

The pro-supersolvable topology on a free group: deciding denseness

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May 30, 2023

2020 Mathematics Subject Classification: 20E05, 20E10, 20F10, 20F16, 11D79

Keywords: subgroups of the free group, pro-supersolvable topology, denseness, polynomial congruences

Abstract. Let F be a free group of arbitrary rank and let H be a finitely generated subgroup of F . Given a pseudovariety \mathbf{V} of finite groups, i.e. a class of finite groups closed under taking subgroups, quotients and finitary direct products, we endow F with its pro- \mathbf{V} topology. Our main result states that it is decidable whether H is pro- \mathbf{Su} dense, where $\mathbf{Su} \subset \mathbf{S}$ denote respectively the pseudovarieties of all finite supersolvable groups and all finite solvable groups. Our motivation stems from the following open problem: is it decidable whether H is pro- \mathbf{S} dense?

1 Introduction

The use of topology to deduce group theoretic properties of free groups has been a hugely fruitful endeavour. In Marshall Hall's seminal paper [6], a number of natural topologies on a group G are introduced, and used to prove a series of interesting results.

Let \mathbf{V} be a pseudovariety of finite groups and let G be any group. Then G can be endowed with the pro- \mathbf{V} topology. In other words, G is a topological group where the normal subgroups K of G with $G/K \in \mathbf{V}$ form a basis of neighbourhoods of the identity. Hall's work suggests that working with such topologies can lead to surprising group theoretic results, particularly when G is a free group. For example, by working with the pseudovariety of all finite groups (referred to in the literature as the profinite topology), Hall proves that the property of a subgroup of a free group F being the intersection of subgroups of finite index is a topological property, namely being closed in the profinite topology [6, Theorem 3.4].

For this reason, the past 70 years or so have seen considerable effort poured into five key questions that have arisen from Hall's work. These questions, for a subgroup H of a free group $F = F(X)$, are as follows:

Question 1. Does there exist an algorithm deciding whether or not H is \mathbf{V} -dense?

Question 2. Does there exist an algorithm deciding whether or not H is \mathbf{V} -closed?

Question 3. Is it decidable whether or not the pro- \mathbf{V} closure $\text{Cl}_{\mathbf{V}}^F(H)$ of H is finitely generated?

Question 4. Given $w \in F$, is there an algorithm that decides whether or not w belongs to $\text{Cl}_{\mathbf{V}}(H)$?

Question 5. Suppose $\text{Cl}_{\mathbf{V}}^F(H)$ is finitely generated. Does there exist an algorithm that computes a basis (in terms of X) for $\text{Cl}_{\mathbf{V}}^F(H)$?

Here, the subgroup H is said to be $\mathbf{V}\mathcal{P}$, for a topological property \mathcal{P} , if H has property \mathcal{P} in the pro- \mathbf{V} topology.

When considering Questions 1-5, if F is residually \mathbf{V} , i.e. $\text{Cl}_{\mathbf{V}}^F(1) = 1$, then by [11, Corollary 2.4] one can reduce to the case where F is of finite rank. In particular this holds for \mathbf{V} nontrivial and extension-closed, or when \mathbf{V} is the pseudovariety of all finite nilpotent groups; or the pseudovariety of all finite supersolvable groups.

Significant progress has been made on these questions in the case where \mathbf{V} is extension-closed (see [13, 9, 10]). In particular, in this case, Question 3 has a positive answer [13], while Questions 4 and 5 are equivalent [9]. Moreover, Questions 4 and 5 have positive answers when \mathbf{V} is either the pseudovariety of all finite p -groups for a fixed prime p (which is extension-closed); or the pseudovariety of all finite nilpotent groups (which is not extension-closed). These facts are proved in [13] and [9], respectively. We also mention that in [10], it is explicitly proved that Questions 1, 2 and 5 are equivalent when \mathbf{V} is extension-closed.

In [10] we show that Questions 1-5 have positive answers when \mathbf{V} is the pseudovariety of all finite abelian groups; or the pseudovariety of all finite metabelian groups (and neither of them is extension-closed).

For applications of these questions to graph theory and monoid theory, see [7, Section 6] and [9, Section 5].

Apart from the fact that they are equivalent, little is known about Questions 1, 2 and 5 when \mathbf{V} is the pseudovariety of all finite solvable groups. Motivated by this, we focus, in this paper, on the subclass \mathbf{Su} of all finite supersolvable groups. Recall that a finite group is supersolvable if every of its chief factors is cyclic (of prime order). In this case, we can answer Question 1 in the affirmative:

Theorem 1. *Let F be a free group of arbitrary rank. Given a finitely generated subgroup H of F , it is decidable whether or not H is **Su**-dense.*

The layout is as follows. In Section 2, we fix some notation, collect some useful facts about pro- \mathbf{V} topologies, and give some preliminary group-theoretical results. In Section 3, we give some results on the set of prime numbers p for which a given system of polynomial equations in several variables over the integers has a common solution modulo p . In Section 4, we prove a reduction lemma due to Martino Garonzi and define p -hypersolvable groups (a useful tool in the proof of Theorem 1). In Section 5, we prove a technical lemma needed in the proof of Theorem 1. Finally, we prove Theorem 1 in Section 6.

We are grateful to Martino Garonzi for his insight and Lemma 4.4.

2 Preliminaries

We first fix some notation used throughout the paper unless otherwise stated. We let F be a free group of finite rank d and let H be a given finitely generated subgroup of F . We let $X = \{x_1, \dots, x_d\}$ be a basis for F and $W = \{w_1, \dots, w_e\}$ be a basis for H where, for each $1 \leq i \leq e$, w_i is a reduced word over $X \cup X^{-1}$. Recall that if F is a free group of possibly arbitrary rank, an element of F is called a primitive element if it belongs to a basis of F . Given a pseudovariety \mathbf{V} of finite groups, we endow F with its pro- \mathbf{V} topology.

Let G be a group. We let $d(G)$ denote the minimal number of generators for G , $|G|$ denote the order of G , and for an element $g \in G$ we let $\text{ord}(g)$ denote the order of g ($d(G)$, $|G|$ and $\text{ord}(g)$ are possibly infinite). If $g, h \in G$, we let $[g, h] = ghg^{-1}h^{-1}$. We also denote by $\text{Aut}(G)$ the group of automorphisms of G . If H is a subgroup of G then the core $\text{Core}_G(H)$ of H in G is the largest normal subgroup of G contained in H and is equal to $\bigcap_{u \in G} u^{-1}Hu$.

Given a positive integer a , we let $\mathbb{Z}/a\mathbb{Z}$ denote a cyclic group of order a . When clear from the context, we consider $\mathbb{Z}/a\mathbb{Z}$ as a ring and denote by $(\mathbb{Z}/a\mathbb{Z})^*$ its group of units. Moreover given a prime p and a positive integer n , we let $V_n := (\mathbb{Z}/p\mathbb{Z})^n$ denote the elementary abelian p -group of order p^n that can also be seen as a vector space over $\mathbb{Z}/p\mathbb{Z}$ of dimension n .

We let \mathbf{Ab} denote the pseudovariety of all finite abelian groups. Given a positive integer a , $\mathbf{Ab}(a)$ is the pseudovariety of all finite abelian groups of exponent dividing a . In particular, given a prime p , $\mathbf{Ab}(p)$ is the pseudovariety of all finite elementary abelian p -groups.

Given integers a and b not both equal to 0, $\text{gcd}(a, b)$ denotes the greatest common divisor of a and b , while if a and b are nonzero $\text{lcm}(a, b)$ denotes the lowest positive common multiple of a and b . We naturally extend the notation to tuples of integers. More precisely, if (a_1, \dots, a_n) is a tuple of integers not all equal to 0, then $\text{gcd}(a_1, \dots, a_n)$ is the greatest positive integer dividing each of a_1, \dots, a_n , whereas if (b_1, \dots, b_n) is a tuple of nonzero integers then $\text{lcm}(b_1, \dots, b_n)$ is smallest positive integer which is a multiple of each of b_1, \dots, b_n .

For $k \in \mathbb{N}$ and $1 \leq i \leq k$, let $\iota_i \in \mathbb{Z}^k$ have 1 in the i th coordinate and 0 everywhere else. Given a prime p , we sometimes view ι_i as an element of $(\mathbb{Z}/p\mathbb{Z})^k$. We denote by \mathbb{P} the set of

all primes.

We now collect some facts on pro- \mathbf{V} topologies. For a pseudovariety \mathbf{V} , considering finite groups endowed with the discrete topology, the pro- \mathbf{V} topology on a group G is defined as the coarsest topology which makes all homomorphisms from G into elements of \mathbf{V} continuous. Equivalently G is a topological group and the normal subgroups K of G such that $G/K \in \mathbf{V}$ form a basis of neighbourhoods of the identity. A pseudovariety \mathbf{V} is extension-closed if whenever G is a finite group with a normal subgroup N such that N and G/N belong to \mathbf{V} then G also belongs to \mathbf{V} . The *trivial pseudovariety* consists of all trivial groups. A subgroup H of G is pro- \mathbf{V} open if and only if it is pro- \mathbf{V} clopen, and if and only if $G/\text{Core}_G(H)$ belongs to \mathbf{V} . Note that H is pro- \mathbf{V} closed if and only if, for every $g \in G \setminus H$, there exists some pro- \mathbf{V} clopen $K \leq G$ such that $H \leq K$ and $g \notin K$. Moreover a subgroup H of G is pro- \mathbf{V} dense if and only if $HN = G$ for every normal subgroup N of G with $G/N \in \mathbf{V}$. Given $S \subseteq G$, we also denote by $\text{Cl}_{\mathbf{V}}^G(S)$ the \mathbf{V} -closure of S in G . If $H \leq G$, then also $\text{Cl}_{\mathbf{V}}^G(H) \leq G$ [6, Theorem 3.3].

