Rate of escape of the conditioned two-dimensional simple random walk

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June 16, 2023

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Abstract

We prove sharp asymptotic estimates for the rate of escape of the two-dimensional simple random walk conditioned to avoid a fixed finite set. We derive it from asymptotics available for the continuous analogue of this process [1], with the help of a KMT-type coupling adapted to this setup.

Keywords: Brownian motion, conditioning, KMT approximation, transience, Doob's h-transform

AMS 2010 subject classifications: 60J10, 60J60, 60J65

1 Introduction and main results

In this paper we study two-dimensional simple random walk (also abbreviated as SRW) conditioned on not hitting the origin. It appeared in [3] (see also [2, 4]) as the main ingredient in the construction of two-dimensional random interlacement process. Besides taking part in that construction, it has a number of remarkable properties itself (cf. e.g. [3, 6, 10]). Just to mention some of these (in the following, the conditioned SRW is denoted by $\widehat{S}$):

- the walk $\widehat{S}$ is transient; however, *any* infinite subset of $\mathbb{Z}^2$ is recurrent (i.e., visited infinitely many times a.s.);

- a (distant) site y is eventually visited by $\widehat{S}$ with probability converging to $\frac{1}{2}$, as $|y| \to \infty$.
- if one considers a "typical" large subset of $\mathbb{Z}^2$ (i.e., a box or a segment), then the proportion of its sites which are eventually visited by $\widehat{S}$ is a random variable that converges to Uniform$[0,1]$ in distribution (as the size of that subset grows).
- two independent copies of $\widehat{S}$ a.s. meet, and, moreover, $\widehat{S}$ a.s. meets an independent SRW S infinitely many times.

Here, our aim is to study the rate of escape of $(\widehat{S}_n)_{n\geq 0}$ to infinity, which we measure via the so-called *future minima process*:

$$M_n = \inf_{m\geq n} |\widehat{S}_m|, \qquad n \geq 0, \tag{1}$$

where $|\cdot|$ denotes the Euclidean norm (in $\mathbb{Z}^2$ or $\mathbb{R}^2$). One of the main results of [10] is that M "oscillates a lot": for every $0 < \delta < \frac{1}{2}$ we have, almost surely,

$$M_n \leq n^\delta \text{ i.o.}, \quad \text{but} \quad M_n \geq \tfrac{\sqrt{n}}{\ln^\delta n} \text{ i.o.}. \tag{2}$$

We aim to obtaining a finer version of the above result, via a comparison to the corresponding continuous model (which we define and discuss later).

Let us now pass to the formal discussion. Let $S = (S_n)_{n\geq 0}$ be the simple random walk on $\mathbb{Z}^2$. In the sequel, we will write $\mathbb{P}_x$ and $\mathbb{E}_x$ for the probability and the expectation with respect to the process (SRW or another one) started at x, when no confusion can arise. The SRW's *potential kernel* is the unique function (up to a constant factor) on $\mathbb{Z}^2$ that is 0 at the origin and harmonic on $\mathbb{Z}^2 \setminus \{0\}$. It is defined by:

$$a(x) = \sum_{n\geq 0} \big(\mathbb{P}_0[S_n = 0] - \mathbb{P}_0[S_n = x]\big), \qquad x \in \mathbb{Z}^2.$$

Harmonicity of a outside the origin implies that $(a(S_{n\wedge \tau_0^S}))_{n\geq 0}$, where τ_0^S is the time of the first visit of S to the origin, is a martingale (we therefore say that a is a scale function of S). See section 4.4 of [8] for more details on the potential kernel. Here, we only recall a useful asymptotic expression for a:

$$\frac{\pi a(x)}{2} = \ln |x| + \gamma_{\text{EM}} + \frac{3}{2}\ln 2 + O(|x|^{-2}), \tag{3}$$

as $|x|$ goes to infinity, where γ_{EM} is the Euler-Mascheroni constant.[1]

[1] $\gamma_{\text{EM}} = \lim_{n\to\infty} \big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\big) \approx 0.577$

We now consider another Markov chain on $\mathbb{Z}^2 \setminus \{0\}$, denoted by $\widehat{S} = (\widehat{S}_n)_{n\geq 0}$, which is obtained by applying the Doob's transform to $(S_n)_{n\geq 0}$ with respect to the potential kernel a. Explicitly, it is a Markov chain on $\mathbb{Z}^2 \setminus \{0\}$ with transition probabilities:

$$p_{x,y} = \frac{a(y)}{4a(x)}\mathbf{1}_{|x-y|=1}.$$

This Markov chain is called *SRW on $\mathbb{Z}^2$ conditioned on not hitting the origin* since it appears as a natural limiting process, as r goes to infinity, of the law of S conditioned on reaching distance to the origin larger than r, before visiting the origin. We have already mentioned some remarkable properties of $\widehat{S}$; let us refer to section 4.1 of [11] for an elementary introduction to Doob's h-transforms, and to section 4.2 of [11] for the detailed discussion of this definition and for derivation of some basic properties of $\widehat{S}$.

The main result of this paper is the following theorem about the future minimum process M (recall (1)). It can be seen as the discrete counterpart of the corresponding results for the analogous "continuous model" (which is the two-dimensional Brownian motion conditioned on staying outside of a disk, as explained in section 1.2 below), see Theorems 1.1 and 1.2 of [1].

Theorem 1. *Independently of the starting point of $\widehat{S}$ in $\mathbb{Z}^2\setminus\{0\}$, the two following results hold.*

- *For $g : \mathbb{R}_+ \to \mathbb{R}_+$ non-increasing such that $t \mapsto (\ln t)g(\ln\ln t)$ is non-decreasing,*

$$\mathbb{P}\big[M_n \leq e^{(\ln n)g(\ln\ln n)} \text{ i.o.}\big] = 0 \text{ or } 1 \text{ according to } \int^{\infty} g(u)du < \infty \text{ or } = \infty.$$

- *The limit*

$$\limsup_{t\to\infty} \frac{M_n}{\sqrt{n \ln\ln\ln n}}$$

is a.s. finite and positive constant, equal to the value K^* provided by Theorem 1.2 of [1].

It is straightforward to see that the above result indeed generalizes (2); in particular, take $g \equiv \delta$ to obtain that $M_n \leq n^{\delta}$ i.o.. The main idea of the proof will be using the dyadic coupling (also known as KMT approximation, or Hungarian embedding) to be able to "compare" the conditioned SRW to the conditioned Brownian motion; then, it will be possible to derive Theorem 1 from the corresponding results in the continuous case (see Theorem 3 and Remark 4 in section 1.2). Note that the actual value of K^* is not known; in the above result, we only claim that here we obtain the same constant as for the corresponding continuous model.

1.1 A generalization: SRW conditioned on not hitting a finite set

A natural question that one may pose is the following one: what if, instead of conditioning the SRW on not hitting the origin, we condition it on not hitting some *finite* set $A \subset \mathbb{Z}^2$, will it change anything? In this section we argue that the answer to this question is essentially "no". While one may take this fact as evident since (assuming without loss of generality that $0 \in A$) S conditioned on not hitting A "should be" $\widehat{S}$ conditioned on not hitting A and a transient walk (such as $\widehat{S}$) will eventually escape from A anyway, we still decided to discuss this question here since conditioning on zero-probability events can be tricky, so, in principle, one should not always blindly believe one's intuition in that respect.

We also remark that conditioning the SRW on not hitting an *infinite* set A' can indeed be a quite different story (in the sense that the resulting process will be "drastically different"[2] from $\widehat{S}$); however, questions of this sort are beyond the scope of this paper.

Let us fix a finite set A containing the origin and study the SRW conditioned on not hitting A, which we define now. Consider the function $q_A : \mathbb{Z}^2 \to \mathbb{R}$ defined by

$$q_A(x) = a(x) - \mathbb{E}_x[a(S_{\tau_A})], \qquad x \in \mathbb{Z}^2,$$

with $\mathbb{E}_x$ denoting the expectation under law $\mathbb{P}_x$ and τ_A the hitting time of A by S. Function q_A vanishes on A and is harmonic on $\mathbb{Z}^2 \setminus A$, so q_A is a scale function for $S_{\cdot \wedge \tau_A}$. Furthermore, it has the following asymptotic expression:

$$\frac{\pi q_A(x)}{2} = \ln|x| - \frac{\pi}{2}\operatorname{cap}(A) + \gamma_{\mathrm{EM}} + \frac{3}{2}\ln 2 + O(|x|^{-1}), \tag{4}$$

as $|x|$ goes to infinity, where $\operatorname{cap}(A)$ is the *capacity* of A. We refer to chapter 3 of [11] for all necessary details.

