

# Two-dimensional random interacements: 0-1 law and the vacant set at criticality

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## Abstract

We correct and streamline the proof of the fact that, at the critical point  $\alpha = 1$ , the vacant set of the two-dimensional random interacements is infinite [4]. Also, we prove a zero-one law for a natural class of tail events related to the random interacements.

In this note we discuss the model of two-dimensional random interacements — we refer to [4, 6, 8] for its definition. Here, we will mostly use the notations of [4], and let us quickly recall the main ones:

- SRW stands for simple random walk on  $\mathbb{Z}^2$  or on the torus  $\mathbb{Z}_n^2 := \mathbb{Z}^2/n\mathbb{Z}^2$ ,  $\text{RI}(\alpha)$  stands for two-dimensional random interacements with rate  $\alpha > 0$ , and  $\mathcal{V}^\alpha$  is the vacant set of  $\text{RI}(\alpha)$ ;
- $\partial A = \{x \in A : \text{there is } y \in \mathbb{Z}^2 \setminus A \text{ such that } x \sim y\}$  is the (inner) boundary of  $A \subset \mathbb{Z}^2$ ,  $\text{hm}_A(\cdot)$  is the harmonic measure (with respect to SRW) on  $A$  (it is concentrated on  $\partial A$  and is only well-defined when  $A$  is finite);
- $\mathbb{B}(x, r) = \{y \in \mathbb{Z}^2 : \|y - x\| \leq r\}$  is the (discrete) disk of radius  $r$  centered at  $x$ , and  $\mathbb{B}(r) := \mathbb{B}(0, r)$  (we also sometimes consider such disks placed on tori);
- for  $A \subset A'$ , an *excursion* between  $\partial A$  and  $\partial A'$  is a (finite) piece of a nearest-neighbor trajectory that starts at  $\partial A$  and ends on its first visit to  $\partial A'$ ; in this paper, we will only consider excursions between the boundaries of concentric disks.

One of our main intentions is to correct and to streamline the proof of the main result of [4] (Theorem 1.2):

**Theorem 1.** *The vacant set  $\mathcal{V}^1$  is a.s. infinite.*

In the paper [4], the construction described on the last pages, unfortunately, contains a flaw. This was overlooked by the authors of [4] due to a mistake in the excursion count calculation just before (83) (for the SRW on the torus, before time  $t_k$  defined there): the correct leading term should be  $\frac{1}{2\ln\gamma} \ln^2 b_k$  instead of  $\frac{2}{\ln\gamma} \ln^2 b_k$ . That would ruin the subsequent argument since the number of excursions generated by the interlacements would be more than enough to cover the corresponding disk.

Below, we present a corrected argument. It is possible to modify the construction of [4] “mechanically” to address that issue (basically, in [4]’s notation, considering  $B_k = \mathbf{B}(v_k, \frac{b_k}{\ln^2 b_k})$  instead of  $B_k = \mathbf{B}(v_k, b_k^{1/2})$  and modifying  $B'_k$  and  $b_k$  accordingly would suffice), but we prefer to use this paper to present a substantially simpler and cleaner way of proving the result.

*Proof of Theorem 1.* First, assume that we have the following general fact:

$$\text{For any } \alpha > 0, \quad \mathbb{P}[|\mathcal{V}^\alpha| = \infty] = 0 \text{ or } 1. \quad (1)$$

Then, with (1) to hand, if one assumes that  $|\mathcal{V}^1| < \infty$  a.s., that would mean that for any  $\varepsilon > 0$  there is  $R = R(\varepsilon)$  such that  $\mathbb{P}[\mathcal{V}^1 \subset \mathbf{B}(R)] > 1 - \varepsilon$ . Therefore, to obtain a contradiction, it suffices to prove that, for any fixed  $R > 0$ , the vacant set  $\mathcal{V}^1$  contains a site outside of  $\mathbf{B}(R)$  with uniformly positive probability. This means that one does not need to control the dependencies in the whole sequence of events  $(\{\mathcal{V}^1 \cap B_k \neq \emptyset\})_{k \geq 1}$  (as was done in [4], see the proof of (75)) to prove that an infinite number of these occurs a.s.; just estimating (from below) the probability of a generic event of that sequence would suffice.

We now derive such an estimate. To do this in a cleaner way (compared to [4]), we need another preliminary result (also for future reference).

**Proposition 2.** *Let  $\gamma > 1$  and  $\beta > 0$  be fixed numbers. Consider  $\frac{2\ln^2 n}{\ln\gamma} - (1 + \beta)\frac{\ln n \ln \ln n}{\ln\gamma}$  independent SRW’s excursions between  $\partial\mathbf{B}(n)$  and  $\partial\mathbf{B}(\gamma n)$ , with starting points chosen according to  $\text{hm}_{\mathbf{B}(n)}$ . Then*

$$\mathbb{P}[\mathbf{B}(n) \text{ is not completely covered by these excursions}] \rightarrow 1 \quad \text{as } n \rightarrow \infty. \quad (2)$$