Suppose that \mathbf{V} and \mathbf{W} are pseudovarieties of finite groups. Then:

$$\text{If } \mathbf{W} \subseteq \mathbf{V}, \text{ then } \text{Cl}_{\mathbf{V}}^G(S) \subseteq \text{Cl}_{\mathbf{W}}^G(S) \text{ for every } S \subseteq G. \quad (2.1)$$

This follows from the pro- \mathbf{W} topology being coarser than the pro- \mathbf{V} topology on G .

We will also use the following result.

Lemma 2.1. *Let \mathbf{V} be a pseudovariety of finite groups, let H be a subgroup of a group G and let $\varphi \in \text{Aut}(G)$. If H is \mathbf{V} -dense in G , then so is $\varphi(H)$.*

Proof. Assume that H is \mathbf{V} -dense in G . Let $N \trianglelefteq G$ be such that $G/N \in \mathbf{V}$. Then $\varphi^{-1}(N) \trianglelefteq G$ and $G/\varphi^{-1}(N) \cong G/N \in \mathbf{V}$. Since H is \mathbf{V} -dense in G , we have $H\varphi^{-1}(N) = G$ and so

$$(\varphi(H))N = \varphi(H\varphi^{-1}(N)) = \varphi(G) = G.$$

Therefore $\varphi(H)$ is \mathbf{V} -dense in G . □

In the remainder of this section we give other group-theoretical results. The result below is well known as Gaschutz's lemma.

Lemma 2.2. [4] *Let $\pi : G \rightarrow K$ be a surjective homomorphism of finite groups. Suppose $m \geq d(K)$ and (k_1, \dots, k_m) is a generating tuple for K . Then there exists a generating tuple (g_1, \dots, g_m) for G with $\pi(g_i) = k_i$ for $1 \leq i \leq m$.*

We apply Gaschutz's lemma in the next result.

Lemma 2.3. *Let G be a finite cyclic group and let $\{g_1, \dots, g_m\}$ be a generating set for G . Then for $1 \leq i < m$ there exist integers r_i such that $g_m \prod_{i=1}^{m-1} g_i^{r_i}$ generates G .*

Proof. We prove the result by induction on m . If $m = 1$ the result is trivial since g_1 generates G . We assume $m > 1$. Let $N = \langle g_1 \rangle$. Now G/N is cyclic with generating set $\{Ng_2, \dots, Ng_m\}$. By the induction hypothesis for $1 < i < m$ there exist integers r_i such that G/N is generated by $Ng_m \prod_{i=2}^{m-1} g_i^{r_i}$. Since G is cyclic, by Lemma 2.2, there exists $n = g_1^{r_1} \in N$ where r_1 is an integer such that $g_m \prod_{i=1}^{m-1} g_i^{r_i}$ generates G . This completes the induction. \square

We can now apply Lemma 2.3 to prove the following result.

Lemma 2.4. *Let F be a free group of finite rank d over $X = \{x_1, \dots, x_d\}$ and let c be a positive integer. Let $\varphi : F \rightarrow \mathbb{Z}/c\mathbb{Z}$ be a surjective homomorphism. Then there exists some $\lambda \in \text{Aut}(F)$ such that $\langle \varphi(\lambda(x_1)) \rangle = \mathbb{Z}/c\mathbb{Z}$.*

Proof. Write $C = \mathbb{Z}/c\mathbb{Z}$. The case $d = 1$ is immediate, hence we assume that $d > 1$. For $1 \leq i \leq d$, let $c_i = \varphi(x_i) \in C$. As $\varphi : F \rightarrow C$ is surjective, $\{c_1, \dots, c_d\}$ is a generating set for C . By Lemma 2.3 for $1 \leq i < d$ there exist integers r_i such that $c_d \prod_{i=1}^{d-1} c_i^{r_i}$ generates C . Now since $\{x_1, \dots, x_d\}$ is a basis of F and $\{x_1, \dots, x_{d-1}, x_d \prod_{i=1}^{d-1} x_i^{r_i}\}$ is clearly a generating set for F , the latter set is a basis of F (as F is hopfian). Thus $x_d \prod_{i=1}^{d-1} x_i^{r_i}$ is a primitive word of F and there exists $\lambda \in \text{Aut}(F)$ such that $\lambda(x_1) = x_d \prod_{i=1}^{d-1} x_i^{r_i}$. Now

$$\varphi(\lambda(x_1)) = \varphi \left(x_d \prod_{i=1}^{d-1} x_i^{r_i} \right) = \varphi(x_d) \prod_{i=1}^{d-1} \varphi(x_i)^{r_i} = c_d \prod_{i=1}^{d-1} c_i^{r_i}$$

and so $\varphi(\lambda(x_1))$ generates C . \square

3 Systems of polynomial equations over the integers

In this section, given a system of polynomial equations in several variables over the integers, we are interested in properties of the set of primes p for which the system has a common solution modulo p . We first introduce some notation.

Definition 3.1. *Let $J = \{f_1, \dots, f_m\} \subset \mathbb{Z}[X_1, \dots, X_n]$ where $m, n \geq 1$. We define $S(J)$ to be the set of primes p such that f_1, \dots, f_m have a common root modulo p .*

The result below establishes that one can essentially reduce the original system of polynomials to a single polynomial in one variable.

Theorem 3.2. [8, Theorem 1.2] *Let $J = \{f_1, \dots, f_m\} \subset \mathbb{Z}[X_1, \dots, X_n]$ where $m, n \geq 1$. There is a polynomial $f \in \mathbb{Z}[X]$ such that $S(J) = S(f)$.*

Let $J = \{f_1, \dots, f_m\} \subset \mathbb{Z}[X_1, \dots, X_n]$ where $m, n \geq 1$, and let $f \in \mathbb{Z}[X]$ be such that $S(J) = S(f)$. The existence of such a polynomial f is guaranteed by Theorem 3.2. The aim of this section is to show that one can decide whether f is a nonzero constant polynomial, and moreover whether $f = \pm 1$ (see Corollary 3.7 below).

Definition 3.3. Let $J = \{f_1, \dots, f_m\} \subset \mathbb{C}[X_1, \dots, X_n]$ where $m, n \geq 1$. We let $I = \langle J \rangle$ be the ideal of $\mathbb{C}[X_1, \dots, X_n]$ generated by J and set

$$V(I) = \{(x_1, \dots, x_n) \in \mathbb{C}^n : g(x_1, \dots, x_n) = 0 \text{ for all } g \in I\}.$$

We also let \mathcal{G} be the unique reduced Gröbner basis of I with respect to a fixed monomial order on $\mathbb{C}[X_1, \dots, X_n]$.

We record below three results needed for the proof of Corollary 3.7 below.

Proposition 3.4. [3, Proposition 2.7] Let $J = \{f_1, \dots, f_m\} \subset \mathbb{Z}[X_1, \dots, X_n]$ where $m, n \geq 1$. Then $S(J)$ is finite if and only if $V(\langle J \rangle) = \emptyset$.

Lemma 3.5. Let $J = \{f_1, \dots, f_m\} \subset \mathbb{Z}[X_1, \dots, X_n]$ where $m, n \geq 1$. Then it is decidable whether $V(\langle J \rangle) = \emptyset$. Consequently, it is also decidable whether $S(J)$ is finite.

Proof. Fix a monomial order on $\mathbb{C}[X_1, \dots, X_n]$. There is an algorithm to compute the (unique) reduced Gröbner basis \mathcal{G} of $\langle J \rangle$ (see [2]). Since $\langle \mathcal{G} \rangle = \langle J \rangle$, Hilbert's Nullstellensatz implies that $V(\langle J \rangle) = \emptyset$ if and only if $\mathcal{G} = \{1\}$. Thus it is decidable whether $V(\langle J \rangle) = \emptyset$, and Proposition 3.4 yields that is decidable whether $S(J)$ is finite. \square

Proposition 3.6. Let $f \in \mathbb{Z}[X]$. Then $S(f)$ is finite if and only if f is a nonzero constant polynomial.

