 0, 1), \qquad \text{and} \qquad \mathbf{v}_- = (0, 0, -1),$$

where the derivative of $F_{(0,0)}$ is

$$DF_{(0,0)}(0, 0, \sigma) = \begin{pmatrix} \beta - \sigma\alpha & 0 & 0 \\ 0 & \beta + \sigma\alpha & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \text{where} \qquad \sigma = \pm 1.$$

On $\mathbf{S}^2$, there are also four unstable foci $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}, 0\right)$ on the lines $Fix(\kappa_2 \circ \kappa_1^3)$ and $Fix(\kappa_2 \circ \kappa_1)$. The intersections of the sphere $\mathbf{S}^2$ with the planes $Fix(\kappa_2 \circ \kappa_1^2)$ and $Fix(\kappa_2)$ generate two pairs of one-dimensional heteroclinic connections linking the equilibria $\mathbf{v}_+$ and $\mathbf{v}_-$ — see Figure 1.

The union of these equilibria and connections forms four heteroclinic cycles and, taken together, a heteroclinic network denoted Γ. The two-dimensional coordinate subspaces $Fix(\kappa_2 \circ \kappa_1^2)$ and $Fix(\kappa_2)$ are flow-invariant, preventing visits to more than one cycle in the network, hence there is no switching in the sense of [3].

Within each of the invariant planes defined by $x = 0$ and $y = 0$, the connecting orbit is a saddle-sink connection. Therefore the network Γ is robust in the class of $\mathcal{G}$-symmetric vector fields.

The constant $\delta = \dfrac{\alpha - \beta}{\alpha + \beta} > 1$ measures the *strength of attraction* of the cycle. By the construction in [2] the network Γ is *globally asymptotically stable*: typical trajectories starting near Γ accumulate on one of the cycles in the network and remain near the equilibria for increasing periods of time. These trajectories make fast transitions from one equilibrium point to the next. In particular, there are no periodic solutions near Γ. If either $\nu > 0$ or $\mu > 0$, only the symmetry κ_2 remains.

The case $\nu > \mu = 0$ and $\delta \gtrsim 1$ has been studied in [13], where the authors found the emergence of a discrete-time Bogdanov-Takens bifurcation. The autonomous case $\mu > \nu = 0$ may be studied as in [9, §5.2.1] to conclude that the dynamics is governed by an attracting periodic solution.

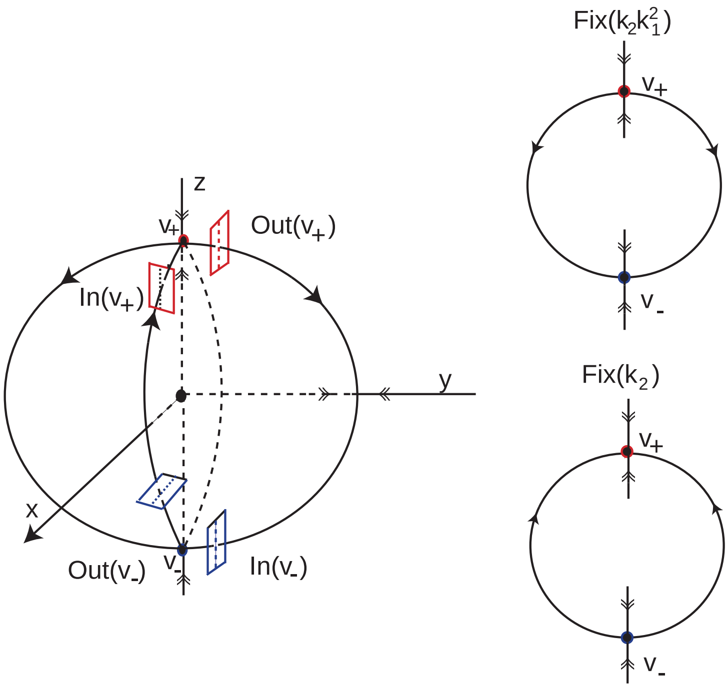

FIGURE 1. Sketch of the heteroclinic connections when $\mu = \nu = 0$.

3.2. **Cross-sections.** Our results will be obtained analysing the first return map to a suitable cross-section to the flow of (2.1), obtained from the transitions between four cross-sections for the unperturbed equation. To do this, consider cubic neighbourhoods $V_\pm$ in $\mathbf{R}^3$ of $\mathbf{v}_\pm$:

$$V_\sigma = \{(x, y, w), |x| < \varepsilon, |y| < \varepsilon, |w| < \varepsilon\} \qquad w = z - \sigma \qquad \sigma = \pm 1$$

for $\varepsilon > 0$ small. As suggested by Figure 2, we use the following cross-sections contained in the boundary of V_+.

- $In(\mathbf{v}_+) = \{(\varepsilon, y, w), |y| < \varepsilon, |w| < \varepsilon\}$ with coordinates (y_1, w_1) It consists of points whose trajectories go into V_+ in small positive time.
- $Out(\mathbf{v}_+) = \{(x, \varepsilon, w), |x| < \varepsilon, |w| < \varepsilon\}$ with coordinates $(\hat{x}_1, \hat{w}_1)$ It consists of points whose trajectories go out of V_+ in small positive time.

The cross-sections contained in the boundary of V_- are:

- $In(\mathbf{v}_-) = \{(x, \varepsilon, w), |x| < \varepsilon, |w| < \varepsilon\}$ with coordinates (x_2, w_2) with points whose trajectories go into V_- in small positive time,
- $Out(\mathbf{v}_-) = \{(\varepsilon, y, w), |y| < \varepsilon, |w| < \varepsilon\}$ with coordinates $(\hat{y}_2, \hat{w}_2)$ containing points whose trajectories go out of V_- in small positive time.