*Proof.* First, we give an outline of the proof: Let us consider the torus  $\mathbb{Z}_m^2$  with  $m = \lfloor 3\gamma n \rfloor$ , with concentric disks  $\mathbf{B}(n)$  and  $\mathbf{B}(\gamma n)$  embedded there in a natural way. It is known that, if one runs the SRW on that torus up to time  $t_{m,\beta} := \frac{4}{\pi} m^2 \ln^2 m - (1 + \beta)\frac{2}{\pi} m^2 \ln m \ln \ln m$ , the number of excursions between  $\partial\mathbf{B}(n)$

and  $\partial\mathbf{B}(\gamma n)$  will be well concentrated around  $\frac{2\ln^2 n}{\ln \gamma} - (1 + \beta)\frac{\ln n \ln \ln n}{\ln \gamma}$ . It is also known [1] that with high probability there will still be uncovered sites on the torus by time  $t_{m,\beta}$ , and that it is possible to relate SRW's excursions to independent excursions via e.g. soft local times (as explained below); unfortunately, this is not yet enough to obtain (2) since one cannot a priori exclude the possibility that the uncovered sites are "spatially concentrated". (This way, one can only obtain that the probability in (2) is uniformly bounded from below, which was still enough for the argument of [4] at the cost of some additional technical difficulties.) In fact, one can still obtain (2) (actually, (8) below, that would by its turn imply (2)) by a suitable modification of the arguments of [1] (or, in the continuous setting, by a modification of the arguments of [2]) as the authors were able to find out thanks to private communications with Y. Abe and O. Zeitouni. However, we prefer to present a "softer" way to obtain (2), which only uses the main cover time result and does not require plunging into the technical details of [1, 2].

We prove (2) in three steps: first, as mentioned above, we use the known results for the cover time of the torus to prove that the probability of the corresponding event for independent excursions is uniformly positive. Next, we use that uniform positivity to refine a bit the torus covering result: we will show that a fixed disk of radius proportional to the size of the torus will contain an uncovered site with high probability. Then, we translate this "improved" result back to independent excursions. It is also worth mentioning that the harmonic measure  $\text{hm}_{\mathbf{B}(n)}$  is not exactly the "correct" one for choosing the starting points of the excursions. With  $A \subset A'$  and  $y \in \partial A$ , define

$$\text{hm}_{A'}^A(y) = \mathbb{P}_y[\tau_1(\partial A') < \tau_1(A)] \left( \sum_{z \in \partial A} \mathbb{P}_z[\tau_1(\partial A') < \tau_1(A)] \right)^{-1}, \quad (3)$$

where  $\tau_1(A) = \min\{k \geq 1 : S_k \in A\}$  is the hitting time of  $A$  by the SRW  $(S_n, n \geq 0)$ . For example, for the excursion process (between  $\partial\mathbf{B}(n)$  and  $\partial\mathbf{B}(\gamma n)$ ) generated by the SRW on the torus, it is the measure  $\text{hm}_{\mathbf{B}(n)}^{\mathbf{B}(\gamma n)}$  that will be invariant for it. It is, however, quite close to  $\text{hm}_{\mathbf{B}(n)}$ : due to Lemma 2.5 of [4], we have

$$\text{hm}_{\mathbf{B}(n)}^{\mathbf{B}(\gamma n)}(y) = \text{hm}_{\mathbf{B}(n)}(y) (1 + O(n^{-1})); \quad (4)$$

since we only need to deal with  $O(\ln^2 n)$  excursions, there will be no essential difference (in particular, Proposition 2 also holds with the starting points chosen according to  $\text{hm}_{\mathbf{B}(n)}^{\mathbf{B}(\gamma n)}$ ).

*Step 1.* Denote  $\psi_{n,\beta} = \frac{2\ln^2 n}{\ln \gamma} - (1 + \beta)\frac{\ln n \ln \ln n}{\ln \gamma}$ , and let us first prove that

$$\mathbb{P}[\mathbf{B}(n) \text{ is not completely covered by } \psi_{n,\beta} \text{ independent excursions}] \geq c_\beta \quad (5)$$

for some  $c_\beta > 0$ . For this, we note that, on the torus  $\mathbb{Z}_m^2$ , the number of SRW's excursions between  $\partial\mathbf{B}(n)$  and  $\partial\mathbf{B}(\gamma n)$  up to time  $t_{m,\beta/3}$  will be at most  $\psi_{n,\beta/2}$  with probability at least  $1 - c_1 \exp(-c_2(\ln \ln n)^2)$  (this follows e.g. from Lemma 2.11 of [4], take  $\delta = \frac{\varepsilon \ln \ln n}{\ln n}$  with small enough  $\varepsilon > 0$ ). Next, we use soft local times [8, 9], to construct a coupling of the SRW's excursions with independent excursions. More specifically, we use the construction with marked Poisson process (with excursions as marks) described in Section 2.2 of [4]; let us recall the notation  $L_k(\cdot)$  for the soft local time on  $\partial\mathbf{B}(n)$  generated by  $k$  excursions of the SRW on the torus, and let  $\tilde{L}_k(y) = (\tilde{\xi}_1 + \dots + \tilde{\xi}_k) \text{hm}_{\mathbf{B}(n)}(y)$  be the soft local times for the independent excursions.<sup>1</sup> Using (4) and Lemma 2.9 of [4] (with  $\theta = \frac{\varepsilon \ln \ln n}{\ln n}$ , where  $\varepsilon > 0$  is small enough), we obtain

$$\mathbb{P}[\tilde{L}_{\psi_{n,\beta}}(y) \leq L_{\psi_{n,\beta/2}}(y) \text{ for all } y \in \partial\mathbf{B}(n)] \geq 1 - c'_1 \exp(-c'_2(\ln \ln n)^2), \quad (6)$$

meaning that the set of independent excursions will be contained in the set of the SRW's excursions with high probability. Theorem 1.1 of [1] implies that at time  $t_{m,\beta/3}$  there will be an uncovered site in the torus with high probability, so (since  $\mathbf{B}(n)$  occupies a constant fraction of the volume of the torus) there will be an uncovered site in  $\mathbf{B}(n)$  with at least a constant probability. This shows (5).