Proof. Clearly if f is a nonzero constant polynomial then $S(f)$ is finite. Also if $f = 0$ then $S(f) = \mathbb{P}$. Suppose now that f is not a constant polynomial. Write $f(X) = \sum_{i=0}^n a_i X^i$. If $a_0 = 0$, then $f(p) \equiv 0 \pmod{p}$ for every prime p , and so $S(f) = \mathbb{P}$. We can therefore assume that $a_0 \neq 0$. For any nonzero $t \in \mathbb{Z}$, let

$$\begin{aligned} g_t(X) &= f(a_0 t X) \\ &= \sum_{i=0}^n a_i (a_0 t X)^i \\ &= a_0 \left[1 + \sum_{i=1}^n a_i a_0^{i-1} t^i X^i \right] \end{aligned}$$

and consider

$$h_t(X) = 1 + \sum_{i=1}^n a_i a_0^{i-1} t^i X^i.$$

Since $h_t(X)$ can take the values -1 , 0 and 1 only at finitely many points, there exists a prime p such that $h_t(n) \equiv 0 \pmod p$ for some integer n . Note that $h_t(n) \equiv 1 \pmod t$, and so $\gcd(t, p) = 1$. Also, since $f(a_0tn) = a_0h_t(n)$, then $f(a_0tn) \equiv 0 \pmod p$ and a_0tn is a root of f modulo p . Starting with $t = 1$, we get a prime p_1 such that f has a root modulo p_1 (and $\gcd(t, p_1) = 1$). Then, setting $t = p_1$, we get a prime p_2 such that f has a root modulo p_2 and $\gcd(p_1, p_2) = 1$. In particular p_1 and p_2 are distinct. Suppose we have obtained a sequence of distinct primes p_1, \dots, p_m such that f has a root modulo p_i for $1 \leq i \leq m$. Setting $t = \prod_{i=1}^m p_i$, we get a prime p_{m+1} such that f has a root modulo p_{m+1} and $\gcd(\prod_{i=1}^m p_i, p_{m+1}) = 1$. In particular, p_1, \dots, p_{m+1} are distinct primes. In this way, we get an infinite sequence (p_i) of primes such that f has a root modulo p_i for every i . \square

We can now derive the result aforementioned above.

Corollary 3.7. *Let $J = \{f_1, \dots, f_m\} \subset \mathbb{Z}[X_1, \dots, X_n]$ where $m, n \geq 1$. Let $f \in \mathbb{Z}[X]$ be such that $S(J) = S(f)$. The following assertions hold.*

- (i) *It is decidable whether f is a nonzero constant polynomial.*
- (ii) *It is decidable whether $f = \pm 1$.*

Proof. The first part follows from Lemma 3.5 and Proposition 3.6. We now consider the second part. If f is not a nonzero constant polynomial, then $f \neq \pm 1$ and we are done. We can therefore suppose that f is a nonzero constant polynomial and we would like to decide whether $f = \pm 1$, i.e whether $S(J)$ is empty. Since $S(J) = S(f)$ is finite, it follows from Proposition 3.4 that $V(\langle J \rangle) = \emptyset$ and by Hilbert's Nullstellensatz $\langle J \rangle = \mathbb{C}[X_1, \dots, X_n]$. In particular, there exist polynomials $g_1, \dots, g_m \in \mathbb{C}[X_1, \dots, X_n]$ such that $\sum_{i=1}^m f_i g_i = 1$. The equation $\sum_{i=1}^m f_i g_i = 1$ amounts to a system of linear equations with integer coefficients (since the m polynomials f_i are in $\mathbb{Z}[X_1, \dots, X_n]$). We know that this system has a (complex) solution, so it must also have a rational solution. Scaling up, there exist an integer a and polynomials $\ell_1, \dots, \ell_m \in \mathbb{Z}[X_1, \dots, X_n]$ such $\sum_{i=1}^m f_i \ell_i = a$.

By [12, Chapter 4, Theorem IV], which gives an Effective Hilbert's Nullstellensatz, a and ℓ_1, \dots, ℓ_m are computable. Note that if $a \in \pm\{1\}$ then $S(J)$ is empty and $f = \pm 1$.

Suppose that $a \notin \{\pm 1\}$. Now if $f \neq \pm 1$, that is, if $S(J)$ is nonempty, then every prime $p \in S(J)$ must divide a . So we just need to check whether the polynomials in J have a common root modulo the primes dividing a . Then $f \neq \pm 1$ if and only if $S(J)$ is nonempty if and only if the polynomials in J have a common root modulo a prime dividing a . This latter condition is decidable. \square

4 A reduction lemma and p -hypersolvable groups

In this section we prove the reduction lemma due to Martino Garonzi, namely Lemma 4.4 below. Given a prime p , we then introduce the concept of a p -hypersolvable group and give

some of its properties that we will use later.

We first consider the following definition generalising the notion of denseness in pro- \mathbf{V} topologies on a group G .

Definition 4.1. *Let G be a group and let H be a subgroup of G . For a class \mathcal{C} of finite groups, let $P_{\mathcal{C}}(H, G)$ be the following property: $HN = G$ whenever N is a normal subgroup of G with $G/N \in \mathcal{C}$.*

We characterise below the finite primitive supersolvable groups. Recall that a finite group L is primitive if it has a maximal subgroup M such that $\text{Core}_L(M)$ is trivial.

Lemma 4.2. *A finite group is primitive supersolvable if and only if it is a semidirect product $\mathbb{Z}/p\mathbb{Z} \rtimes C$ where p is a prime and $C \leq \text{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^*$.*

Proof. We use some elementary facts about finite primitive groups, for more details see [1]. Let G be a finite primitive supersolvable group. Since G is solvable, a minimal normal subgroup N of G is (elementary) abelian. Since G is primitive and N is abelian, N is in fact the unique minimal normal subgroup of G and $C_G(N) = N$. Since G is primitive, G has a core free maximal subgroup C . Therefore $CN = G$. Also $C \cap C_G(N) = 1$ and so $N \cap C = 1$. In particular $G = N \rtimes C$. Finally, since G is supersolvable and N is a minimal normal subgroup of G , we obtain $N \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p . As $C_G(N) = N$, we obtain $C \leq \text{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^*$.

Conversely, let $G = \mathbb{Z}/p\mathbb{Z} \rtimes C$ where p is prime and $C \leq \text{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^*$. It is clear that G is supersolvable. If $C = 1$, then G is primitive (with maximal subgroup 1). Suppose that $C \neq 1$. As G is supersolvable and C has prime index in G , C is a maximal subgroup of G . Moreover $\text{Core}_G(C) = 1$, as $\mathbb{Z}/p\mathbb{Z}$ is the unique nontrivial normal subgroup of G . Hence G is primitive. \square

We now introduce a notion used in Lemma 4.4 below.

Definition 4.3. *Let G be a group and let p be a prime. We say that a subgroup H of G satisfies condition $Q_p(H, G)$ if $HN = G$ for all $N \trianglelefteq G$ with $F/N \cong \mathbb{Z}/p\mathbb{Z} \rtimes C$ for some $C \leq \text{Aut}(\mathbb{Z}/p\mathbb{Z})$.*

Lemma 4.4. *Let G be a nontrivial group and let H be a subgroup of G . Let \mathcal{C} be a nontrivial class of finite groups closed under taking quotients. Let $\mathcal{C}_{\text{prim}} = \{L \in \mathcal{C} : L \text{ primitive}\}$. Let \mathbf{V} be a pseudovariety of finite groups. The following assertions hold.*

- (i) $P_{\mathcal{C}}(H, G)$ holds if and only if $P_{\mathcal{C}_{\text{prim}}}(H, G)$ holds.
- (ii) H is \mathbf{V} -dense if and only if $P_{\mathbf{V}_{\text{prim}}}(H, G)$ holds.
- (iii) H is \mathbf{Su} -dense if and only if H satisfies condition $Q_p(H, G)$ for every prime p .

Proof. We consider the first part. It is clear that if $P_{\mathcal{C}}(H, G)$ holds then $P_{\mathcal{C}_{\text{prim}}}(H, G)$ also holds. Suppose now that $P_{\mathcal{C}_{\text{prim}}}(H, G)$ holds. Assume for a contradiction that $P_{\mathcal{C}}(H, G)$ does not hold. Then there exists $N \trianglelefteq G$ with $G/N \in \mathcal{C}$ such that $HN \neq G$. Let M be a maximal subgroup of G containing HN and set $L = \text{Core}_G(M)$. Note that L contains N . Since \mathcal{C} is closed under taking quotients and $G/N \in \mathcal{C}$, we obtain that $G/L \cong \frac{G/N}{L/N}$ belongs to \mathcal{C} . Now M/L is a maximal subgroup of G/L and $\text{Core}_{G/L}(M/L) = 1$, hence $G/L \in \mathcal{C}_{\text{prim}}$. Since $P_{\mathcal{C}_{\text{prim}}}(H, G)$ holds, $HL = G$. As $L \leq M$, this implies $HM = G$, a contradiction since $HM = M$.

The second part is now immediate and the final part follows from Lemma 4.2 and Definition 4.3. \square

Given a prime p , we now introduce the family of p -hypersolvable groups.

Definition 4.5. Fix a prime p . For a positive integer n , let V_n denote the elementary abelian group of order p^n . A finite group G is said to be p -hypersolvable if G is isomorphic to $V_n \rtimes C$ for some positive integer n and some $C \leq \text{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^*$ acting diagonally on V_n . We let x be a generator for C and identify x with an integer $1 \leq \alpha \leq p-1$. In this way, $V_n \rtimes C = V_n \rtimes_{\alpha} C$ via the action $v^x = \alpha v$ for all $v \in V_n$, endowed with the product $(v, x^a)(w, x^b) = (v + w^{(x^a)}, x^{a+b}) = (v + \alpha^a w, x^{a+b})$. (Note that $\alpha = 1$ if and only if $C = 1$.)