The local stable manifolds of $\mathbf{v}_+$ and $\mathbf{v}_-$ in the cross sections $In(\mathbf{v}_+)$ and $In(\mathbf{v}_-)$ are given by:

$$W^s(\mathbf{v}_+) \cap In(\mathbf{v}_+) = \{(0, w_1) : |w_1| < \varepsilon\} \qquad W^s(\mathbf{v}_-) \cap In(\mathbf{v}_-) = \{(0, w_2) : |w_2| < \varepsilon\}.$$

From now on we restrict our attention to the $y > 0$ component of $In(\mathbf{v}_+)\backslash W^s(\mathbf{v}_+)$ (respectively the $x > 0$ component of $In(\mathbf{v}_-)\backslash W^s(\mathbf{v}_-)$) and we abuse notation by calling it $In(\mathbf{v}_+)$ (respectively $In(\mathbf{v}_-)$). All the results hold on the other component, but follow a different cycle in Γ.

[1]For the terminology of equivariant differential equations we refer the reader to the book [11].

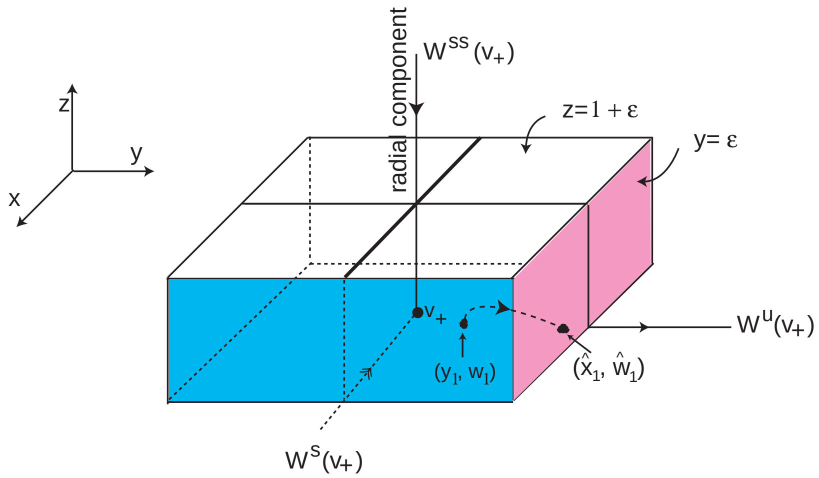

FIGURE 2. Scheme of the cross-sections $In(\mathbf{v}_+)$ and $Out(\mathbf{v}_+)$ in the neighbourhood V_+ of $\mathbf{v}_+$. The set $W^{ss}(\mathbf{v}_+)$, the local strong stable manifold of $\mathbf{v}_+$, corresponds to the radial direction in the invariant sphere $\mathbf{S}^2$.

By rescaling the variables (x, y, z) and the parameters μ and ν we may take $\varepsilon = 1$ in the cross-sections defined above.

4. Local maps

The aim of this section is to obtain expressions that will be used to compute the first return map to the cross-section $In(\mathbf{v}_+)$ in the flow of (2.1). The calculations are similar to those of [13], so we give only an overview. The expression for the first return map to this cross-section to Γ is obtained as the composition of two types of maps: *local maps* between the neighbourhood walls of each $V_\pm$, and *global maps* from one neighbourhood to the other. Here we obtain the local maps (at leading order in ν, μ) by computing the point where a solution hits each cross-section and the time the solution takes to move between cross-sections.

Suspension. For $(\nu, \mu) \neq (0, 0)$ we consider the suspension of (2.1) given by

$$(4.3) \qquad \begin{cases} \dot{X} = F_{(\nu,\mu)}(X, s) \\ \dot{s} = 1 \end{cases} \qquad \text{where} \qquad X = (x, y, z) \quad \text{and} \quad s \in \mathbf{S}^1$$

and the augmented cross-sections $\mathbf{S}^1 \times In(\mathbf{v}_\pm)$ and $\mathbf{S}^1 \times Out(\mathbf{v}_\pm)$ to the suspended flow. Although $\mathbf{v}_+$ and $\mathbf{v}_-$ are no longer constant solutions of (2.1) for $(\nu, \mu) \neq (0, 0)$, for small (ν, μ) the augmented cross-sections defined above are still crossed transversely by trajectories.

4.1. **Linearisation.** The linearisation of (2.1) near $\mathbf{v}_\pm$ is

$$(4.4) \qquad \begin{cases} \dot{x} = (\beta - \sigma\alpha)x - \nu f(2\omega t) - \mu \\ \dot{y} = (\alpha + \sigma\beta)y \\ \dot{w} = -2(w + \sigma) \end{cases} \qquad w = z - \sigma \qquad \sigma = \pm 1$$

Equation (2.1) may be written in the form

$$\dot{X} = \mathcal{M}X + R(X) - [\nu f(2\omega t) + \mu](1, 0, 0)^T$$

for $X^T = (x, y, z)$ where $\dot{X} = \mathcal{M}X - [\nu f(2\omega t) + \mu](1, 0, 0)^T$ is any of the equations (4.4) for $\sigma = \pm 1$.

The constant matrix $\mathcal{M}$ has no eigenvalues with zero real part, the perturbation $\nu f(2\omega t) + \mu$ is limited and the non-linear part $R(X)$ is limited and uniformly Lipschitz in a compact neighbourhood of $\mathbf{S}^2$. Under these conditions, Palmer's Theorem [16, pp 754] implies that there are neighbourhoods of $\mathbf{v}_+$ and $\mathbf{v}_-$ where the vector field is C^1 conjugate to its linearisation.