In the following, we use the general construction of chapter 4.1 of [11] (see also exercise 4.6 there). Consider the Markov chain on $\mathbb{Z}^2 \setminus A$, denoted by $\widehat{S}^{(A)} = (\widehat{S}^{(A)}_n)_{n\geq 0}$, which is obtained by applying the Doob's transform to $(S_n)_{n\geq 0}$ with respect to q_A. Analogously to $\widehat{S}$, it is the Markov chain on $\mathbb{Z}^2 \setminus A$ with transition probabilities:

$$p^{(A)}_{x,y} = \frac{q_A(y)}{4q_A(x)}\mathbf{1}_{|x-y|=1}.$$

We now prove the following fact ($\widehat{\tau}$ denotes the hitting times with respect to the conditioned walk $\widehat{S}$):

[2] Indeed, as we mentioned, $\widehat{S}$ hits any fixed infinite set infinitely many times a.s.

Lemma 2. *For any $x_0 \notin A$ and any nearest-neighbour path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_m)$ which does not touch A and such that $x_0 = \gamma_0$, we have*

$$\mathbb{P}_{x_0}[\widehat{S}_1 = \gamma_1, \dots, \widehat{S}_m = \gamma_m \mid \widehat{\tau}_A = \infty] = \mathbb{P}_{x_0}[\widehat{S}^{(A)}_1 = \gamma_1, \dots, \widehat{S}^{(A)}_m = \gamma_m]; \tag{5}$$

i.e., it holds indeed that $\widehat{S}$ conditioned on never hitting A is $\widehat{S}^{(A)}$.

Proof. Let us first show that, for $x_0 \notin A$,

$$\mathbb{P}_{x_0}[\widehat{\tau}_A = \infty] = \frac{q_A(x_0)}{a(x_0)}. \tag{6}$$

Indeed, take a large R such that[3] $B(0, R)$ contains both x_0 and A, and, with some abuse of notation, abbreviate τ_R (respectively, $\widehat{\tau}_R$) to be the hitting time of $\partial B(0, R)$ by S (respectively, by $\widehat{S}$). By, first, Lemma 4.4 of [11], and, then, Lemma 3.12 and formula (3.52) of [11], we can write (recall that $0 \in A$)

$$\begin{aligned}
\mathbb{P}_{x_0}[\widehat{\tau}_A > \widehat{\tau}_R] &= \mathbb{P}_{x_0}[\tau_A > \tau_R \mid \tau_0 > \tau_R]\big(1 + O((R \ln R)^{-1})\big) \\
&= \frac{\mathbb{P}_{x_0}[\tau_A > \tau_R]}{\mathbb{P}_{x_0}[\tau_0 > \tau_R]}\big(1 + O((R \ln R)^{-1})\big) \\
&= \frac{q_A(x_0)}{a(R) + O(R^{-1})} \times \frac{a(R) + O(R^{-1})}{a(x_0)}\big(1 + O((R \ln R)^{-1})\big);
\end{aligned}$$

sending R to infinity, we obtain (6).

Next, denoting $y = \gamma_m$ we have, using (6) twice

$$\begin{aligned}
&\mathbb{P}_{x_0}[\widehat{S}_1 = \gamma_1, \dots, \widehat{S}_m = \gamma_m \mid \widehat{\tau}_A = \infty] \\
&= \frac{a(x_0)}{q_A(x_0)} \mathbb{P}_{x_0}[\widehat{S}_1 = \gamma_1, \dots, \widehat{S}_m = \gamma_m, \widehat{\tau}_A = \infty] \\
&= \frac{a(x_0)}{q_A(x_0)} \times \mathbb{P}_y[\widehat{\tau}_A = \infty] \times \frac{a(y)}{a(x_0)} \mathbb{P}_{x_0}[S_1 = \gamma_1, \dots, S_m = \gamma_m] \\
&= \frac{q_A(y)}{q_A(x_0)} \mathbb{P}_{x_0}[S_1 = \gamma_1, \dots, S_m = \gamma_m] \\
&= \mathbb{P}_{x_0}[\widehat{S}^{(A)}_1 = \gamma_1, \dots, \widehat{S}^{(A)}_m = \gamma_m],
\end{aligned}$$

and the proof is complete. □

Now, with Lemma 2 to hand, it becomes clear that Theorem 1 should hold with $\widehat{S}^{(A)}$ on the place of $\widehat{S}$, due to the usual excursion decomposition argument. Namely, an $\widehat{S}$-trajectory has an a.s. finite number of excursions from the external boundary of A to A, and then the "escape trajectory" that does not touch A anymore. That escape trajectory has the law of $\widehat{S}^{(A)}$, which implies the preceding claim.

[3] Here and in the sequel, $B(x, r)$ stands for the disk $\{y \in \mathcal{X} : |y - x| \leq r\}$, where $\mathcal{X} = \mathbb{Z}^2$ or $\mathbb{R}^2$ (depending on the context).

1.2 Coupling with the conditioned Brownian motion

As we mentioned above, in [1], the continuous analogue of the process $\widehat{S}$ was studied and sharper bounds on the rate of escape were obtained.

Let $W = (W_t)_{t\geq 0}$ be the Brownian motion on $\mathbb{R}^2$. For any $\rho \in (0, \infty)$, the function $x \mapsto \ln(|x|/\rho)$ vanishes on $\partial B(0, \rho)$ and is a scale function for W. The standard Brownian motion on $\mathbb{R}^2$ *conditioned not to visit* $B(0, \rho)$ is a diffusion on $\mathbb{R}^2 \setminus B(0, \rho)$, denoted $\widehat{W}^\rho = (\widehat{W}^\rho_t)_{t\geq 0}$, which can be defined e.g. via its transition kernel $\hat{p}_\rho$: for $\|x\| > \rho$, $\|y\| \geq \rho$,

$$\hat{p}_\rho(t, x, y) = p_0(t, x, y)\frac{\ln(|y|/\rho)}{\ln(|x|/\rho)}, \tag{7}$$

where p_0 denotes the transition subprobability density of W killed on hitting the disk $\mathsf{B}(0, \rho)$. We refer to [5] for a more detailed discussion.

For now, we consider the case $\rho = 1$. Process $\widehat{W}^1$ is rotationally invariant, its norm is itself a one-dimensional diffusion, which exhibits interesting renewal properties. This rotational invariance is the key difference from the discrete case, which makes it possible to obtain sharp results on the rate of escape by only studying that one-dimensional diffusion, as was done in [1]. Denoting by

$$M^1(t) = \inf_{s\geq t} |\widehat{W}^1_s|, \qquad t \geq 0,$$

the future minima process of $|\widehat{W}^1|$, the main results of [1] are the following.

Theorem 3. *Independently of the starting point of $\widehat{W}^1$ in $\mathbb{R}^2 \setminus B(0, 1)$, the two following results hold.*

- *For $g : \mathbb{R}_+ \to \mathbb{R}_+$ non-increasing such that $t \mapsto (\ln t)g(\ln\ln t)$ is non-decreasing, we have:*

  $$\mathbb{P}\big[M^1(t) \leq e^{(\ln t)g(\ln\ln t)} \text{ } i.o.\big] = 0 \text{ or } 1 \text{ } \textit{according to } \int^\infty g(u)du < \infty \text{ or } = \infty.$$

  (Here, "i.o." means "for an unbounded set of t's".)

- *The limit*

  $$K^* = \limsup_{t\to\infty} \frac{M^1(t)}{\sqrt{t\ln\ln\ln t}}$$

  exists, is a.s. constant and is non-degenerate: $0 < K^* < \infty$.

