SINGULARITIES OF HOLOMORPHIC FOLIATIONS IN DIMENSIONS ≥ 3: RECENT RESULTS AND PROBLEMS

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Abstract. In dimensions greater than or equal to 3, the local structure of a singular holomorphic foliation conceals a globally defined foliation on the projective space of dimension one less. In this paper, we will investigate how the global dynamics of the latter foliation exerts influence on several problems that apparently have a purely local nature. In the course of the discussion, a few recent results and open problems in the area will be reviewed as well.

Contents

1. Introduction

All *foliations* considered in this work are *holomorphic and (possibly) singular*. Whereas our main object are 1-dimensional holomorphic foliations and holomorphic/meromorphic vector fields, foliations of codimension 1 will also play a role in the discussion especially when the ambient is of dimension 3, see Section 2 for accurate definitions. Foliations of dimension 1 defined on some complex manifold M will typically be denoted by $\mathcal{F}$ while $\mathcal{D}$ will stand for codimension 1 foliations. The purpose of this paper is to discuss recent results and open problems in the local theory of 1-dimensional foliations when the ambient manifold M has dimension 3 though, occasionally, results and questions in higher dimensions will also be included.

Foliations defined on complex surfaces, i.e. complex manifolds of dimension 2, are basically left aside in this paper largely due to the fact that their local theory is in a far more advanced stage than their higher dimensional counterparts. Indeed, these singularities are only mentioned in Section 2 and, yet, with the simple purpose of identifying a few issues that make them so special and amenable to very detailed analysis. In doing so, we will be able to single out one of the most fundamental issues guiding the discussion conducted here: the presence of a global dynamical phenomenon intrinsically attached to germs $\mathcal{F}$ of 1-dimensional foliations on $\mathbb{C}^n$ provided that $n \geq 3$. In slightly vague though more incisive words, the understanding of a germ of 1-dimensional foliation $\mathcal{F}$ on $\mathbb{C}^n$, $n \geq 3$, passes through the description of a foliation defined on $\mathbb{CP}^{n-1}$ which, as a global object, may exhibit a wild dynamical behavior (cf. Section 3). The global foliation in question will usually be referred to as the *core foliation* of the (local)

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foliation $\mathcal{F}$. We will also use the phrase *core dynamics of* $\mathcal{F}$ to refer to the dynamics of the core foliation associated with $\mathcal{F}$.

The common thread of this paper is the existence and implications of a global dynamical system inherently attached to the structure of a singularity of a 1-dimensional, holomorphic foliation defined on $(\mathbb{C}^n, 0)$ provided that $n \geq 3$. Basically, we will discuss which types of results can be proved if the above mentioned dynamics can accurately be described and, similarly, which general problems may provide us with the tools to ensure this dynamics is "tame enough" to be described, while bearing in mind that in full generality this dynamics can be extremely wild.

The paper is structured as follows. In Section 2, we introduce standard terminology and recall some basic features of singular foliations, in particular pointing out fundamental issues setting apart foliations of dimension 1 and foliations of codimension 1. Then we focus on the special case of singularities of foliations defined on $(\mathbb{C}^2, 0)$. Whereas this case is clearly distinguished by the fact that the foliations are simultaneously of dimension 1 and of codimension 1, we discuss to a rather non-trivial extent the main 2-dimensional issues allowing for the existence of such a sophisticated theory covering truly fine issues.

In Section 3, we introduce a fundamental object that exists for singularities of (1-dimensional) foliations defined on $(\mathbb{C}^n, 0)$ provided that $n \geq 3$, namely: the *core dynamics*. This is a global foliation defined on $\mathbb{CP}^{n-1}$ whose (global) dynamics poses a fundamental obstacle towards the full understanding of the initial singular point. In particular, we explain how this core dynamics plays a major role in problems about existence of separatrices for dicritical codimension 1 foliations on $(\mathbb{C}^3, 0)$. Also, we show how its very existence basically rules out any hope of achieving a full understanding of large classes of singular points.

The remainder of this survey is devoted to more advanced material, in particular touching on quite a few open problems of current interest. Section 4 contains a detailed review of resolution theorems for singular points of 1-dimensional foliations on $(\mathbb{C}^3, 0)$. The first definitive resolution theorem in this context was obtained by McQuillan and Panazzolo in [36] which, in turn, relies heavily on Panazzolo's algorithm introduced in [38]. More recently, a different proof based on Zariski general point of view was obtained in [48] which can be seen as the completion of the previous work carried out by Cano-Roche-Spivakovsky [9]. Despite the undisputed importance of resolution theorem, it seems these results are still not as widely known as one would expect and, for this reason, we thought useful to conduct a thorough discussion about the content of the resolution theorems in [36] and in [48], highlighting virtues and potential limitations.

In Section 5, we consider the fundamental problem of existence of separatrices that, roughly speaking, concerns the existence of germs of analytic sets invariant by (germs of) singular foliations. The discussion is essentially conducted in $(\mathbb{C}^3, 0)$. Expanding on the discussion of Section 3, we consider the existence of separatrices for codimension 1 foliation spanned by two commuting vector fields and state Theorem 5.3 affirmatively answering this question. We also detail the general strategy for proving this theorem which, in turn, emphasizes a few often overlooked points in resolution theorems for foliations as well as the major role played by the general question of "taming a core dynamics". The second part of this section, we review some results on the existence of separatrices for foliations of dimension 1. The nature of this second problem is far more topological/geometric and "core dynamics" plays a much smaller role. Yet, some of the results will find applications in the last section.

Finally, in Section 6 we discuss a particular class of singularities of foliations of dimension 1, namely those supporting *semicomplete vector fields*. Albeit small in an appropriate sense, this class of singularities has rather distinguished properties and quite a few applications that make it worth studying. The section will precisely begin with proper definitions and a general discussion of applications. Once the basic setting is in place, we will go on to discuss two

fundamental problems on the area: the first problem can vaguely be stated by asking *how wild can the core dynamics be in this class of foliations*? The main results here stem from the seminal paper of A. Guillot about Halphen vector fields and their role in $\mathrm{SL}(2,\mathbb{C})$-actions, see [17]. The second problem aims at quantifying how "degenerate" a singularity in this class can be. This second problem stems from a well known question raised long ago by E. Ghys and the topic has applications in the study of automorphism groups of compact complex manifolds.
Acknowledgments. We are grateful to many of our colleagues for several discussion over the years. A special thanks is due to D. Panazzolo who has helped us to understand many subtle points of his fundamental work on desingularization theorems. We also thank J.-F. Mattei for many discussions and explanations about the vast and fundamental work in singularity theory he has accomplished with his collaborators.

2. Basics in the local theory of foliations and the special case of dimension 2

It is convenient to begin by recalling the definition of 1-dimensional singular holomorphic foliation. First, let $X = f_1\partial/\partial x_1 + \cdots + f_n\partial/\partial x_n$ be a non-trivial holomorphic vector field defined on an open set V of $\mathbb{C}^n$. The singular set $\mathrm{Sing}\,(X)$ of X is then given by $\bigcap_{i=1}^n \{f_i = 0\}$. It is a (proper) analytic subset of V and it is well known that $\mathrm{Sing}\,(X)$ has codimension 1 if and only if the coordinate functions f_i admit a non-trivial common factor.

We are then able to define singular holomorphic foliations as they will be considered throughout this work. Let M be a complex manifold and consider a covering $\{(U_k, \varphi_k)\}$ of M by coordinate charts. We denote by n the dimension of M so that $\varphi_k(U_k)$ is an open set of $\mathbb{C}^n$.

Definition 2.1. Let M and $\{(U_k, \varphi_k)\}$ be as above. A singular holomorphic foliation $\mathcal{F}$ of dimension 1 on M consists of a collection of holomorphic vector fields Y_k satisfying the following conditions:

- For every k, Y_k is a holomorphic vector field defined on $\varphi_k(U_k) \subset \mathbb{C}^n$ whose singular set has codimension at least 2.
- Whenever $U_{k_1} \cap U_{k_2} \neq \emptyset$, we have $Y_{k_1} = g_{k_1k_2}.(\varphi_{k_2} \circ \varphi_{k_1}^{-1})^* Y_{k_2}$ for some nowhere vanishing holomorphic function $g_{k_1k_2}$.

The *singular set* $\mathrm{Sing}\,(\mathcal{F})$ of a foliation $\mathcal{F}$ is then defined as the union over k of the sets $\varphi_k^{-1}(\mathrm{Sing}\,(Y_k)) \subset M$. Therefore the singular set of any holomorphic foliation has codimension at least two. In particular, contrasting with the case of vector fields, a foliation *has no divisor of zeros*. The accurate content of this assertion can be explained as follows. If X is a vector field on M, then the set $\{X = 0\}$ formed by all points where X vanishes may contain codimension 1 irreducible components. Furthermore, the *vanishing order of X* at any of these irreducible components is well defined so that the data consisting of the codimension 1 irreducible components of $\{X = 0\}$ counted with the multiplicity given by the corresponding vanishing order of X naturally defines a divisor on M. However, there is no analogue of this notion for a foliation since the only possible notion of zero-set for a foliation coincides with its singular set which has codimension at least 2. Similarly, there is no relevant notion of "meromorphic foliation" since Definition 2.1 requires the foliation to be holomorphic away from an analytic set of codimension at least 2 which forces the foliation to be holomorphic over the entire manifold where it is defined.

Conversely, we say that a holomorphic vector field Y is a *local representative* of the 1-dimensional foliation $\mathcal{F}$ if Y is tangent to $\mathcal{F}$ and the singular set of Y has codimension at least 2. It is clear that representative vector fields are locally unique up to multiplication by an invertible holomorphic function.

Analogously, we might also consider a differential 1-form ω on $V \subseteq \mathbb{C}^n$, $\omega = g_1 dx_1 + \cdots + g_n dx_n$. Again the singular set $\text{Sing}\,(\omega)$ of ω is given by the intersection $\bigcap_{i=1}^n \{g_i = 0\}$ and it is an analytic set of V which has codimension 1 if and only if there is a non-trivial common factor for the functions $g_1, \ldots, g_n$. Away from its singular points, the kernel of ω defines a distribution of complex hyperplanes on V. If in Definition 2.1 we replace "local vector fields" by "integrable local 1-forms", we obtain the notion codimension 1 foliations. More precisely, we have:

Definition 2.2. Let M be a complex manifold and $\{(U_k, \varphi_k)\}$ a covering of M by coordinate charts. A singular holomorphic foliation $\mathcal{D}$ of codimension 1 on M consists of a collection of differential 1-forms Ω_k satisfying the following conditions:

- For every k, Ω_k is a differential 1-form defined on $\varphi_k(U_k) \subset \mathbb{C}^n$ with singular set of codimension at least 2 and such that $\Omega_k \wedge d\Omega_k$ vanishes identically.
- Whenever $U_{k_1} \cap U_{k_2} \neq \emptyset$, we have $\Omega_{k_1} = g_{k_1 k_2}.(\varphi_{k_2} \circ \varphi_{k_1}^{-1})^* \Omega_{k_2}$ for some nowhere vanishing holomorphic function $g_{k_1 k_2}$.

In particular the singular set $\text{Sing}\,(\mathcal{D})$ of $\mathcal{D}$ again has codimension at least 2. The notion of *representative* 1-*form* for a codimension 1 foliation $\mathcal{D}$ is defined analogously to the notion of representative vector fields in the case of foliations with dimension 1.

Whereas our main focus here is on germs of foliations, or in slightly more concrete terms, on foliations defined on a neighborhood of the origin in $\mathbb{C}^n$, the reader will notice that the global point of view considered in Definitions 2.1 and 2.2 is really indispensable to investigate the local structure of the singular point. Indeed, globally defined foliations - and in particular the "global dynamical phenomenon" mentioned in the Introduction - will come to fore in the context of birational theory of foliations which is needed, for example, if one seeks to "resolve singular points".

It is also convenient to complement the above definitions with a couple of comments.

Remark 2.3. Already on $(\mathbb{C}^n, 0)$, $n \geq 3$, it is easy to see a first fundamental difference between foliations of dimension 1 and foliations of codimension 1 arising from the Frobenius condition. To formulate this, note that any choice of local holomorphic functions $f_1, \ldots, f_n$ naturally gives rise to two (singular) distributions: one of lines and one of hyperplanes. In fact, to the collection of functions $f_1, \ldots, f_n$, we may associate the vector field $Y = f_1 \partial/\partial x_1 + \cdots + f_n \partial/\partial x_n$ or the 1-form $\Omega = f_1 dx_1 + \cdots + f_n dx_n$. Whereas the local integral curves of Y always yield a 1-dimensional foliation $\mathcal{F}$, the Frobenius condition for Ω to yield a codimension 1 foliation is non-trivial and amounts to requiring the 3-form $\Omega \wedge d\Omega$ to vanish identically which, in turn, leads to a highly non-trivial PDE system involving the functions $f_1, \ldots, f_n$.

Remark 2.4. In general, foliations of dimension 1 are very abundant, at least in algebraic manifolds, and they may have an extremely complicated dynamical behavior, more on this in Section 3, see also [24], [27]. This contrasts with the case of codimension 1 foliations that are far more rigid and in several cases amenable to classification, at least at conjectural level, all codimension 1 foliations on, say, $\mathbb{CP}^n$ should be transversely homogeneous or can be obtained as a suitable pull-back of a foliation defined on a surface. For an interesting discussion of several global aspects of codimension 1 foliations, we refer the reader to [57].

A basic object in the local theory of foliations that has largely motivated its early development is the notion of *separatrix*. Although the definition of *separatrix* depends on the dimension of the foliation, the cases of foliations of dimension 1 and of codimension 1 can naturally be formulated together.

Definition 2.5. Let $\mathcal{F}$ (resp. $\mathcal{D}$) be a foliation of dimension 1 (resp. codimension 1) on $(\mathbb{C}^n, 0)$. A separatrix S for $\mathcal{F}$ (resp. $\mathcal{D}$) is the germ of an irreducible analytic set of dimension 1 (resp. codimension 1) containing $0 \in \mathbb{C}^n$ and invariant by $\mathcal{F}$ (resp. $\mathcal{D}$).

Separatrices are objects of natural interest since they fit the framework of "invariant manifolds" in dynamical systems. In particular, up to (Hironaka) desingularization, their presence allows us to consider leaves that (locally) are contained in a manifold of smaller dimension and, hence, can be easier to understand. For example, a separatrix for a foliation of dimension 1 yields a very special leaf which coincides with an analytic curve and hence can be captured by algebraic/analytic methods that are eluded by other leaves with complicate dynamical behavior. Some very concrete applications of theorems related to existence of separatrices can be found in Section 6. This said, in the local theory of foliations as discussed here, the notion of separatrix first appeared in the classical work of Briot and Bouquet [1] that established the existence of separatrices in several non-trivial cases. Much later, R. Thom has sought to generalize the existence of separatrices for codimension 1 foliations. The first example of a codimension 1 foliation on $(\mathbb{C}^3, 0)$ without separatrix was, however, found by J.-P. Jouanolou [25] and its likely that his example was motivated by Thom's question. As it will be seen in the next section, Jouanolou's counterexample relies on the core dynamics of certain 1-dimensional foliations on $(\mathbb{C}^3, 0)$. Let us also point out that the existence of separatrices for foliations on $(\mathbb{C}^2, 0)$ was established as a general phenomenon by Camacho and Sad in [4].

Another fundamental notion in the theory of singularities of foliations is the notion of *eigenvalues of a foliation at a singular point*. For the discussion conducted in this paper, it suffices to consider *foliations of dimension* 1. Since the notion is local, we just need to consider foliations of dimension 1 defined on $(\mathbb{C}^n, 0)$. Given $\mathcal{F}$ as above, up to reducing the neighborhood of the origin, there is a holomorphic vector field Y whose zero-set has codimension at least 2 and such that $\mathcal{F}$ is nothing but the foliation induced by the local orbits of Y. As mentioned, Y is said to be a representative of $\mathcal{F}$ and, while Y is not unique, two representative vector fields for the same foliation $\mathcal{F}$ differ by multiplication by an invertible holomorphic function.