4.2. **Local map near $\mathbf{v}_+$.** The calculation of the first return map will use the form of the general solution of (4.4). For $z = 1 + w$ we get $\dot{w} = -2(1 + w)$. Using the Lagrange method of variation of parameters as described in [21, pp 842], the solution of the linearised system (4.4) near $\mathbf{v}_+$, with initial condition $(x, y, w)(s) = (1, y_1, w_1) \in \mathbf{S}^1 \times In(\mathbf{v}_+) \backslash W^s(\mathbf{v}_+)$ at time s, is:

$$(4.5)\qquad \begin{cases} x(t,s) = \mathrm{e}^{(\beta-\alpha)(t-s)} \left(1 - \displaystyle\int_s^t \mathrm{e}^{-(\beta-\alpha)(\tau-s)} (\nu f(2\omega\tau) + \mu) d\tau \right) \\ y(t,s) = y_1 \mathrm{e}^{(\alpha+\beta)(t-s)} \\ w(t,s) = (w_1 + 1)\mathrm{e}^{-2(t-s)} - 1. \end{cases}$$

For $y_1 > 0$, the time of arrival at $Out(\mathbf{v}_+)$ is the solution of $y(T_1) = 1$, hence:

$$y(T_1) = 1 \quad \Leftrightarrow \quad y_1 \mathrm{e}^{(\alpha+\beta)(T_1-s)} = 1 \quad \Leftrightarrow \quad \ln\left(\frac{1}{y_1}\right) = (\alpha+\beta)(T_1 - s).$$

In this case, T_1 does not depend on ν nor on μ. These solutions arrive at $Out(\mathbf{v}_+)$ at a time

$$T_1 = s + \ln\left(\frac{1}{y_1}\right)^{\frac{1}{\alpha+\beta}} = s - \frac{1}{\alpha+\beta} \ln y_1.$$

Replacing t by T_1 in the first and third equations of (4.5), we get the map

$$\Phi_{\mathbf{v}_+} : \mathbf{S}^1 \times In(\mathbf{v}_+) \to \mathbf{S}^1 \times Out(\mathbf{v}_+)$$

$$(4.6)\qquad \Phi_{\mathbf{v}_+}(s, y_1, w_1) = \begin{pmatrix} s - \frac{1}{\alpha+\beta} \ln y_1 \\ y_1^\delta \left(1 - \displaystyle\int_s^{T_1} \mathrm{e}^{-(\beta-\alpha)(\tau-s)} (\nu f(2\omega\tau) + \mu) d\tau \right) \\ (w_1 + 1) y_1^{\frac{2}{\alpha+\beta}} - 1 \end{pmatrix} = (T_1, \hat{x}_1, \hat{w}_1).$$

4.3. **Local map near $\mathbf{v}_-$.** The treatment of (4.4) for $\sigma = -1$ is similar to Subsection 4.2, although the computations involve more steps. The solution of (4.4), with initial condition

$$(x, y, w)(s) = (x_2, 1, w_2) \in \mathbf{S}^1 \times In(\mathbf{v}_-)$$

at time s, is:

$$(4.7)\qquad \begin{cases} x(t) = x_2 \mathrm{e}^{(\alpha+\beta)(t-s)} \left(1 - \dfrac{1}{x_2} \displaystyle\int_s^t \mathrm{e}^{-(\alpha+\beta)(\tau-s)} (\nu f(2\omega\tau) + \mu) d\tau \right) \\ y(t) = \mathrm{e}^{(\beta-\alpha)(t-s)} \\ w(t) = (w_2 - 1)\mathrm{e}^{-2(t-s)} + 1. \end{cases}$$

The time T_2 of arrival at $Out(\mathbf{v}_-)$, starting at $In(\mathbf{v}_-)$ is more difficult to compute than T_1. This is why we use its Taylor expansion at $(0, 0)$ truncated at second order of ν and μ. We write $T_2(\nu, \mu)$ to stress its dependence on the bifurcation parameters.

Lemma 1. *The time of flight T_2 inside V_- may be written as:*

$$T_2(\nu,\mu) = s - \frac{1}{\alpha+\beta}\ln x_2 + \left[\frac{1}{(\alpha+\beta)}\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right] + \mathcal{O}(||(\nu,\mu)||^2),$$

where $\mathcal{O}(||(\nu,\mu)||^2)$ denotes the usual Landau notation and $T_2(0,0) = s - \dfrac{\ln x_2}{\alpha+\beta}$.

Proof. Let us derive the Taylor expression of $T_2(\nu,\mu)$ of degree 1:

$$T_2(\nu,\mu) = T_2(0,0) + \frac{\partial T_2}{\partial \nu}(0,0) + \frac{\partial T_2}{\partial \mu}(0,0) + \mathcal{O}(||(\nu,\mu)||^2). \tag{4.8}$$

By definition of time of flight in V_-, we may write:

$$\begin{aligned} 1 &\equiv x(T_2(\nu,\mu),\nu,\mu) \\ &\overset{(4.7)}{=} \mathrm{e}^{(\alpha+\beta)(T_2(\nu,\mu)-s)}\left(x_2 - \nu\int_s^{T_2(\nu,\mu)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}f(2\omega\tau)d\tau - \mu\int_s^{T_2(\nu,\mu)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}d\tau\right). \end{aligned}$$

For $\nu=\mu=0$, we get:

$$x_2\mathrm{e}^{(\alpha+\beta)(T_2(0,0)-s)} = 1 \quad \Rightarrow \quad T_2(0,0) = s - \frac{\ln x_2}{\alpha+\beta}. \tag{4.9}$$