*Step 2.* Now, take a large  $h > \gamma$ . Let  $\kappa_h$  be the maximal number of nonoverlapping disks of radius  $\gamma h^{-1}n$  and with centers at integer points that fit inside  $\mathbf{B}(n)$ ; clearly,  $\kappa_h$  is of order  $h^2$ . Let  $x_1, \dots, x_{\kappa_h}$  be these centers, and denote  $B_j = \mathbf{B}(x_j, h^{-1}n)$ ,  $B'_j = \mathbf{B}(x_j, \gamma h^{-1}n)$ ,  $j = 1, \dots, \kappa_h$ . For  $j = 1, \dots, \kappa_h$ , let  $\tilde{Z}^{(j),k}$ ,  $k \geq 1$  be the independent excursions between  $\partial B_j$  and  $\partial B'_j$  (again, with the initial points chosen according to  $\text{hm}_{B_j}$ ). Consider *independent* events

$$U_j = \left\{ B_j \text{ is not fully covered by } \tilde{Z}^{(j),1}, \dots, \tilde{Z}^{(j),\psi_{h^{-1}n,\beta/2}} \right\},$$

and let  $U = U_1 \cup \dots \cup U_{\kappa_h}$ ; then, (5) implies that

$$\mathbb{P}[U] \geq 1 - (1 - c_{\beta/2})^{\kappa_h}, \quad (7)$$

which can be made arbitrarily close to 1 by the choice of  $h$ . But, similarly to (6) of Step 1, we can argue (also using Lemma 2.10 of [4]) that with probability at least  $1 - c''_1 \kappa_h \exp(-c''_2(\ln \ln n)^2)$  the soft local time of real SRW's excursions up to time  $t_{m,\beta}$  is below the soft local time of the above independent excursions, meaning that  $\mathbf{B}(n)$  contains an uncovered site at time  $t_{m,\beta}$  with probability at least  $1 - (1 - c_{\beta/2})^{\kappa_h} - c''_1 \kappa_h \exp(-c''_2(\ln \ln n)^2)$ . One can then choose  $h = \ln \ln n$  (so that  $\kappa_h \asymp (\ln \ln n)^2$ ) to obtain that, for any fixed  $\beta > 0$ ,

$$\mathbb{P}[\mathbf{B}(n) \text{ is not fully covered by SRW on } \mathbb{Z}_m^2 \text{ at time } t_{m,\beta}] \rightarrow 1 \quad \text{as } n \rightarrow \infty. \quad (8)$$

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<sup>1</sup> $\tilde{\xi}$ 's are i.i.d. Exponential(1) random variables, used in the definition of the soft local time.

*Step 3.* Now, to obtain (2), we just repeat what was done at Step 1, but with (8) to hand (instead of the main cover time result of [1]). This concludes the proof of Proposition 2.  $\square$

We continue proving Theorem 1. Fix a large (integer)  $s > 0$  and a site  $x_s \in \mathbb{Z}^2$  such that  $\|x_s\| = s$  (for example,  $x_s = (s, 0)$ ); then, define  $B = \mathbf{B}(x_s, \frac{s}{\ln^2 s})$ ,  $B' = \mathbf{B}(x_s, e \frac{s}{\ln^2 s})$  (we took  $\gamma = e$  just to get rid of  $\ln \gamma$  terms in the formulas). In the following, RI stands for RI(1), and we remind the reader that the RI's excursions are governed by the conditioned SRW. Then, Lemma 2.7 of [4] implies that

$$\text{cap}(\{0\} \cup B) = \frac{2}{\pi} \ln s \times \left(1 + O\left(\frac{\ln \ln s}{\ln s}\right)\right), \quad (9)$$

and Lemma 2.6 of [4] implies that, for any  $x \in \partial B'$

$$\mathbb{P}_x[\widehat{\tau}(B) < \infty] = 1 - \frac{1}{\ln s} \left(1 + O\left(\frac{\ln \ln s}{\ln s}\right)\right), \quad (10)$$

where  $\widehat{\tau}(B)$  is the hitting time by the conditioned SRW. So, similarly to the argument in [4], the number  $N$  of RI's excursions between  $\partial B$  and  $\partial B'$  is compound Poisson with rate  $\pi \text{cap}(\{0\} \cup B) = 2 \ln s \times \left(1 + O\left(\frac{\ln \ln s}{\ln s}\right)\right)$  and (approximately) exponentially distributed summands of mean  $\ln s \times \left(1 + O\left(\frac{\ln \ln s}{\ln s}\right)\right)$ . Therefore, the expected number of these excursions is  $2 \ln^2 s \times \left(1 + O\left(\frac{\ln \ln s}{\ln s}\right)\right)$ , and, moreover, it is straightforward to argue that

$$\frac{N - 2 \ln^2 s}{2 \ln^{3/2} s} \xrightarrow{\text{law}} \text{standard Normal}. \quad (11)$$