Definition 2.6. Let $\mathcal{F}$ be a 1-dimensional holomorphic foliation on $(\mathbb{C}^n, 0)$ and let Y denote a representative vector field for $\mathcal{F}$. Assume that $\mathcal{F}$ is singular at the origin, i.e., $Y(0) = 0$. Then the eigenvalues of $\mathcal{F}$ at $0 \in \mathbb{C}^n$ are the eigenvalues of the Jacobian matrix D_0Y.

Since Y is well defined up to multiplication by an invertible holomorphic function, there follows that the eigenvalues of $\mathcal{F}$ at $0 \in \mathbb{C}^n$ are well defined only up to simultaneous multiplication by a non-zero constant.

2.1. **Blow-ups and dicritical singularities.**

Blow-ups are a standard tool to produce non-trivial birational maps and to understand the local structures of singular points, whether these are "singularities of the ambient space" or "singularities of a foliation on a smooth space". The transform of a foliation under a blow-up map is called the *blow-up* of the foliation. The blown-up space, however, contains an exceptional divisor which may or may not be invariant by the transformed foliation. This issue gives rise to the notion of *dicritical foliation at a singular point*.

Definition 2.7. Let M be a complex manifold equipped with a holomorphic foliation $\mathcal{F}$ of dimension 1 and consider a blow-up map $\pi : \widetilde{M} \to M$ centered at $C \subset \mathrm{Sing}(\mathcal{F})$, where $\mathrm{Sing}(\mathcal{F})$ stands for the singular set of $\mathcal{F}$. The foliation $\mathcal{F}$ is said to be dicritical with respect to π if its corresponding blow-up $\widetilde{\mathcal{F}}$ does not leave the exceptional divisor $\pi^{-1}(C)$ invariant.

Analogously, if $\mathcal{D}$ is a codimension 1 foliation on M and $\pi : \widetilde{M} \to M$ is a blow-up map centered at $C \subset \mathrm{Sing}(\mathcal{D})$, then $\mathcal{D}$ is said to be dicritical with respect to π if its blow-up $\widetilde{\mathcal{D}}$ does not leave the exceptional divisor $\pi^{-1}(C)$ invariant.

Whenever no misunderstanding is possible, we will simply say that a given foliation is, or is not, dicritical without specifically mentioning to the blow-up map. Also, for most of our discussion, it will suffice to consider blow-ups centered at single points (sometimes called one-point blow-ups). For this type of blow-up, the characterization of 1-dimensional dicritical foliations is very simple. More precisely, let $\mathcal{F}$ be a 1-dimensional foliation on $(\mathbb{C}^n, 0)$ and fix a representative vector field Y of $\mathcal{F}$. Denote by Y_k the non-zero homogeneous component of least degree in the Taylor series of Y based at $0 \in \mathbb{C}^n$. Then, we have:

Lemma 2.8. *The foliation $\mathcal{F}$ is dicritical with respect to the blow-up centered at $0 \in \mathbb{C}^n$ if and only if Y_k is a multiple of the Radial vector field $R = x_1\partial/\partial x_1 + \cdots + x_n\partial/\partial x_n$.*

Proof. It suffices to compute the pull-back of Y in the coordinates $(x_1, u_2, \ldots, u_n)$ for the blow-up of $\mathbb{C}^n$ where the blow-up map π is given by $\pi(x_1, u_2, \ldots, u_n) = (x_1, x_1u_2, \ldots, x_1u_n)$, cf. for example [23] (page 121) or [42] (page 74). □

In more general terms, a foliation $\mathcal{F}$ is said to be *dicritical* at a center C if there exists a sequence of blow-ups beginning at C and leading to a foliation which does not leave all the irreducible components of the global exceptional divisor invariant. Now, by considering a parameterization for a leaf of the foliation $\widetilde{\mathcal{F}}$ transverse to (a component of) the exceptional divisor and the projection of this parameterization, it follows that the mentioned leaf of $\widetilde{\mathcal{F}}$ projects to a *separatrix* for the initial foliation $\mathcal{F}$. Thus, we derive the following property of 1-dimensional dicritical foliations:

(for details see Section 4). Since a blow-up map is proper, and therefore so is a composition of blow-up maps, there follows from Remmert's theorem that a leaf of the foliation $\widetilde{\mathcal{F}}$ transverse to (a component of) the exceptional divisor must project to a separatrix for the initial foliation $\mathcal{F}$. Thus we derive the following property of 1-dimensional dicritical foliations:

Lemma 2.9. *If the foliation $\mathcal{F}$ at the center C is dicritical, then the union of separatrices of $\mathcal{F}$ through points of C yields a set with non-empty interior.* □

The converse to Lemma 2.9 is known to hold for ambient spaces of dimension up to 3, and it is a simple consequence of "resolution theorems", cf. Section 4. Whereas the result is likely to hold in general, a proof of this statement dispensing with "resolution" results seems to still be lacking in the literature.

Remark 2.10. In general, and regardless of the dimension of the foliation, a separatrix for a blown-up foliation always project to a separatrix for the initial foliation. This is, indeed, a consequence of Remmert's theorem since every blow-up map is proper, and therefore any composition of blow-up maps is proper as well.

The above lemmas show that 1-dimensional dicritical foliations are, somehow, *very special*. In particular, a "generic foliation" is not dicritical at their singular points. Also, owing to Lemma 2.9, for most of the problems discussed in this paper involving 1-dimensional foliations, we can assume without loss of generality that the foliation in question is not dicritical.

Remark 2.11. On $(\mathbb{C}^2, 0)$, where all foliations are of dimension 1 and of codimension 1, all examples of dicritical foliations stem from foliations represented by vector field whose Taylor series begins with a vector field that is a multiple of radial vector field on $\mathbb{C}^2$. In other other, if to abridge notation we restrict ourselves to *homogeneous foliations*, i.e. by foliations invariant under the radial vector field, the only examples of dicritical foliations are those provided by the actual multiples of this radial vector field.

In this sense, it is interesting to point out that there are many more examples of *codimension* 1 foliation in ambient spaces of *dimension* 3. Indeed, every homogeneous polynomial vector field X on $\mathbb{C}^3$ *that is not* a multiple of the radial vector field yields an example of a homogeneous codimension 1 foliation. In fact, the homogeneous vector field X and the radial vector field R on $\mathbb{C}^3$ generate the Lie algebra of the affine group of $\mathbb{C}$ so that the plane distribution on $\mathbb{C}^3$ spanned by X and by R is integrable, see Section 3 for detail. By construction the resulting codimension 1 foliation $\mathcal{D}$ is dicritical at the origin.

It is, however, not clear if there is any reasonable sense in claiming that a "generic foliation of codimension 1 on $(\mathbb{C}^3, 0)$" is not dicritical. This happens because the integrability condition of Frobenius - which is automatically satisfied for line distrubtions - becomes very non-trivial once we are dealing with distributions of 2-planes. More precisely, whereas there is probably a way to make sense of the following loose assertion "a generic distribution of 2-planes on $(\mathbb{C}^3, 0)$ is non-dicritical", the problem of deciding how many of these distributions actually give rise to a codimension 1 foliation is a very subtle one.

2.2. **Singularities of foliations on** $(\mathbb{C}^2, 0)$**.** As mentioned, singularities of foliations on $(\mathbb{C}^2, 0)$ are the object of a highly developed theory, at least in the very general setting of *non-dicritical foliations*. In this paragraph, we shall collect some reasons that allowed so much progress in this topic and compare them with the situation of foliations on $(\mathbb{C}^3, 0)$.

(A) Seidenberg theorem. It is commonly accepted that no general theorem in singularity theory can be proved without relying on a suitable "desingularization theorem". In the theory of foliations, however, it is not possible in general to actually *desingularize a foliation*, i.e., to obtain a non-singular model of the foliation up to birational transformations. In fact, whereas the phrase *desingularization theorem* is sometimes used as an abuse of language, a more accurate terminology would be *reduction of singularities theorem*. In other words, rather than looking for a non-singular foliation, we look for a foliation whose singular points are as "well behaved as possible". Typically, we will look for a foliation all of whose singular points are *elementary*, i.e., all of them have *at least one eigenvalue different from zero*.

Seidenberg theorem [54] provides a suitable procedure to reduce the singularities of holomorphic foliations on $(\mathbb{C}^2, 0)$. Let $\mathcal{F}$ denote a singular holomorphic foliation defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$. Seidenberg theorem asserts the existence of a finite sequence of blow-up maps, along with transformed foliations $\mathcal{F}_i$ $(i = 1, \ldots, n)$

$$\mathcal{F} = \mathcal{F}_0 \stackrel{\pi_1}{\longleftarrow} \mathcal{F}_1 \stackrel{\pi_2}{\longleftarrow} \cdots \stackrel{\pi_n}{\longleftarrow} \mathcal{F}_n$$

such that the following holds:

- Each blow-up map π_i $(i = 1, \ldots, n)$ is centered at a singular point of $\mathcal{F}_{i-1}$.
- All singular points of $\mathcal{F}_n$ are elementary, i.e., the foliation $\mathcal{F}_n$ possesses at least one eigenvalue different from zero at each of them.

Denote by $D_1, \ldots, D_n$ the irreducible components of the total exceptional divisor associated with $\mathcal{F}_n$. Each D_i is therefore a rational curve with strictly negative self-intersection and the corresponding dual graph is a tree.

(B) A global pseudogroup - Mattei-Moussu technique. Assume next that the foliation $\mathcal{F}$ is not dicritical. Then, for each $i = 1, \ldots, n$, $D_i \setminus \mathrm{Sing}\,(\mathcal{F}_n)$ is a regular leaf of $\mathcal{F}_n$, where $\mathrm{Sing}\,(\mathcal{F}_n)$ stands for the singular set of $\mathcal{F}_n$. In particular, all non-trivial dynamics associated with the foliation $\mathcal{F}_n$ is of transverse nature. Moreover this transverse dynamics naturally arises from the holonomy representations of each of the leaves $D_i \setminus \mathrm{Sing}\,(\mathcal{F}_n)$, $i = 1, \ldots, n$. In turn, at least to a considerable extent, the dynamics of these representations can be merged together through the argument of "passage of corners" (a.k.a. "Dulac transform"), whenever $D_i \cap D_j \neq \emptyset$.

The preceding can be summarized by saying that all singular points of $\mathcal{F}_n$ are "dynamically connected" in the sense that their local dynamics blend together in a nice pseudogroup of maps of $(\mathbb{C}, 0)$. Furthermore, the dynamics of this pseudogroup encodes virtually all the information on the local structure of the initial foliation $\mathcal{F}$.

The method described above to investigate the singularities of foliations on $(\mathbb{C}^2, 0)$ was very much set up in the seminal paper by Mattei and Moussu [34]. This technique has proven time and again to be extremely effective in a variety of situations in dimension 2, see [4] and [19] for two examples of problems whose solutions have involved this type of setup. In the next section we will discuss how far this approach can be generalized to higher dimensions.

(C) Dynamics of pseudogroups acting on $(\mathbb{C}, 0)$. Although for many problems this issue plays a relatively minor role, let us still point out that the dynamics of pseudogroups acting on $(\mathbb{C}, 0)$ is itself a highly developed topic. This type of dynamics was first investigate by Huddai-Verenov [22] and then by Il'yashenko in [24] where a "generic situation" of groups generated by hyperbolic diffeomorphisms was considered. In contrast, in [34], the authors have dealt with subgroups all of whose orbits are finite. An absolute breakthrough then came with the works of Shcherbakov and of Nakai about general non-solvable subgroups, see [52], [53], [37]. The reader may consult [41] and references therein for a more complete account of these dynamics in the non-solvable case whereas solvable pseudogroups are discussed in detail in [12].

Remark 2.12. It should be pointed out that much progress in terms of construction of moduli spaces for foliations on $(\mathbb{C}^2, 0)$ and in describing the topology of leaves has been made in recent years, chiefly by Marín, Mattei, and Salem. While these aspects will not be discussed in this survey which is mostly devoted to higher dimensional situations. Yet, the reader interested in the topology of leaves will find more up-to-date information in [29], [30], and [56]. As to the construction of moduli spaces, we refer to [31] and to the preprints [32] and [33].

3. Splitting the problem: core dynamics and resolution

The main object of this section are 1-dimensional foliations defined around the origin of $\mathbb{C}^n$, for $n \geq 3$. Moreover, most of the discussion can be conducted without loss of generality in the case $n = 3$.

In light of the undeniable success of the strategy described in Section 2.2 in the study of singularities of foliations on $(\mathbb{C}^2, 0)$, it is natural to envisage a similar point of view for 1-dimensional foliations $\mathcal{F}$ defined on $(\mathbb{C}^n, 0)$ or, simply, on $(\mathbb{C}^3, 0)$. The first ingredient would of course be a reduction of singularities procedure to replace the foliation $\mathcal{F}$ by a birationally equivalent foliation having only "simple singular points", up to properly formulating a notion of "simple singular point". Whereas such procedure is an undisputed necessity to develop a comprehensive theory, for $n \geq 3$ there is a new phenomenon, which has global nature and no analogue on $(\mathbb{C}^2, 0)$, that basically rules out by itself any possibility of achieving a fully general description of the singularities in question, regardless of desingularization issues. This phenomenon is called here "core dynamics" and its existence and its little connection with desingularization difficulties suggests that the problem of describing the structure of a foliation $\mathcal{F}$ as above around the origin should be split into two parts. Namely, we need to be able to reduce its singular point by means of suitable blow ups but we also need to understand the core dynamics, roughly speaking the dynamics of the foliation restricted to the invariant divisors.

Once "core dynamics" is controlled and singularities can be reduced, then we are in good shape to work out a detailed descrition of the full structure of initial singular point.

It is useful to begin by recalling some well known facts about foliations on complex projective spaces. Let $\mathbb{CP}^n$ be viewed as the space of Radial lines through the origin of $\mathbb{C}^{n+1}$ and denote by $\Pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ the canonical projection. Also, for $\lambda \in \mathbb{C}^*$, denote by $h_\lambda : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ the homothety defined by $h_\lambda(x_1, \ldots, x_{n+1}) = (\lambda x_1, \ldots, \lambda x_{n+1})$. Finally let R denote the Radial vector field $R = x_1\partial/\partial x_1 + \cdots + x_{n+1}\partial/\partial x_{n+1}$ and consider a homogeneous polynomial vector field

$$X = P_1 \frac{\partial}{\partial x_1} + \cdots + P_{n+1} \frac{\partial}{\partial x_{n+1}}$$

of degree d on $\mathbb{C}^{n+1}$. In other words, each P_i is a degree d homogeneous polynomial, for every $i = 1, \ldots, n+1$. In what follows X is always assumed to satisfy the following conditions:

(1) The singular set of X on $\mathbb{C}^{n+1}$ has codimension at least 2.
(2) The vector fields X and R are linearly independent at generic points.

Next note that we have

$$h_\lambda^* X = \lambda^{d-1} X$$

so that the vector fields $h_\lambda^* X$ and X are everywhere parallel for any fixed value of $\lambda \in \mathbb{C}^*$. In particular, if $p \in \mathbb{C}^{n+1}$ is a point at which $X(p)$ and $R(p)$ are linearly independent, then $X(p)$ induces a direction in $T_{q=\Pi(p)}\mathbb{CP}^n$ which is well defined in the sense that it does not depend on $p \in \Pi^{-1}(q)$. From this, it easily follows that X induces a singular holomorphic foliation $\mathcal{F}$ on $\mathbb{CP}^n$ in the sense of Definition 2.1. A standard application of Serre's GAGA principle yields a type of converse for the above construction, namely the following proposition holds, cf. for example [23], [42].

Proposition 3.1. *Let $\mathcal{F}$ denote a singular holomorphic foliation on $\mathbb{CP}^n$. Then, there exists a homogeneous polynomial vector field X on $\mathbb{C}^{n+1}$ having singular set of codimension at least 2 and inducing the foliation $\mathcal{F}$ on $\mathbb{CP}^n$ by means of the above described construction.* □

Whereas, given $\mathcal{F}$, the mentioned homogeneous vector field X of Proposition 3.1 is not uniquely defined, two homogeneous polynomial vector fields having singular set of codimension at least 2 and inducing the same foliation on $\mathbb{CP}^n$ must have the same degree. Thus we can talk about the *degree of a foliation on* $\mathbb{CP}^n$ as follows:

Definition 3.2. The degree of a foliation $\mathcal{F}$ on $\mathbb{CP}^n$ is the degree of a homogeneous polynomial vector field on $\mathbb{C}^{n+1}$ having singular set of codimension at least 2 and inducing $\mathcal{F}$ in $\mathbb{CP}^n$ viewed as the space of Radial lines of $\mathbb{C}^{n+1}$.