Using the Chain Rule applied to the equality $x(T_2(\nu,\mu),\nu,\mu)=1$ at $(\nu,\mu)=(0,0)$, we get

$$\frac{\partial x}{\partial t}(T_2(0,0),0,0)\frac{\partial T_2}{\partial \nu}(0,0) + \frac{\partial x}{\partial \nu}(T_2(0,0),0,0) = 0$$

and thus:

$$(\alpha+\beta)x_2\mathrm{e}^{(\alpha+\beta)(T_2(0,0)-s)}\frac{\partial T_2}{\partial \nu}(0,0) - \int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}f(2\omega\tau)d\tau = 0.$$

According to the left hand side of (4.9), the previous equality may be simplified as:

$$\begin{aligned} &(\alpha+\beta)\frac{\partial T_2}{\partial \nu}(0,0) - \int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}f(2\omega\tau)d\tau = 0, \\ \Leftrightarrow\quad &\frac{\partial T_2}{\partial \nu}(0,0) = \frac{1}{(\alpha+\beta)}\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}f(2\omega\tau)d\tau. \end{aligned}$$

Analogously, we have

$$(\alpha+\beta)\frac{\partial T_2}{\partial \mu}(0,0) - \int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}d\tau = 0$$

and hence

$$\frac{\partial T_2}{\partial \mu}(0,0) = \frac{1}{(\alpha+\beta)}\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}d\tau.$$

Replacing $T_2(0,0)$, $\frac{\partial T_2}{\partial \nu}(0,0)$ and $\frac{\partial T_2}{\partial \mu}(0,0)$ in (4.8), it yields:

$$T_2(\nu,\mu) = s - \frac{1}{\alpha+\beta}\ln x_2 + \left[\frac{1}{(\alpha+\beta)}\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right] + \mathcal{O}(||(\nu,\mu)||^2)$$

and the result follows. □

From now on, we omit the remainder $\mathcal{O}(||(\nu,\mu)||^2)$ of T_2 in the computations. Using the expression of T_2 obtained in Lemma 1 in (4.7), we may deduce that:

$$\begin{array}{rcl} y(T_2) & = & e^{(\beta-\alpha)(T_2-s)} \\ & = & \exp\left(-\dfrac{(\beta-\alpha)}{\alpha+\beta}\ln x_2 + \left[\dfrac{(\beta-\alpha)}{(\alpha+\beta)}\displaystyle\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right]\right) \\ & = & x_2^{\delta}\exp\left(-\delta\displaystyle\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right). \end{array}$$

Analogously, we may write:

$$\begin{array}{rcl} w(T_2) & = & 1+(w_2-1)\mathrm{e}^{-2(T_2-s)} \\ & = & 1+(w_2-1)\exp\left(\dfrac{2}{\alpha+\beta}\ln x_2 + \left[\dfrac{-2}{(\alpha+\beta)}\displaystyle\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right]\right) \\ & = & 1+(w_2-1)x_2^{2/(\alpha+\beta)}\exp\left(\dfrac{-2}{(\alpha+\beta)}\displaystyle\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right). \end{array}$$

Therefore, we define the local map as:

$$\Phi_{\mathbf{v}_-} : \mathbf{S}^1 \times In(\mathbf{v}_-) \to \mathbf{S}^1 \times Out(\mathbf{v}_-)$$

(4.10)

$$\Phi_{\mathbf{v}_-} = \begin{pmatrix} s - \frac{1}{\alpha+\beta}\ln x_2 + \frac{1}{(\alpha+\beta)}\displaystyle\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau \\ \\ x_2^{\delta}\exp\left(-\delta\displaystyle\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right) \\ \\ 1+(w_2-1)x_2^{2/(\alpha+\beta)}\exp\left(\frac{-2}{\alpha+\beta}\displaystyle\int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right) \end{pmatrix} = (T_2, \hat{y}_2, \hat{w}_2).$$

It follows that if $x_2 > 0$ then $\hat{y}_2 > 0$. To establish a similar statement for $\Phi_{\mathbf{v}_+}$ we will need the information of Lemma 2 below.

4.4. **Summary.** The expressions for $\Phi_{\mathbf{v}_+}$ (cf. (4.6)) and $\Phi_{\mathbf{v}_-}$ (cf. (4.10)) may be written as:

$$\Phi_{\mathbf{v}_+}(s, y_1, w_1) = \begin{pmatrix} s - \dfrac{1}{\alpha+\beta}\ln y_1 \\ \\ y_1^{\delta}(1-K_1) \\ (w_1+1)y_1^{\frac{2}{\alpha+\beta}} \end{pmatrix} = (T_1, \hat{x}_1, \hat{w}_1)$$

and

$$\Phi_{\mathbf{v}_-}(s,x_2,w_2)=\begin{pmatrix} s-\dfrac{1}{\alpha+\beta}\ln x_2+\dfrac{K_2}{(\alpha+\beta)} \\ \\ x_2^{\delta}\exp(-\delta K_2) \\ \\ 1+(w_2-1)\left(x_2^{2/(\alpha+\beta)}\exp\left(\dfrac{-2K_2}{\alpha+\beta}\right)\right) \end{pmatrix}=(T_2,\hat{y}_2,\hat{w}_2).$$

where

$$\begin{aligned} K_1 &= \int_s^{T_1} \mathrm{e}^{-(\beta-\alpha)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau \\ K_2 &= \int_s^{T_2(0,0)} \mathrm{e}^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau. \end{aligned}$$