Remark 4. *A consequence of scaling invariance of the Brownian motion is that for any $\rho \in (0, \infty)$, process $\widehat{W}^\rho$ is equal in law to process $(\rho\widehat{W}^1_{t\rho^{-2}})_{t\geq 0}$. Hence, Theorem 3 holds with M^1 replaced by $(M^\rho_t)_{t\geq 0} = (\inf_{s\geq t} |\widehat{W}^\rho_s|)_{t\geq 0}$, the future minima process of $(|\widehat{W}^\rho_t|)_{t\geq 0}$, with the same value of K^*.*

Next, denote $\rho_0 = \exp(-\gamma_{\text{EM}} - \frac{3}{2}\ln 2) \approx 0.1985$ and write $\widehat{W}$ for $\widehat{W}^{\rho_0}$. In order to prove Theorem 1, we will construct a coupling of processes $\widehat{W}$ and $\widehat{S}$. In sections 3 and 4, we are going to show the following

Theorem 5. *There exists a positive constant α and for any $\varepsilon > 0$ a coupling of $\widehat{S}$ and $\widehat{W}$ started at any prescribed points in $\mathbb{Z}^2 \setminus \{0\}$ and $\mathbb{R}^2 \setminus B(0, \rho_0)$ respectively, such that with probability not smaller than $1 - \varepsilon$:*

$$\big||\widehat{S}_t| - |\widehat{W}_t|\big| \le \ln^\alpha t \qquad \text{eventually, as } t \to \infty, \tag{8}$$

where $(\widehat{S}_t)_{t\ge 0}$ is the process defined by linear interpolation of the sequence $(\widehat{S}_n)_{n\ge 0}$.

Also, with probability not smaller than $1 - \varepsilon$:

$$|\widehat{S}_t| \sim |\widehat{W}_t|, \text{ as } t \to \infty. \tag{9}$$

Remark 6. *We can be more precise about the value of constant α in Theorem 5. Indeed, its proof (which is to be found in section 4) shows that any $\alpha > 27$ works. However, we stress that we have no reason to believe that the above would have to be sharp.*

Remark 7. *The choice of $\rho_0 = \exp(-\gamma_{\text{EM}} - \frac{3}{2}\ln 2)$ comes from the asymptotic expression (3). Indeed, it is the only choice which guarantees that*

$$\frac{\pi a(x)}{2} = \ln \frac{|x|}{\rho_0} + O(|x|^{-1}), \tag{10}$$

as $|x|$ goes to infinity. Even though, essentially in the spirit of section 1.1, it is possible to show that, even without space-time rescaling, all processes $\widehat{W}^\rho$ have the same large-t behaviour due to transience, we will see in the proofs that $\widehat{W} = \widehat{W}^{\rho_0}$ is the "right one" for comparison with $\widehat{S}$, since it has almost the same transition probabilities between certain concentric "levels".

2 Sampling $\widehat{W}$ and $\widehat{S}$

As a first step towards the construction of the coupling, we describe how a realisation of $\widehat{W}$ (respectively, $\widehat{S}$) can be sampled using an i.i.d. family of W-trajectories (respectively S-trajectories).

2.1 Notations

For $m \ge 0$, we set $r_m = \rho_0\, e^m$ and denote $C_m = \{x \in \mathbb{R}^2 : |x| = r_m\}$ the circle of radius r_m centered at the origin. Consider a trajectory of $\widehat{W}$ started at some point $x \in C_h$, where h is some positive integer. It can be described in terms of excursions between the levels $(C_m, m \geq 1)$. Indeed, define the sequence of stopping times $(t_k)_{k\geq 0}$ and the *level sequence* $(m_k)_{k\geq 0}$ such that: $t_0 = 0, m_0 = h$ and for $k \geq 0$,

$$t_{k+1} = \inf\{t > t_k : \widehat{W}_t \in C_{m_k-1} \cup C_{m_k+1}\}, \tag{11}$$

$$m_{k+1} = \ln(|\widehat{W}_{t_{k+1}}|/\rho_0). \tag{12}$$

Also, denote by $x_k = \widehat{W}_{t_k}, k \geq 0$ the hitting points of the circles.

We need to be more careful when introducing the discrete counterparts of the above notations. Let us define $\Gamma_0 = \{0\}$, $\Gamma_1 = \{x \in \mathbb{Z}^2 : |x| = 1\}$, $\Gamma_2 = \{x \in \mathbb{Z}^2 : 1 < |x| \leq 2\}$, and, for $m \geq 3$ (note that $r_3 \approx 3.9871$),

$$\Gamma_m = \{x \in \mathbb{Z}^2 \colon r_m - 1 < |x| \leq r_m\}.$$

We can then decompose any trajectory started from $\xi \in \Gamma_h$, with some positive integer h, into excursions between the levels $(\Gamma_m, m \geq 0)$. Define sequences $(\tau_k)_{k\geq 0}$ and $(\mu_k)_{k\geq 0}$ such that: $\tau_0 = 0, \mu_0 = h$, and for $k \geq 0$,

$$\tau_{k+1} = \inf\big\{n > \tau_k : \widehat{S}_n \in \Gamma_{\mu_k-1} \cup \Gamma_{\mu_k+1}\big\}, \tag{13}$$

$$\mu_{k+1} = m \text{ on } \big\{\widehat{S}_{\tau_{k+1}} \in \Gamma_m\big\}, m = 1, 2, 3, \ldots. \tag{14}$$

Finally, denote by $\xi_k = \widehat{S}_{\tau_k}, k \geq 0$ the hitting points of these "discrete circles".

For any k, $(\widehat{W}_t)_{t\in(t_k,t_{k+1}]}$ is called an *excursion* between *levels* m_k and m_{k+1}, and $(\widehat{S}_n)_{n\in(\tau_k,\tau_{k+1}]}$ is called an excursion between levels μ_k and μ_{k+1}. By construction, the level sequences $(m_k)_{k\geq 0}$ and $(\mu_k)_{k\geq 0}$ have increments in $\{-1, +1\}$, and by transience of $\widehat{W}$ and $\widehat{S}$ these sequences tend to infinity.

2.2 Constructing $\widehat{W}$ from W

Take a positive integer m and any point x in C_m. We consider the first excursion of $\widehat{W}$ started from level m, at point x. By definition of $\widehat{W}$ (recall section 1.2), we have

$$\mathbb{P}_x\big[(\widehat{W}_t)_{0\leq t\leq t_1} \in \cdot\,, \widehat{W}_{t_1} \in dy\big] = \frac{\ln(|y|/\rho_0)}{\ln(|x|/\rho_0)}\mathbb{P}_x\big[\,(W_t)_{0\leq t\leq t_1} \in \cdot\,, W_{t_1} \in dy\big] \tag{15}$$

(defining t_1 for process W in the same way as for process $\widehat{W}$).

On one hand, using that W has scaling function $h : x \mapsto \ln|x|$, we have that $\mathbb{P}^W_x\big[W_{t_1} \in C_{m-1}\big] = \mathbb{P}^W_x\big[W_{t_1} \in C_{m+1}\big] = \frac{1}{2}$, hence (15) implies that

$$\mathbb{P}_x\big[\widehat{W}_{t_1} \in C_{m-1}\big] = \frac{m-1}{2m}, \tag{16}$$

$$\mathbb{P}_x\big[\widehat{W}_{t_1} \in C_{m+1}\big] = \frac{m+1}{2m}. \tag{17}$$

Applying the strong Markov property, we see that the random sequence $(m_k)_{k\geq 0}$ follows the law of a Markov chain on $\mathbb{N}$ with transition probabilities $\frac{m-1}{2m}$ to the left and $\frac{m+1}{2m}$ to the right. In the sequel, we will refer to this Markov chain as *conditioned simple random walk in one dimension*, or 1-CSRW for short. The reason for this name is that it is indeed the simple random walk on $\mathbb{Z}$ conditioned to stay positive, because it is obtained by applying the Doob's transform (with respect to $x \mapsto |x|$) to the simple random walk on $\mathbb{Z}$. It can be interpreted, once again, as the limit of the simple random walk (started in the positive domain) conditioned to visit site N before the origin, as N goes to infinity, see section 4.1 of [11]. We remark also that, as 1-CSRW has the drift of order x^{-1} at x, it belongs to the class of *Lamperti processes*, cf. Chapter 3 of [9].

On the other hand, (15) implies that, conditionally on the sequence $(m_k)_{k\geq 0}$, the excursions of $\widehat{W}$ are simply Brownian excursions conditioned on the exiting the annuli via the corresponding (inner or outer) boundaries.

Therefore, one way to sample $\widehat{W}$ would be to sample the level sequence $(m_k)_{k\geq 0}$ as a realisation of 1-CSRW, and then to sample excursions of W conditioned on hitting the inner or outer circle first, according to the level sequence. However, we prefer to use the construction described below, which has the virtue of avoiding this conditioning of excursions. The procedure we use is the usual acceptance-rejection technique of generating random variables.