Naturally blow-ups provide an alternative way to realize the foliation induced on $\mathbb{CP}^n$ by a homogeneous polynomial vector field X on $\mathbb{C}^{n+1}$. Let $\widetilde{\mathbb{C}}^{n+1}$ stand for the blow-up of $\mathbb{C}^{n+1}$ at the origin and consider a homogeneous vector field X as above on $\mathbb{C}^{n+1}$. The blow-up $\widetilde{X}$ of X induces on $\widetilde{\mathbb{C}}^{n+1}$ the blow-up $\widetilde{\mathcal{F}}$ of the the foliation $\mathcal{F}$ induced by X on $\mathbb{C}^n$. Since, by assumption, X is not everywhere parallel to the Radial vector field R, there follows that the foliation $\widetilde{\mathcal{F}}$ leaves invariant the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{CP}^n$. The restriction of $\widetilde{\mathcal{F}}$ to the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{CP}^n$ can then naturally be identified with the foliation induced by X on $\mathbb{CP}^n$ - viewed as the space of Radial lines of $\mathbb{C}^{n+1}$ - by means of the preceding construction.

Note that the blow-up construction does not really requires the vector field to be homogeneous. In fact, as in Lemma 2.8, consider a holomorphic vector field Y defined around the origin of $\mathbb{C}^{n+1}$ whose Taylor series takes on the form $Y = \sum_{i=k}^{\infty} Y_i$, where Y_i stands for the homogeneous component of degree i of this Taylor series and Y_k is not identically zero. As in Lemma 2.8, we assume that Y_k is not everywhere parallel to the Radial vector field R. The blow-up of Y induces a holomorphic foliation $\widetilde{\mathcal{F}}$ on a neighborhood of the exceptional divisor $\pi^{-1}(0) \subset \widetilde{\mathbb{C}}^{n+1}$. Moreover, since Y_k is not a multiple of R, this foliation leaves $\pi^{-1}(0) \simeq \mathbb{CP}^n$ invariant and, in addition, it is immediate to check that the restriction of $\widetilde{\mathcal{F}}$ to $\pi^{-1}(0)$ coincides with the restriction to $\pi^{-1}(0)$ of the foliation induced on $\widetilde{\mathbb{C}}^{n+1}$ by the blow-up of Y_k (alone). In particular, the restriction of $\widetilde{\mathcal{F}}$ to $\pi^{-1}(0)$ is identified with the foliation induced by the homogeneous vector field Y_k on $\mathbb{CP}^n$ viewed as the space of lines of $\mathbb{C}^{n+1}$.

The preceding motivates the following definition.

Definition 3.3. Let $\mathcal{F}$ be a 1-dimensional holomorphic foliation defined around the origin of $\mathbb{C}^n$ and assume that the blow-up $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ at the origin leaves the exceptional divisor $\pi^{-1}(0)$ invariant. Then the foliation induced on $\mathbb{CP}^{n-1} \simeq \pi^{-1}(0)$ by the restriction of $\widetilde{\mathcal{F}}$ is called the *core foliation* of $\mathcal{F}$ and its global dynamics is referred to as the *core dynamics* of $\mathcal{F}$.

Again, if Y is a representative of $\mathcal{F}$ and Y_k is as above ($Y = \sum_{i=k}^{\infty} Y_i$), the preceding then shows that the core foliation of $\mathcal{F}$ is nothing but the foliation induced on $\mathbb{CP}^{n-1}$ by the homogeneous vector field Y_k.

3.1. **1-dimensional foliations and dicritical codimension 1 foliations on $\mathbb{C}^3$.** The preceding discussion about foliations on projective spaces also applies to the case of codimension 1 foliations on $(\mathbb{C}^n, 0)$ *provided that they are dicritical for the blow up centered at the origin*. Throughout this section, the phrase *codimension 1 dicritical foliation $\mathcal{D}$ on $(\mathbb{C}^n, 0)$* will refer to a codimension 1 foliation that is dicritical for the blow up of $\mathbb{C}^n$ centered at the origin. In particular, the corresponding exceptional divisor can be identified with $\mathbb{CP}^{n-1}$ and, through this identification, the (dicritical) foliation $\mathcal{D}$ induces a foliation on on $\mathbb{CP}^{n-1}$ that will also be called the *core foliation of $\mathcal{D}$*.

In his famous book [25], Jouanolou studied some specific foliations on $\mathbb{CP}^2$, nowadays known as *Jouanolou foliations*, and proved that they left no algebraic curve in $\mathbb{CP}^2$ invariant. From there, he was able to provide an example of a codimension 1 foliation on $(\mathbb{C}^3, 0)$ without separatrices, thus answering a question of Thom. Using his result on the existence of foliations on $\mathbb{CP}^2$ leaving no algebraic curve invariant to derive a counterexample to Thom's question is arguably the first manifestation of the role played by core dynamics into the understanding of singular points. We shall review this issue below and go somewhat further by exploiting the results in [27] to see how difficult the situation may become.

In the sequel, we set $n = 3$ to abridge notation. First, let us characterize codimension 1 foliations that are *dicritical* for the blow-up of $\mathbb{C}^3$ centered at the origin. Since the lemma below does not seem to be accurately stated in the literature, a detailed - albeit straightforward - proof is included below.

Lemma 3.4. *Assume that $\mathcal{D}$ is a singular codimension 1 foliation defined on $(\mathbb{C}^3, 0)$ and denote by $\widetilde{\mathcal{D}}$ its blow-up centered at the origin. Then the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{CP}^2$ is invariant under $\widetilde{\mathcal{D}}$ if and only if no holomorphic vector field Z tangent to $\mathcal{D}$ admits a first non-zero homogeneous component (at the origin) that is a multiple of the Radial vector field R.*

Proof. Let $\mathcal{D}$ be given by a holomorphic 1-form $\Omega = F\,dx + G\,dy + H\,dz$ whose singular set has codimension at least 2. Denote by Ω_k the first non-zero homogeneous component of Ω at the origin, where k stands for the degree of Ω_k. Next, let $\Omega_k = F^k dx + G^k dy + H^k dz$. A direct inspection shows that $\pi^{-1}(0)$ *is not invariant* by $\widetilde{\mathcal{D}}$ if and only if

$$xF^k + yG^k + zH^k = 0\,. \tag{1}$$

Now, let Z be a vector field tangent to $\mathcal{D}$ and denote by Z^l its first non-zero homogeneous component at the origin. Since Z is tangent to $\mathcal{D}$, Z^l naturally provides a solution for $\{\Omega_k = 0\}$, i.e., we have $\Omega_k.Z^l = 0$. Next, if in addition, there is Z tangent to $\mathcal{D}$ for which Z^l is a multiple of the Radial vector field, then $\Omega_k.Z^l = 0$ is tantamount to Equation (1) which is thus satisfied. Hence, the exceptional divisor is not invariant by $\widetilde{\mathcal{D}}$. In other words, we have proved that the existence of a vector field Z as in the statement ensures that $\mathcal{D}$ is dicritical.

For the converse, let us assume that $\mathcal{D}$ is dicritical, i.e., that Equation (1) holds. The following argument was suggested to us by the anonymous referee. Next, note that for a vector field Z as above, the fact that its first non-zero homogeneous component at the origin Z^l is a multiple of the radial vector field is invariant by linear change of coordinates. In particular, our statement is invariant by linear change of coordinates and we can therefore assume without loss of generality that F^k is not identically zero. Next, we define

$$X_1 = H\partial/\partial x - F\partial/\partial z \quad \text{and} \quad X_2 = G\partial/\partial z - F\partial/\partial y \,.$$

We then have $\Omega(X_1) = \Omega(X_2) = 0$ so that X_1 and X_2 are tangent to $\mathcal{D}$. Thus the vector field $Z = xX_1 + yX_2$ is tangent to $\mathcal{D}$ as well. Finally, a direct inspection shows that the first non-zero homogeneous component of Z at the origin is given by $-F^k(x\partial/\partial x + y\partial/\partial y + z\partial/\partial z)$. It follows that Z satisfies the required conditions and this completes the proof of the lemma. □

Next, let us consider again a homogeneous polynomial vector field X on $\mathbb{C}^3$ satisfying conditions (1) and (2) in the previous subsection, i.e. the singular set of X has codimension at least 2 and the vector fields X and R are linearly independent at generic points (note that in the case of homogeneous vector fields of degree at least 2, conditions (1) and (2) are equivalent). Since X is homogeneous, we have

$$[R, X] = (d-1)X$$

where d stands for the degree of X. Thus the pair X and R generates the Lie algebra of the affine group. In particular the distribution of planes (of dimension 2) spanned by X and R is involutive and hence integrable. Let us then denote by $\mathcal{D}$ the codimension 1 foliation spanned by X and R.

Let $\widetilde{\mathcal{D}}$ stands for the blow-up of $\mathcal{D}$ centered at the origin so that $\widetilde{\mathcal{D}}$ is defined on $\widetilde{\mathbb{C}}^3$. Owing to Lemma 3.4, the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{CP}^2$ *is not invariant* under $\widetilde{\mathcal{D}}$. Furthermore, the structure of the foliation $\widetilde{\mathcal{D}}$ (and hence that of $\mathcal{D}$) is essentially as complicated as the structure of the blow-up $\widetilde{\mathcal{F}}$ of $\mathcal{F}$, where $\mathcal{F}$ denotes the foliation induced by X. This observation deserves further comments.

To begin with, recall that $\widetilde{\mathbb{C}}^3$ can also be seen as the tautological line bundle over $\pi^{-1}(0) \simeq \mathbb{CP}^2$. The bundle projection will be denoted by $\widetilde{\Pi} : \widetilde{\mathbb{C}}^3 \to \pi^{-1}(0)$ since it can naturally be identified with the canonical projection $\Pi : \mathbb{C}^3 \setminus \{(0,0,0)\} \to \mathbb{CP}^2$. Next, recall that, unlike $\widetilde{\mathcal{D}}$, the foliation $\widetilde{\mathcal{F}}$ leaves the exceptional divisor invariant. In particular, at a point p of $\pi^{-1}(0) \simeq \mathbb{CP}^2$ that is regular for $\widetilde{\mathcal{F}}$, this foliation defines a direction $u_p \in T_p\pi^{-1}(0)$. Next, for p "sufficiently generic", the leaf of $\widetilde{\mathcal{D}}$ intersects transversely $\pi^{-1}(0)$. This transverse intersection naturally defines a direction $v_p \in T_p\pi^{-1}(0)$. It is immediate to check that the directions of v_p coincides with the one defined by $\widetilde{\mathcal{F}}$. Denoting by $\widetilde{\mathcal{F}}_{|\pi^{-1}(0)}$ the foliation on $\pi^{-1}(0)$ obtained by restriction of $\widetilde{\mathcal{F}}$, we have the following:

Lemma 3.5. *The leaves of the foliation $\widetilde{\mathcal{D}}$ are of the form $\widetilde{\pi}^{-1}(L)$ where L is a leaf of $\widetilde{\mathcal{F}}_{|\pi^{-1}(0)}$. Similarly, every leaf of $\widetilde{\mathcal{D}}$ is invariant by the foliation $\widetilde{\mathcal{F}}$.* □

Recalling that every foliation on a projective space is induced by a homogeneous polynomial vector field, the interest of Lemma 3.5 is actually captured by the following slightly loose statement: *every foliation on $\mathbb{CP}^2$ is naturally the core foliation for singularities of both dimension 1 and codimension 1 foliations on $(\mathbb{C}^3, 0)$.*

Before considering some concrete applications of the previous remark, let us close this section by pointing out that the above construction allows us to define the core of a dicritical codimension 1 foliation on $(\mathbb{C}^3, 0)$ as follows.

Definition 3.6. Let $\mathcal{D}$ be a codimension 1 holomorphic foliation defined around the origin of $\mathbb{C}^3$ and assume that the blow-up $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ at the origin does not leave the exceptional divisor $\pi^{-1}(0)$ invariant. Then the foliation induced on $\mathbb{CP}^{n-1} \simeq \pi^{-1}(0)$ by the restriction of $\widetilde{\mathcal{D}}$ is called the *core foliation* of $\mathcal{D}$ and its global dynamics is referred to as the *core dynamics* of $\mathcal{D}$.

3.2. **Jouanolou's example, chaotic dynamics, and their meaning for singularity theory.** Let us go back to R. Thom's question on the existence of separatrices for codimension 1 foliations on $(\mathbb{C}^3, 0)$, cf. Definition 2.5. As pointed out in Section 2, it is not always easy to construct codimension 1 foliations due to Frobenius integrability condition that has to be satisfied by the distribution of planes in question. Yet, the discussion revolving around Lemma 3.5 also indicates a simple way to construct lots of *dicritical* codimension 1 foliations on $\mathbb{C}^3$. More precisely, every foliation on $\mathbb{CP}^2$ yields one such dicritical codimension 1 foliation.

Let then $\mathcal{D}$ be a dicritical codimension 1 foliation as above and assume that $\mathcal{D}$ admits separatrices. Let then S denote a germ of an irreducible separatrix for $\mathcal{D}$. Since S has codimension 1, there follows the existence of a germ of an irreducible holomorphic function $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that S coincides with the set $\{f = 0\}$. In terms of Taylor series, we set $f = \sum_{i \geq l}^{\infty} f_i$ where l is the degree of the first non-zero homogeneous component of f. Let $\mathcal{C} \subset \mathbb{CP}^2$ be the curve defined on the projective plane by the homogeneous equation $\{f_l = 0\}$ (the tangent cone to S). If we denote by $\mathcal{F}$ the *core* of $\mathcal{D}$ (recall than that $\mathcal{F}$ is a 1-dimensional foliation on $\mathbb{CP}^2$), then the following can be said:

Lemma 3.7. *With the preceding notation, the curve $\mathcal{C} \subset \mathbb{CP}^2$ is invariant by $\mathcal{F}$.*

Proof. The foliation $\mathcal{D}$ is defined by an integrable 1-form Ω whose Taylor series takes on the form $\Omega = \sum_{i=k}^{\infty} \Omega_i$ where k stands again for the first non-zero homogeneous component of Ω. A simple argument based on degrees shows that the 1-form Ω_k is integrable as well, i.e., it satisfies Frobenius equation $\Omega_k \wedge d\Omega_k = 0$. Similarly, one checks that the (homogeneous) surface defined by $\{f_l = 0\}$ yields a separatrix for the codimension-1 foliation $\mathcal{D}_k$ induced by Ω_k. Set $\Omega_k = F^k dx + G^k dy + H^k dz$ and, as usual, let R denote the Radial vector field on $\mathbb{C}^3$.

Next recall that a homogeneous vector field of $\mathbb{C}^3$ representing $\mathcal{F}$ is well defined only up to a multiplicative constant and addition of a multiple of the Radial vector field. Now, since Ω_k is homogeneous, the vector $R(p)$ is contained in the plane defined by the kernel of $\Omega_k(p)$ at the point p. Hence, up to eliminating multiplicative factors, a representative vector field X for $\mathcal{F}$ can be obtained by letting $X(p) = R(p) \wedge (F^k(p), G^k(p), H^k(p))$. In particular $X(p)$ lies in the kernel of $\Omega_k(p)$, i.e., X is tangent to the foliation $\mathcal{D}_k$. Finally, since at regular points $p \in \{f_l = 0\}$ the tangent space at $\{f_l = 0\}$ and the kernel of $\Omega_k(p)$ coincide, we conclude that X is tangent to the surface $\{f_l = 0\}$. The lemma then follows immediately. □

In view of Lemma 3.7, the basic remark of Jouanolou concerning Thom's conjecture was the following one: if we can find a foliation $\mathcal{F}$ on $\mathbb{CP}^2$ leaving invariant no algebraic curve, then the (dicritical) codimension 1 foliation $\mathcal{D}$ arising from combining the Radial vector field of $\mathbb{C}^3$ and a representative of homogeneous vector field for $\mathcal{F}$ will admit *no separatrix.*

Jouanolou's remark is possibly the first instance where the existence of the *core dynamics* actually impacts the study of singularities of foliations. From this point of view, the main result of Jouanolou in [26] can be stated as follows:

Theorem 3.8. [25] *For every $d \geq 2$, the foliation induced on $\mathbb{CP}^2$ by the vector field*

$$X_d = y^d \partial/\partial x + z^d \partial/\partial y + x^d \partial/\partial z$$

leaves no algebraic curve invariant.