Both K_1 and K_2 depend on s,ν and μ. Furthermore, when $\mu=\nu=0$, we get:

$$\begin{aligned} (T_1,\hat{x}_1,\hat{w}_1) &= \left(s-\frac{1}{\alpha+\beta}\ln y_1,\ y_1^{\delta},\ (w_1+1)y_1^{\frac{2}{\alpha+\beta}}\right) \\ (T_2,\hat{y}_2,\hat{w}_2) &= \left(s-\frac{1}{\alpha+\beta}\ln x_2,\ x_2^{\delta},\ 1+(w_2-1)x_2^{2/(\alpha+\beta)}\right), \end{aligned}$$

corresponding to the expressions of the local maps for the unperturbed case. Note that, although the second and third coordinates of $\Phi_{\mathbf{v}_+}$ are well defined and equal to zero at $(s,y_1,w_1)=(s,0,0)$, the first coordinate tends to infinity as y_1 goes to zero, since this point corresponds to the heteroclinic connection from $\mathbf{v}_+$ to $\mathbf{v}_-$. In other words, $(s,0,0)\in In(\mathbf{v}_+)$ is a point that never returns. The same remark applies to $\Phi_{\mathbf{v}_-}$.

4.5. **Auxiliary result.** The integrals K_1 and K_2 are linear on ν and μ and may be computed explicitly.

Lemma 2. *The following equalities hold:*

(1)

$$\begin{aligned} K_1 &= \frac{\nu y_1^{-\delta}}{(\beta-\alpha)^2+4\omega^2}\left[(\beta-\alpha)\sin(2\omega(T_1-s))+2\omega\cos(2\omega(T_1-s))\right] \\ &\quad -\frac{\nu}{(\beta-\alpha)^2+4\omega^2}\left[(\beta-\alpha)\sin(2\omega s)+2\omega\cos(2\omega s)\right]+\mu\frac{y_1^{-\delta}-1}{\alpha-\beta}. \end{aligned}$$

(2)

$$\begin{aligned} K_2 &= \frac{\nu x_2}{(\alpha+\beta)^2+4\omega^2}\left[(\alpha+\beta)\sin(2\omega(T_2(0,0)-s))+2\omega\cos(2\omega(T_2(0,0)-s))\right] \\ &\quad -\frac{\nu}{(\alpha+\beta)^2+4\omega^2}\left[(\alpha+\beta)\sin(2\omega s)+2\omega\cos(2\omega s)\right]+\mu\frac{1-x_2}{\alpha+\beta}. \end{aligned}$$

Proof. (1) Using the linearity of the integral, K_1 may be written as

$$\nu\int_s^{T_1}\mathrm{e}^{-(\beta-\alpha)(\tau-s)}f(2\omega\tau)d\tau+\mu\int_s^{T_1}\mathrm{e}^{-(\beta-\alpha)(\tau-s)}d\tau.$$

Each summand may be computed explicitly. For the second one we have

$$\int e^{-(\beta-\alpha)(\tau-s)}d\tau = \frac{-e^{-(\beta-\alpha)(\tau-s)}}{(\beta-\alpha)}.$$

From [13, Lemma 6] (integrating by parts twice) and since $f(t)=\sin t$, we get

$$\int e^{-(\beta-\alpha)(\tau-s)}\sin(2\omega\tau)d\tau = \frac{e^{-(\beta-\alpha)(\tau-s)}}{(\beta-\alpha)^2+4\omega^2}\left[(\beta-\alpha)\sin(2\omega\tau)+2\omega\cos(2\omega\tau)\right]$$

To compute K_1 we need $T_1 - s = -\dfrac{\ln y_1}{\beta+\alpha}$, hence

$$e^{-(\beta-\alpha)(T_1-s)} = y_1^{(\beta-\alpha)/(\beta+\alpha)} = y_1^{-\delta}$$

and the expression for K_1 follows.

(2) Analogous computations.

□

In particular it follows from this lemma that $K_1 < 1$ for sufficiently large ω, since in this case the terms with ν are small and the term with μ is negative. Hence, for large ω, if $y_1 > 0$ and $\Phi_{\mathbf{v}_+}(s, y_1, w_1) = (T_1, \hat{x}_1, \hat{w}_1)$ then $\hat{x}_1 > 0$.

4.6. **Global and first return maps.** The first return map to $\mathbf{S}^1 \times In(\mathbf{v}_-)$ will be

$$\mathcal{R}_{(\nu,\mu)} := (\Psi_{\mathbf{v}_+\to\mathbf{v}_-})\circ\Phi_{\mathbf{v}_+}\circ(\Psi_{\mathbf{v}_-\to\mathbf{v}_+})\circ\Phi_{\mathbf{v}_-} \tag{4.11}$$

where

$$\Psi_{\mathbf{v}_+\to\mathbf{v}_-} : \mathbf{S}^1\times Out(\mathbf{v}_+) \to \mathbf{S}^1\times In(\mathbf{v}_-) \quad \text{and} \quad \Psi_{\mathbf{v}_-\to\mathbf{v}_+} : \mathbf{S}^1\times Out(\mathbf{v}_-) \to \mathbf{S}^1\times In(\mathbf{v}_+)$$

are the global maps whose expressions are given below.

Trajectories that remain close to the network Γ spend long times near the equilibria and make fast transitions from each neighbourhood $V_\pm$ to the next one. Thus we may take the time transitions (first component of $\Psi_{\mathbf{v}_+\to\mathbf{v}_-}$ and $\Psi_{\mathbf{v}_-\to\mathbf{v}_+}$) to be instantaneous. Since the symmetry κ_2 remains for $\mu,\nu > 0$, we may assume that $\Psi_{\mathbf{v}_-\to\mathbf{v}_+}$ is the identity. Finally, since the cycle is broken for $\mu > 0$, the map $\Psi_{\mathbf{v}_+\to\mathbf{v}_-}$ depends on μ in an affine way. Therefore, for $a > 0$, we take the transition maps as:

$$\begin{aligned}\Psi_{\mathbf{v}_+\to\mathbf{v}_-}(s,\hat{x}_1,\hat{w}_1) &\mapsto (s, \hat{x}_1 + a\mu, \hat{w}_1) = (s, x_2, w_2)\\ \Psi_{\mathbf{v}_-\to\mathbf{v}_+}(s,\hat{x}_2,\hat{w}_2) &\mapsto (s, \hat{x}_2, \hat{w}_2) = (s, x_1, w_1).\end{aligned}$$

5. Main result

We are now in a position to show that when the frequency ω of the forcing tends to infinity, the first return map for $F_{(\nu,\mu)}$ approaches that of the averaged system $F_{(0,\mu)}$.