Clearly the p -hypersolvable groups are supersolvable. The p -hypersolvable groups in fact have a number of other useful properties, which we collect in the next lemma.

Lemma 4.6. Fix a prime p and a positive integer n . Let $G = V_n \rtimes_{\alpha} C$ be a p -hypersolvable group. Set $c = |C|$. Let $g, h \in G$ and write g, h uniquely in the form $g = vx^{c_1}$ and $h = wx^{c_2}$, where $v, w \in V_n$, and $0 \leq c_1, c_2 \leq c-1$. Then

- (i) We have $g^{-1} = (-\alpha^{-c_1}v)x^{-c_1}$.
- (ii) We have $[g, h] = (1 - \alpha^{c_2})v + (\alpha^{c_1} - 1)w$.
- (iii) If $g \neq 1$ then $\text{ord}(g) = p$ if $c_1 = 0$, and $\text{ord}(g) = c/\text{gcd}(c, c_1)$ otherwise.
- (iv) If $c_1 \neq 0$ then there exists $h \in G$ such that $hgh^{-1} = x^{c_1} \in C$.
- (v) If $u \in V_n$ then $gug^{-1} = \alpha^{c_1}u$.

Proof. Part (i) is clear. Indeed $(-\alpha^{-c_1}v)x^{-c_1} \in G$,

$$g \cdot ((-\alpha^{-c_1}v)x^{-c_1}) = (vx^{c_1}) \cdot ((-\alpha^{-c_1}v)x^{-c_1}) = (v + \alpha^{c_1}(-\alpha^{-c_1}v))(x^{c_1}x^{-c_1}) = 1.$$

We now consider part (ii). Using part (i), we have

$$\begin{aligned}
[g, h] &= ghg^{-1}h^{-1} \\
&= (vx^{c_1}) \cdot (wx^{c_2}) \cdot ((-\alpha^{-c_1}v)x^{-c_1}) \cdot ((-\alpha^{-c_2}w)x^{-c_2}) \\
&= ((v + \alpha^{c_1}w)x^{c_1+c_2}) \cdot ((-\alpha^{-c_1}v - \alpha^{-c_1-c_2}w)x^{-c_1-c_2}) \\
&= (v + \alpha^{c_1}w) + \alpha^{c_1+c_2}(-\alpha^{-c_1}v - \alpha^{-c_1-c_2}w) \\
&= (1 - \alpha^{c_2})v + (\alpha^{c_1} - 1)w.
\end{aligned}$$

We consider part (iii). Given a positive integer i , arguing by induction yields

$$g^i = \sum_{j=0}^{i-1} \alpha^{jc_1} vx^{ic_1}.$$

Suppose that $g \in V_n$, so that $c_1 = 0$. As g is nontrivial and V_n is an elementary abelian p -group, g has order p . Suppose now that $g \notin V$. Then $c_1 \neq 0$ and for $i \geq 1$

$$g^i = \frac{\alpha^{ic_1} - 1}{\alpha^{c_1} - 1} vx^{ic_1}.$$

As elements of $(\mathbb{Z}/p\mathbb{Z})^*$, x and α have order c , and so x^{c_1} and α^{c_1} have order $c/\gcd(c, c_1)$. It now follows that g has order $c/\gcd(c, c_1)$. We now consider part (iv). Setting $h := v/(\alpha^{c_1} - 1)$, one checks that $hgh^{-1} = x^{c_1} \in C$. We finally consider part (v). By part (i), $g^{-1} = (-\alpha^{-c_1}v)x^{-c_1}$ and so

$$\begin{aligned}
gug^{-1} &= (vx^{c_1}) \cdot u \cdot ((-\alpha^{-c_1}v)x^{-c_1}) \\
&= ((v + \alpha^{c_1}u)x^{c_1}) \cdot ((-\alpha^{-c_1}v)x^{-c_1}) \\
&= v + \alpha^{c_1}u + \alpha^{c_1}(-\alpha^{-c_1}v) \\
&= \alpha^{c_1}u.
\end{aligned}$$

□

We characterise below generating sets in (non elementary abelian) p -hypersolvable groups.

Lemma 4.7. *Fix a prime p and an integer $d \geq 2$. Let $G := V_{d-1} \rtimes C$ be an associated p -hypersolvable group with $C \neq 1$ and let $c = |C|$. For an element $g \in G$, write g uniquely in the form $g = v(g)x^{c(g)}$, where $v(g) \in V_d - 1$ and $0 \leq c(g) \leq c - 1$. Then a subset $Z := \{g_1, \dots, g_d\}$ of G of cardinality d is a generating set for G if and only if each of the following holds:*

$$(i) \text{ lcm} \left\{ \frac{c}{\gcd(c, c(g_i))} : 1 \leq i \leq d, c(g_i) \neq 0 \right\} = c.$$

(ii) $\{[g_i, g_j] : g_i, g_j \in Z\}$ spans $V_d - 1$.

Proof. If (i) and (ii) hold, then it is clear that Z generates G . So assume that Z is a generating set for G . Then the reduction of Z modulo V_{d-1} is a generating set for the cyclic group C . It follows that Z satisfies (i). Next, we prove (ii). Since $V_{d-1} = [G, G]$ and Z generates G , it follows easily from the commutator identities and induction that V_{d-1} is generated by the normal closure of the set $\{[g_i, g_j] : g_i, g_j \in Z\}$. Since every subgroup of V_{d-1} is normal in G , part (ii) follows. \square

Finally, using Gaschutz's lemma, we determine the minimal number of generators of a p -hypersolvable group.

Lemma 4.8. *Fix a prime p and a positive integer n . Let $G := V_n \rtimes C$ be an associated p -hypersolvable group. Then*

$$d(G) = \begin{cases} n & \text{if } C = 1 \\ n + 1 & \text{otherwise.} \end{cases}$$

Proof. The case $C = 1$ is well known, hence we assume that $C \neq 1$. We have $C \cong \mathbb{Z}/c\mathbb{Z}$ for some divisor $c > 1$ of $p-1$. Let x be a generator of C . It is clear that $d(G) \leq n+1$. We prove that $d(G) = n+1$ by induction on n . The case $n = 1$ is trivial since $V_1 \rtimes C$ is not cyclic. Suppose $n > 1$. Let $\langle v_1, \dots, v_n \rangle$ be a basis of V_n and let $U = \langle v_n \rangle$. Note that $U \triangleleft G$. By the induction hypothesis, we have $d(G/U) = n$. Thus since $d(G/U) \leq d(G) \leq n+1$, we may assume (in pursuit of a contradiction) that $d(G) = n = d(G/U)$. Since $\{Uv_1, \dots, Uv_{n-1}, Ux\}$ is a generating set of G/U , Lemma 2.2 yields that for $1 \leq i \leq n$, there exist $u_i \in U$ such that $G = \langle u_1v_1, \dots, u_{n-1}v_{n-1}, u_nx \rangle$. Let $N = \langle u_i v_i : 1 \leq i \leq n-1 \rangle$. Then $N \leq V_n$ and so $N \triangleleft G$. Since u_nx has order c , we obtain

$$G = N \rtimes \langle u_nx \rangle.$$

This is a contradiction, as $|G| = p^n c$ but $N \rtimes \langle u_nx \rangle$ has order at most $p^{n-1}c$. Hence $d(G) = n+1$. \square

5 A technical lemma

The aim of this section is to prove the technical lemma below. Let F be a free group of finite rank $d > 1$ over $X = \{x_1, \dots, x_d\}$. For $i = 1, \dots, d-1$, let $\tau_i \in S_d$ denote the transposition $(i d)$, and let τ_d be the identity. We also denote by τ_i the automorphism of F which switches the letters x_i and x_d and fixes the remaining ones. For $i = 1, \dots, d-1$, let $\iota_i \in \mathbb{Z}^{d-1}$ have 1 in the i th coordinate and 0 everywhere else. We may also view ι_i as an element of V_{d-1} if convenient.

Lemma 5.1. *Let F be a free group of finite rank $d > 1$ over $X = \{x_1, \dots, x_d\}$ and let G be a nonabelian p -hypersolvable group. Let $\varphi : F \rightarrow G$ be a surjective homomorphism. Then there exist some $t \in \{1, \dots, d\}$, $C \leq \mathbb{Z}/(p-1)\mathbb{Z}$, $\alpha \in \{2, \dots, p-1\}$ and surjective homomorphisms $\psi : F \rightarrow V_{d-1} \rtimes_{\alpha} C$ and $\theta : V_{d-1} \rtimes_{\alpha} C \rightarrow G$ such that the diagram*

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & G \\ \pi_t \downarrow & & \uparrow \theta \\ F & \xrightarrow{\psi} & V_{d-1} \rtimes_{\alpha} C \end{array}$$

commutes and, for $i = 1, \dots, d-1$, $\psi(x_i) = (v_i, c_i)$ for some $c_i \in C$.