The fact that typical deviations of the excursion count are rather large (of order of the mean to power  $\frac{3}{4}$ ) plays the key role in the infiniteness of the critical vacant set: it does not cost much to have a downward fluctuation in the number of RI's excursions between  $\partial B$  and  $\partial B'$  which makes it “subcritical”. Note that, from the above discussion it follows that

$$\mathbb{P}[N \leq 2 \ln^2 s - \ln^{3/2} s] \geq \frac{1}{4} \quad (12)$$

for all large enough  $s$ . Abbreviate  $m_0 = \frac{s}{\ln^2 s}$ . Next, again by means of soft local times, we construct a coupling of the RI excursions with (say)  $\psi^* := 2 \ln^2 m_0 - 3 \ln m_0 \ln \ln m_0$  independent ones (generated by a SRW with starting points chosen by  $\text{hm}_B$ ). Let us denote by  $\widehat{L}_k(\cdot)$  the soft local time generated by  $k$  RI's excursions (again, see Section 2.2 of [4] for definitions). First, observe that due to Lemma 3.3 (ii) of [6], one can successfully couple one SRW's excursion with one conditioned SRW's excursion started at the same site with probability at

least  $1 - O(\ln^{-3} s)$ . Since there are only  $O(\ln^2 s)$  of these, we see that it is possible to couple the “marks” (of the marked Poisson process) with high probability; so, we only need to couple the initial points now. Note that, due to Lemma 2.5 of [4],

$$\widehat{\text{hm}}_B^{B'}(y) = \text{hm}_B(y)(1 + O(\ln^{-3} s)), \quad (13)$$

for all  $y \in \partial B$ . Using the above with Lemma 2.9 of [4] (take  $\theta = \ln^{-3/4} s$  there), one obtains that

$$\mathbb{P}[\widehat{L}_{2\ln^2 s - \ln^{3/2} s}(y) \leq L_{\psi^*}(y) \text{ for all } y \in \partial B] \geq 1 - c \exp(-c' \ln^{1/2} s). \quad (14)$$

Therefore, by Proposition 2 and (12), RI's excursions leave a vacant site in  $B$  with probability at least  $\frac{1}{4} - c \exp(-c' \ln^{1/2} s)$ . The above shows that  $\mathcal{V}^1 \cap B \neq \emptyset$  with uniformly positive probability, as desired. This concludes the proof of Theorem 1 (under the assumption that (1) holds).  $\square$

**Remark 3.** *To prove the above result, another possible route would be using Theorem 2.6 of [3] together with Proposition 2: since we know that  $2\ln^2 s - C\ln^{3/2} s$  independent excursions do not cover  $B$  with probability close to 1, the same number of RI's excursions also will not do that at least with a constant probability.*

We are left with the task of proving (1). The event  $\{|\mathcal{V}^\alpha| = \infty\}$  looks like a tail event, in the sense that it is not affected by what happens in any finite region. Our intuition then says that a 0-1 law should hold for it; it is however not immediate to obtain such a law for the model of random interacements since even one trajectory can (and will) affect what happens in arbitrarily remote regions. In fact, one can still prove (1) in a direct way (basically, for this one needs to argue that a finite number of conditioned SRW's trajectories cannot cover almost all sites of a fixed infinite subset of  $\mathbb{Z}^2$ ), but we prefer to prove a more general 0-1 law, also for future reference. We remark also that [10, Section 2] contains a version of 0-1 law for the “classical” random interacements (i.e., in dimensions  $d \geq 3$ ); however, it makes use of the translational invariance property which is absent in two dimensions.

We need to introduce more notations. Consider the sequence  $\Lambda_n = \{x \in \mathbb{Z}^2 : n - 1 < \|x\| \leq n\}$  of disjoint and nonempty subsets of  $\mathbb{Z}^2$ ; we then have  $\mathbb{Z}^2 \setminus \{0\} = \bigcup_{n \geq 1} \Lambda_n$ . Note also that if an unbounded nearest-neighbor trajectory passes through  $\Lambda_{n_0}$ , then it has to pass through all  $\Lambda_n$  for  $n \geq n_0$  (i.e., the trajectories cannot overjump these subsets; this is because the norms of neighboring sites cannot differ by more than one unit). In addition, we denote  $\Theta_n = \bigcup_{j \geq n} \Lambda_j$ . Being  $\mathcal{I}$  an interlacement configuration (seen as a countable set of trajectories) and  $K$  a finite subset of  $\mathbb{Z}^2$ , we denote by  $\theta_K \mathcal{I}$  the set of trajectories obtained from  $\mathcal{I}$  by removing all trajectories that intersect  $K$  (i.e., we keep only the trajectories that are fully inside  $\mathbb{Z}^2 \setminus K$ ), and by  $\sigma_K \mathcal{I}$  the (a.s. finite) set of trajectories

that intersect  $K$ ; clearly,  $\mathcal{I} = \theta_K \mathcal{I} \cup \sigma_K \mathcal{I}$  and  $\theta_K \mathcal{I}$  is independent of  $\sigma_K \mathcal{I}$ . For finite  $K$ , we denote by  $|\sigma_K \mathcal{I}|$  the number of  $\mathcal{I}$ 's trajectories that intersect  $K$ ; note that  $|\sigma_K \mathcal{I}| \sim \text{Poisson}(\pi\alpha \text{cap}(\{0\} \cup K))$ .