Jouanoulou theorem implies, in particular, that for every fixed $d \geq 2$, there exist foliations of degree d that are not tangent to any algebraic curve of $\mathbb{CP}^2$.

Armed with the above theorem, there follows from what precedes that the codimension 1 *Jouanolou foliation* J_d, $d \geq 2$, of $\mathbb{C}^3$ which is defined as the singular foliation spanned by X_d and the Radial vector field is a *counterexample to Thom's question.* The well-known explicit 1-form Ω,

$$\Omega = (yx^d - z^{d+1})\, dx \ + \ (zy^d - x^{d+1})\, dy \ + \ (xz^d - y^{d+1})\, dz \ ,$$

defining the foliation J_d can promptly be obtained by taking the vector product of X_d and R.

The next question is to wonder how far the *core dynamics* can influence the study of singularities of foliations, say of dimension 1 on $\mathbb{C}^n$, $n \geq 3$. In other words, owing to the discussion in this section, the detailed understanding of the local structure of one such foliation arguably passes through the global description of its *core foliation.* This understanding would require, in particular, a (global) control of the dynamics of the core foliation. At this point, we might wonder whether it is possible to obtain such an accurate local description of all 1-dimensional foliations on, say, $(\mathbb{C}^3, 0)$. From the standpoint emphasized above, an easier question would be to provide a reasonable global description of all or nearly all foliations on $\mathbb{CP}^2$. Unfortunately, the latter question does not seem to admit an affirmative answer as it follows from Loray-Rebelo theorem [27] as stated below.

Fix positive integers n and d, with $\min\{n, d\} \geq 2$. A straightforward counting of parameters shows that the space $\mathrm{Fol}_{\mathbb{CP}}^{(d)(n)}$ of degree d foliations on $\mathbb{CP}^n$ can be identified with a Zariski-open set of the complex projective space of dimension

$$(d+n+1)\frac{(d+n-1)!}{d!(n-1)!} - 1.$$

This space of foliation can then be furthered moduled out by the action of the automorphism group $\mathrm{PSL}\,(n+1, \mathbb{C})$ of $\mathbb{CP}^n$ but this will not be needed in the sequel. The main upshot here is that $\mathrm{Fol}_{\mathbb{CP}}^{(d)(n)}$ can be parameterized by a finite dimensional complex manifold and, in particular, inherits of a natural topology. With this notation, the main result of [27] reads as follows:

Theorem 3.9. [27] *Fixed $n, d \geq 2$, there exists a non-empty open subset $\mathcal{U} \subset \mathrm{Fol}_{\mathbb{CP}}^{(d)(n)}$ such that every foliation $\mathcal{F}$ lying in $\mathcal{U}$ satisfies all the conditions below:*

1. *All singular points of $\mathcal{F}$ are hyperbolic. In particular, they form a finite set.*
2. *Minimality: Every leaf of $\mathcal{F}$ is dense in $\mathbb{CP}^n$.*
3. *Ergodicity: Every measurable set of leaves has either zero or total Lebesgue measure.*
4. *Rigidity: If $\mathcal{F}' \in \mathrm{Fol}_{\mathbb{CP}}^{(d)(n)}$ is conjugate to $\mathcal{F}$ by a homeomorphism $h : \mathbb{CP}^n \to \mathbb{CP}^n$ that is close to the identity, then $\mathcal{F}$ and $\mathcal{F}'$ are also conjugate by an element of* $\mathrm{PSL}\,(n+1, \mathbb{C})$.

The level of dynamical complication exhibited by the foliations indicated above puts any accurate description of them basically out of reach. Moreover, even up to topological conjugation, it is not possible to achieve a reasonable list of "models" or "normal forms" owing to the above indicated rigidity phenomenon.

It is convenient to expound a bit on the consequences of Theorem 3.9 from the point of view of singularity theory for 1-dimensional foliations on dimensions 3 and greater. Consider then a foliation lying in the set $\mathcal{U} \subset \mathrm{Fol}_{\mathbb{CP}}^{(d)(n)}$ provided by Theorem 3.9. As a foliation defined on $\mathbb{CP}^n$, it can be represented by some homogeneous polynomial vector field X on $\mathbb{C}^{n+1}$. In other words, if $\mathcal{F}$ is the foliation on $\mathbb{C}^{n+1}$ induced by the local orbits of X then $\widetilde{\mathcal{F}}$, its (one-point) blow-up at the origin, leaves the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{CP}^n$ invariant and is such that the restriction of $\widetilde{\mathcal{F}}$ to $\pi^{-1}(0) \simeq \mathbb{CP}^n$ is naturally identified with the initial foliation in $\mathcal{U} \subset \mathrm{Fol}_{\mathbb{CP}}^{(d)(n)}$.

Now recall that the vector field X is not uniquely defined: most notably, we can add to X any multiple of the Radial vector field by a homogeneous polynomial of degree $d-1$. Since, in addition, the singularities of the initial foliation in $\mathcal{U}$ are all hyperbolic, it is easy to conclude that the vector field X can be chosen so as to fulfill the following conditions:

(1) The foliation $\mathcal{F}$ has an isolated singularity at the origin of $\mathbb{C}^{n+1}$.
(2) The foliation $\widetilde{\mathcal{F}}$, viewed as foliation on a manifold of dimension $n+1$, still have only hyperbolic singularities.

Furthermore, a generic choice of the initial foliation in $\mathcal{U}$ and of the vector field X allows us to rule out the existence of resonances at the singular points of $\widetilde{\mathcal{F}}$ as well. Thus, all the singularities of $\widetilde{\mathcal{F}}$ are, in fact, linearizable. Also, all the above mentioned characteristic are stable under higher order perturbations of a representative vector field. The situation can then be summarized as a statement in itself.

Theorem 3.10. (Corollary of [27]) *For every degree $d \geq 2$, there exists a non-empty open set V of homogeneous vector fields of degree d in $\mathfrak{X}(\mathbb{C}^{n+1}, 0)$ such that every germ of foliation $\mathcal{F}$ represented by a holomorphic vector field X having the form $X = X^d + \mathrm{h.o.t.}$, with $X^d \in V$ and where* h.o.t. *stands for higher order terms, satisfy all of the following conditions:*

(1) *The one-point blow-up $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ at the origin leaves the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{CP}^n$ invariant.*
(2) *All singular points of $\widetilde{\mathcal{F}}$ are hyperbolic and linearizable. In particular, $\widetilde{\mathcal{F}}$ has exactly*
$$\frac{d^{n+1}-1}{d-1}$$
singular points and all of them lie in $\pi^{-1}(0) \simeq \mathbb{CP}^n$.
(3) *The restriction of $\widetilde{\mathcal{F}}$ to $\pi^{-1}(0) \simeq \mathbb{CP}^n$ defines a degree d foliation of $\mathbb{CP}^n$ lying in the open set $\mathcal{U} \subset \mathrm{Fol}_{\mathbb{CP}}^{(d)(n)}$ given by Theorem 3.9.*

Let us point out that the formula in item (2) for the number of singular points of $\widetilde{\mathcal{F}}$, i.e., for a degree d foliation on $\mathbb{CP}^n$ all of whose singular points are hyperbolic is well known and can be proved in a variety of ways. For example, by choosing affine coordinates yielding a "hyperplane at infinity" on which the foliation has no singular point and then applying Bézout theorem to the corresponding polynomial vector field representing the foliation in the above indicated affine coordinates.

Remark 3.11. Naturally the content of Theorem 3.10 can be adapted to germs of codimension 1 dicritical foliations on $(\mathbb{C}^3, 0)$.

To close this section, it is convenient to make a parallel with the discussion in Section 2.2 for singularities of foliations on $(\mathbb{C}^2, 0)$ so as to better appreciate the difficulties arising from the existence of wild core dynamics as stated in Proposition 3.10.

(A') Generalizations of Seidenberg theorem to $(\mathbb{C}^n, 0)$. The problem is wide open for $n \geq 4$ though sharp desingularization theorems are now established for $n = 3$. The topic is of clear interest since virtually all general statements about singularities rely, directly or indirectly, on a suitable "resolution theorem". Yet, for $n \geq 3$, the ability to obtain a birational model of the foliation where all singular points are "simple enough" might still be a long way off of providing an accurate description of the singularity in question.

To substantiate the above claim, it suffices to consider the local foliations $\mathcal{F}$ on $(\mathbb{C}^n, 0)$ provided by Theorem 3.10. The blow-up $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ at the origin provides a birational model for $\mathcal{F}$ possessing only "simple singular points": in fact, all singularities of $\widetilde{\mathcal{F}}$ are hyperbolic and linearizable. In other words, the local behavior of $\widetilde{\mathcal{F}}$ around each of its singular points is essentially trivial and promptly available. Nonetheless, the very complicated dynamical behavior of $\mathcal{F}$ around the origin encoded in its core dynamics (cf. Lemma 3.5) puts a detailed description of the corresponding singular point essentially out of reach.

(B') Taming the core dynamics. If one is to fully understand the structure of a foliation around a singular point, then an accurate description of its core dynamics needs to be envisaged. If Proposition 3.10 tells us this is a kind of unrealistic goal, it also raises the question of "selecting" those classes of singular points allowing a more detailed description. This is a very interesting point as it hints at considering the connections between singularity theory and the remainder of Mathematics or, even, Physics. Singularities playing a special role in problems from Geometry, Complex Analysis and/or Integrable Systems are likely to be amenable to a more complete analysis. Examples of these situations will be discussed in the forthcoming sections.

In terms of "taming core dynamics", of course the ideal situation would be to have a core foliation defining an "integrable system" in some suitable sense. Alternatively, for a number of problems, it might be enough to ensure the existence of ("sufficiently many") algebraic invariant curves. An important issue involving invariant curves is that more often than not the dynamics of the foliation in question can be investigated in more details on a neighborhood of them, especially when their fundamental group contains more than a single generator. This study, whereas of more global nature, is somehow akin to "Mattei-Moussu pseudogroup technique" mentioned in Section 2.2. Interesting examples where this point of view have successfully been employed - even outside the scope of singularity theory - include [24], [27], and [19].

(C') Dynamics of pseudogroups acting on $(\mathbb{C}^n, 0)$. The perspective of focusing in the local dynamics arising from the holonomy of an invariant algebraic curve in higher dimensions naturally leads us towards considering the dynamics of subgroups of $\text{Diff}\,(\mathbb{C}^n, 0)$, $n \geq 2$. As was to be expected, many new dynamical phenomena arise for $n \geq 2$ compared to the situation $n = 1$. As pointed out in Section 2.2 much is known about the dynamics of subgroups of $\text{Diff}\,(\mathbb{C}, 0)$ whereas for subgroups of $\text{Diff}\,(\mathbb{C}^n, 0)$, the theory is still in its early stages.

Nonetheless, we mention that generalizations to higher dimensions of Mattei-Moussu's theorem on groups with finite orbits is by now well understood, see [45], [47], [51]. These results are likely to have impact in problems about existence of first integrals but they might also provide insight in higher dimensional versions of the so-called "analytic limit set", see [2].

Finally a major issue in the theory is to find sharp conditions to extend to higher dimensions the Shcherbakov-Nakai theory of local vector fields in the "closure of the group" [52], [53], [37]. Very little is known about this question aside from some results in [27] which rely on the existence of a hyperbolic contraction for the group in question. This assumption looking rather far from sharp, the topic appears to be ripe for significant progress.

4. Resolution theorems in dimension 3

In the remainder of this survey, we will discuss relatively recent progress in some of the several aspects of singularity theory. This section is devoted to "resolution theorems" while the next two sections will basically review the general problem of invariant varieties and the study of a particular and important class of singular points, namely the semicomplete ones. In the course of these discussions , theorems providing - at various degrees - some control on the core dynamics in question will play a prominent role.

As previously indicated, theorems on reductions of singular points are always of paramount importance in the theory whether or not there are major difficulties lying out of their reach (e.g. complicated core dynamics). For foliations defined on complex 2-dimensional manifolds (or varieties), Seidenberg's theorem provides a sharp *reduction of singularities theorem* (a.k.a. "resolution theorem") that is particularly easy to manipulate. Beyond dimension 2, decisive results exist only in dimension 3. This section is devoted to reviewing and explaining the main "resolution theorems" for 1-dimensional foliations in dimension 3. First, a standard terminology.

Definition 4.1. Let $\mathcal{F}$ be a holomorphic foliation of dimension 1 defined on a complex manifold M (smooth complex space). A singular point p of $\mathcal{F}$ is said to be elementary if $\mathcal{F}$ possesses at least one eigenvalue different from zero at p.

In principle, the purpose of theorems about reduction of singular points for foliations of dimension 1 is to provide a birational model for the foliation in question where all singular points are elementary. Along these lines, we say that a singularity *cannot be reduced* by means of a sequence of blow ups (possibly satisfying certain conditions), if no sequence of blow ups verifying the conditions in question leads to a birationally equivalent foliation all of whose singular points are elementary.

Owing to the classical Hironaka resolution theorem, we can assume that our singular foliations are always defined on manifolds. Furthermore, since the problems are local, we may assume them to be defined on a neighborhood of the origin of $\mathbb{C}^n$. The case $n = 2$ being settled by Seidenberg theorem, we assume from now on that $n = 3$, i.e., our foliations are defined on a neighborhood of the origin of $\mathbb{C}^3$.

We note that a reduction of singularities procedure for *codimension* 1 *foliations* on $(\mathbb{C}^3, 0)$ was earlier obtained in [7]. However, the story involving foliations of dimension 1 - the main object of this survey - is longer and more elusive.

Resolution results for foliations of dimension 1 on $(\mathbb{C}^3, 0)$ have first appeared in [5], where the author proves the existence of a formal local uniformization theorem. In this work, the author also hints at the existence of a new phenomenon involving singularities possessing a certain formal separatrix (i.e. a formal curve invariant by the foliation) which posed serious difficulties to resolve the singularity by means of standard blow-ups. The issue was made clear by Sancho and Sanz who provided explicit examples of foliations in $(\mathbb{C}^3, 0)$ that cannot be reduced by sequences of standard blow-ups centered at sets contained in the singular loci of the initial foliations and its transforms.

After the examples found by Sancho and Sanz, the next truly major result in the area is due to D. Panazzolo [38]. In [38], Panazzolo considers singularities of real foliations in (real) dimension 3. He works in the real setting, rather than in the complex one, mostly due to the fact that his original motivation lied in Hilbert's problem about the number of limit cycles of a polynomial vector field on $\mathbb{R}^2$. In his work, Panazzolo shows that the corresponding germs of foliations can always be turned into a foliation all of whose singular points are elementary *by means of a finite sequence of weighted blow-ups centered at singular sets*. The proof is constructive and actually provides an algorithm to obtain the desired reduction of singularities. Later, relying on Panazzolo's algorithm introduced in [38], McQuillan and Panazzolo were able to provide a very satisfactory answer to the generalization of Seidenberg's theorem for foliations on $(\mathbb{C}^3, 0)$ in [35], [36].

The preprint [35] was made available in 2009 and a few years later, Cano, Roche, and Spivakovsky revisited the topic from the point of view of valuation theory, see [9]. Their strategy is in line with Zariski's general approach to desingularization problems and, hence, is essentially divided in two parts. First, for a given foliation, we seek to "simplify" only the singularities lying in the center of a given valuation (identified with its transforms, or extensions, through blow-ups). Resolution results for singularities lying in the center of a valuation are often referred to as *local uniformization theorems* and the first part of Zariski approach aims at obtaining this type of statement. Once a convenient local uniformization result is obtained, the second part of Zariski approach deals with its *globalization*. More precisely, once it is proved that for every valuation ν, the singularities lying in the center of ν can be simplified (in some appropriate sense), we try to conclude that, in fact, all singularities of the foliation can simultaneously be simplified in the same sense. When it comes to applying this point of view to singularities of foliations most of the difficulties related to the globalization procedure are handled pretty well by a very general axiomatic argument due to O. Piltant [39], c.f., the last section of [9].