Proof. By Lemma 4.8, G must be of the form $V_m \rtimes_{\alpha} C$, with $m \leq d-1$ and $1 \neq C \leq \text{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^*$ acting diagonally (and nontrivially) on V_m . Let $\pi : G \rightarrow C$ be the canonical homomorphism. By Lemma 2.4, there exists some $\lambda \in \text{Aut}(F)$ such that $\pi(\varphi(\lambda(x_1)))$ generates C . Write $x = \pi(\varphi(\lambda(x_1)))$. Then there exists some $\alpha \in \{2, \dots, p-1\}$ such that $v^x = \alpha v$ for every $v \in V_m$.

Write $G' = V_{d-1} \rtimes_{\alpha} C$. We build a surjective homomorphism $\theta' : G' \rightarrow G$ by considering the projection $V_{d-1} \rightarrow V_m$ on the first m coordinates and sending x to x . The action is certainly preserved, hence θ' is a well-defined homomorphism. Naturally, we may view G as a subgroup of G' by identifying V_m with the elements of V_{d-1} with the last $d-1-m$ components equal to 0.

Since θ' is surjective, it follows from the universal property that there exists some homomorphism $\varphi' : F \rightarrow G'$ such that $\theta' \circ \varphi' = \varphi$. The question is: can we make φ' surjective? We prove it next.

For $i = 2, \dots, d$, we have $\pi(\varphi(\lambda(x_i))) = x^{q_i}$ for some $0 \leq q_i \leq |C| - 1$. Let $\lambda' : F \rightarrow F$ be the homomorphism defined by

$$\lambda'(x_i) = \begin{cases} x_1 & \text{if } i = 1 \\ x_i x_1^{-q_i} & \text{if } 1 < i \leq d \end{cases}$$

Since λ' is surjective and F is hopfian, then $\lambda' \in \text{Aut}(F)$. Let $\eta = \varphi \circ \lambda \circ \lambda'$. Then there exist $u_1, \dots, u_d \in V_m$ such that $\eta(x_1) = u_1 x$ and $\eta(x_i) = u_i$ for $i = 2, \dots, d$. Since η is onto, it follows from condition (ii) of Lemma 4.7 that

$$\langle [u_1 x, u_2], \dots, [u_1 x, u_d] \rangle = V_m.$$

Now $[u_1 x, u_i] = (\alpha - 1)u_i$ by Lemma 4.6(ii) and so $V_m = \langle u_2, \dots, u_d \rangle$.

Since V_m is a vector space of dimension m over $\mathbb{Z}/p\mathbb{Z}$, the spanning subset $\{u_2, \dots, u_d\}$ must contain some basis. Hence there exists some $\lambda'' \in \text{Aut}(F)$ induced by a permutation $\sigma \in S_d$ such that $\{u_{\sigma(1)}, \dots, u_{\sigma(m)}\}$ is such a basis and $\sigma(d) = 1$. Let $\eta' = \eta \circ \lambda''$. We show

that there exists some surjective homomorphism $\eta'' : F \rightarrow G'$ such that $\theta' \circ \eta'' = \eta'$, i.e., making the diagram

$$\begin{array}{ccc}
 & F & \\
 & \uparrow \lambda \circ \lambda' & \searrow \varphi \\
 & F & \xrightarrow{\eta} G \\
 & \uparrow \lambda'' & \nearrow \eta' \\
 F & \xrightarrow{\eta''} & G' \\
 & & \uparrow \theta'
 \end{array}$$

commute.

Let $\eta'' : F \rightarrow G'$ be the homomorphism defined by

$$\eta''(x_i) = \begin{cases} \eta'(x_i) & \text{if } 1 \leq i \leq m \text{ or } i = d \\ \eta'(x_i) + \iota_i & \text{if } m < i < d \end{cases}$$

It is routine to check that $\theta' \circ \eta'' = \eta'$ through the image of the letters. We show that η'' is onto.

For $i = 1, \dots, m, d$, we have $\eta''(x_i) = \eta'(x_i) = \eta(\lambda''(x_i)) = \eta(x_{\sigma(i)})$, hence

$$\langle \eta''(x_1), \dots, \eta''(x_m) \rangle = \langle \eta(x_{\sigma(1)}), \dots, \eta(x_{\sigma(m)}) \rangle = \langle u_{\sigma(1)}, \dots, u_{\sigma(m)} \rangle = V_m$$

and $\eta''(x_d) = \eta(x_{\sigma(d)}) = \eta(x_1) = u_1x$. It follows that $\iota_1, \dots, \iota_m, x \in \text{Im } \eta''$ and so $G = \text{Im } \eta' \leq \text{Im } \eta''$. Now it follows that $\iota_{m+1}, \dots, \iota_{d-1} \in \text{Im } \eta''$ as well. Thus η'' is surjective.

Let $\varphi' = \eta'' \circ (\lambda \circ \lambda' \circ \lambda'')^{-1}$. Then $\varphi' : F \rightarrow G'$ is a surjective homomorphism such that $\theta' \circ \varphi' = \varphi$.

For $i = 1, \dots, d$, write $\varphi'(x_i) = v_i x^{c_i}$ with $v_i \in V_{d-1}$ and $c_i \in C$. By Lemmas 4.6(ii) and 4.7, $\{v_1, \dots, v_d\}$ spans V_{d-1} . Since V_{d-1} is a vector space of dimension $d-1$ over $\mathbb{Z}/p\mathbb{Z}$, the spanning subset $\{v_1, \dots, v_d\}$ must contain some basis. Hence there exists some $t \in \{1, \dots, d\}$ such that $\{v_{\tau_t(1)}, \dots, v_{\tau_t(d-1)}\}$ is a basis of V_{d-1} .

Now we define a homomorphism $\zeta : G' \rightarrow G'$ by $\zeta(x) = x$ and $\zeta(\iota_i) = v_{\tau_t(i)}$ ($i = 1, \dots, d-1$). It is routine to check that ζ is a surjective homomorphism, hence an automorphism of G' . Finally, let $y_1, \dots, y_{d-1} \in \mathbb{Z}/p\mathbb{Z}$ be such that $v_t = \sum_{i=1}^{d-1} y_i v_{\tau_t(i)}$. We define a homomorphism $\psi : F \rightarrow G'$ by

$$\psi(x_i) = \begin{cases} \iota_i x^{c_{\tau_t(i)}} & \text{if } i \in \{1, \dots, d-1\} \\ (y_1, \dots, y_{d-1}) x^{c_t} & \text{if } i = d. \end{cases}$$

We claim that $\zeta \circ \psi \circ \tau_t = \varphi'$. Indeed, if $i \neq t$ we get

$$\zeta(\psi(\tau_t(x_i))) = \zeta(\psi(x_{\tau_t(i)})) = \zeta(\iota_{\tau_t(i)} x^{c_i}) = v_i x^{c_i} = \varphi'(x_i).$$

as well as

$$\zeta(\psi(\tau_t(x_t))) = \zeta(\psi(x_d)) = \zeta\left(\left(\sum_{i=1}^{d-1} y_i \iota_i\right) x^{c_t}\right) = \left(\sum_{i=1}^{d-1} y_i v_{\tau_t(i)}\right) x^{c_t} = v_t x^{c_t} = \varphi'(x_t),$$

thus $\zeta \circ \psi \circ \tau_t = \varphi'$. Since ζ and τ_t are both automorphisms and φ' is surjective, so is ψ .

Let $\theta = \theta' \circ \zeta$ so that we have the following commutative diagram

$$\begin{array}{ccc}
 F & \xrightarrow{\varphi} & G \\
 \downarrow \tau_t & \searrow \varphi' & \nearrow \theta' \\
 & G' & \\
 & \swarrow \zeta & \uparrow \theta \\
 F & \xrightarrow{\psi} & G'
 \end{array}$$

Then

$$\theta \circ \psi \circ \tau_t = \theta' \circ \zeta \circ \psi \circ \tau_t = \theta' \circ \varphi' = \varphi.$$

Since $\zeta \in \text{Aut}(G')$ and θ' is surjective, so is θ , and the lemma is proved. \square

6 The proof of Theorem 1

For an integer $d > 1$ we consider the monoid ring

$$R = \mathbb{Z}[\mathbb{Z}^d \times \mathbb{N}^{d-1}] = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_d, \alpha_d^{-1}, \beta_1, \dots, \beta_{d-1}],$$

which we choose to represent as some sort of polynomial ring over the indeterminates $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_{d-1}$, where the α_i admit inverses.

Definition 6.1. *Let p be a prime and $\mathcal{P} \subseteq R$. If there exists some nonzero ring homomorphism $\varphi : R \rightarrow \mathbb{Z}/p\mathbb{Z}$ such that $\varphi(P) = 0$ for every $P \in \mathcal{P}$, we say that*

$$(\varphi(\alpha_1), \varphi(\alpha_1^{-1}), \dots, \varphi(\alpha_d), \varphi(\alpha_d^{-1}), \varphi(\beta_1), \dots, \varphi(\beta_{d-1}))$$

is a common root modulo p for the polynomials in \mathcal{P} .