Now, what is the correct way to define a tail event, for random interlacements? Informally, if one wants to know if such an event occurs, it is enough to look at the interlacement configuration outside of any finite set, i.e., it only depends on “what happens at infinity”. The precise definition of tail events depends on what exactly is meant by “interlacement configuration on a set” — one can just keep track of vacant/occupied sites, or the field of local times, or somehow keep track of (finite or infinite) pieces of trajectories that belong to that set (possibly even specifying which of these pieces are parts of the same infinite trajectory and in which order), etc. Here we adopt a rather general approach which still permits us to keep the notations relatively simple.

Let  $A$  be any subset of  $\mathbb{Z}^2 \setminus \{0\}$  and let  $I$  be a (discrete) interval in  $\mathbb{Z}$ , finite or infinite (i.e.,  $I = (a, b) \cap \mathbb{Z}$ , where  $a, b \in \mathbb{R} \cup \{\pm\infty\}$ ). A *noodle*  $w$  on  $A$  is simply a (finite or infinite) nearest-neighbor sequence of sites of  $A$  indexed by some interval  $I$ :  $w = (x_k, k \in I)$  such that  $x_k \sim x_{k+1}$  when  $k, k+1 \in I$ . Then, the interlacement configuration on  $A$  is the multiset (i.e., a collection without ordering but possibly with repetitions) of noodles generated by the interlacement trajectories in a natural way: if  $\varrho = (\varrho(k))_{k \in \mathbb{Z}}$  is a RI's trajectory, then the index set  $\{k : \varrho(k) \in A\}$  is uniquely represented as  $I_1 \cup I_2 \cup \dots$ , a (finite or infinite) union of nonadjacent discrete intervals. The noodles generated by that trajectory are  $w_1^{(\varrho)} = (\varrho(i), i \in I_1)$ ,  $w_2^{(\varrho)} = (\varrho(j), j \in I_2)$ , and so on; we then take the noodles generated by all the trajectories, and “store” them in the multiset. Note that, unless a noodle is indexed by  $\mathbb{Z}$  (i.e., it is a whole RI's trajectory), its first (last) site should be a neighbor of  $\mathbb{Z}^2 \setminus A$ . It is also clear that the interlacement configuration defined in this way is “informative enough”: having a multiset of noodles on  $A$ , one can figure out which sites of  $A$  are vacant, calculate the local times, etc. Also, we will use the following notation: if  $\mathcal{I}$  is the whole  $\text{RI}(\alpha)$  configuration (i.e., the set of all its trajectories), then  $\mathcal{I}(A)$  will denote the interlacement configuration (or “noodle configuration”) on  $A$ . Note also that  $\mathcal{I}(A) = \sigma_A \mathcal{I}(A)$ .

Now, let  $\mathcal{F}_n$  be the sigma-algebra generated by the cylinder events

$$\begin{aligned} &(\{\mathcal{I}(K) \in \mathcal{N}\}, K \text{ is a finite subset of } \Theta_n, \\ &\mathcal{N} \text{ is a set of finite noodle configurations on } K); \end{aligned}$$

define  $\mathcal{T} = \bigcap_{n \geq 1} \mathcal{F}_n$ , the sigma-algebra of tail events. The key result (which implies (1)) is:

**Theorem 4.** *If  $E \in \mathcal{T}$ , then  $\mathbb{P}[E] = 0$  or  $1$ .*

*Proof.* Some preparations are necessary. First, we need to recall an elementary result on the total variation distance  $\text{dist}_{TV}$  between Poisson distributions of dif-

ferent (but relatively close) rates, as well as between a Poisson distribution and its shifted version:

**Lemma 5.** *There exists a universal constant  $c$  such that, for all  $\lambda, h > 0$ ,*

$$\text{dist}_{TV}(\text{Poisson}(\lambda + h), \text{Poisson}(\lambda)) \leq c \frac{h}{\sqrt{\lambda}}. \quad (15)$$

Also, we have

$$\text{dist}_{TV}(\text{Poisson}(\lambda), \text{Poisson}(\lambda) + 1) \leq \frac{1}{2\sqrt{\lambda}}. \quad (16)$$