Theorem 3. *For initial conditions close to any of the cycles in the network* Γ, *the first return map*

$$\mathcal{R}_{(\nu,\mu)} : \mathbf{S}^1\times In(\mathbf{v}_-) \to \mathbf{S}^1\times In(\mathbf{v}_-)$$

for (2.1) *satisfies*

$$\lim_{\omega\to\infty}\mathcal{R}_{(\nu,\mu)} = \mathcal{R}_{(0,\mu)}.$$

Proof. To prove the result we do not need to write explicitly the analytical expression of $\mathcal{R}_{(\nu,\mu)}$. First of all, note that the global maps defined in Subsection 4.6 do not depend on ν. From the expressions (4.6) and (4.10) derived in Subsection 4.4 it is clear that $\Phi_{\mathbf{v}_+}$ and $\Phi_{\mathbf{v}_-}$ only depend on ω, ν and μ

through the integrals K_1 and K_2. Their expressions are computed explicitly in Lemma 2 above, where we show them to be of the form

$$K_i(\omega, s, \nu, \mu) = \nu H_{i,\nu}(\omega, s) + \mu H_{i,\mu}(\omega, s), \quad i = 1, 2.$$

In both cases the term $H_{i,\nu}(\omega, s)$ contains one of the factors

$$1/[(\beta \pm \alpha)^2 + 4\omega^2]$$

multiplying a combination of sines and cossines, the last ones multiplied by ω. Hence, $H_{i,\nu}(\omega, s)$ consists of a term that goes to zero with ω^2 multiplying functions that are either limited or ω times something limited. Therefore $\lim_{\omega\to\infty} H_{i,\nu}(\omega, s) = 0$ and the result follows. □

6. Dynamics for ω large

We have proved that the dynamics associated to $\lim_{\omega\to+\infty} F_{(\nu,\mu)}$ is qualitatively the same as that of the autonomous system $F_{(0,\mu)}$. In this section we explore this feature to obtain some dynamical information. Before going further, note that if $\nu = 0$ then:

$$K_1 = \mu\frac{y_1^{-\delta} - 1}{\alpha - \beta} \qquad \text{and} \qquad K_2 = \mu\frac{1 - x_2}{\alpha + \beta}.$$

Our first result is technical and provides an explicit expression for the components of $\mathcal{R}_{(0,\mu)}$.

Lemma 4. *For $\nu = 0$ and $(s, x_2, w_2) \in \mathbf{S}^1 \times In(\mathbf{v}_-)$ writing $\mathcal{R}_{(0,\mu)} = (h_1, h_2, h_3)$ the following equalities hold:*

(1) $h_1(s, x_2, w_2) = s - \dfrac{1+\delta}{\alpha+\beta}\ln x_2 + \dfrac{K_2(1+\delta)}{\alpha+\beta}$;

(2) $h_2(s, x_2, w_2) = x_2^{\delta^2}\exp\left(-\delta^2 K_2\right)\left[1 - \dfrac{\mu}{\alpha-\beta}\right] - \dfrac{\mu}{\alpha-\beta} + a\mu$;

(3) $h_3(s, x_2, w_2) = \left[2 + (w_2 - 1)x_2^{\frac{2}{\alpha+\beta}}\exp\left(\dfrac{-2K_2}{\alpha+\beta}\right)\right]\left[x_2^{\delta}\exp(-\delta K_2)\right]^{\frac{2}{\alpha+\beta}}$.

Proof. The proof follows by composing of the local maps (4.6) and (4.10) derived in Subsection 4.4 and the global maps $\Psi_{\mathbf{v}_+\to\mathbf{v}_-}$ and $\Psi_{\mathbf{v}_-\to\mathbf{v}_+}$ in the order prescribed by

$$\mathcal{R}_{(0,\mu)} = (\Psi_{\mathbf{v}_+\to\mathbf{v}_-}) \circ \Phi_{\mathbf{v}_+} \circ (\Psi_{\mathbf{v}_-\to\mathbf{v}_+}) \circ \Phi_{\mathbf{v}_-}$$

cf. (4.11). We start by obtaining for $\nu = 0$ the expressions:

$$y_1 = x_2^{\delta}\exp(-\delta K_2) \qquad K_2 = \mu\frac{1 - x_2}{\alpha+\beta} \qquad K_1 = \mu\frac{y_1^{-\delta} - 1}{\alpha-\beta} = \mu\frac{x_2^{-\delta^2}\exp(\delta^2 K_2) - 1}{\alpha-\beta}.$$

Hence, for (2) we get:

$$\begin{aligned}
h_2(s, x_2, w_2) &= y_1^{\delta}(1 - K_1) \\
&= \left(x_2^{\delta}\exp(-\delta K_2)\right)^{\delta}\left(1 - \mu\frac{(x_2^{-\delta}\exp(-\delta K_2))^{-\delta} - 1}{\alpha-\beta}\right) + a\mu \\
&= x_2^{\delta^2}\exp(-\delta^2 K_2) - \frac{\mu}{\alpha-\beta} + \mu\frac{x_2^{\delta^2}\exp(-\delta^2 K_2)}{\alpha-\beta} + a\mu \\
&= x_2^{\delta^2}\exp(-\delta^2 K_2)\left[1 - \frac{\mu}{\alpha-\beta}\right] - \frac{\mu}{\alpha-\beta} + a\mu.
\end{aligned}$$