To construct $(\widehat{W}_t)_{t\geq 0}$, we need the following ingredients:

- a collection of i.i.d. stochastic processes $(\widetilde{W}_\cdot^{(k,\ell)}, k, \ell \geq 0)$, which are all two-dimensional Brownian motions (started at the origin).
- a collection of i.i.d. random variables $(U_{k,\ell}, k, \ell \geq 0)$ with Uniform$[0,1]$ distribution;

Suppose that, initially, the process is at $x \in C_h$, with some positive integer h. Now, what we aim to construct is a sequence of excursions (as defined above), i.e., a sequence of time moments $(t_k, k \geq 0)$ with $t_0 = 0$, and a sequence of integers $(m_k, k \geq 0)$ with $m_0 = h$ such that $\widehat{W}_{t_k} \in C_{m_k}$ and such that on the interval $(t_k, t_{k+1}]$, $\widehat{W}$ travels between the levels m_k and m_{k+1}.

So, assume that we have constructed the process up to time t_k and that $x_k = \widehat{W}_{t_k} \in C_{m_k}$. We consider successively for all ℓ, the trajectory $x_k + \widetilde{W}_\cdot^{(k+1,\ell)}$, stopped when it first hits $C_{m_k-1} \cup C_{m_k+1}$, say at time $s_{k+1}^{(\ell)}$.

- if the trajectory first hits C_{m_k+1}, we keep it: we set $m_{k+1} = m_k + 1$, $t_{k+1} = t_k + s_{k+1}^{(\ell)}$ and $\widehat{W}_{\tau_k+t} = \widehat{W}_{\tau_k} + \widetilde{W}_t^{(k+1,l)}$ for $0 < t \leq s_{k+1}^{(\ell)}$;

- if the trajectory first hits C_{m_k-1},
  - provided $U_{k+1,\ell} \le 1 - \frac{2}{m_k+1}$, we keep it: we set $m_{k+1} = m_k - 1$, $t_{k+1} = t_k + s_{k+1}^{(\ell)}$, and $\widehat{W}_{t_k+t} = x_k + \widetilde{W}_t^{(k+1,\ell)}$ for $0 < t \le s_{k+1}^{(\ell)}$;
  - otherwise, we discard it, and restart the procedure using the trajectory $\widetilde{W}_\cdot^{(k+1,\ell+1)}$ together with the next variable $U_{k+1,\ell+1}$.

Once an excursion has been accepted (which eventually happens a.s.), we stop the iteration (in ℓ) and construct the next excursion starting from $x_{k+1} = \widehat{W}_{t_{k+1}} \in C_{m_{k+1}}$.

This construction relies on two key observations:

- the sequence of jumps $(m_k, k \ge 0)$ follows the law of 1-CSRW. Indeed, assume that $m_k = m$ and denote by p_m (respectively, q_m) the probability that the process we constructed jumps to $m+1$ (respectively, to $m-1$). As already observed, using the harmonicity of $x \mapsto \ln(|x|/\rho_0)$, for each ℓ, the trajectory $x_k + \widetilde{W}_\cdot^{(k+1,\ell)}$ hits C_{m+1} before C_{m-1} with probability $1/2$. Therefore, p_m and q_m are proportional to, respectively, $\frac{1}{2}$ and $\frac{1}{2}(1-\frac{2}{m+1})$. Using the equality $p_m + q_m = 1$, we obtain the correct probabilities.
- by construction, the excursions are excursions of the Brownian motion W, as required.

2.3 Constructing $\widehat{S}$ from S

We are going to present a similar construction for process $\widehat{S}$, but let us first investigate the decomposition into excursions, as we did for the continuous process. Take a positive integer m and any point x in Γ_m. We consider the first excursion of $\widehat{S}$ started from level m, at point x. By definition of $\widehat{S}$ as the Doob's transform of S, we have:

$$\mathbb{P}_x\big[(\widehat{S}_n)_{0\le n\le\tau_1} \in \cdot\,, \widehat{S}_{\tau_1} = y\big] = \frac{a(y)}{a(x)}\,\mathbb{P}_x[\,(S_n)_{0\le n\le\tau_1} \in \cdot\,, S_{\tau_1} = y] \tag{18}$$

(defining τ_1 for process S in the same manner as for process $\widehat{S}$). In other terms, the law of $\widehat{S}_{\tau_1}$ is given by:

$$\mathbb{P}_x\big[\widehat{S}_{\tau_1} = y\big] = \frac{a(y)}{a(x)}\,\mathbb{P}_x[S_{\tau_1} = y]\,, \tag{19}$$

and, conditionally on $\{\widehat{S}_{\tau_1} = y\}$, $(\widehat{S}_n)_{0\le n\le\tau_1}$ is an excursion of S started from x and conditioned to hit $\Gamma_{m-1} \cup \Gamma_{m+1}$ at y.

To sample a realisation of $\widehat{S}$ started at some initial point $\xi \in \Gamma_h$, with some positive integer h, we use the following ingredients:

- a collection of i.i.d. stochastic processes $(\widetilde{S}^{(k,\ell)}_{\cdot}, k, \ell \geq 0)$, which are two-dimensional simple random walks (started at the origin).

- a collection of i.i.d. random variables $(U_{k,\ell}, k, \ell \geq 0)$ with Uniform$[0,1]$ distribution.[4]

We will also use the following notation: for $m \geq 1$ and $y \in \Gamma_{m-1} \cup \Gamma_{m+1}$, define

$$\pi_{m,y} = \frac{a(y)}{\max_{z \in \Gamma_{m-1} \cup \Gamma_{m+1}} a(z)}, \tag{20}$$

which is a real number in $[0,1]$.

We proceed as before, i.e., we construct recursively a sequence of stopping times $(\tau_k, k \geq 0)$ with $\tau_0 = 0$, and a sequence of integers $(\mu_k, k \geq 0)$ with $\mu_0 = h$ such that $\widehat{S}_{\tau_k} \in \Gamma_{\mu_k}$ and such that on the interval $(\tau_k, \tau_{k+1}]$, $\widehat{S}$ travels between the two levels Γ_{μ_k} and $\Gamma_{\mu_{k+1}}$. Recall that the sequence of hitting points is denoted by $(\xi_k)_{k \geq 0}$, so, in particular, $\xi_0 = \xi$.

Now, assume that we have constructed the process up to time τ_k and that $\xi_k = \widehat{S}_{\tau_k} \in \Gamma_{\mu_k}$. Successively, for all values of ℓ, we consider the trajectory $\xi_k + \widetilde{S}^{(k+1,\ell)}_{\cdot}$, stopped when it first hits $\Gamma_{\mu_k - 1} \cup \Gamma_{\mu_k + 1}$, say at (integer) time $\sigma^{(\ell)}_{k+1}$, and denote the hitting point by $\xi^{(\ell)}_{k+1}$. Then we do the following:

- if $U_{k+1,\ell} \leq \pi_{m_k, \xi^{(\ell)}_{k+1}}$, we keep the trajectory: we set $\xi_{k+1} = \xi^{(\ell)}_{k+1}$, $m_{k+1} = m_k + 1$ or $m_k - 1$ according to ξ_{k+1}, $\tau_{k+1} = \tau_k + \sigma^{(\ell)}_{k+1}$ and $\widehat{S}_{\tau_k + n} = \xi_k + \widetilde{S}^{(k+1,\ell)}_n$ for $n = 1, \ldots, \sigma^{(\ell)}_{k+1}$.

- otherwise, we discard it, and restart the procedure using the next trajectory $\widetilde{S}^{(k+1,\ell+1)}_{\cdot}$ and the next variable $U_{k+1,\ell+1}$.

Once an excursion has been accepted (which eventually happens a.s.), we stop the iteration (in ℓ) and construct the next excursion starting from $\xi_{k+1} = \widehat{S}_{\tau_{k+1}} \in \Gamma_{\mu_{k+1}}$.

The construction we described indeed gives rise to process $\widehat{S}$ since

- the sequence of hitting points $(\xi_k)_{k \geq 0}$ obeys the law described in (19). Indeed, assume that $\xi_k = x$; then the probability that the process we constructed satisfies $\xi_{k+1} = y$ is proportional to $\pi_{m,y} \mathbb{P}_x[S_{\tau_1} = y]$, hence it is equal to $\frac{a(y)}{a(x)} \mathbb{P}_x[S_{\tau_1} = y]$.