*Proof.* Being  $X \sim \text{Poisson}(\lambda)$ , write (using  $\mathbb{E}a^X = e^{\lambda(a-1)}$ )

$$\begin{aligned} 2 \text{dist}_{TV}(\text{Poisson}(\lambda + h), \text{Poisson}(\lambda)) &= \sum_{k=0}^{\infty} \left| e^{-(\lambda+h)} \frac{(\lambda+h)^k}{k!} - e^{-\lambda} \frac{\lambda^k}{k!} \right| \\ &= \mathbb{E} \left| e^{-h} \left(1 + \frac{h}{\lambda}\right)^X - 1 \right| \\ &\leq \sqrt{\mathbb{E} \left( e^{-h} \left(1 + \frac{h}{\lambda}\right)^X - 1 \right)^2} \\ &= \sqrt{e^{h^2/\lambda} - 1}, \end{aligned}$$

which implies (15). Similarly, we have

$$2 \text{dist}_{TV}(\text{Poisson}(\lambda), \text{Poisson}(\lambda) + 1) = \mathbb{E} \left| \frac{X}{\lambda} - 1 \right| \leq \frac{1}{\lambda} \sqrt{\mathbb{E}(X - \lambda)^2} = \frac{1}{\sqrt{\lambda}},$$

which shows (16). □

Next, we argue that the trajectories that pass close to the origin are “not important at infinity”, in the following sense:

**Lemma 6.** *Fix a finite  $K \subset \mathbb{Z}^2$  and let  $\mathcal{N}$  be a set of finite noodle configurations on  $K$ . Then, for any  $\varepsilon > 0$  there exists a coupling of two copies  $\mathcal{I}$  and  $\mathcal{J}$  of  $\text{RI}(\alpha)$  such that*

- (i)  $\mathcal{I}$  and  $\mathcal{J}(K)$  are independent;
- (ii) there exists (large enough)  $n$  such that

$$\mathbb{P}[\mathcal{I}(\Theta_n) = \mathcal{J}(\Theta_n) \mid \mathcal{J}(K) \in \mathcal{N}] \geq 1 - \varepsilon. \quad (17)$$



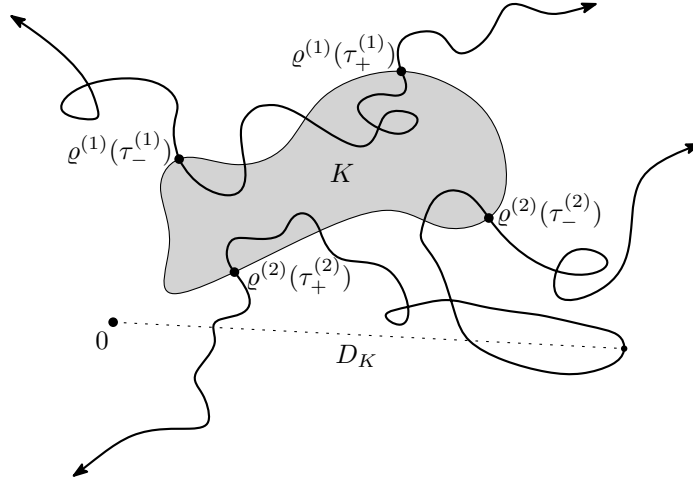


Figure 1: On the definition of the auxiliary random variables (here,  $\xi_K = 2$ ).

*Proof.* Let  $\xi_K = |\sigma_K \mathcal{J}|$  be the number of  $\mathcal{J}$ 's trajectories that intersect  $K$ . We denote these trajectories by  $\varrho^{(1)}, \dots, \varrho^{(\xi_K)}$ , where  $\varrho^{(j)} = (\varrho^{(j)}(m), m \in \mathbb{Z})$ . Let (see Figure 1)

$$\begin{aligned}\tau_-^{(j)} &= \min \{m : \varrho^{(j)}(m) \in K\}, \\ \tau_+^{(j)} &= \max \{m : \varrho^{(j)}(m) \in K\}\end{aligned}$$

be the times when the  $j$ th trajectory first enters  $K$  and leaves  $K$  for good; then,  $(\varrho^{(j)}(\tau_-^{(j)} - k), k = 0, 1, 2, \dots)$  and  $(\varrho^{(j)}(\tau_+^{(j)} + k), k = 0, 1, 2, \dots)$  are conditioned SRW's trajectories also conditioned on not re-entering  $K$ ; we will refer to these as *escape trajectories*. An important observation is that the escape trajectories are conditionally independent from the “inner part”  $(\varrho^{(j)}(m), j = 1, \dots, \xi_K, \tau_-^{(j)} \leq m \leq \tau_+^{(j)})$ : to obtain the interlacement configuration on  $K$ , it suffices to generate the above collection of finite pieces of trajectories; to obtain the whole trajectories of  $\sigma_K \mathcal{J}$ , one has to further run  $2\xi_K$  independent conditioned (on not hitting  $K \cup \{0\}$ ) random walks started at  $(\varrho^{(j)}(\tau_\pm^{(j)}), j = 1, \dots, \xi_K)$ , but this can be done at a later stage of the coupling's construction. Define (again, see Figure 1)

$$D_K = \max_{\substack{j=1, \dots, \xi_K \\ \tau_-^{(j)} \leq m \leq \tau_+^{(j)}}} \|\varrho^{(j)}(m)\|$$

to be the maximal distance from the origin achieved by these finite pieces of the “inner part”.