Let T denote the additive group of R . We define an action of \mathbb{Z}^d on T^{d-1} by group automorphisms through

$$\begin{aligned}
 \mathbb{Z}^d \times T^{d-1} &\rightarrow T^{d-1} \\
 ((k_1, \dots, k_d), (P_1, \dots, P_{d-1})) &\mapsto (\alpha_1^{k_1} \dots \alpha_d^{k_d} P_1, \dots, \alpha_1^{k_1} \dots \alpha_d^{k_d} P_{d-1})
 \end{aligned}$$

Since the α_i are invertible, this is indeed a well-defined action, inducing a semidirect product $T^{d-1} \rtimes \mathbb{Z}^d$.

Recall that F is a free group of rank d with basis $\{x_1, \dots, x_d\}$. Also given a positive integer k and an integer $1 \leq i \leq k$, we let ι_i be the element of \mathbb{Z}^k (or T^k) having 1 in the i th coordinate and 0 elsewhere. We define a homomorphism $\xi : F \rightarrow T^{d-1} \rtimes \mathbb{Z}^d$ by

$$\xi(x_i) = \begin{cases} (\iota_i, \iota_i) & \text{if } 1 \leq i < d \\ (\beta_1, \dots, \beta_{d-1}, \iota_d) & \text{if } i = d. \end{cases}$$

It follows that

$$\xi(x_i^{-1}) = \begin{cases} (-\alpha_i^{-1}\iota_i, -\iota_i) & \text{if } 1 \leq i < d \\ (-\alpha_d^{-1}\beta_1, \dots, -\alpha_d^{-1}\beta_{d-1}, -\iota_d) & \text{if } i = d \end{cases}$$

thus we do not need to consider inverses for the β_i . We write $\xi(w) = (\xi_1(w), \xi_2(w))$ for every $w \in F$.

For every prime $p > 2$, we define

$$\begin{aligned} \mathcal{Q}(p) = \{ & (c, \alpha, u, m) \mid c = (c_1, \dots, c_d) \in \{0, \dots, p-2\}^d; 2 \leq \alpha, m \leq p-1; \alpha^m \equiv 1 \pmod{p}; \\ & u = (u_1, \dots, u_{d-1}) \in \{0, \dots, p-1\}^{d-1} \}. \end{aligned}$$

Given $Q = (c, \alpha, u, m) \in \mathcal{Q}(p)$, let $\varphi_{p,Q} : R \rightarrow \mathbb{Z}/p\mathbb{Z}$ be the ring homomorphism extending the canonical homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ and such that

$$\varphi_{p,Q}(\alpha_i) = \alpha^{c_i}, \quad \varphi_{p,Q}(\beta_j) = u_j \quad (1 \leq i \leq d, 1 \leq j \leq d-1).$$

Since $\alpha^m \equiv 1 \pmod{p}$ and $\alpha \neq 1$, then $V_{d-1} \rtimes_{\alpha} \mathbb{Z}/m\mathbb{Z}$ is a well-defined p -hypersolvable group.

Lemma 6.2. *Let p be a prime and let $d > 1$ be an integer. Given $Q = (c, \alpha, u, m) \in \mathcal{Q}(p)$, let $\Phi_{p,Q} : T^{d-1} \rtimes \mathbb{Z}^d \rightarrow V_{d-1} \rtimes_{\alpha} \mathbb{Z}/m\mathbb{Z}$ be defined by*

$$\Phi_{p,Q}(P_1, \dots, P_{d-1}, k_1, \dots, k_d) = (\varphi_{p,Q}(P_1), \dots, \varphi_{p,Q}(P_{d-1}), \sum_{i=1}^d c_i k_i).$$

Then $\Phi_{p,Q}$ is a group homomorphism.

Proof. Let $Y, Y' \in T^{d-1} \rtimes \mathbb{Z}^d$, say $Y = (P_1, \dots, P_{d-1}, k_1, \dots, k_d)$ and $Y' = (P'_1, \dots, P'_{d-1}, k'_1, \dots, k'_d)$. Write $\alpha' = \alpha_1^{k_1} \dots \alpha_d^{k_d}$. Then:

$$\begin{aligned} \Phi_{p,Q}(YY') &= \Phi_{p,Q}(P_1 + \alpha' P'_1, \dots, P_{d-1} + \alpha' P'_{d-1}, k_1 + k'_1, \dots, k_d + k'_d) \\ &= (\varphi_{p,Q}(P_1 + \alpha' P'_1), \dots, \varphi_{p,Q}(P_{d-1} + \alpha' P'_{d-1}), \sum_{i=1}^d c_i (k_i + k'_i)) \\ \Phi_{p,Q}(Y)\Phi_{p,Q}(Y') &= (\varphi_{p,Q}(P_1), \dots, \varphi_{p,Q}(P_{d-1}), \sum_{i=1}^d c_i k_i) \\ &\quad \cdot (\varphi_{p,Q}(P'_1), \dots, \varphi_{p,Q}(P'_{d-1}), \sum_{i=1}^d c_i k'_i) \\ &= (\varphi_{p,Q}(P_1) + \alpha^{\sum_{i=1}^d c_i k_i} \varphi_{p,Q}(P'_1), \\ &\quad \dots, \varphi_{p,Q}(P_{d-1}) + \alpha^{\sum_{i=1}^d c_i k_i} \varphi_{p,Q}(P'_{d-1}), \sum_{i=1}^d c_i (k_i + k'_i)), \end{aligned}$$

hence it suffices to show that

$$\varphi_{p,Q}(P_j + \alpha' P'_j) = \varphi_{p,Q}(P_j) + \alpha^{\sum_{i=1}^d c_i k_i} \varphi_{p,Q}(P'_j)$$

holds for $j = 1, \dots, d-1$. Indeed, this follows easily from $\varphi_{p,Q}$ being a ring homomorphism and

$$\varphi_{p,Q}(\alpha') = \varphi_{p,Q}(\alpha_1^{k_1} \dots \alpha_d^{k_d}) = \alpha^{c_1 k_1} \dots \alpha^{c_d k_d} = \alpha^{\sum_{i=1}^d c_i k_i}.$$

Thus $\Phi_{p,Q}$ is a group homomorphism. \square

Now $\Phi_{p,Q} : T^{d-1} \rtimes \mathbb{Z}^d \rightarrow V_{d-1} \rtimes_{\alpha} \mathbb{Z}/m\mathbb{Z}$ induces a group homomorphism $\Phi'_{p,Q} : T^{d-1} \rightarrow V_{d-1}$ by

$$\Phi'_{p,Q}(P_1, \dots, P_{d-1}) = \Phi_{p,Q}(P_1, \dots, P_{d-1}, 0, \dots, 0).$$

If the rows of a matrix M belong to T^{d-1} , then $\Phi'_{p,Q}(M)$ denotes the matrix obtained from M by replacing each row by its image under $\Phi'_{p,Q}$.

Suppose that F is a free group of finite rank $d > 1$ over $X = \{x_1, \dots, x_d\}$ and H is a finitely generated subgroup of F . Let $\{w_1, \dots, w_e\}$ be a basis of H . Then $\{\tau_t(w_1), \dots, \tau_t(w_e)\}$ is a basis of $\tau_t(H)$ for $t = 1, \dots, d$. We define an $\binom{e}{2} \times (d-1)$ matrix $\mathcal{M}(\tau_t(H))$ over T by taking $\xi_1([\tau_t(w_r), \tau_t(w_s)])$ as row vectors for all $1 \leq r < s \leq e$. If $H = F$, we consider the basis $\{x_1, \dots, x_d\}$ of F , and define a $\binom{d}{2} \times (d-1)$ matrix $\mathcal{M}(F)$ over T by taking $\xi_1([x_r, x_s])$ as row vectors for all $1 \leq r < s \leq d$.

We now introduce extra indeterminates $\gamma_1, \gamma_2, \gamma_3$. Write

$$R' = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_d, \alpha_d^{-1}, \beta_1, \dots, \beta_{d-1}, \gamma_1, \gamma_2, \gamma_3].$$

Let $\mathcal{Y}(F)$ denote the set of all $(d-1) \times (d-1)$ minors of $\mathcal{M}(F)$. For every $Y \in \mathcal{Y}(F)$, let

$$\mathcal{P}_Y = \{\gamma_1 \alpha_1 \dots \alpha_d \det(Y) - 1, \alpha_1 + \dots + \alpha_d - d - \gamma_2, \gamma_2 \gamma_3 - 1\} \subset R'.$$

The purpose of γ_1 and γ_3 is to ensure that, for any common root of these polynomials, $\alpha_1, \dots, \alpha_d, \det(Y)$ and γ_2 are all nonzero, and similarly the purpose of γ_2 is to ensure that, for any common root of these polynomials, $\alpha_i \neq 1$ for some i .

For $t = 1, \dots, d$, let $\mathcal{Y}(\tau_t(H))$ denote the set of all $(d-1) \times (d-1)$ minors of $\mathcal{M}(\tau_t(H))$ and for $Y_0 \in \mathcal{Y}(F)$, let

$$\mathcal{P}_{Y_0}(\tau_t(H)) = \mathcal{P}_{Y_0} \cup \{\det(Y) \mid Y \in \mathcal{Y}(\tau_t(H))\} \subset R'.$$

Lemma 6.3. *Let F be a free group of finite rank d over $X = \{x_1, \dots, x_d\}$. Let H be a finitely generated subgroup of F . Suppose that H is **Ab**-dense. Then the following conditions are equivalent:*

- (i) H is not **Su**-dense;
- (ii) there exist some $p \in \mathbb{P}$, $t \in \{1, \dots, d\}$ and $Y_0 \in \mathcal{Y}(F)$ such that the polynomials in $\mathcal{P}_{Y_0}(\tau_t(H))$ admit a common root modulo p .