In view of what precedes, the content of [9] can roughly be summarized by claiming *the existence of a birational model for the initial foliation where the singular points are log-elementary*. The reader is referred to [9] for the definition of log-elementary singular points. For our purposes, it suffices to know that such singularities are, at worst, *quadratic* in the sense that they are locally given by a representative vector fields with non-zero second-jet at the singular point in question. One of the goals of [48] was to complete the work of Cano-Roche-Spivakovsky by deriving "final models" similar to those of [36], in order to obtain a global resolution theorem comparable to [35], [36] through Zariski classical approach.

We will compare the resolution theorems for foliations obtained by McQuillan-Panazzolo in [36] and by ourselves in [48], they correspond to Theorem 2 and Theorem A of the respective papers. In particular, it will be seen that in the context of foliations the two results are pretty much equivalent and can be summarized by the following assertion: *given a singular holomorphic* 1*-dimensional foliation* $\mathcal{F}$ *on* $(\mathbb{C}^3, 0)$*, there exists a birational model of* $\mathcal{F}$ *where all singular points are elementary*. In this sense, the only difference between the theorems in question will be down to the way in which the desired birational model is constructed.

4.1. **Persistent nilpotent singularities.** As already mentioned, Sancho and Sanz have showed the existence of foliations in $(\mathbb{C}^3, 0)$ that cannot be reduced by sequences of standard blow-ups with centers contained in the singular set of the initial foliation and its transforms. In fact, their result is slightly more general in the sense that we may allow for blow-ups of invariant centers not necessarily contained in the singular locus. As a matter of fact, they have provided a 3-parameter family of foliations whose elements cannot be turned into a foliation all of whose singularities are elementary by means of blow-ups centered in the singular loci and whose generic element cannot be turned into a foliation with elementary singular points even if invariant centers are allowed. This family of foliations is represented by the family of vector fields $X_{\alpha,\beta,\lambda}$ taking on the form

$$X_{\alpha,\beta,\lambda} = x\left(x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y} - \beta z\frac{\partial}{\partial z}\right) + xz\frac{\partial}{\partial y} + (y - \lambda x)\frac{\partial}{\partial z}\,. \tag{2}$$

Accordingly, foliations in this family will be denoted by $\mathcal{F}_{\alpha,\beta,\lambda}$. The foliations $\mathcal{F}_{\alpha,\beta,\lambda}$ are *nilpotent* at the origin in the sense that so are the vector fields $X_{\alpha,\beta,\lambda}$. For reference, it is convenient to make accurate the notion of *nilpotent foliation*.

Definition 4.2. A (1-dimensional) holomorphic foliation is said to have a nilpotent singularity at a singular point p if its representative vector field around p has non-zero nilpotent linear part at p.

The above notion of nilpotent singularity is well defined since it does not depend on the choice of the representative vector field. Also, whenever no misunderstanding about the singular point in question is possible, we will abridge notation by simply saying that $\mathcal{F}$ *is a nilpotent foliation*. Going back to the nilpotent foliations $\mathcal{F}_{\alpha,\beta,\lambda}$, we note that the plane $\{x = 0\}$ is invariant by them and that it contains the singular set of $\mathcal{F}_{\alpha,\beta,\lambda}$ which coincides with the axis $\{x = y = 0\}$. We now have the following:

Proposition 4.3. *The foliations in the family $\mathcal{F}_{\alpha,\beta,\lambda}$ cannot be turned into a foliation all of whose singular points are elementary by means of a sequence of standard blow-ups with centers contained in singular sets.*

Sketch of Proof. Consider the one-point blow-up centered at the origin of $(\mathbb{C}^3, 0)$ and let π stands for the blow-up map. Let then (x, u, v) be the affine coordinates for the blown-up space where $y = ux$ and $z = vx$. The pull-back $\pi^* X_{\alpha,\beta,\lambda}$ of the vector field $X_{\alpha,\beta,\lambda}$ is given by

$$\pi^* X_{\alpha,\beta,\lambda} = x\left(x\frac{\partial}{\partial x} - (\alpha+1)u\frac{\partial}{\partial u} - (\beta+1)v\frac{\partial}{\partial v}\right) + xv\frac{\partial}{\partial u} + (u-\lambda)\frac{\partial}{\partial v}\,,$$

whose expression is similar to the expression of $X_{\alpha,\beta,\lambda}$. In fact, the main difference between the two expressions concerns the last term. Note, however, that the origin of the present coordinates is not contained in the singular set of the induced foliation, which is given by $\{x = 0, u = \lambda\}$. Thus, if we consider the translation $T(x, \overline{u}, \overline{v}) = (x, \overline{u} + \lambda, \overline{v} + \mu)$, the pull-back of $\pi^* X_{\alpha,\beta,\lambda}$ through T is given by

$$x\left(x\frac{\partial}{\partial x} - (\alpha+1)\overline{u}\frac{\partial}{\partial \overline{u}} - (\beta+1)\overline{v}\frac{\partial}{\partial \overline{v}}\right) + x(\overline{v} + \mu - \lambda(\alpha+1))\frac{\partial}{\partial \overline{u}} + (\overline{u} - \mu(\beta+1)x)\frac{\partial}{\partial \overline{v}}\,.$$

In the particular, if we choose $\mu = \lambda(\alpha + 1)$, the vector field in question coincides with the vector field $X_{\alpha+1,\beta+1,\lambda(\alpha+1)(\beta+1)}$. In other words, the transformed foliation of $\mathcal{F}_{\alpha,\beta,\lambda}$ contains a nilpotent singular point belonging to the (initial) Sancho-Sanz family. It can be checked that the same issue occurs if the blow-up centered at the curve of singular points of $\mathcal{F}_{\alpha,\beta,\lambda}$ is considered. □

Summarizing what precedes, every sequence of blow-ups as above applied to a foliation in Sancho-Sanz family lead to a foliation having a singular point where the foliation is locally conjugate to another foliation in the initial family. In particular, all transformed foliations will exhibit a nilpotent singular point. This nilpotent singular point has a geometric interpretation naturally related to the issues raised by Cano in [5] for a resolution by standard blow-ups. In fact, by elaborating in the above indicated argument, Sancho and Sanz have shown that the parameters α, β, λ can be chosen so that the foliation associated with the vector field $X_{\alpha,\beta,\lambda}$ possesses a *strictly formal separatrix $S = S_0$ through the origin*. Moreover, given a sequence of blow-ups as before, the sequence of points $\{p_n\}$ in the exceptional divisors corresponding to the position of the (persistent) nilpotent singularity is determined by the sequence of transforms $\{S_n\}$ of the formal separatrix $S = S_n$. We should still note that, the fact that every separatrix S_n is stricty formal says that even in the case we allow blow-ups to be centered at analytic invariant curves that *are not* necessarily contained in singular set of the foliation, a resolution procedure still does not exist.

In terms of the relation between foliations and - possibly formal - separatrices, a natural object that plays an important role is the notion of *multiplicity of the foliation along the separatrix*. Let X be a representative vector field of $\mathcal{F}$ and φ the Puiseux parametrization of S. Let $\varphi^* X|_S$ stands for the pull-back of the restriction of X to S. If $\varphi^* X|_S = g(t)\partial/\partial t$, then the multiplicity of $\mathcal{F}$ along S, that will be denoted by $\text{mult}(\mathcal{F}, S)$, is defined as the order of g at $0 \in \mathbb{C}$ (assuming that $\varphi(0)$ coincides with the singular point).

In [48], we introduced the notion of *persistent nilpotent singular point* which is as follows.

Definition 4.4. A nilpotent singular point p_0 of a foliation $\mathcal{F}_0$ is said to be persistent if there exists a formal separatrix S_0 for $\mathcal{F}_0$ through p_0 such that for every sequence of blowing-ups

$$\mathcal{F}_0 \overset{\pi_1}{\longleftarrow} \mathcal{F}_1 \overset{\pi_2}{\longleftarrow} \cdots \overset{\pi_l}{\longleftarrow} \mathcal{F}_n$$

where $\mathcal{F}_i$ stands for the transformed of $\mathcal{F}_{i-1}$ through the (standard) blow-up centered at some $C_{i-i} \subseteq \mathrm{Sing}\,(\mathcal{F}_{i-1})$ containing the point p_{i-1} (selected by the transformed separatrix S_{i-1}, in the sense that it corresponds to the intersection of S_{i-1} with the excetional divisor), the following conditions are satisfied

(a) the singular points p_i are all nilpotent singular points for the corresponding foliations;
(b) the multiplicity of $\mathcal{F}_i$ along S_i does not depend on i.

The role played by the notion of multiplicity of a foliation along a separatrix is closely related to its natural behavior under blow-ups. Recall that the *order of a foliation at a singular point* is nothing but the order of a representative vector field X , i.e. the degree of the first non-zero jet of X, at the singular point in question. With this notation, assume that $\widetilde{\mathcal{F}}$ is obtained by blowing-up $\mathcal{F}$ at a singular point p. Assume also that S is a (formal) separatrix of $\mathcal{F}$ at p and denoted by $\widetilde{S}$ the transform of S which yields a (formal) separatrix for $\widetilde{\mathcal{F}}$ at the point $\widetilde{p}$. Then we have:

$$\mathrm{mult}(\widetilde{\mathcal{F}}, \widetilde{S}) \leq \mathrm{mult}(\mathcal{F}, S) \tag{3}$$

with equality holding if and only if the order of $\mathcal{F}$ at p equals 1. The same formula holds for blowing-ups centered at a curve contained in the singular set of $\mathcal{F}$, up to considering a variant of the notion of "order of the foliation" that is adapted to the center of the blow-up, for details see [48] or the discussion at the end of Section 5.1.

It is easy to understand the interest of the multiplicity of a foliation $\mathcal{F}$ along a separatrix from the above perspective: if the existence of a (formal) separatrix S is ensured, then its multiplicity will drop strictly providing that the order of $\mathcal{F}$ at the singular point in question is greater than or equal to 2. Moreover, once this decreasing sequence stabilizes, then the corresponding singular point is either elementary or *nilpotent*. From this it also follows that it is useful to understand *persistent nilpotent singular points* in order to establish resolution theorems for foliations.

Clearly, in dimension 2, persistent nilpotent singular points do not exist as follows from Seidenberg theorem. In dimension 3, their existence is established by the above discussed examples due to Sancho and Sanz. A characterization of these points in dimension 3 in terms of normal forms can be formulated as follows.

Theorem 4.5. [48] *Assume that $\mathcal{F}$ cannot be resolved by a finite sequence of standard blow-ups centered at singular sets. Then there exists a sequence of one-point blow-ups (centered at singular points) leading to a foliation $\mathcal{F}'$ with a singular point p around which $\mathcal{F}'$ is given by a vector field of the form*

$$(y + zf(x,y,z))\frac{\partial}{\partial x} + zg(x,y,z)\frac{\partial}{\partial y} + z^n\frac{\partial}{\partial z}$$

for some $n \geq 2$ and holomorphic functions f and g of order at least 1 *with* $\partial g/\partial x(0,0,0) \neq 0$. *Furthermore we have:*

(1) *The resulting foliation $\mathcal{F}'$ admits a formal separatrix at p which is tangent to the z-axis;*
(2) *The exceptional divisor is locally contained in the plane $\{z = 0\}$.*

Theorem 4.5 deserves a couple of comments as it has a natural analogue in [36], namely:

- In [36], the authors obtain an alternative characterization of persistent nilpotent singularities which are presented as singular points arising from elementary ones by means of a $\mathbb{Z}/2\mathbb{Z}$-orbifold singularity, see Section 4.2 for more details. It is relatively straightforward to establish the equivalence between their characterization and the normal forms provided by Theorem 4.5.
- As in [36], an immediate consequence of the normal forms in Theorem 4.5 is that every persistent nilpotent singular point can immediately be turned into elementary ones by means of a single blow-up of weight 2, see [48].

In closing this section, let us point out the family of vector fields described in Theorem 4.5 is a genuine extension of the Sancho-Sanz family, albeit one naturally obtained by following their construction. Indeed, for persistent nilpotent singular points, the multiplicity of the foliation along the corresponding (formal) separatrix is fully invariant under blow-ups whose centers are contained in singular sets, cf, Formula 3. In the Sancho-Sanz family, all multiplicities are equal to 2 so that for $n \geq 3$, Theorem 4.5 yields examples that cannot be turned in Sancho-Sanz examples by means of successive blow-ups (and conversely). For example, vector fields in the family

$$X_\lambda = (y - \lambda z)\frac{\partial}{\partial x} + zx\frac{\partial}{\partial y} + z^3\frac{\partial}{\partial z}\,,$$

with $\lambda \neq 0$, yield foliations $\mathcal{F}_\lambda$ with persistent nilpotent singularities arising from a (strictly) formal separatrix S_λ. The multiplicity of $\mathcal{F}_\lambda$ along S_λ being equal to 3.

4.2. **The desingularization theorem of McQuillan-Panazzolo.** The purpose of this paragraph is to explain in detail the desingularization theorem proved in [36]. As previously mentioned, McQuillan and Panazzolo work from the start in the category of weighted blow-ups, thus not limiting themselves to stardard ones. Unlike standard blow-ups, that keep the smooth nature of the space, weighted blow-ups lead to singular ambient spaces. Yet, the singularities in question are of orbifold-type and hence of a rather simple nature. Whereas singular, it should be pointed, that the ambient space obtained after a sequence of finitely many weighted blow-ups still is *birationally equivalent to the initial one*. In particular, foliations can be transformed without any restrictions under weighted blow-ups to yield new birational models for them.

Keeping in mind the issues pointed out above, let us summarize the contents of [36]. The paper [36] is essentially divided into two parts. Its first part is devoted to prove that the algorithm of [38] - leading to a resolution of singularities by means of a sequence of weighted blow-ups for real analytic foliations on $(\mathbb{R}^3, 0)$ - applies equally well in the general case of holomorphic foliations on $(\mathbb{C}^3, 0)$. The algorithm in question thus provides a birational model for the foliation on a space possessing orbifold-type singular points. Note that, since we are dealing with (singular) foliations on spaces with singular points of orbifold type, a word is needed about the meaning of "elementary singular points". In this regard, the singular point of the foliation is said to be *elementary* if it is given by an elementary singular point in a *orbifold coordinate* for the space. In particular, there are an open set $U \subset \mathbb{C}^3$ and finitely ramified map from U to a neighborhood of the orbifold singular point such that when the foliation is pulled-back to the open set $U \subset \mathbb{C}^3$ only elementary singular points are obtained.

In the second part of [36], the authors consider the problem of resolving the singular points of the ambient space while keeping the singular points of the foliation elementary. They prove that a resolution for such singularities exists except when the singular point correspond to a $\mathbb{Z}/2\mathbb{Z}$-orbifold. In other words, they have shown that, given a foliation $\mathcal{F}$, it is always possible to obtain a birational model for $\mathcal{F}$ possessing only $\mathbb{Z}/2\mathbb{Z}$-orbifold singular points and where all the singular points of the foliation in question are elementary. These singularities associated with

$\mathbb{Z}/2\mathbb{Z}$-orbifolds actually correspond to the previously described persistent nilpotent singular points.

The result in [36] can thus be stated as follows:

Theorem 4.6. [36] *Let $\mathcal{F}$ be a singular holomorphic foliation on $(\mathbb{C}^3, 0)$. There is a sequence of weighted blow-ups*

$$\mathcal{F}_0 \overset{\pi_1}{\longleftarrow} \mathcal{F}_1 \overset{\pi_2}{\longleftarrow} \cdots \overset{\pi_l}{\longleftarrow} \mathcal{F}_l \tag{4}$$

satisfying the following conditions:

(i) *The center of each weighted-blow-up is strictly invariant with respect to the "quasi-homogeneous filtration" in question.*
(ii) *The ambient space is an analytic space of dimension 3 whose singular points are $\mathbb{Z}/2\mathbb{Z}$-orbifold type and the total blow-up map $\pi_1 \circ \cdots \circ \pi_l$ is birational.*
(iii) *The singular points of $\mathcal{F}_l$ are elementary in orbifold coordinates.*

Let us close this paragraph with a comment concerning item (i) of Theorem 4.6. Note that this item is not emphasized in [36] though it is a characteristic property of Panazzolo's algorithm in [38]. Whereas, as far as foliations are concerned, this is a minor issue - as it would also be the case of blow-ups centered away from the singular locus (whether or not the blow-ups are weighted) - the issue *becomes relevant* when our main interest lies in vector fields, rather than foliations, see Section 4.4.