- by construction, the excursions are SRW's excursions, as required.

[4] We use the same notation as in Section 2.2 for these random variables because they are actually the same random variables; i.e., we will couple $\widehat{S}$ and $\widehat{W}$ by using the same collection of U's.

3 Construction of the coupling

In this section, we will present the construction of the coupling mentioned in Theorem 5. But first, let us make an observation on the range of initial points we have to handle. Fix some $\varepsilon \in (0,1)$, some $x \in \mathbb{R}^2 \setminus B(0,\rho_0)$, and some $\xi \in \mathbb{Z}^2 \setminus \{0\}$. Assume that we want to build the coupling with $\widehat{W}$ started from x and $\widehat{S}$ started from ξ. Choose some (large) h such that $r_h \geq |x| \vee |\xi|$, then let $\widehat{S}$ and $\widehat{W}$ evolve independently under their respective laws until times

$$t_0 = \inf\{t \geq 0 : \widehat{W}_t \in C_h\}, \tag{21}$$

$$\tau_0 = \inf\{n \geq 0 : \widehat{S}_n \in \Gamma_h\}. \tag{22}$$

Then, we observe that

$$\begin{aligned} \big||\widehat{S}_t| - |\widehat{W}_t|\big| &\leq \big||\widehat{S}_t| - |\widehat{S}_{t+\tau_0-t_0}|\big| + \big||\widehat{S}_{\tau_0+t-t_0}| - |\widehat{W}_t|\big| \\ &\leq |\tau_0 - t_0| + \big||\widehat{S}_{\tau_0+t-t_0}| - |\widehat{W}_t|\big|, \end{aligned} \tag{23}$$

where we have used in the last inequality that $\widehat{S}$ makes jumps of size 1. Now, if we manage to construct $\widehat{S}$ and $\widehat{W}$ after times τ_0 and t_0 such that with probability not smaller than $1-\varepsilon$

$$\big||\widehat{S}_{\tau_0+t}| - |\widehat{W}_{t_0+t}|\big| \leq \ln^\alpha t \quad \text{eventually}, \tag{24}$$

then we will have eventually, plugging this bound into (23), that

$$\big||\widehat{S}_t| - |\widehat{W}_t|\big| \leq |\tau_0 - t_0| + \ln^\alpha(t - \tau_0), \tag{25}$$

which, for any $\alpha' > \alpha$, is eventually not larger than $\ln^{\alpha'} t$.

Hence we are left with the task of obtaining (24), which means constructing the coupling starting from any prescribed points $\xi \in \Gamma_h$ and $x \in C_h$, with h chosen as large as we desire.

In [7], J. Komlós, P. Major, and G. Tusnády proved the existence of a coupling between the simple random walk $(S_n)_{n\geq 0}$ and the Brownian motion $(W_t)_{t\geq 0}$, both started at the origin, and constants C, K, λ such that for all $n \in \mathbb{N}$ and $x > 0$

$$\mathbb{P}\Big[\max_{k=1,\dots,n} |S_k - W_k| \geq C\ln n + x\Big] \leq Ke^{-\lambda x}. \tag{26}$$

From the Borel-Cantelli lemma, it is immediate to derive that for any $\varepsilon > 0$, almost surely $|S_n - W_n| \leq (C + 1/\lambda + \varepsilon)\ln n$ eventually, as n goes to infinity. We refer to Chapter 7 of [8] for a modern introduction to this coupling, which is known under the denomination of *dyadic coupling*, *KMT approximation*, or *Hungarian embedding*. Using it, we are going to construct a coupling between $\widehat{S}$, the simple random walk conditioned on not hitting the origin, and $\widehat{W}$, the Brownian motion conditioned on not hitting the ball $B(0, \rho_0)$.

Our aim is to achieve in parallel the constructions of $\widehat{W}$ and $\widehat{S}$ described in previous section, by using KMT-coupled trajectories of W and S, and common variables $(U_{k,\ell}, k, \ell \geq 0)$, in such a way that the moduli stay close. To construct the coupling, we need the following ingredients:

- a collection of i.i.d. pairs of stochastic processes $\big((\widetilde{W}^{(k,\ell)}_{\cdot}, \widetilde{S}^{(k,\ell)}_{\cdot}), k, \ell \geq 0\big)$, where $\widetilde{W}$ is a two-dimensional Brownian motion (started at the origin), $\widetilde{S}$ is a two-dimensional SRW (also started at the origin), and they are KMT-coupled (within each pair);
- a collection of i.i.d. random variables $(U_{k,\ell}, k, \ell \geq 0)$ with Uniform$[0, 1]$ distribution.

Suppose that, initially, both processes are at level h, with h large, i.e., $x_0 = \widehat{W}_0 \in C_h$ and $\xi_0 = \widehat{S}_0 \in \Gamma_h$. What we intend to construct are two sequences of time moments $(t_k, \tau_k, k \geq 0)$, such that $\widehat{W}_{t_k} \in C_{m_k}$, $\widehat{S}_{\tau_k} \in \Gamma_{\mu_k}$, and with probability not smaller than $1-\varepsilon$ the following happens: for all $k \geq 0$, $\widehat{W}$ on the interval $(t_k, t_{k+1}]$ and $\widehat{S}$ on the interval $(\tau_k, \tau_{k+1}]$ "travel together" between the same levels (i.e., the level sequences m and μ coincide) and the difference $|t_k - \tau_k|$ does not grow too much.

So, assume that we have constructed both processes up to t_k and τ_k, and both $x_k = \widetilde{W}_{t_k}$ and $\xi_k = \widehat{S}_{\tau_k}$ belong to C_{m_k} and, respectively, to Γ_{m_k}.

We now intend to simultaneously construct $\widehat{W}$ on the interval $(t_k, t_{k+1}]$ and $\widehat{S}$ on the interval $(\tau_k, \tau_{k+1}]$; we will do that using the U's and the $(\widetilde{W}, \widetilde{S})$'s with the first index k. We successively try $\ell = 1, 2, 3, \ldots$, and each step will be a *success* (in which case we have what we wanted, and can pass from k to $k + 1$), a *failure* (this means that the candidate excursions were discarded, so we pass to the next ℓ and try to obtain a success), or a *catastrophe* (which means that the processes have really decoupled). A technical operation is needed, which is to rotate the trajectory $\widetilde{W}^{(k+1,\ell)}_{\cdot}$. Take θ_k some angle so that r_{θ_k}, the rotation by angle θ_k, sends x_k near ξ_k; more precisely, we require that $|\xi_k - r_{\theta_k}(x_k)| \leq 1$. Denote by $s^{(\ell)}_{k+1}$ the hitting time of $C_{m-1} \cup C_{m+1}$ by $x_k + r_{-\theta_k}\big(\widetilde{W}^{(k+1,\ell)}_{\cdot}\big)$, and by $\sigma^{(\ell)}_k$ the hitting time of $\Gamma_{m-1} \cup \Gamma_{m+1}$ by $\xi_k + \widetilde{S}^{(k+1,\ell)}_{\cdot}$, and let $x^{(\ell)}_{k+1}$ and $\xi^{(\ell)}_{k+1}$ be the corresponding hitting points. At the ℓ-th try, we declare it to be a

- *success*, if the four following assumptions hold
  - (Assumption 1) the KMT coupling event occurred up to time r_m^3, and

$\widetilde{W}^{(k+1,\ell)}$ is "controlled" between integer times, i.e.,

$$\max_{n=1,\dots,r_m^3} \max_{n\le t\le n+1} |\widetilde{S}_n^{(k+1,\ell)} - \widetilde{W}_t^{(k+1,\ell)}| \le Dm, \tag{27}$$

with some $D > 0$ to be chosen later;

– (Assumption 2) $s_{k+1}^{(\ell)}, \sigma_{k+1}^{(\ell)} \in [r_m, r_m^3]$;

– (Assumption 3) $|s_{k+1}^{(\ell)} - \sigma_{k+1}^{(\ell)}| \le m^\beta$ with some $\beta > 0$ to be chosen later;

– (Assumption 4a) one of these two conditions applies (recall the definition of $\pi_{m,y}$ in (20)):

  * $x_{k+1}^{(\ell)} \in C_{m+1}$, $\xi_{k+1}^{(\ell)} \in \Gamma_{m+1}$, and $U_{k+1,\ell} \le \pi_{m_k,\xi_{k+1}^{(\ell)}}$ ("both went outside and we decided to keep both");
  * $x_{k+1}^{(\ell)} \in C_{m-1}$, $\xi_{k+1}^{(\ell)} \in \Gamma_{m-1}$ and $U_{k+1,\ell} \le \pi_{m_k,\xi_{k+1}^{(\ell)}} \wedge (1 - \frac{2}{m+1})$ ("both went inside and we decided to keep both");

- *failure*, if the above Assumptions 1, 2 and 3 hold and the following assumption holds:

  – (Assumption 4b) $x_{k+1}^{(\ell)} \in C_{m-1}$, $\xi_{k+1}^{(\ell)} \in \Gamma_{m-1}$ and $U_{k,\ell} \ge \pi_{m_k,\xi_k} \vee (1 - \frac{2}{m+1})$ ("both went inside and we decided to discard both");

- *catastrophe*, otherwise.