Proof. We are given a basis $W = \{w_1, \dots, w_e\}$ of H where for $1 \leq i \leq e$, w_i is a reduced word over $X \cup X^{-1}$. Note that if $d = 1$ then $F \cong \mathbb{Z}$ is abelian and so H is **Su**-dense (as it is **Ab**-dense). We therefore assume that $d > 1$. Since H is **Ab**-dense and $\mathbf{Ab}(2) \subset \mathbf{Ab}$, H is **Ab**(2)-dense by (2.1), and so H satisfies condition $Q_2(H, F)$. (i) \Rightarrow (ii). By Lemma 4.4(iii), H fails condition $Q_p(H, F)$ for some prime p which must be odd. Hence there exists some $N \trianglelefteq F$ such that $HN < F$ and $F/N \cong \mathbb{Z}/p\mathbb{Z} \rtimes_{\alpha} C$ for some $C \leq \text{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^*$ and some $\alpha \in \{1, \dots, p-1\}$. Since H is **Ab**-dense, then F/N must be nonabelian and so $C = \mathbb{Z}/m\mathbb{Z}$ for some $m > 1$ dividing $p-1$ and $\alpha \in \{2, \dots, p-1\}$. Let $\varphi : F \rightarrow F/N$ be the canonical homomorphism. By Lemma 5.1, there exist some $t \in \{1, \dots, d\}$, and surjective homomorphisms $\psi : F \rightarrow V_{d-1} \rtimes_{\alpha} C$ and $\theta : V_{d-1} \rtimes_{\alpha} C \rightarrow F/N$ such that $\theta \circ \psi \circ \tau_t = \varphi$ and, for $i = 1, \dots, d-1$, $\psi(x_i) = (\iota_i, c_i)$ for some $c_i \in C$. Write $\psi(x_d) = (u_1, \dots, u_{d-1}, c_d)$, $c = (c_1, \dots, c_d)$ and $u = (u_1, \dots, u_{d-1})$. Since $\alpha^m \equiv 1 \pmod{p}$ is a requirement for the action being well defined, we get $Q = (c, \alpha, u, m) \in \mathcal{Q}(p)$.

Consider now the ring homomorphism $\varphi_{p,Q} : R \rightarrow \mathbb{Z}/p\mathbb{Z}$. We use this homomorphism to find a common root modulo p for all the polynomials in $\mathcal{P}_{Y_0}(\tau_t(H))$ for some $Y_0 \in \mathcal{Y}(F)$. We do not need to worry about the auxiliary indeterminates γ_i if we understand their purpose. That is, it suffices to show that:

- (A1) $\varphi_{p,Q}(\det(Y_0)) \neq 0$ for some $Y_0 \in \mathcal{Y}(F)$;
- (A2) $\varphi_{p,Q}(\alpha_i) \neq 0$ for $i = 1, \dots, d$;
- (A3) $\varphi_{p,Q}(\alpha_i) \neq 1$ for some $1 \leq i \leq d$;
- (A4) $\varphi_{p,Q}(\det(Y)) = 0$ for every $Y \in \mathcal{Y}(\tau_t(H))$.

Recall now $\Phi_{p,Q}$, $\Phi'_{p,Q}$, ξ and $\mathcal{M} := \mathcal{M}(F)$. We consider the composition

$$\Phi_{p,Q} \circ \xi : F \rightarrow V_{d-1} \rtimes_{\alpha} \mathbb{Z}/m\mathbb{Z} = V_{d-1} \rtimes_{\alpha} C.$$

It follows from the definitions that $\Phi_{p,Q}(\xi(x_i)) = \psi(x_i)$ for $i = 1, \dots, d$, hence $\Phi_{p,Q} \circ \xi = \psi$.

Since ψ is surjective, it is easy to see that $C = \langle c_1, \dots, c_d \rangle$ and Lemma 4.7 yields

$$\langle [\Phi_{p,Q}(\xi(x_i)), \Phi_{p,Q}(\xi(x_j))] \mid 1 \leq i < j \leq d \rangle = V_{d-1},$$

hence

$$\langle \Phi'_{p,Q}(\xi_1([x_i, x_j])) \mid 1 \leq i < j \leq d \rangle = V_{d-1}.$$

This is equivalent to saying that the row vectors of $\Phi'_{p,Q}(\mathcal{M})$ generate V_{d-1} , hence this matrix has necessarily rank $d-1$. If

$$\mathcal{M} = \begin{pmatrix} P_{1,1} & \cdots & P_{1,d-1} \\ \cdots & \cdots & \cdots \\ P_{m,1} & \cdots & P_{m,d-1} \end{pmatrix}$$

then

$$\Phi'_{p,Q}(\mathcal{M}) = \begin{pmatrix} \varphi_{p,Q}(P_{1,1}) & \cdots & \varphi_{p,Q}(P_{1,d-1}) \\ \cdots & \cdots & \cdots \\ \varphi_{p,Q}(P_{m,1}) & \cdots & \varphi_{p,Q}(P_{m,d-1}) \end{pmatrix}$$

Since $\Phi'_{p,Q}(\mathcal{M})$ has rank $d-1$ and $\varphi_{p,Q}$ is a ring homomorphism, there exists some $Y_0 \in \mathcal{Y}(F)$ such that $\varphi_{p,Q}(\det(Y_0)) \neq 0$. Thus (A1) holds.

On one hand, $\varphi_{p,Q}(\alpha_i) = \alpha^{c_i} \neq 0$ since $2 \leq \alpha \leq p-1$, hence (A2) holds. On the other hand, $C = \langle c_1, \dots, c_d \rangle$ nontrivial implies $c_i \neq 0$ for some i , therefore (A3) holds as well.

Finally, let $Y \in \mathcal{Y}(\tau_t(H))$. Suppose that $\varphi_{p,Q}(\det(Y)) \neq 0$. The same argument used above shows that the row vectors of $\Phi'_{p,Q}(\mathcal{M}(\tau_t(H)))$ generate V_{d-1} . Since $\Phi_{p,Q} \circ \xi = \psi$, we get

$$\langle [\psi(\tau_t(w_i)), \psi(\tau_t(w_j))] \mid 1 \leq i < j \leq e \rangle = V_{d-1},$$

hence $V_{d-1} \leq \psi(\tau_t(H))$. On the other hand, since $\tau_t(H)$ is **Ab**-dense by Lemma 2.1, we can compose ψ with the canonical homomorphism $\pi : V_{d-1} \rtimes_{\alpha} C \rightarrow C$ to derive

$$\tau_t(H)\ker(\pi \circ \psi) = F$$

and consequently $\pi(\psi(\tau_t(H))) = C$. Together with $V_{d-1} \leq \psi(\tau_t(H))$, this yields $\psi(\tau_t(H)) = V_{d-1} \rtimes_{\alpha} C$ and so $(\tau_t(H))\ker \psi = F$.

Let $g \in F$. Then $\tau_t(g) = (\tau_t(h))z$ for some $h \in H$ and $z \in \ker \psi$. Hence

$$g = \tau_t^2(g) = h\tau_t(z).$$

Since $\psi(\tau_t(\tau_t(z))) = \psi(z) = 1$, we get $H\ker(\psi \circ \tau_t) = F$. Since $\theta \circ \psi \circ \tau_t = \varphi$, then $\ker(\psi \circ \tau_t) \leq \ker \varphi = N$ and so $HN = F$, a contradiction. Therefore (A4) holds and so does condition (ii).

(ii) \Rightarrow (i). Assume that condition (ii) holds for $p \in \mathbb{P}$, $Y_0 \in \mathcal{Y}(F)$ and $t \in \{1, \dots, d\}$. Let $a_1, \dots, a_d, u_1, \dots, u_{d-1}, g_1, g_2, g_3$ be a common root modulo p for the polynomials in $\mathcal{P}_{Y_0}(\tau_t(H))$.

Since $a_i \not\equiv 0 \pmod{p}$ for $i = 1, \dots, d$, we can write $C = \langle a_1, \dots, a_d \rangle \leq (\mathbb{Z}/p\mathbb{Z})^*$. Note that $a_i \not\equiv 1 \pmod{p}$ for some i , hence $|C| > 1$. Let x denote a generator of C and let $\alpha \in \{2, \dots, p-1\}$ represent x modulo p . For $i = 1, \dots, d$, we have $a_i = \alpha^{c_i}$ for some $0 \leq c_i \leq |C| - 1$. Write $c = (c_1, \dots, c_d)$ and $u = (u_1, \dots, u_{d-1})$. Since $\alpha^{|C|} \equiv 1 \pmod{p}$, we can define $Q = (c, \alpha, u, |C|) \in \mathcal{Q}(p)$.