4.3. **Resolution following** [48]**.** In [48], we also establish the existence of a birational model for $\mathcal{F}$ where all singularities of $\mathcal{F}$ are elementary except for finitely many ones that can be turned into elementary singular points by means of a single blow-up of weight 2. To be more precise, our resolution result for foliations can be stated as follows.

Theorem 4.7. [48] *Let $\mathcal{F}$ denote a singular holomorphic foliation defined on a neighborhood of $(0, 0, 0) \in \mathbb{C}^3$. Then there exists a finite sequence of blow-up maps along with transformed foliations*

$$\mathcal{F} = \mathcal{F}_0 \overset{\pi_1}{\longleftarrow} \mathcal{F}_1 \overset{\pi_2}{\longleftarrow} \cdots \overset{\pi_l}{\longleftarrow} \mathcal{F}_n \tag{5}$$

satisfying all of the following conditions:

(1) *The center of the blow-up map π_i is (smooth and) contained in the singular set of $\mathcal{F}_{i-1}$, $i = 1, \ldots, n$.*
(2) *The singularities of $\mathcal{F}_n$ are either elementary or persistently nilpotent.*
(3) *The number of persistently nilpotent singularities of $\mathcal{F}_n$ is finite and each of them can be turned into elementary singular points by performing a single weighted blow-up of weight 2.*

The proof of this theorem has essentially two main ideas. The first one concerns a (personal) comment by F. Cano claiming that "if a foliation cannot be resolved by standard blow-ups, then there must exist a formal separatrix giving rise to a sequence of infinitely near singular points that never becomes elementary". This assertion harks back to his earlier works on resolutions of 1-dimensional foliations [5] and is used in his joint paper with C. Roche [8]. Also some important results in this direction can be found in [9]. To provide a complete proof of Cano's assertion was therefore a crucial point in the proof of Theorem 4.7 and the corresponding result is the content of Proposition 4 in [48]. Interestingly enough, the argument provided in [48] is rather different from the one envisaged by F. Cano.

With Proposition 4 of [48] in place, the main idea to derive Theorem 4.7 is to argue from the notion of multiplicity of a foliation along a separatrix, as defined in Subsection 4.1. The sequence formed by a separatrix and its transforms is decreasing so that it stabilizes after finitely steps. When the sequence becomes stable, the order of the singular point of the foliation must be 1. Thus either the singularity has become elementary or we can resort to Theorem 4.5 to characterize it as a persistent nilpotent singularity, which is necessarily isolated among other possible persistent nilpotent singular points. Therefore this yields a *local uniformization theorem* in the sense of Zariski. At this point, O. Piltant "gluing theorem" [39] allows one to conclude Theorem 4.7, c.f., the corresponding discussion in [9].

Recalling that persistent nilpotent singular points are in correspondence with $\mathbb{Z}/2\mathbb{Z}$-orbifold type singular points, the differences between Theorem 4.6 and Theorem 4.7 are down to the way the corresponding birational models are constructed. Unlike Panazzolo [38], our proof of Theorem 4.7 does not provide any effective algorithm to resolve singularities. In some problems, however, it might simplify discussions/arguments by sticking to a single type of blow-up, the standard one, provided that the problem in question requires only a theorem asserting the existence of a resolution, as opposed to an effective manner to obtain the resolution in question.

Remark 4.8. It might be convenient to point out that local uniformization statements concern valuations of the field of rational functions so that they are specific of the algebraic setting used in [9] and in [39]. However, in practice, this hardly limits the application of these results since the difficulties of reducing singular points in dimension 3 are typically localized at specific points and these can be identified with a neighborhood of the origin in $\mathbb{C}^3$, see [9], [48].

4.4. **A final comment on transforming vector fields.** In close this section, let us point out a virtue of standard blow-ups, as used as in Theorem 4.7, that is also present in Theorem 4.6 thanks to item (i) in the corresponding statement. This concerns vector fields as opposed to 1-dimensional foliations.

In fact, it is not a foliation but rather some holomorphic vector field that is the object of primary interest in many problems and applications of singularity theory. Examples of this situation are provided in Section 5.1 and throughout Section 6. Naturally a vector field X gives rise to an 1-dimensional foliation $\mathcal{F}$ of which a birational model whose all singular points are elementary may be useful. Nonetheless, if the vector field X is the object of primary interest, then the transforms of X have to be considered as well. At this point, the difference between vector fields and 1-dimensional foliations is summarized by the following self-evident statement: the transform of a 1-dimensional holomorphic foliation under a rational map is another 1-dimensional holomorphic foliation, however, the transform of a holomorphic vector field under a rational map *is, in general, a meromorphic vector field.*

When applying resolution theorems for 1-dimensional foliations to the study of vector fields it is therefore relevant to seek to retain the "good" analytic properties of them as much as possible (again concrete examples are provided in Sections 5 and 6). For example, if we start with a holomorphic vector field X, we might hope that its transform at the end of a resolution procedure still is a holomorphic vector field. In this context, we have:

Claim. The transform of a holomorphic vector field under a resolution procedure as in Theorem 4.6 or in Theorem 4.7 still is a holomorphic vector field.

It is a basic fact that a (standard or unramified) blow-up of a holomorphic vector field is again holomorphic provided that the blow-up is centered at the singular locus of the vector field in question. Slightly more generally, for the blow-up of a holomorphic vector field to be holomorphic again it suffices to have the blow-up center invariant by the initial vector field. Thus the claim holds in the case of Theorem 4.7.

In the case of Theorem 4.6 this is not immediate as a weighted blow-up may turn a holomorphic vector field into a *meromorphic one even if its center is contained in the singular set of the initial vector field.* For example, consider the holomorphic vector field $X = y\partial/\partial x + G(x, y, z)\partial/\partial y + H(x, y, z)\partial/\partial z$ where G and H are such that the z-axis $\{x = y = 0\}$ coincides with the singular set of X. Let Π be the blow-up of weight 2 centered at the z-axis. Let (x, t, z) be coordinates where Π becomes $\Pi(x, t, z) = (x^2, tx, z)$. Then the transform Π^*X of X is given by

$$\Pi^* X = \frac{t}{2}\frac{\partial}{\partial x} + \left[\frac{-t^2}{2x} + \frac{1}{x} G(x^2, tx, z)\right]\frac{\partial}{\partial t} + H(x^2, tx, z)\frac{\partial}{\partial z}\,.$$

Since $G(x^2, tx, z)/x$ is *holomorphic*, and so is $H(x^2, tx, z)$, it follows that the vector field X has poles over the exceptional divisor locally given by $\{x = 0\}$.

It is exactly in the context of the difficulty illustrated by the above example that *the condition of having centers that are called strictly invariant with respect to the quasi-homogeneous filtration in question, as used in* [38] *and reproduced in the first part of* [36]*, comes into play (see item (i) in Theorem 4.6).* If slightly technical, this condition ensures that the transforms of holomorphic vector fields remain holomorphic.

5. Invariant analytic sets

The problem of existence of *invariant manifolds* has always been a central theme in the theory of dynamical systems. Among the many reasons for this, there is the simple fact that they single out orbits that are contained in a manifold of smaller dimension and, in this sense, potentially easier to understand. In particular, invariant manifold of dimension 1 actually encode special solutions of the system that are amenable to detailed analysis and that may shed light into the behavior of other solutions. For example, in the general theory of hyperbolic systems, the so-called stable manifolds are examples of invariant manifolds and, in fact, their existence form a cornerstone of the theory of hyperbolic dynamical systems. The existence of stable manifolds for hyperbolic singular points is a consequence of the general theory and ensured by the well-known Stable Manifold Theorem. However, "stable" invariant manifolds may fail to exist if the singular point is no longer hyperbolic. The simplest example is provided by the vector field

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

all of whose integral curves are circles around the origin of $\mathbb{R}^2$. Clearly there is no invariant manifold in this case.

The general problem of existence of invariant manifolds may also be considered in the context of holomorphic dynamics. In this case, we look for invariant complex-analytic objects, which is a much stronger regularity condition. We allow, however, these objects to be singular in the sense of analytic sets. In other words, we look for *invariant varieties*, as opposed to actual *manifolds.* In the sequel, the word "manifold" will be saved for smooth objects.

As mentioned in section 2, Briot and Bouquet were the first to consider the problem of existence of separatrices for holomorphic vector fields defined on a neighborhood of the origin of $\mathbb{C}^2$ in [1]. However, they were not able to establish the existence of separatrices for all holomorphic vector fields on $(\mathbb{C}^2, 0)$. This question was settled only much later by Camacho and Sad in their remarkable paper [4] where the following is proved:

Theorem 5.1. [4] *Let $\mathcal{F}$ be a singular holomorphic foliation defined on a neighborhood of the origin of $\mathbb{C}^2$. Then there exists an analytic invariant curve passing through $(0, 0)$ and invariant by $\mathcal{F}$.*

Theorem 5.1 is well worth a few additional comments, namely:

- It is somehow surprising that separatrices for holomorphic vector fields on $(\mathbb{C}^2, 0)$ always exist despite the much stronger regularity condition for the invariant curve. For example, for the holomorphic vector field $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ defined on $(\mathbb{C}^2, 0)$, the separatrices are given by the two complex lines $y = \pm ix$ and hence are totally contained in the non-real part of $\mathbb{C}^2$ (bar the singular point itself).
- However, as mentioned, we do not require the separatrices to be smooth invariant curves otherwise no general existence statement would hold. An example of this is provided by the holomorphic vector field $2y\partial/\partial x + 3x^2\partial/\partial y$ on $(\mathbb{C}^2, 0)$. Since this vector field admits $f(x, y) = x^3 - y^2$ as first integral, it follows that the only separatrix of X is the cusp of equation $\{x^3 - y^2 = 0\}$ which is singular at the origin. Fortunately, allowing separatrices to be singular is not a problem, since they can always be desingularized.
- Also it is important to emphasize that Theorem 5.1 applies only to foliations defined on smooth ambients. In fact, if germs of foliations defined on singular surfaces are considered, then separatrices may fail to exist as shown by Camacho in [3].

The existence of separatrices is, however, no longer a general phenomenon in dimension 3, regardless of the dimension of the foliation. As already said, a first example of codimension 1 foliation on $(\mathbb{C}^3, 0)$ without separatrices was provided by Jouanolou in [25]. Jouanolou's example essentially hinging from the core dynamics of (dicritical) foliations on $(\mathbb{C}^3, 0)$, the same idea enables us to construct plenty of additional examples of codimension 1 foliations without separatrices (cf. Section 3.2 or the summary below).

Concerning 1-dimensional foliations, examples of foliations without separatrices in dimension 3 were found by Gomez-Mont and Luengo, [15]. Their work will be discussed in Section 5.2. For the time being, we will focus on the problem of invariant manifolds for codimension 1 foliations.

5.1. **Separatrices for codimension 1 foliations induced by pairs of commuting vector fields.** Let us begin by recalling/summarizing the discussion in Section 3.2 where it was shown how Jouanolou's method can be used to produce many examples of codimension 1 foliations without separatrix on $(\mathbb{C}^3, 0)$. This is as follows.

(i) Every homogeneous polynomial vector field X on $\mathbb{C}^3$ that is not a multiple of the Radial vector field induces a foliation on $\mathbb{CP}^2$ corresponding to the so-called core foliation associated with X. Conversely, given a foliation on $\mathbb{CP}^2$, there exists a homogeneous polynomial vector field on $\mathbb{C}^3$ whose core foliation is the given one.

(ii) Let X be a homogeneous vector fields which is not a multiple of the Radial vector field R. There follows from the well-known Euler relation that X and R generates a Lie algebra isomorphic to the Lie algebra of the affine group. In particular, the distribution generated by X and R can be integrated to yield a dicritical codimension 1 foliation $\mathcal{D}$. Furthermore, the core foliation associated with $\mathcal{D}$ coincides with the core foliation associated with X.

(iii) Finally, for every fixed degree, Theorem 3.9 ensures the existence of a (non-empty) open set of foliations on $\mathbb{CP}^2$ such that all leaves of each foliation $\mathcal{F}$ in this set are dense. In particular, no foliation in this set admits algebraic invariant curves. The codimension 1 foliations generated by R and by the homogeneous vector field having one such foliation as core foliation has no separatrix. Note that this conclusion is essentially Jounolou's theorem in [25] except for the fact that Jounolou's original argument does not immediately imply that the set in question has non-empty interior.

In view of what precedes, it is natural to wonder if all example of codimension 1 foliations without separatrices are to be found among dicritical ones. In ambient spaces of dimension 3 this, in fact, holds as proved by Cano and Cerveau in [6]. Their result can be stated as follows.

Theorem 5.2. [6] *Let $\mathcal{D}$ be a germ of a holomorphic singular codimension 1 foliation on $(\mathbb{C}^3, 0)$. If $\mathcal{D}$ is not dicritical, then it admits a separatrix.*

The proof of Theorem 5.2 relies heavily on a resolution theorem for non-dicritical codimension 1 foliations obtained by the authors in the same paper. Note, however, that the non-dicritical assumption, implies that the transforms of the initial codimension 1 foliation leave every irreducible component of the exceptional divisor invariant. In other words, away from singular points, every irreducible component of the exceptional divisor is a leaf of the foliations in question. This rules out the existence of any meaningful core dynamics and makes the problem very much comparable to the 2-dimensional situation handled by Camacho and Sad in [4]: a problem of more geometric nature (and relatively little dynamical content).

In a different direction, experts including F. Cano, D. Cerveau, and L. Stolovitch have since long wondered what would be the "correct generalization" of Camacho-Sad theorem for $(\mathbb{C}^3, 0)$, already at level of codimension 1 foliations. In particular, the idea that a codimension 1 foliation spanned by a pair of commuting vector fields (not everywhere parallel) might necessarily admit separatrices was advanced. The question is settled by the theorem below which confirms their intuition.

Theorem 5.3. [43] *Consider holomorphic vector fields X, Y defined on a neighborhood of the origin of $\mathbb{C}^3$. Suppose that they commute and are linearly independent at generic points (so that they span a codimension 1 foliation denoted by $\mathcal{D}$). Then $\mathcal{D}$ possesses a separatrix.*

The remainder of this paragraph is devoted to single out a few issues involved in the proof of Theorem 5.3. This illustrates several points made in the preceding sections, including the usefulness of "taming" core dynamics (and how "symmetries" may be exploited to this effect) and the role of resolutions theorems. Concerning the latter, the argument will also highlight the importance of having actual vector fields - rather than mere foliations - being "nicely" transformed during the resolution procedure.

The first ingredient in the proof of Theorem 5.3 is therefore a general resolution of singularities theorem for codimension 1 foliations in dimension 3. Compared to Theorem 5.2, the main result in [6] is arguably a theorem of reduction of singularities for the foliations in question *under the additional condition* that the foliation should be non-dicritical. Fortunately, Cano has obtained in [7] a general resolution theorem for codimension 1 foliations on $(\mathbb{C}^3, 0)$ which applies equally well to dicritical foliations.

Armed with Cano's theorem [7], we see that the basic obstacle for the existence of separatrices lies in the core dynamics by means of the phenomenon already pointed out in Jouanolou examples, cf. Section 3. The central point in the proof of Theorem 5.3 is therefore to "tame" the core dynamics arising from dicritical divisors the resolution procedure applied to $\mathcal{D}$ will have "plenty of algebraic curves". In the sequel, we shall indicate some simple ideas used to show that the mentioned core dynamics cannot be "too wild".

Let us consider the simplest case where we want to blow-up the origin (a degenerate singular point of $\mathcal{D}$). The first lemma shows that at least one between the vector fields X and Y have to induce a foliation on the resulting exceptional divisor, unless we have a truly very special situation that is essentially "linear" (which can directly be handled).

Recalling that $\mathcal{D}$ is spanned by the commuting vector fields X and Y, let $\mathcal{F}_X$ (resp. $\mathcal{F}_Y$) denote the 1-dimensional singular foliation associated with X (resp. Y).