For (1):

$$\begin{aligned} h_1(s,x_2,w_2) &= s - \frac{1}{\alpha+\beta}\ln x_2 + \frac{K_2}{\alpha+\beta} - \frac{1}{\alpha+\beta}\ln y_1 \\ &= s - \frac{1+\delta}{\alpha+\beta}\ln x_2 + \frac{K_2(1+\delta)}{\alpha+\beta}. \end{aligned}$$

Finally, for (3):

$$w1 = 1 + (w_2-1)\left(x_2^{2/(\alpha+\beta)}\exp\left(\frac{-2K_2}{\alpha+\beta}\right)\right)$$

$$\begin{aligned} h_3(s,x_2,w_2) &= (w_1+1)y_1^{\frac{2}{\alpha+\beta}} \\ &= \left[2 + (w_2-1)x_2^{\frac{2}{\alpha+\beta}}\exp\left(\frac{-2K_2}{\alpha+\beta}\right)\right]\left[x_2^{\delta}\exp(-\delta K_2)\right]^{\frac{2}{\alpha+\beta}}. \end{aligned}$$

□

From now on, we are interested in the dynamics when $\nu = 0$. The map h_2 just depends on x_2; this is why we may define $h_2 : [0,1] \to \mathbf{R}$ as

$$h_2(x_2) := h_2(s,x_2,w_2).$$

Lemma 5. *The following assertions are valid for* $(s,x_2,w_2) \in \mathbf{S}^1 \times In(\mathbf{v}_-)$*:*

(1) *For* $\mu = 0$, $x_2 = 0$ *is a hyperbolic attracting fixed point of* $h_2(x_2) = x_2^{\delta^2}$*;*

(2) *If* $\mu > 0$*, then:*

  (a) *if* $a - \frac{1}{\alpha-\beta} > 0$*, then* h_2 *has a hyperbolic attracting fixed point* $x^\star$ *of order* $\mathcal{O}(\mu)$*;*

  (b) *if* $\frac{4\alpha}{(\alpha+\beta)^2} > 1$*, then for any* $s \in \mathbf{S}^1$ *and any* $x_2 \in [0,1]$ *the map* $w_2 \mapsto h_3(s,x_2,w_2)$ *is a Lipschitz contraction in the variable* w_2*.*

Proof. (1) For $\mu = 0$, we have $K_2 = 0$ and then $h_2(x_2) = x_2^{\delta^2}$ (cf. Item (2) of Lemma 4), whose fixed point is 0. Since $h_2'(0) = 0$, the hyperbolicity and attractiveness follow.

(2) (a) For $\mu = 0$ the graph of $h_2(x_2)$ crosses transversely the graph of the identity at $x_2 = 0$, then for small $\mu \neq 0$ the two graphs still cross transversely at a nearby value $x^*(\mu)$ of x_2, the hyperbolic continuation of the fixed point found in (1). Since $h_2(0) = \mu\left(a - \frac{1}{\alpha-\beta}\right)$ then $x^*(\mu) > 0$ if and only if $h_2(0) > 0$ and in this case the fixed point is a hyperbolic attractor with $\lim_{\mu\to 0} x^\star(\mu) = 0$. The location of this fixed point is sketched in Figure 3.

  (b) By item (3) of Lemma 4, the map h_3 does not depend on s and may be written as

$$h_3(s,x_2,w_2) = C_1(x_2) + C_2(x_2)w_2$$

where $C_1, C_2 : [0,1] \to \mathbf{R}$ are smooth maps with

$$C_2(x_2) = x_2^{\frac{2+2\delta}{\alpha+\beta}}\exp\left(\frac{-2K_2(1+\delta)}{\alpha+\beta}\right).$$

Since

$$\frac{2+2\delta}{\alpha+\beta} = \frac{4\alpha}{(\alpha+\beta)^2} \overset{Hyp}{>} 1$$

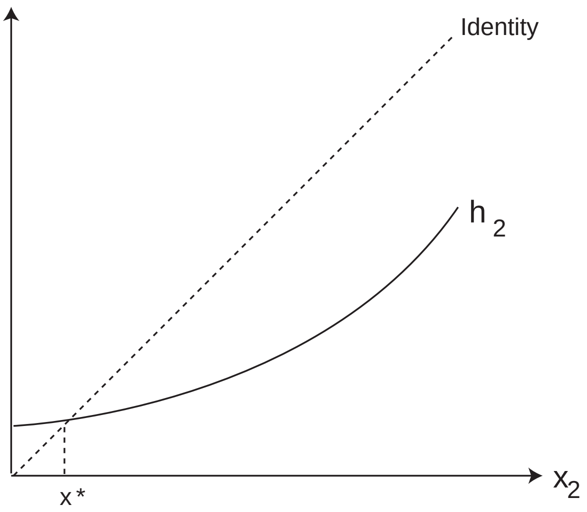

FIGURE 3. The graph of h_2 intersects the Identity map once near the origin.

and $\left|\exp\left(\frac{-2K_2(1+\delta)}{\alpha+\beta}\right)\right| < 1$, then h_3 is a Lipschitz contraction in the variable w_2.