We proceed as follows. In case of failure, we repeat the procedure with the next ℓ, until we meet a success or a catastrophe. In case of success, we declare the sampled excursions to be the next excursions of our processes $\widehat{W}$ and $\widehat{S}$, i.e., we set $m_{k+1} = \mu_{k+1} = \ln |x_{k+1}^{(\ell)}|$; then on one hand,

$$t_{k+1} = t_k + s_{k+1}^{(\ell)}, \quad \text{and} \quad \widehat{W}_{t_k+t} = x_k + r_{-\theta_k}\left(\widetilde{W}_t^{(k+1,\ell)}\right), \quad \text{for } 0 < t \le s_{k+1}^{(\ell)};$$

and, on the other hand,

$$\tau_{k+1} = \tau_k + \sigma_{k+1}^{(\ell)}, \quad \text{and} \quad \widehat{S}_{\tau_k+n} = \xi_k + \widetilde{S}_n^{(k+1,\ell)}, \quad \text{for } n = 1, \dots, \sigma_{k+1}^{(\ell)}.$$

Finally, we denote the exiting points by $x_{k+1} = x_{k+1}^{(\ell)}$ and $\xi_{k+1} = \xi_{k+1}^{(\ell)}$ before moving on to the next step of the coupling (constructing the $(k+2)$-th excursions). Instead, if a catastrophe occurs, we abandon the procedure and let both processes $\widehat{W}$ and $\widehat{S}$ evolve independently (according to their respective laws) forever after times t_k and τ_k. However, for formal reasons we still define level sequences $(m_k)_{k\ge0}$ and $(\mu_k)_{k\ge0}$: almost surely, they will cease to coincide at some point.

4 Proof of Theorem 5

In this section, we will prove that the above-described coupling satisfies statements (8) and (9) of theorem 5. We begin with two lemmas which control the probability of an occurrence of a catastrophe.

First, we define and estimate the probability of an occurrence of a catastrophe in one step of the coupling.

Definition 8. *For $(x, \xi) \in C_m \times \Gamma_m$, denote by $\psi_{x,\xi}$ the probability that a catastrophe occurs during the construction of the first excursion of the coupling started at (x, ξ). We set*

$$p_m = \max_{(x,\xi)\in C_m\times\Gamma_m} \psi_{x,\xi}.$$

Next, recall the above Assumptions 1 and 3. We have

Lemma 9. *If D and β are large enough, then*

$$\sum_m m^2 p_m < \infty. \tag{28}$$

Proof. Let us bound $\psi_{x,\xi}$ uniformly in (x, ξ) belonging to $C_m \times \Gamma_m$. Since the probability that one try in ℓ ends up in a failure is not larger than 1/2 (the probability that the Brownian trajectory hits the inner circle), it is enough to bound the probability that a catastrophe occurs in one try in ℓ. Say we look at the first one, we denote $\widetilde{S} = \widetilde{S}^{(1,1)}$ and $\widetilde{W} = \widetilde{W}^{(1,1)}$. We handle the different assumptions that appear in the definition of a success or a failure.

But first, let us recall the following facts. There exist real positive numbers c and c' such that for all $t \geq 0$ and all $\kappa > 0$ we have

$$\mathbb{P}\left[\max_{0\leq s\leq t} |\widetilde{W}_s| > \kappa\sqrt{t}\right] \leq c\exp(-c'\kappa^2), \tag{29}$$

$$\mathbb{P}\left[\max_{0\leq s\leq \kappa t} |\widetilde{W}_s| < \sqrt{t}\right] \leq c\exp(-c'\kappa), \tag{30}$$

and the same bounds hold with real times $t \geq 0$ replaced by integer times n and $\max_{0\leq s\leq t} |\widetilde{W}_s|$ replaced by $\max_{0\leq j\leq n} |\widetilde{S}_j|$. These claims are simple consequences of the similar statements for the one-dimensional Brownian motion and the one-dimensional simple random walk, which are classical results.

1. Pick $\delta > 0$. The KMT bound (26) gives

$$\mathbb{P}\left[\max_{n=1,\dots,r_m^3} |\widetilde{S}_n - \widetilde{W}_n| \geq (3C+\delta)m\right] \leq K\rho_0^{3C\lambda}e^{-\lambda\delta m}.$$

Furthermore using an union bound and (29) we have

$$\mathbb{P}\left[\max_{n=1,\dots,r_m^3}\max_{n\le t\le n+1}|\widetilde{W}_t-\widetilde{W}_n|\ge\delta m\right]\le r_m^3\times\mathbb{P}\left[\sup_{0\le t\le 1}|\widetilde{W}_t|\ge\delta m\right]$$
$$\le \rho_0^3 e^{3m}\times ce^{-c'(\delta m)^2}.$$

Hence, for any $D>3C$, denoting by $p_m^{(1)}$ the probability that Assumption 1 does not hold, we have $\sum_m m^2 p_m^{(1)}<\infty$.

2. Recall the definitions of $s_1:=s_1^{(1)}$ and $\sigma_1:=\sigma_1^{(1)}$. We are going to bound the probability that Assumption 2 does not hold, uniformly in (x,ξ). We use the following implications.

   - If $s_1<r_m$ (respectively, if $\sigma_1<r_m$), then process $\widetilde{W}$ (respectively, process $\widetilde{S}$) has left the ball $B(0,r_m/4)$ before time r_m.
   - If $s_1>r_m^3$ (respectively, if $\sigma_1>r_m^3$), then process $\widetilde{W}$ (respectively, process $\widetilde{S}$) has stayed inside the ball $B(0,4r_m)$ up to time r_m^3.

   Choosing $(t,\kappa)=(r_m,\sqrt{r_m}/4)$ in (29) and $(t,\kappa)=(16r_m^2,r_m/16)$ in (30) (and similarly for the process $\widetilde{S}$), we derive a bound on the probability $p_m^{(2)}$ that Assumption 2 does not hold, which is uniform in (x,ξ) and satisfies $\sum_m m^2p_m^{(2)}<\infty$.

3. Now, we are going to estimate the probability that $|s_1-\sigma_1|>m^\beta$. We work on the event that Assumption 1 holds and also Assumption 4a holds, i.e., both excursions go in the outward direction. Denote by T_r the hitting time of $\partial B(0,r)$ by process $x+r_{-\theta_0}(\widetilde{W}_\cdot)$ and consider also the times $T_-:=T_{r_{m+1}-Dm-1}$ and $T_+:=T_{r_{m+1}+Dm}$. Obviously, $s_1=T_{r_{m+1}}$ belongs to $[T_-,T_+]$. On the event $\{T_+\le r_m^3\}$ (whose complement has exponentially decaying probability arguing as in point 2 of the present proof), Assumption 1 ensures that σ_1 also belongs to $[T_-,T_+]$. Hence, $|s_1-\sigma_1|\le T_+-T_-$ and we are interested in bounding the difference T_+-T_- with large probability.