Lemma 5.4. *Assume that the first jets of X and of Y at the origin are equal to zero. Then none of the foliations $\mathcal{F}_X$ or $\mathcal{F}_Y$ is dicritical for the blow-up π of $\mathbb{C}^3$ at the origin.*

Proof. Denote by $\widetilde{X}$ and $\widetilde{Y}$ the blow-ups of X and Y at the origin. Similarly, $\widetilde{\mathcal{F}}_X$ and $\widetilde{\mathcal{F}}_Y$ will stand for the blow-ups of the foliations $\mathcal{F}_X$ and $\mathcal{F}_Y$. Since the vector fields X and Y have zero linear part at the origin, there follows that both $\widetilde{X}$ and $\widetilde{Y}$ vanish identically over the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{CP}^2$. Now assume that, say, X is dicritical for π. Then the leaf of $\widetilde{\mathcal{F}}_X$ is regular and transverse to $\pi^{-1}(0)$ at generic points of $\pi^{-1}(0)$. Therefore, around one such point, we can choose local coordinates (u, v, w) such that $\{u = 0\} \subset \pi^{-1}(0)$ and where $\widetilde{\mathcal{F}}_X$ is represented by the (regular) vector field $\partial/\partial u$. In particular the blow-up $\widetilde{X}$ takes on the form $f(u, v, w)\partial/\partial u$ where f is a holomorphic function (divisible by u). In these coordinates, let the blow-up $\widetilde{Y}$ be given by $\widetilde{Y} = f_1\partial/\partial u + f_2\partial/\partial v + f_3\partial/\partial w$. Since $[\widetilde{X}, \widetilde{Y}] = 0$, there follows that f_2 and f_3 do not depend on the variable u. However, these functions must vanish identically for $u = 0$ since $\widetilde{Y}$ vanishes identically over $\pi^{-1}(0)$ (locally given by $\{u = 0\}$). Thus they must vanish identically over an open set and this contradicts the fact that X and Y span a codimension 1 foliation. □

Remark 5.5. The argument above shows the importance of transforming vector fields, as opposed to foliations, in certain cases. In fact, the proof of Lemma 5.4 hinges from the fact that the transform of the vector field Y vanishes identically over the exceptional divisor $\pi^{-1}(0)$ - something that does not make sense for a foliation since the singular set of the latter has codimension at least 2.

Along similar lines, to ensure that the transformed vector field $\widetilde{Y}$ vanishes identically over $\pi^{-1}(0)$, the fact that the origin (center of the blow-up) is contained in the singular set of $\mathcal{F}_X$ (or more generally, the center of the blow-up is invariant under the foliation) was implicitly used. This is in line with the discussion in Section 4.4. It is often important that the transformed vector field retains its holomorphic character. In addition, in quite a few cases, it is also important that the zero-divisor of the transformed vector field contains all components of the exceptional divisor arising from the resolution procedure.

Plenty of additional examples of this issue can be found in the theory of semicomplete vector fields, see for example [14], [18], or [19].

Unless otherwise mentioned, in the sequel we systematically assume that blow-ups are dicritical for the codimension 1 foliation $\mathcal{D}$ spanned by X and Y.

Lemma 5.4 shows that both $\mathcal{F}_X$ or $\mathcal{F}_Y$ must induce a foliation on $\mathbb{CP}^2$. If the foliations induced on $\mathbb{CP}^2$ are different, then Lemma 3.4 in Section 3 can be used to prove that $\pi^{-1}(0)$ is invariant by $\mathcal{D}$. Hence we can assume without loss of generality that they do coincide. Recalling that the *order* of a vector field at a singular point p is nothing but the degree of the first non-zero homogeneous component of its Taylor series based at the point in question, the preceding implies:

Lemma 5.6. *The orders at the origin of X and Y can be assumed to be different.*

Proof. Assume that X and Y have the same order at the origin. Because they induce the same foliation on $\mathbb{CP}^2$, they will differ by a multiple of the Radial vector field (up to multiplying, say X, by a non-zero constant). Hence, by considering $Z = X - Y$, there follows that the foliation $\mathcal{D}$ is still spanned by X and Z. Moreover X and Z satisfy $[X, Z] = 0$. Moreover the order of Z at the origin is equal to or greater than the order of X at the origin, since X and Y have the same order. We claim that the order of Z must be strictly greater than the order of X which, in turn, establishes the lemma up to replacing Y by Z. To check the claim, just notice that if the order of Z were equal to the order of X, then the first non-zero homogeneous component of Z would be a multiple of the radial vector field. Thus Z would be diritical which contradicts Lemma 5.4 and completes our proof. □

Denote by X^H (resp. Y^H) the first non-zero homogeneous component of X (resp. Y) at the origin. Owing to the above lemma, we can assume that the degree of Y^H is *strictly greater* than the degree of X^H. Regardless of degree conditions, by assumption, the blow-up π of the origin of $\mathbb{C}^3$ is dicritical for $\mathcal{D}$ so that the foliations induced on $\pi^{-1}(0) \simeq \mathbb{CP}^2$ coincide. In other words, the core dynamics of X^H coincides with the core dynamics of Y^H and they both coincide with the core dynamics of the codimension 1 foliation $\mathcal{D}$. Now, Proposition 5.7 below provides some serious control on the core dynamics in question and, along with its analogue for blow-ups centered at curves, constitutes a fundamental starting point of the discussion conducted in [43].

Proposition 5.7. *The vector field X^H admits a non-constant meromorphic/holomorphic first integral.*

Proof. Owing again to the discussion in Section 3, the dicritical nature of $\mathcal{D}$ ensures the existence of holomorphic functions f and g such that

$$fX + gY = Z\,, \tag{6}$$

with Z being a holomorphic vector field whose first non-zero homogeneous component at the origin is a multiple of the Radial vector field R. Denoting by f^H, g^H the first non-zero homogeneous components of f, g, there follows from the preceding that $f^H X^H$ and $g^H Y^H$ must have the same degree. Furthermore, we have a homogeneous equation

$$f^H X^H + g^H Y^H = h^H R \tag{7}$$

where h^H is a homogeneous polynomial - possibly identically zero. In the sequel we assume that h^H does not vanish identically since it is easy to adapt the discussion below to cover this case as well.

Because X, Y commute, so do X^H, Y^H. Thus we have

$$\begin{aligned}
[X^H, Y^H] &= \left[X^H, \frac{h^H}{g^H} R - \frac{f^H}{g^H} X^H\right] \\
&= \left[X^H . \left(\frac{h^H}{g^H}\right)\right] R - \frac{h^H}{g^H}[R, X^H] - \left[X^H . \left(\frac{f^H}{g^H}\right)\right] X^H \\
&= \left[X^H . \left(\frac{h^H}{g^H}\right)\right] R - \left[(d-1)\frac{h^H}{g^H} - X^H . \left(\frac{f^H}{g^H}\right)\right] X^H \\
&= 0
\end{aligned}$$

where d stands for the degree of X^H. In particular

$$\left[X^H . \left(\frac{h^H}{g^H}\right)\right] R = \left[(d-1)\frac{h^H}{g^H} - X^H . \left(\frac{f^H}{g^H}\right)\right] X^H \,.$$

The expression between brackets on the left hand side (i.e. the expression multiplying R) must vanish identically for otherwise X^H would be a multiple of the Radial vector field R. It then follows that

$$X^H . \left(\frac{h^H}{g^H}\right) = 0\,.$$

In other words, h^H/g^H is a meromorphic (possibly holomorphic) first integral for X^H.

It only remains to prove that h^H/g^H is not constant. However, if this function is constant (different from zero since h^H does not vanish identically), then we can assume $h^H/g^H = 1$ without loss of generality. Hence dividing (7) by g^H, it would follow

$$\frac{f^H}{g^H} X^H + Y^H = R.$$

This last equation is however impossible since Y^H has degree at least 2 and the expression $f^H X^H/g^H$ is homogeneous. Therefore h^H/g^H cannot be constant. Since the argument is symmetric in the vector fields X, Y, the last assertion completes our proof. □

The key to prove Theorem 5.3 is to observe that the core dynamics of dicritical components of a foliation like $\mathcal{D}$ must leave invariant certain algebraic curves. Clearly Proposition 5.7 along with some refinements play a role in this proof.

However, it is also clear that the discussion leading to Proposition 5.7 depends heavily on the vanishing assumption for the first jet of X, Y at the origin. This issue requires to consider separately some special situations that are referred to as "linear foliations" in the sense that there is a *non-zero first jet involved.* Not surprisingly, "linear foliations" can be dealt with through rather direct methods.

Finally, as it is inevitable in dimension 3, every desingularization procedure requires *two types of blow-ups*: beyond blow-ups centered at points, blow-ups centered at curves are needed as well. In particular, another basic ingredient in the proof of Theorem 5.3 will be analogues, both in "linear" and "non-linear" settings, of the previous results. This issue has already appeared in Section 4 (cf. the discussion about Equation 3) and can easily lead to misunderstandings so that it seems convenient to close this paragraph by carefully explaining the appropriate formulations.

The following example was pointed out to us by D. Cerveau. It helps to explain the notion of "zero first jet" in the case of blow-ups centered at smooth curves. The example also highlights difficulties related to the existence of first integrals (cf. Proposition 5.7 and a few other intermediary results used in [43] and not explicitly mentioned here).

Example 1. Consider the pair of vector fields X, Y given by

$$X = zy\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z} \qquad \text{and} \qquad Y = x^2\frac{\partial}{\partial x} + axy\frac{\partial}{\partial y}$$

which are quadratic at the origin. Note that these two vector fields commute so that they span a codimension 1 foliation denoted by $\mathcal{D}$.

The axis $\{y = z = 0\}$ is invariant by both X and Y. This axis is also contained in the singular set of $\mathcal{D}$. Let us then consider the blow-up of $\mathbb{C}^3$ centered at the axis $\{y = z = 0\}$ along with the corresponding transforms of $\mathcal{D}$, X, and Y. It is immediate to check $\mathcal{D}$ is dicritical for the blow-up in question. Similarly the foliation $\mathcal{F}_X$ associated with X is also dicritical for this blow-up which might lead to some confusion with Lemma 5.4.

The explanation for this example lies in the fact that the vector field Y is regular (non-zero) at generic points of the axis $\{y = z = 0\}$. Similarly, its transform under the previous blow-up is regular at generic points of the exceptional divisor. In other words, this case must be considered as a "linear one" and the order of Y with respect to this blow-up must, indeed, be equal to *zero*. An adequate definition of the order of a vector field with respect to the center of a blow-up is included below.

Let us then provide an accurate definition of *order* of a vector field when a curve, as opposed to a single point, is blown-up. To explain the idea, consider first a holomorphic vector field X with a singular point at the origin along with the corresponding Taylor series. The order of X

at the origin is said to be the degree of the first non-zero homogeneous component of its Taylor expansion. This can also be viewed as the integer d for which the limit

$$\lim_{\lambda\to 0} \frac{1}{\lambda^{d-1}} \Gamma_\lambda^* X$$

yields a (non identically zero) holomorphic vector field. Here $\Gamma_\lambda^* X$ stands for the pull-back of X by the homothety $\Gamma_\lambda : (x,y,z) \mapsto (\lambda x, \lambda y, \lambda z)$. Note that the limit above corresponds to the first non-zero homogeneous component of the Taylor's expansion of X at the origin. Next, assume now that $C = \{y = z = 0\}$ is contained in the singular set of X so that the blow-up centered along this curve of singular points will be considered. The order of X with respect to C is defined as the integer d for which

$$\lim_{\lambda\to 0} \frac{1}{\lambda^{d-1}} \Lambda_\lambda^* X$$

is a (non identically zero) holomorphic vector field, where $\Lambda_\lambda^* X$ denotes the pull-back of X by the homothety $\Lambda_\lambda : (x,y,z) \mapsto (x, \lambda y, \lambda z)$. The limit above, for the appropriate choice of d, is said to be the first non-zero homogeneous component of X with respect to the variables y, z. In general, the cases in [43] that are called *linear* are those cases in which the vector field has order 1 or zero, with respect to the center of the blow-up in question. In particular, with the above definition, it can immediately be checked that the vector field Y of Example 1 has order *zero* with respect to $C = \{y = z = 0\}$, although its order at the origin is 2.

5.2. **Separatrices for foliations of dimension** 1. This paragraph is devoted to discussing in detail the problem about existence of separatrices *for* 1*-dimensional foliations*. Contrasting with the case of codimension 1 foliations, it will soon be seen that the influence of core dynamics in the existence of these separatrices is rather limited. In fact, the existence of separatrices for foliations of dimension 1 is an phenomenon having, in a suitable sense, a *very local nature*: it essentially hinges from two basic ingredients, namely:

- The analysis of simple singularities which is basically conducted by direct methods involving normal forms and divergent series.
- Geometric considerations involving the relative positions of the simple singularities in question.

In this regard, and provided that a convenient resolution of singularities theorem is available, the problem somehow retains the same nature regardless of the dimension. More precisely, the difficulties arising from increasing the dimension stem either from the evident fact that simple singularities are not always easy to describe (e.g. saddle-nodes of high codimension) and from the fact that the number of possible arrangements of their relative positions increase as well.

As previously said, after Camacho-Sad theorem in [4] establishing the existence of separatrices for every foliation on $(\mathbb{C}^2, 0)$, Gomez-Mont and Luengo found a foliation on $(\mathbb{C}^3, 0)$ that admits no separatrix. Let us begin by providing an outline of their construction.

5.3. **On Gomez-Mont and Luengo counterexample.** Their example of foliation without separatrix on $(\mathbb{C}^3, 0)$ relies on two simple remarks. Consider a foliation $\mathcal{F}$ on $(\mathbb{C}^3, 0)$ given by a holomorphic vector field satisfying the following conditions

1. The origin $(0,0,0) \in \mathbb{C}^3$ is an isolated singularity of X
2. $J^1 X(0,0,0) = 0$ but $J^2 X(0,0,0) \neq 0$, where $J^k X(0,0,0)$ stands for the jet of order k of X at the origin ($k = 1, 2$).
3. The quadratic part X^2 of X at $(0,0,0)$ is a vector field whose singular set has codimension 2. Also X^2 is not a multiple of the Radial vector field $x\partial/\partial x + y\partial/\partial y + z\partial/\partial z$.

Assume that $\mathcal{F}$ has a separatrix C and consider the blow-up $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ centered at the origin. Denote by π the blow-up map so that $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ and let $\pi^{-1}(0)$ denote the exceptional divisor isomorphic to $\mathbb{CP}^2$. Since X^2 is not a multiple of the Radial vector field, there follows that $\pi^{-1}(0)$ is invariant by $\widetilde{\mathcal{F}}$ so that the restriction of $\widetilde{\mathcal{F}}$ to $\pi^{-1}(0)$ can be seen as a foliation of degree 2 on $\mathbb{CP}^2$ (cf. item (3)).

Because $\pi^{-1}(0)$ is invariant by $\widetilde{\mathcal{F}}$, the transform $\pi^{-1}(C)$ of the separatrix C can only intersect $\pi^{-1}(0)$ at singular points of $\widetilde{\mathcal{F}}$. Furthermore, all of these singular points lie in $\pi^{-1}(0)$ since X has an isolated singularity at the origin. In other words, $\pi^{-1}(C)$ must be a separatrix (not contained in $\pi^{-1}(0)$) for one of the singular points of $\widetilde{\mathcal{F}}$.

Now, the second ingredient is as follows: as a foliation of degree 2 on $\mathbb{CP}^2$, the restriction $\widetilde{\mathcal{F}}|_{\pi^{-1}(0)}$ of $\widetilde{\mathcal{F}}$ to $\pi^{-1}(0)$ has at most (and generically) 7 singular points. Since it is hard to control the position of 7 points in $\mathbb{CP}^2$, the authors of [15] started from a foliation satisfying the following conditions:

(A) The foliation of degree 2 has only 3 singular points (we can think of the foliation as obtained by letting some of the 7 singular points of a generic quadratic foliation to "collide in groups"). Naturally the position of 3 points in $\mathbb{CP}^2$ can easily be controlled.
(B) Each of the 3 singular points will have an eigenvalue equal to zero in the direction transverse to $\pi^{-1}(0)$. The 3 singular points are therefore saddle-node singularities (in dimension 3).
(C) Furthermore, the authors arrange for the saddle-node singularities to have two equal (and non-zero) eigenvalues tangent to $\pi^{-1}(0)$. In other words, the singular points in question are (codimension 1) resonant saddle-nodes with weak direction transverse to $\pi^{-1}(0)$.
(D) As is well known, it is easy to produce examples of codimension 1 saddle-nodes all of whose separatrices are included in an invariant (2-dimensional) plane tangent to the directions of the non-zero eigenvalues.