□

Although $x_2 = 1$ is also a fixed point of h_2 (for $\mu = 0$), we are disregarding this point from the statement of Lemma 5 by two reasons: first is is repelling and second it lies on the boundary of the domain of $\mathcal{R}_{(0,0)}$. Please note that the restriction $\frac{4\alpha}{(\alpha+\beta)^2} > 1$ of Lemma 5 (2b) is also used in [13, §4.6] for the reduced map.

Theorem 6. *For $\nu = 0$ and small $\mu > 0$, if $\dfrac{4\alpha}{(\alpha+\beta)^2} > 1$ and $a > \dfrac{1}{\alpha-\beta}$, then the flow associated to $F_{(0,\mu)}$ has an attracting periodic solution whose period P satisfies $P = \mathcal{O}(-\ln\mu)$.*

Proof. Let $x^\star$ be the hyperbolic and attracting fixed point of h_2 obtained in Lemma 5. Since the map $w_2 \mapsto h_3(s, x^\star, w_2)$ is a Lipschitz contraction in the variable w_2, it has an attracting fixed point $w^\star$. that depends smoothly on x_2.

This means that there exists $P > 0$ such that

$$\mathcal{R}_{(0,\mu)}(s, x^\star, w^\star) = \mathcal{R}_{(0,\mu)}(s + P, x^\star, w^\star),$$

therefore the flow associated to $F_{(0,\mu)}$ has an attracting periodic solution of period P. This periodic solution spends a time $(T_2(s, x^\star, w^\star) - s)$ inside V_-, followed, inside V_+, by a flight time $(T_1(s^+, y^+, w^+) - s^+)$ with

$$(s^+, y^+, w^+) = \Psi_{\mathbf{v}_- \to \mathbf{v}_+} \circ \Phi_{\mathbf{v}_-}(s, y^\star, w^\star).$$

Using Item (1) of Lemma 4, this solution has period given by:

$$\begin{aligned} P &= s - \frac{1+\delta}{\alpha+\beta}\ln x^\star + \frac{K_2(1+\delta)}{\alpha+\beta} \\ &= s - \frac{1+\delta}{\alpha+\beta}\ln x^\star + \mu\frac{1-x^\star}{\alpha+\beta}\frac{(1+\delta)}{\alpha+\beta} \\ &\overset{x^\star=\mathcal{O}(\mu)}{=} s - \frac{1+\delta}{\alpha+\beta}\ln x^\star + \mathcal{O}(\mu). \end{aligned}$$

Since $x^\star = \mathcal{O}(\mu)$ (by Lemma 5) then the period P is of the order $\mathcal{O}(-\ln\mu)$ completing the proof. □

From Theorems 3 and 6 it follows immediately that:

Corollary 7. *For small $\nu, \mu > 0$ and very large ω, if $\dfrac{4\alpha}{(\alpha+\beta)^2} > 1$ and $a > \dfrac{1}{\alpha-\beta}$, then the flow associated to $F_{(\nu,\mu)}$ has an attracting periodic solution whose period P satisfies $P = \mathcal{O}(-\ln\mu)$.*

7. Discussion

In this work, we investigate the influence of high frequency forcing on a differential equation exhibiting a clean attracting heteroclinic network − *clean* in the sense that the unstable manifolds of all the nodes lie in the network. Our result says that if the frequency of the non-autonomous perturbation goes to infinity, then the dynamics of the vector field is governed by the averaged system: the non-autonomous equation (2.1) behaves like an autonomous one. Our main result has been motivated by Tsai and Dawes [9, 22, 23] in the context of the Guckenheimer and Holmes example. They claim without proof that the time-periodic forcing term has an effect equivalent to that of the time-averaged perturbation term.

Our findings agree well with the theory developed by Cheng-Gui *et al* [8]. They considered a system of the form

$$\dot{x} = f(x) + B\cos(\omega t)H \tag{7.12}$$

where $x \in \mathbf{R}^n$ represents the state vector of the nonlinear system, $H = (1, 1, ..., 1)^T$, $f : \mathbf{R}^n \to \mathbf{R}^n$ is a nonlinear vector field and $B\cos(\omega t)$ denotes a forcing with frequency ω and amplitude B. The unforced system ($H = \overline{0}$) may exhibit stationary, periodic or chaotic behaviour for different system parameters. They state that a general solution of (7.12), say $x(t)$, may be written as the sum of a slow motion $X(t)$ and a fast motion $\Psi(t)$:

$$x(t) = X(t) + \frac{1}{\omega}\Psi(t, \omega)$$

where $\Psi : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is a $2\pi/\omega$ t–periodic function with zero mean. If $\omega \to +\infty$, then the solution is governed by the slow-motion which is the solution of the original unperturbed system. The effect of high-frequency forcing becomes apparent. Stating our result in their terms, as the forcing frequency tends to infinity the equation for the fast motion drops out completely.

In our analysis, we have used the map $f(t) = \sin t$ in (2) and $H = (1, 0, ..., 0)^T$ but our work is still valid for any smooth t–periodic map f with zero average. We may conclude that when $\omega = +\infty$, in the extended phase space (Equation (4.3)) we cannot write the global map as in Section 4.3 of [14]. More specifically, in the limit case $\omega = +\infty$, the set $W^u(\mathbf{S}^1 \times \{\mathbf{v}_-\}) \cap In(\mathbf{S}^1 \times \{\mathbf{v}_+\})$ is not a non-degenerate graph of a multimodal function. When $\omega = +\infty$ the necessary distortion to obtain chaos (rotational horseshoes) does not hold to guarantee the torus-breakdown effects. This limit case was left open by Wang [24, pp. 4391].

We conjecture that our result holds for any attracting and clean heteroclinic network where the connections are one-dimensional. Since nonlinear systems driven by high frequency forcing are prevalent in nature and engineering, we expect that these results are valuable and helpful to those applications.

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