We consider now the p -hypersolvable group $G = V_{d-1} \rtimes_{\alpha} C$, viewing C as $\mathbb{Z}/|C|\mathbb{Z}$ (hence $C = \langle c_1, \dots, c_d \rangle$). Let $\psi : F \rightarrow G$ be the homomorphism defined by

$$\psi(x_i) = \begin{cases} (\iota_i, c_i) & \text{if } 1 \leq i < d \\ (u_1, \dots, u_{d-1}, c_d) & \text{if } i = d. \end{cases}$$

Once again, we have $\Phi_{p,Q} \circ \xi = \psi$. We prove that ψ is surjective, by showing that $\psi(x_1), \dots, \psi(x_d)$ satisfy the conditions of Lemma 4.7.

On the one hand,

$$\gcd(|C|, c_1, \dots, c_d) = 1$$

follows from $\langle c_1, \dots, c_d \rangle = C = \mathbb{Z}/|C|\mathbb{Z}$, and so condition (i) of Lemma 4.7 is satisfied.

On the other hand,

$$[\psi(x_i), \psi(x_j)] = [\Phi_{p,Q}(\xi(x_i)), \Phi_{p,Q}(\xi(x_j))] = \Phi'_{p,Q}(\xi_1([x_i, x_j]))$$

holds for all $1 \leq i < j \leq d$, and once again we are applying the ring homomorphism $\varphi_{p,Q}$ to each of the entries of the matrix $\mathcal{M}(F)$. By our assumption, the corresponding equality of type (A1) is satisfied, that is $\varphi_{p,Q}(\det(Y_0)) \neq 0$, which implies that the matrix with rows $[\psi(x_i), \psi(x_j)]$ has rank $d - 1$. Therefore condition (ii) of Lemma 4.7 is satisfied and so ψ is surjective.

Let $N = \text{Ker } \psi \trianglelefteq F$. Next we prove that $\tau_t(H)N < F$, which implies that $\tau_t(H)$ is not **Su**-dense. By Lemma 2.1, H is not **Su**-dense either.

Indeed, suppose that $\tau_t(H)N = F$. Then $\psi(\tau_t(H)) = G$ and so Lemma 4.7 implies that

$$\langle [\psi(\tau_t(w_i)), \psi(\tau_t(w_j))] \mid 1 \leq i < j \leq e \rangle = V_{d-1},$$

yielding

$$\langle [\Phi_{p,Q}(\xi(\tau_t(w_i))), \Phi_{p,Q}(\xi(\tau_t(w_j)))] \mid 1 \leq i < j \leq e \rangle = V_{d-1}$$

and consequently

$$\langle \Phi'_{p,Q}(\xi_1([\tau_t(w_i), \tau_t(w_j)])) \mid 1 \leq i < j \leq e \rangle = V_{d-1}.$$

It follows that the matrix obtained by applying $\varphi_{p,Q}$ to every entry of $\mathcal{M}(\tau_t(H))$ has rank $d - 1$, contradicting condition (A4), which follows from our assumption. Therefore $\tau_t(H)N < F$ and so H is not **Su**-dense. \square

We can now prove Theorem 1.

Proof of Theorem 1. Since **Ab** \subset **Su**, it follows from (2.1) that being **Ab**-dense is a necessary condition for being **Su**-dense. Since **Ab**-denseness is decidable by [10, Theorem 4.4], we may assume that H is **Ab**-dense. Also by [11, Corollary 2.4], if F is not of finite rank then H is not **Su**-dense. We can therefore assume that F is of finite rank d . We assume that $d > 1$, as if $d = 1$ then $F \cong \mathbb{Z}$ is abelian and so H is **Su**-dense (as it is **Ab**-dense). By Lemma 6.3 H is not **Su**-dense if and only if there exist some $p \in \mathbb{P}$, $t \in \{1, \dots, d\}$ and $Y_0 \in \mathcal{Y}(F)$ such that the polynomials in $\mathcal{P}_{Y_0}(\tau_t(H))$ admit a common root modulo p .

We fix $t \in \{1, \dots, d\}$ and $Y_0 \in \mathcal{Y}(F)$. To complete the proof of the theorem, it suffices to show that it is decidable whether or not there is a prime p such that the polynomials in $\mathcal{P}_{Y_0}(\tau_t(H))$ admit a common root modulo p . The first thing we need to do is to replace $\mathcal{P}_{Y_0}(\tau_t(H)) \subset R'$ by some equivalent subset of polynomials

$$\mathcal{P}'_{Y_0}(\tau_t(H)) \subset R'' = \mathbb{Z}[\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_{d-1}, \gamma_1, \gamma_2, \gamma_3].$$

Indeed, there exists some $N \geq 1$ such that the matrices

$$\alpha_1^N \dots \alpha_d^N \mathcal{M}(F) \quad \text{and} \quad \alpha_1^N \dots \alpha_d^N \mathcal{M}(\tau_t(H))$$

have all the entries in R'' . Let

$$\begin{aligned} \mathcal{P}'_{Y_0}(\tau_t(H)) &= \{\gamma_1 \alpha_1^N \dots \alpha_d^N \det(Y_0) - 1, \alpha_1 + \dots + \alpha_d - d - \gamma_2, \gamma_2 \gamma_3 - 1\} \\ &\cup \{\det(Y) \mid Y \in \alpha_1^N \dots \alpha_d^N \mathcal{Y}(\tau_t(H))\} \subset R''. \end{aligned}$$

Any common root modulo p for the polynomials in $\mathcal{P}'_{Y_0}(\tau_t(H))$ induces by restriction a common root modulo p for the polynomials in $\mathcal{P}'_{Y_0}(\tau_t(H))$ (adapting the value of γ_1 , which unique purpose is to force $\alpha_1, \dots, \alpha_d, \det(Y)$ to assume nonzero values). The converse holds because all the α_i are forced to assume nonzero values, so all we need is to interpret α_i^{-1} as the inverse of α_i (resetting appropriately the value of γ_1). It now suffices to show that it is decidable whether or not

$$S(\mathcal{P}'_{Y_0}(\tau_t(H))) \neq \emptyset.$$

Since $\mathcal{P}'_{Y_0}(\tau_t(H))$ is a finite subset of R'' , it follows from Theorem 3.2 that there exists some $f \in \mathbb{Z}[\alpha]$ such that $S(\mathcal{P}'_{Y_0}(\tau_t(H))) = S(f)$. Since $S(f) = \emptyset$ if and only if $f = \pm 1$, Corollary 3.7 yields the result. \square

Thus we have answered positively Question 1 for the pseudovariety $\mathbf{V} = \mathbf{Su}$, however Questions 2-5 remain open at this stage (for $\mathbf{V} = \mathbf{Su}$).

Acknowledgements

The first author acknowledges support from the Centre of Mathematics of the University of Porto, which is financed by national funds through the Fundação para a Ciência e a Tecnologia, I.P., under the project with references UIDB/00144/2020 and UIDP/00144/2020.

The second author acknowledges support from the Centre of Mathematics of the University of Porto, which is financed by national funds through the Fundação para a Ciência e a Tecnologia, I.P., under the project with reference UIDB/00144/2020.

The third author was supported by the Engineering and Physical Sciences Research Council, grant number EP/T017619/1.

References

- [1] R. Baer. Classes of finite groups and their properties. *Illinois J. Math.* **1** (1957), 115–187.
- [2] B. Buchberger. An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. Translated from the 1965 German original by Michael P. Abramson. *J. Symbolic Comput.* **41** (2006), 475–511.

- [3] L. van den Dries. A remark on Ax’s theorem on solvability modulo primes. *Math. Z.* **208** (1991), 65–70.
- [4] W. Gaschütz. Zu einem von B.H. und H. Neumann gestellten Problem. *Math. Nachr.* **14** (1956), 249–252.
- [5] M. Hall. Coset representations in free groups. *Trans. Amer. Math. Soc.* **67** (1949), 421–432.
- [6] M. Hall. A topology for free groups and related groups. *Annals of Mathematics* **52** (1950), 127–139.
- [7] B. Herwig and D. Lascar. Extending partial automorphisms and the profinite topology on free groups. *Trans. Amer. Math. Soc.* **352** (2000), 1985–2021.
- [8] O. Jarviniemi. Solvability of a system of polynomial equations modulo primes. *Bull. Aust. Math. Soc.* **106** (2022), 404–407.
- [9] S. Margolis, M. Sapir and P. Weil. Closed subgroups in pro- V topologies and the extension problem for inverse automata. *Internat. J. Algebra Comput.* **11** (2001), 405–445.
- [10] C. Marion, P. V. Silva and G. Tracey. The pro- k -solvable topology on a free group, arXiv:2304.10235. Preprint, 2023.
- [11] C. Marion, P. V. Silva and G. Tracey. On the closure of cyclic subgroups of a free group in pro- \mathbf{V} topologies, arXiv:2304.10230. Preprint, 2023.
- [12] D. Masser, G. Wüstholz. Fields of Large Transcendence Degree Generated by Values of Elliptic Functions. *Invent. Math.* **72** (1983), 407–464.
- [13] L. Ribes and P. Zalesskii. The pro- p topology of a free group and algorithmic problems in semigroups. *Internat. J. Algebra Comput.* **4** (1994), 359–374.

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