   Using the strong Markov property and the rotational invariance of W, it is just a matter of bounding $T^X_{r_{m+1}+Dm}$ where X is a Bessel process of order 2 started at $r_{m+1}-Dm-1$ and where T^X_r is the hitting time of r by X. In order to do so, we are going to show that, as soon as $\delta>0$, the term

$$m^2\,\mathbb{P}_{r_{m+1}-Dm-1}\left[T^X_{r_{m+1}-m^{4+\delta}}<T^X_{r_{m+1}+Dm}\right]$$

   and the term

$$m^2\,\mathbb{P}_{r_{m+1}-Dm-1}\left[T^X_{r_{m+1}-m^{4+\delta}}\wedge T^X_{r_{m+1}+Dm}\ge m^{8+3\delta}\right],$$

are the general terms of convergent series. Observe that if $T^X_{r_{m+1}+Dm} \geq m^{8+3\delta}$ then at least one of the two events appearing in the above probabilities holds.

So let us prove the convergence of the two above-mentioned series. The first probability can be computed explicitely using that $r \mapsto \ln r$ is a scale function for X and we obtain that it is $O(m^{-(3+\delta)})$. For the second probability, we consider the stochatic differential equation equation satisfied by X, i.e.:

$$dX_t = d\beta_t + \frac{1}{2X_t} dt, \qquad t \geq 0,$$

with $\beta.$ a standard Brownian motion. In integral form this yields that

$$\beta_t = X_t - X_0 - \frac{1}{2} \int_0^t \frac{1}{2X_s} ds, \qquad t \geq 0. \tag{31}$$

We make the following observation for any $0 < a < b$ and $T > 0$. If X is started inside $[a, b]$ and stays there until time T, then (31) implies that for all $t \in [0, T]$:

$$a - b - \frac{T}{2a} \leq \beta_t \leq b - a.$$

For our purpose, we take $a = r_{m+1} - m^{4+\delta}$ and $b = r_{m+1} + Dm$ and $T = m^{8+3\delta}$, so we bound the probability we are interested in by the probability that a standard Brownian motion remains in $[a - b - \frac{T}{2a}, b - a]$ up to time T and using (30) this decays as $\exp(-cm^\delta)$, hence the convergence of the series.

Thus, as soon as $\beta > 8$, the probability $p_m^{(3a)}$ that Assumption 1 and Assumption 4a do hold but Assumption 3 does not hold satisfies $\sum_m m^2 p_m^{(3a)} < \infty$. Similarly, one can check that, as soon as $\beta > 8$, the probability $p_m^{(3b)}$ that Assumption 1 and Assumption 4b do hold but Assumption 3 does not hold satisfies $\sum_m m^2 p_m^{(3b)} < \infty$.

4. The asymptotic expression (3) for a yields that

$$\sup_{\xi \in \Gamma_{m+1}} 1 - \pi_{m,\xi} = O\left(\frac{1}{me^m}\right) \quad \text{and} \quad \sup_{\xi \in \Gamma_{m-1}} \left| \left(1 - \frac{2}{m+1}\right) - \pi_{m,\xi} \right| = O\left(\frac{1}{me^m}\right).$$

These suprema control the probability $p_m^{(4)}$ that neither Assumption 4a nor Assumption 4b hold, and we have $\sum_m m^2 p_m^{(4)} < \infty$.

Putting everything together, we have proved the convergence of the series $\sum_m m^2 p_m$, when $D > 3C$ and $\beta > 8$. □

In the following, we assume that D and β have been chosen large enough so that (28) holds. Recall the discussion in the beginning of section 3: it is enough to construct the coupling with some *large* starting level h.

Lemma 10. *As h increases to infinity, the probability that a catastrophe occurs in the construction of the coupling converges to* 0.

Proof. Take $(x, \xi) \in C_m \times \Gamma_m$ and consider the coupling starting from (x, ξ). Conditionally on $\{m_k = m\}$, the probability that a catastrophe occurs in the $(k+1)$-th step of the construction of the coupling is bounded by p_m. Therefore, the probability of an occurrence of a catastrophe is bounded by

$$\sum_{k\geq 0}\sum_{m\geq 1} \mathbb{P}_x[m_k = m]p_m = \sum_{m\geq 1} \mathbb{E}_x(\#\{k \geq 0 : m_k = m\})p_m.$$

Now, to estimate the expectation in the above sum, recall that $(m_k)_{k\geq 0}$ follows the law of 1-CSRW (even after a possible catastrophe). Using the fact that 1-CSRW has scaling function $x \mapsto 1/x$, we compute:

$$\mathbb{E}_x[\#\{k \geq 0 : m_k = m\}] = 2m \times \Big(\frac{m}{h} \wedge 1\Big).$$

Since $\sum m^2 p_m < \infty$, we have derived a bound that converges to 0 as h goes to infinity. □

We are now ready to prove (8). Consider some fixed $\varepsilon \in (0, 1)$. Take some h large enough so that with probability not smaller than $1-\varepsilon$, the coupling starting from level h (i.e., from any prescribed points $x \in C_h, \xi \in \Gamma_h$) sees no catastrophe. From now on, we work on this event.

Consider some integer k and some time $t \in [t_k, t_{k+1}]$, so that $\widehat{W}_t$ belongs to the $(k+1)$-th excursion of $\widehat{W}$. We use the following bound:

$$\big||\widehat{S}_t| - |\widehat{W}_t|\big| \leq \big||\widehat{S}_t| - |\widehat{S}_{t-(t_k-\tau_k)}|\big| + \big||\widehat{S}_{t-(t_k-\tau_k)}| - |\widehat{W}_t|\big|. \tag{32}$$

Since $\widehat{S}$ makes jumps of size 1, the first term in the right hand-side of (32) is bounded by $|t_k - \tau_k|$. For the second term, we proceed by distinguishing cases. We abbreviate $t^* = t - (t_k - \tau_k)$. Note that t^* is larger than τ_k. We distinguish cases according to the fact that t^* belongs to the $(k+1)$-th excursion of $\widehat{S}$, or not.

- *case 1:* $\tau_k \leq t^* \leq \tau_{k+1}$. Since t and t^* both belong to the $(k+1)$-th excursion of $\widehat{W}$ (respectively, $\widehat{S}$), and $t^* - \tau_k = t - t_k$, the second term in the right-hand side of (32) can be controlled using the KMT bound. Recall the role of the angle θ_k in the construction of the coupling. If ℓ is the index of the try where the first success happened during the $(k+1)$-th step of the coupling, then $\widehat{S}_{t^*} = \widehat{S}_{\tau_k} + \widetilde{S}^{(k+1,\ell)}_{t^*-\tau_k}$, where $(\widetilde{S}^{(k+1,\ell)}_t)_{t\geq 0}$ is the linear interpolation of $(\widetilde{S}^{(k+1,\ell)}_n)_{n\geq 0}$. Therefore

  $$\big||\widehat{S}_{t^*}| - |\widehat{W}_t|\big| = \big||\widehat{S}_{t^*}| - |r_{\theta_k}(\widehat{W}_t)|\big|$$

$$
\begin{aligned}
&\leq \big|\widehat{S}_{t^*} - r_{\theta_k}(\widehat{W}_t)\big| \\
&\leq \big|\widehat{S}_{t^*} - \widehat{S}_{\tau_k} - r_{\theta_k}(\widehat{W}_t - \widehat{W}_{t_k})\big| + \big|\widehat{S}_{\tau_k} - r_{\theta_k}(\widehat{W}_{t_k})\big| \\
&= \big|\widetilde{S}^{(k+1,\ell)}_{t^*-\tau_k} - \widetilde{W}^{(k+1,\ell)}_{t-t_k}\big| + \big|\widehat{S}_{\tau_k} - r_{\theta_k}\widehat{W}_{t_k}\big| \\
&\leq Dm_k + 1.
\end{aligned}
$$

- *case 2.* If $t^* > \tau_{k+1}$, we need to adapt this argument. We can still bound the distance between $r_{\theta_k}\widehat{W}_t$ and $\widehat{S}_{\tau_k} + \widetilde{S}^{(k+1,\ell)}_{t^*-\tau_k}$, but this time $\widehat{S}_{\tau_k} + \widetilde{S}^{(\ell)}_{t^*-\tau_k}$ is (a priori) not equal to $\widehat{S}_{t^*}$, so we also need to bound the difference between them. We do that by "backtracking" to time τ_{k+1}. Observe that

  $$\widehat{S}_{\tau_{k+1}} = (\widehat{S}_{\tau_k} + \widetilde{S}^{(k+1,\ell)}_{t^*-\tau_k}) + \widetilde{S}^{(k+1,\ell)}_{\tau_{k+1}-\tau_k} - \widetilde{S}^{(k+1,\ell)}_{t^*-\tau_k},$$

  so that, by the triangle inequality,

  $$
  \begin{aligned}
  \big|\widehat{S}_{t^*} - (\widehat{S}_{\tau_k} + \widetilde{S}^{(k+1,\ell)}_{t^*-\tau_k})\big| &\leq \big|\widehat{S}_{t^*} - \widehat{S}_{\tau_{k+1}}\big| + \big|\widehat{S}_{\tau_{k+1}} - (\widehat{S}_{\tau_k} + \widetilde{S}^{(k+1,\ell)}_{t^*-\tau_k})\big| \\
  &\leq 2(t^* - \tau_{k+1}) \\
  &\leq 2|t_{k+1} - \tau_{k+1}|.
  \end{aligned}
  $$