For a given  $\varepsilon > 0$ , let us choose  $m_0, \gamma_0$  in such a way that

$$\mathbb{P}[\xi_K \leq m_0, D_K \leq \gamma_0 \mid \mathcal{J}(K) \in \mathcal{N}] \geq 1 - \frac{\varepsilon}{2}. \quad (18)$$

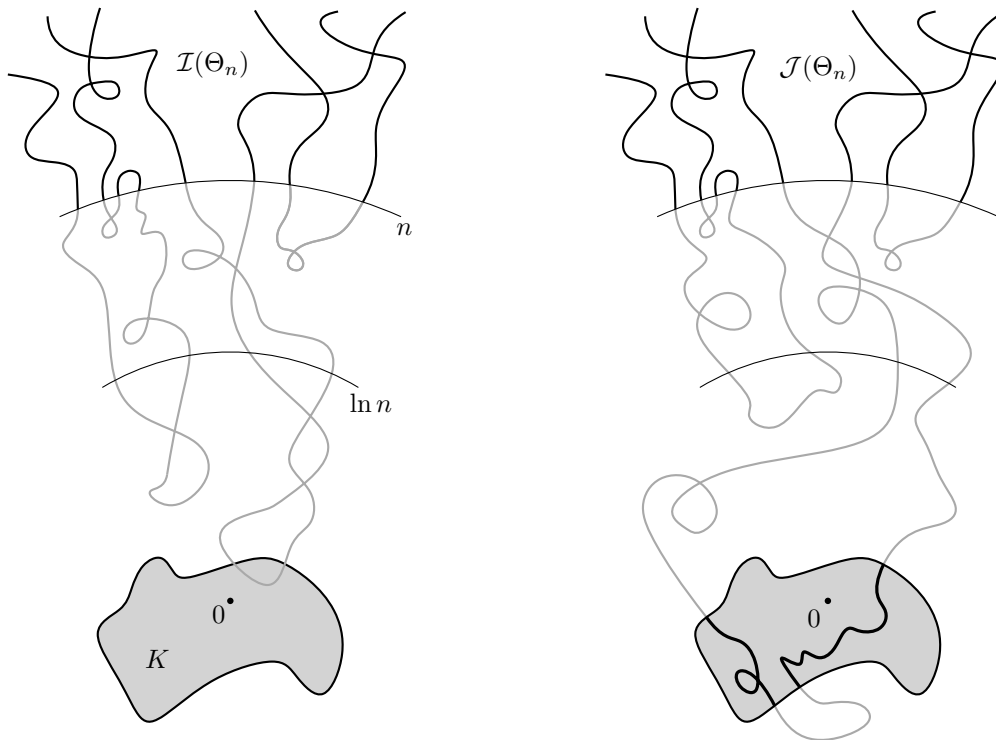


Figure 2: Construction of the coupling.

The idea of the proof is illustrated on Figure 2: we keep the trajectories outside  $\Lambda_{\ln n}$  the same in both the interlacement processes  $\mathcal{I}$  and  $\mathcal{J}$ , but (after obtaining the value of  $\xi_K$ ) resample those that intersect  $\Lambda_{\ln n}$  in such a way that, with high probability with respect to  $\mathbb{P}[\cdot \mid \mathcal{J}(K) \in \mathcal{N}]$ , the total numbers of such trajectories in  $\mathcal{I}$  and  $\mathcal{J}$  are equal; then, we couple these trajectories on their first entrances to  $\Theta_n$  and argue that, with high probability, these trajectories will remain coupled forever.

We now describe the construction in a more detailed way. First, we sample the “inner part” (see above) of the trajectories of  $\sigma_K \mathcal{J}$  in such a way that the event

$$\{\mathcal{J}(K) \in \mathcal{N}, \xi_K \leq m_0, D_K \leq \gamma_0\}$$

occurs; this happens with  $\mathbb{P}[\cdot \mid \mathcal{J}(K) \in \mathcal{N}]$ -probability at least  $1 - \frac{\varepsilon}{2}$  (so, from now on we can treat  $\xi_K$  as a fixed number not exceeding  $m_0$ ). Let  $n$  be such that  $K \subset \mathbf{B}(\ln n)$  and  $D_K \leq n - 1$ . Then, we need to couple the numbers of particles that touch  $\Lambda_{\ln n}$  in both processes; it is  $\text{Poisson}(\pi\alpha \text{cap}(\mathbf{B}(\ln n)))$  in  $\mathcal{I}$  and  $\xi_K + \text{Poisson}(\pi\alpha(\text{cap}(\mathbf{B}(\ln n)) - \text{cap}(\{0\} \cup K)))$  in  $\mathcal{J}$ . By Lemma 5 (note that  $\text{cap}(\mathbf{B}(\ln n)) \asymp \ln \ln n$ ), this coupling will be successful with probability at

least  $1 - O\left(\frac{m_0}{\sqrt{\ln \ln n}}\right)$ . So, given that the coupling is a success and abbreviating  $Y = |\sigma_{\Lambda_{\ln n}} \mathcal{I}|$ , we then have

- in  $\mathcal{I}$ , there are  $Y$  (independent) random walks originating somewhere at  $\Lambda_{\ln n}$  and conditioned on not hitting  $\{0\}$ , and  $Y$  random walks originating at  $\Lambda_{\ln n}$  and conditioned on not re-entering  $\Lambda_{\ln n}$ ;
- in  $\mathcal{J}$ , there are  $2\xi_K$  random walks (which are the “escape trajectories” discussed above) originating at  $\partial K$  and conditioned on not hitting  $\{0\} \cup K$ ,  $Y - \xi_K$  random walks originating at  $\Lambda_{\ln n}$  and conditioned on not hitting  $\{0\} \cup K$ , and  $Y - \xi_K$  random walks originating at  $\Lambda_{\ln n}$  and conditioned on not re-entering  $\Lambda_{\ln n}$ .