  Proceeding as in case 1, we obtain

  $$\big||\widehat{S}_{t^*}| - |\widehat{W}_t|\big| \leq Dm_k + 1 + 2|t_{k+1} - \tau_{k+1}|.$$

Overall, the two terms in the right-hand side of (32) are bounded respectively by $|t_k - \tau_k|$ and $Dm_k + 1 + 2|t_{k+1} - \tau_{k+1}|$. To conclude the proofs we bound these quantities when t is large, i.e., when k is large.

Recall the notations $s_k^{(\ell)}$ and $\sigma_k^{(\ell)}$ from the construction of the coupling. If ℓ is the index of the first success in the k-th step of the construction of the coupling, we denote these quantities by s_k and σ_k, so that for all $k \geq 1$

$$t_k = \sum_{j=1}^{k} s_j \quad \text{and} \quad \tau_k = \sum_{j=1}^{k} \sigma_j.$$

We recall Assumptions 2 and 3 in the definition of a success. We have for all $k \geq 1$

$$t_k \geq s_k \geq r_{m_k} = \rho_0\, e^{m_k}, \tag{33}$$

so

$$m_k \leq \ln(t_k/\rho_0) \leq \ln(t/\rho_0).$$

Besides, as a property of 1-CSRW, almost surely, we have eventually

$$m_k \geq k^{\frac{1}{3}}$$

(in fact, the exponent $\frac{1}{3}$ can be replaced by any exponent in $(0, \frac{1}{2})$, see for example Theorem 3.2.7 in [9] and also Example 3.2.8 there); hence, if k is large enough, then

$$k \leq m_k^3 \leq \ln^3(t/\rho_0). \tag{34}$$

Furthermore, for all $k \geq 1$:

$$|t_k - \tau_k| \leq \sum_{j=1}^{k} |s_j - \sigma_j| \leq \sum_{j=0}^{k-1} m_j^\beta \leq k \max_{j=1,\dots,k} m_j^\beta \leq k(k+h)^\beta \leq (k+h)^{\beta+1}. \tag{35}$$

Using (34) and (35), we obtain for large k that

$$|t_k - \tau_k| \leq (k+h)^{\beta+1} \leq (\ln^3(t/\rho_0) + h)^{\beta+1}.$$

$$|t_{k+1} - \tau_{k+1}| \leq (k+1+h)^{\beta+1} \leq (\ln^3(t/\rho_0) + 1 + h)^{\beta+1}.$$

Hence, we have shown (8) for any $\alpha > 3(\beta+1)$.

To finish the proof of Theorem 5, we argue why (8) implies (9). We use the result (apply Theorem 3 with $g(t) = e^{-t/2}$) that eventually $e^{\sqrt{\ln t}} \leq |\widehat{W}_t|$, so that

$$\ln^\alpha t \leq \ln^{2\alpha} |\widehat{W}_t| = o(|\widehat{W}_t|).$$

The asymptotic equivalence in (9) therefore follows from (8). □

Remark 11. *For the sake of completeness, let us remark that it is possible to build stronger versions of the coupling.*

- *It is possible to build a version of the coupling where the coupling event holds with probability* 1*. In order to do so, we describe a way to "patch" the trajectories after a catastrophe, instead of simply letting them evolve independently. When a catastrophe occurs, treat both trajectories separately according to their respective construction procedures: each excursion may be accepted or discarded and this extends (or not) processes $\widehat{S}$ and $\widehat{W}$ by one excursion. At this point, the two processes may be lying at different levels. To solve this, let the process with the smaller level run until it reaches the larger level. Then, the procedure can begin again, starting from this common level.*

  Due to catastrophes, the level sequences of $\widehat{S}$ and $\widehat{W}$ do not necessarily coincide. This makes it more difficult to write it cleanly, which is why we chose not to present this almost sure construction in the article.

  It is straightforward to adapt the proof of Lemma 9 in order to show that the number of catastrophes occurring during this construction has a finite expectation, hence this number is almost surely finite. What happens before the last catastrophe can be disregarded as it is eventually absorbed in the bound $\ln^\alpha t$, so the proof of (8) *can be conducted without modifications.*

- *It is likely possible to build a version of the coupling satisfying a stronger version of* (8)*, obtained by removing the moduli there:*

$$\left|\widehat{S}_t - \widehat{W}_t\right| \le \ln^\alpha t \qquad \textit{eventually, as } t \to \infty. \tag{36}$$

This, however, would require some finer control on the exit point locations (of the $\widehat{S}$- and $\widehat{W}$-excursions between the levels); since (8) *is already enough for the purposes of this paper, we decided to leave proving* (36) *as a possible topic for future research.*

5 Proof of Theorem 1

As announced in the introduction, we now use Theorem 3 and Remark 4, as well as the coupling of Theorem 5, to prove Theorem 1.

Proof of the first statement of Theorem 1. Take g satisfying the conditions of the theorem, we may assume without loss of generality that $(\ln t)g(\ln\ln t)$ goes to infinity as t goes to infinity. Take any $\varepsilon \in (0,1)$ and consider the coupling $(\widehat{W}, \widehat{S})$ associated to ε. With probability not smaller than $1-\varepsilon$, we have

$$|\widehat{W}_t|/2 \le |\widehat{S}_t| \le 2|\widehat{W}_t| \qquad \text{eventually.}$$

Observe that $g/2$ and $2g$ satisfy the conditions of Theorem 3 and therefore, using Remark 4, we obtain on the event that the coupling is successful:

- if $\int^\infty g = \infty$, then for an unbounded set of t's:

$$M_{\lceil t\rceil} \le |\widehat{S}_{\lceil t\rceil}| \le |\widehat{S}_t| + 1 \le 2|\widehat{W}_t| + 1 \le 2e^{(\ln t)g(\ln\ln t)/2} + 1, \tag{37}$$

  which is eventually not larger than $e^{(\ln\lceil t\rceil)g(\ln\ln\lceil t\rceil)}$;

- if $\int^\infty g < \infty$, then eventually:

$$|\widehat{S}_n| \ge \frac{|\widehat{W}_n|}{2} \ge \frac{e^{2(\ln n)g(\ln\ln n)}}{2}, \tag{38}$$

  which is eventually not smaller than $e^{(\ln n)g(\ln\ln n)}$. Since $n \mapsto e^{(\ln n)g(\ln\ln n)}$ is non-decreasing, it follows that $M_n \ge e^{(\ln n)g(\ln\ln n)}$ eventually.

We have shown this to be true with probability at least $1-\varepsilon$, for any $\varepsilon \in (0,1)$, so it is true almost surely. The first part of Theorem 1 follows. □

Proof of the second statement of Theorem 1. Consider some $\varepsilon \in (0,1)$ and the associated coupling $(\widehat{S}, \widehat{W})$ having success probability not smaller than $1-\varepsilon$. On this event, we obtain using $|\widehat{S}_t| \sim |\widehat{W}_t|$, Theorem 3 and Remark 4, that

$$\limsup_{t\to\infty} \frac{M_n}{\sqrt{n \ln\ln\ln n}} = K^*$$

(we also made use of the fact that $|\widehat{S}_t - \widehat{S}_{\lfloor t \rfloor}| \leq 1$). The latter is true with probability not smaller than $1-\varepsilon$, for any $\varepsilon \in (0,1)$, so it is true almost surely. The second part of the theorem follows. □

Acknowledgements

The authors were partially supported by CMUP, member of LASI, which is financed by national funds through FCT — Fundação para a Ciência e a Tecnologia, I.P., under the project with reference UIDB/00144/2020.

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