Next, by [7, Proposition 6.4.5] (which controls the conditional exit measure to the boundary of a large disk and can be used to obtain that all the above random walkers have essentially the same entrance measure to  $\Theta_n$ ), the probability of successfully coupling two conditioned random walkers originating in  $\mathbb{B}(\ln n)$  to have the common entry point to  $\Theta_n$  is  $1 - O\left(\frac{\ln n \ln \ln n}{n}\right)$  and we need to take care of only  $O(\ln \ln n)$  trajectories. Now, it remains to assure that with high probability these pairs of trajectories will remain coupled (note that they are conditioned on different things). For this, note that (for a random walk  $\widehat{S}$  conditioned on not hitting the origin) the probability of ever reaching  $\Lambda_{\ln n}$  starting somewhere at  $\Lambda_n$  is  $O\left(\frac{\ln \ln n}{\ln n}\right)$  (by, e.g., Lemma 3.4 of [6]), and we have to deal with  $O(\ln \ln n)$  walkers. Therefore, on arrival to  $\Theta_n$ , we can just use the same  $\widehat{S}$ -trajectories for each pair, and this will be successful with probability at least  $1 - O\left(\frac{(\ln \ln n)^2}{\ln n}\right)$ . Gathering the pieces, we obtain that the overall probability of success of the coupling (when the event in (18) occurs) is at least  $1 - \frac{cm_0}{\sqrt{\ln \ln n}}$ , for a large enough  $c$ . We can then choose  $n$  in such a way that  $\frac{cm_0}{\sqrt{\ln \ln n}} \leq \frac{\varepsilon}{2}$ , and this concludes the proof of Lemma 6.  $\square$

Now, we are ready to finish the proof of Theorem 4. Let  $E$  be a tail event. For a finite  $K \subset \mathbb{Z}^2$  and  $\mathcal{N}$  a set of noodle configurations on  $K$ , consider the two copies  $\mathcal{I}$  and  $\mathcal{J}$  of  $\text{RI}(\alpha)$  as in Lemma 6; let  $\mathcal{C}$  be the event that the coupling is successful. We can write

$$\mathbb{P}[\mathcal{I} \in E]$$

(since  $E$  is a tail event and by (i) of Lemma 6)

$$= \mathbb{P}[\mathcal{I}(\Theta_n) \in E \mid \mathcal{J}(K) \in \mathcal{N}]$$

(since, on  $\mathcal{C}$ , we have  $\mathcal{I}(\Theta_n) = \mathcal{J}(\Theta_n)$ )

$$= \mathbb{P}[\mathcal{J}(\Theta_n) \in E, \mathcal{C} \mid \mathcal{J}(K) \in \mathcal{N}] + \mathbb{P}[\mathcal{I}(\Theta_n) \in E, \mathcal{C}^c \mid \mathcal{J}(K) \in \mathcal{N}]$$

(again, since  $E$  is a tail event)

$$\begin{aligned} &= \mathbb{P}[\mathcal{J} \in E \mid \mathcal{J}(K) \in \mathcal{N}] - \mathbb{P}[\mathcal{J} \in E, \mathcal{C}^{\mathfrak{G}} \mid \mathcal{J}(K) \in \mathcal{N}] \\ &\quad + \mathbb{P}[\mathcal{I} \in E, \mathcal{C}^{\mathfrak{G}} \mid \mathcal{J}(K) \in \mathcal{N}], \end{aligned}$$

which means that, since  $\mathcal{J}$  is a copy of  $\mathcal{I}$  and by (17)

$$\left| \mathbb{P}[\mathcal{I} \in E] - \mathbb{P}[\mathcal{I} \in E \mid \mathcal{I}(K) \in \mathcal{N}] \right| \leq \mathbb{P}[\mathcal{C}^{\mathfrak{G}} \mid \mathcal{J}(K) \in \mathcal{N}] \leq \varepsilon.$$

Using that  $\varepsilon$  is arbitrary, we obtain that the events  $\{\mathcal{I} \in E\}$  and  $\{\mathcal{I}(K) \in \mathcal{N}\}$  are independent for any choice of (finite)  $K$  and  $\mathcal{N}$ . In a standard way,<sup>2</sup> Dynkin's  $\pi$ - $\lambda$  theorem then implies that the event  $\{\mathcal{I} \in E\}$  is independent of itself, and so its probability must be equal to 0 or 1. This concludes the proof of Theorem 4.  $\square$

**Remark 7.** *It is worth mentioning that all the discussion of this paper also applies to two-dimensional Brownian random interlacements [5]. In particular, one can define the tail events and prove the 0-1 law, and essentially the same proof of the fact that the  $s$ -interior of the critical vacant set  $\mathcal{D}_s(\mathcal{V}^1)$  is a.s. unbounded works in the continuous setting.*

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<sup>2</sup>Observe that all events independent to a given event form a  $\lambda$ -system, and  $(\{\mathcal{I}(K) \in \mathcal{N}\})$  is a  $\pi$ -system.

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