The remainder of the proof in [15] consists of showing that it is, indeed, possible to prescribe a quadratic X^2 and a cubic X^3 homogeneous components for the vector field X so as to satisfy all of the preceding conditions. In this respect, note that conditions (A), (B), and (C) depend only on the quadratic part X^2. The role played by the appropriately chosen cubic parte X^3 can be summarized as follows.

- it ensures that each of the singular points of $\widetilde{\mathcal{F}}$ are isolated, hence coinciding with the corresponding singular points of $\widetilde{\mathcal{F}}|_{\pi^{-1}(0)}$. Here the reader may note that the homogeneous foliation associated with X^2 has singularities all along the fibers of $\widetilde{\mathbb{C}}^3 \to \widetilde{\pi}^{-1}(0)$ sitting over the singular points of $\widetilde{\mathcal{F}}|_{\pi^{-1}(0)}$. A higher order perturbation of X^2 is thus needed to provide isolated singular points for the blown-up foliation.
- having ensured the singular points are isolated, the cubic part X^3 of X also takes care of condition (D)

As mentioned, the verification that all these conditions are compatible is conducted in [15] with the assistance of suitable software to deal with formal computations.

5.4. **Vector fields and 2-dimensional Lie algebras.** In [43], codimension 1 foliations spanned by pairs of commuting vector fields were considered and it was shown that this condition imposes strong constraints on the core dynamics of dicritical components of the codimension 1 foliation in question. In particular, these constraints have proved to be strong enough to yield the existence of separatrices for the foliation in question.

In view of the preceding, it was natural to wonder if the 1-dimensional foliations arising from the vector fields in question would have separatrices themselves. While the answer turned out to be affirmative, the assumption of having two commuting vector fields can be weakened to encompass also the case of pairs of vector fields generating the Lie algebra of the affine group. In fact, the following theorem was proved in [46]:

Theorem 5.8. [46] *Let X and Y be two holomorphic vector fields defined on a neighborhood U of $(0,0,0) \in \mathbb{C}^3$ which are not linearly dependent on all of U. Suppose that X and Y vanish at the origin and that one of the following conditions holds:*

- $[X,Y]=0$*;*
- $[X,Y]=c\,Y$*, for a certain* $c \in \mathbb{C}^*$*.*

Then there exists a germ of analytic curve $\mathcal{C} \subset \mathbb{C}^3$ passing through the origin and simultaneously invariant under X and Y.

The theorem above deserves a few additional comments.

- First, the fact that Theorem 5.8 applies to pair of vector fields generating the Lie algebra of the affine group is in stark contrast with the analogous problem for codimension 1 foliations. Indeed, every homogeneous vector field of degree at least 2 together with the Radial vector field generate the Lie algebra of the affine group. In particular, Jouanolou's and similar examples of codimension 1 foliation without separatrices arise from pairs of vector fields generating the Lie algebra of the affine group.
- Whereas theorems asserting the existence of separatrices for foliations of dimension 1 holds interest in their own right, they also have non-trivial applications in the general problem of understanding globally defined holomorphic vector fields on compact complex manifolds, see Section 6.2. In particular, the paper [46] also includes a non-trivial applications of Theorem 5.8 in this direction.
- Finally, a relatively minor but yet subtle issue that is worth pointing out is that Theorem 5.8 claims that X and Y possess a common invariant curve without *asserting that the curve in question is invariant by the foliations associated with X and Y*. To further clarify the issue, it is enough to think of the 2-dimensional vector field $x\partial/\partial x$: the axis $\{x=0\}$ is invariant by the vector field but does not constitute a separatrix for the associated foliation. In turn, it might be asked if the foliations associated with X and Y share an actual separatrix, possibly enlarging the notion of "separatrix" to include curves fully constituted by singular points of the corresponding foliation. In particular, it is easy to check that the existence of "common separatrices" always holds when X is a homogeneous vector field and Y is the Radial vector field. Indeed, in this case the leaves of $\mathcal{F}_Y$ are simply the Radial lines. Concerning $\mathcal{F}_X$, since it is not a multiple of the Radial vector field, it induces a 1-dimensional foliation on $\mathbb{CP}^2$ by means of the one-point blow-up of $\mathbb{C}^3$ at the origin. The foliation in question possesses isolated singular points and it can easily be checked that the Radial line naturally associated with any of these singular points is invariant by $\mathcal{F}_X$ as well. We believe that the existence of a common separatrix for $\mathcal{F}_X$ and $\mathcal{F}_Y$ in the general case can also be established.

To finish the section, let us provide an outline of the proof of Theorem 5.8.

Sketch of Proof of Theorem 5.8. Recall that the foliation associated with X (resp. Y) is denoted by $\mathcal{F}_X$ (resp. $\mathcal{F}_Y$). Let $\mathcal{D}$ denote the codimension 1 foliation spanned by X and Y. We have that $\text{codim}\,(\text{Sing}\,(\mathcal{D})) \geq 2$. In other words, $\text{Sing}\,(\mathcal{D})$ is of one of the following types: the union of a finite number of irreducible curves, a single point (the origin), or simply empty (i.e. $\mathcal{D}$ is regular). Since $\text{Sing}\,(\mathcal{D})$ is naturally invariant by X and by Y, the result immediately holds if $\dim(\mathrm{Sing}(\mathcal{D})) = 1$. Hence we can assume without loss of generality that $\mathrm{Sing}(\mathcal{D})$ has codimension at least 3. In other words, either $\mathrm{Sing}(\mathcal{D})$ is reduced to the origin or it is, in fact, empty.

Since the singular set of $\mathcal{D}$ has codimension at least 3, Malgrange Theorem [28] implies that $\mathcal{D}$ possesses a non-constant holomorphic first integral f. Let then $S = f^{-1}(0)$ so that S is an invariant surface for $\mathcal{D}$, i.e. the irreducible components of S are separatrices for $\mathcal{D}$. In particular, S is invariant by both X and Y. Next, note that S can be assumed to be irreducible. Otherwise, the intersection of any two irreducible components of S yields a curve invariant under both X, Y and the conclusion holds. The surface S can then be assumed either regular or having an isolated singularity at the origin (again if S contains a curve of singular points this curve must be invariant by X and Y). At this point, a couple of remarks are in order:

- In the case where S is smooth, each of the foliations $\mathcal{F}_X$ and $\mathcal{F}_Y$ possesses separatrices owing to Camacho-Sad Theorem [4]. Still it remains to check that these foliations share a common separatrix.
- As previously mentioned, in the case of singular surfaces, there are examples of foliations without separatrix (cf. [3] or [15]). This phenomenon needs thus to be ruled out in the present case.

In general, we proceed as follows. Consider the restrictions of X and Y to S. The tangency locus on $\mathrm{Tang}(X|_S, Y|_S) \subset S$ in S of X and Y is defined to be the set of points $p \in S$ at which the vectors $X(p)$ and $Y(p)$ are linearly dependent. Note that $\mathrm{Tang}(X|_S, Y|_S) \neq \emptyset$ since both X and Y vanish at the origin. Furthermore, the tangency locus $\mathrm{Tang}(X|_S, Y|_S)$ is invariant by both X and Y so that the result holds provided that the dimension of $\mathrm{Tang}(X|_S, Y|_S)$ equals 1. So, we shall consider separately the case where $\mathrm{Tang}(X|_S, Y|_S) = \{(0,0,0)\}$ and the case where $\mathrm{Tang}(X|_S, Y|_S) = S$.

Assuming that $\mathrm{Tang}(X|_S, Y|_S)$ is reduced to the origin. Then S is a surface with singular set of codimension at least 2 and equipped with two vector fields that are linearly independent away from this an analytic set of codimension 2 or greater. This implies that tangent sheaf to S is locally trivial which, in turn, implies that S is smooth since S is a hypersurface in $\mathbb{C}^3$. However, being smooth, S is locally equivalent to $\mathbb{C}^2$ and the tangency locus of two vector fields cannot be reduced to a single point. The resulting contradiction rules out this case.

Assume now that $\mathrm{Tang}(X|_S, Y|_S) = S$, i.e. the restrictions to S of X and Y coincide up to a multiplicative function (defined on S). The existence of the desired common separatrix is then ensured in the case where S is smooth by Camacho-Sad theorem. It only remains to consider the case where S has an isolated singular point at the origin. The argument in this case relies on proving that the (1-dimensional) foliation induced on S by either X or Y possesses a non-constant holomorphic first integral. The level curve of this first integral containing the origin then yields the desired separatrix. Details can be found in [46]. □

6. Semicomplete vector fields, automorphism groups, and separatrices

The object of this last section is a distinguished class of singularities of vector fields, namely the *semicomplete (singularities of) vector fields*. Understanding this class of vector fields, both at global level and at level of germs, is a problem with interesting applications. As an example of application, we will see in Section 6.2 that results on singularities of semicomplete vector fields yield insight in some problems about bounds for the dimension of automorphism group of compact complex manifolds. Another motivation to study these vector fields and their singular points stems from the very fact that the *semicomplete property* is somehow akin to the Painlevé property for differential equations, albeit the two notions are not equivalent. As a matter of fact, as it happens with Painlevé property, semicomplete vector fields are also largely present - sometimes implicitly - in the literature of Mathematical Physics.

The notion of *semicomplete singularity* was introduced in [40]. The idea begins with the definition of semicomplete vector fields on general open sets which is as follows.

Definition 6.1. [40] A holomorphic vector field X defined on an open set U of some complex manifold M is said to be semicomplete (on U) if for every $p \in U$ there exists a connected domain $V_p \subset \mathbb{C}$, with $0 \in V_p$, and a map $\phi_p : V_p \to U$ satisfying the following conditions:

- $\phi_p(0) = p$
- $\phi_p'(T) = X(\phi_p(T))$, for every $T \in V_p$.
- For every sequence $\{T_i\} \subset V_p$ such that $\lim_{i\to\infty} T_i = \hat{T} \in \partial V_p$ the sequence $\{\phi_p(T_i)\}$ escapes from every compact subset of U.

The third condition in Definition 6.1 basically means that $\phi_p : V_p \to U$ is a maximal solution of X in a sense similar to the notion of "maximal solutions" commonly used for real vector field and/or differential equations. In this sense, the definition is equivalent to saying that a vector field is semicomplete if for every $p \in U$ the integral curve ϕ satisfying $\phi(0) = p$ has a *maximal domain of definition in* $\mathbb{C}$. Closely connected to the notion of maximal domain of definition in $\mathbb{C}$, we can think of a local integral curve for a vector field X and then extending it over paths which is always possible as long as we stay in the domain of definition of X. The vector field X is then semicomplete if these extensions do not give rise to any monodromy and hence can be merged together in a single (univalued) solution for X which is naturally defined on a maximal domain in $\mathbb{C}$.

Though global in essence, the above definition has also a *local character* that is singled out by the following assertion: *if a vector field X is semicomplete on U, then the restriction of X to every subset V of U is semicomplete as well.* Thus the notion of *germ of semicomplete vector field*, and hence of *semicomplete singularity*, makes sense. Furthermore, even at level of germs, the condition of being *semicomplete* is far from trivial and, in fact, imposes strong constraints on the singular points of vector fields as pointed out in [40]. As a matter of fact, since its introduction, semicomplete singularities have proved time and again that they capture almost all of the "intrinsic nature" of germs of vector fields admitting actual global realizations as complete vector fields.

Germs of holomorphic semicomplete vector fields on $(\mathbb{C}^2, 0)$ were classified by Ghys and Rebelo in the papers [40] and [14]. In particular, all these vector fields admit a non-constant holomorphic/meromorphic first integral so that the dynamics associated with them is rather simple.

After this brief introduction to semicomplete vector fields, the remainder of the section will focus on two fundamental questions related to them. The first question was somehow motivated by the results of Ghys and Rebelo in dimension 2 and asks the extent to which the condition of being semicomplete may tame the *core dynamics* of the corresponding foliation. In other words, we ask:

- Are there semicomplete vector fields exhibiting a genuinely complicated core dynamics?

The second question was raised by E. Ghys long ago and, roughly speaking, involves deciding "how degenerate" can semicomplete singular points be. A prototypical question along these lines concerns semicomplete vector fields with isolated singular points and can be formulated as follows:

- Is it true that the second jet of a semicomplete vector field at an isolated singular point is necessarily different from zero?

This question is affirmatively answered in dimension 2 in the mentioned works by Ghys and Rebelo. It remains open in higher dimension, though a number of partial results are available in dimension 3.

Whereas the interest in "taming" the core dynamics associated to singularities of vector fields has already been emphasized, let us also point out that the general question raised by E. Ghys has applications to problems about bounds for the dimension of automorphism group of compact complex manifolds. This issue will further be discussed in Section 6.2. For the time being, we will focus on the dynamics associated with semicomplete singularities.

6.1. **Semicomplete vector fields with complicated dynamics - Guillot's work** [17]**.** As previously mentioned, singularities of semicomplete vector fields have very simple dynamics in complex dimension 2. In fact, even the global behavior of semicomplete vector fields is amenable to detailed analysis, see [19], [18]. However, this is no longer the case in dimension 3 as follows from Guillot's deep work on Halphen vector fields. This paragraph is basically devoted to summarizing the main dynamical issues appearing in semicomplete Halphen vector fields while referring to [17] for a more comprehensive discussion.

Halphen vector fields were first considered by Halphen himself [20], [21]. Apart from his contribution, let us make clear that *all remaining results in this paragraph are due to Guillot and can be found in* [17]. Up to linear equivalence, Halphen vector fields form a three parameters family of homogeneous polynomial vector fields of degree 2 on $\mathbb{C}^3$ explicitly described as

$$
\begin{aligned}
X = &\left[\alpha_1 z_1^2 + (1-\alpha_1)(z_1z_2 + z_1z_3 - z_2z_3)\right] \frac{\partial}{\partial z_1} + \\
&\left[\alpha_2 z_2^2 + (1-\alpha_2)(z_1z_2 - z_1z_3 + z_2z_3)\right] \frac{\partial}{\partial z_2} + \\
&\left[\alpha_3 z_3^2 + (1-\alpha_3)(-z_1z_2 + z_1z_3 + z_2z_3)\right] \frac{\partial}{\partial z_3}
\end{aligned} \tag{8}
$$

An alternate definition pointed out in [17] which already sheds some light in the intrinsic nature of these vector fields is as follows.

Definition 6.2. A homogeneous polynomial vector field of degree 2 (a quadratic vector field for short) on $\mathbb{C}^3$ is Halphen if it satisfies the following relation

$$
[C, X] = 2R \,, \tag{9}
$$

where C stands for a constant vector field and R is the Radial vector field.

The normal form indicated in (8) is obtained as the solutions of Equation (9) for $C = \partial/\partial z_1 + \partial/\partial z_2 + \partial/\partial z_3$. Since both C and X are homogeneous, Euler relations imply that we also have $[R, C] = -C$ and $[R, X] = X$. In turn, these three relations together mean that the triplet $\{R, C, X\}$ generates the Lie algebra of $\mathrm{SL}(2, \mathbb{C})$.

Let $\mathcal{F}_X$, $\mathcal{F}_R$, and $\mathcal{F}_C$ denote the 1-dimensional foliations associated to the vector fields X, R and C, respectively. Once again, let $\widetilde{\mathbb{C}}^3$ denote the blow-up of $\mathbb{C}^3$ centered at the origin with projection $\pi : \widetilde{\mathbb{C}}^3 \to \mathbb{C}^3$. The exceptional divisor $\pi^{-1}(0)$ is isomorphic to $\mathbb{CP}^2$ and the blow-ups of $\mathcal{F}_X$, $\mathcal{F}_R$, and $\mathcal{F}_C$ will respectively be denoted by $\widetilde{\mathcal{F}}_X$, $\widetilde{\mathcal{F}}_R$, and $\widetilde{\mathcal{F}}_C$. Similarly, $\widetilde{X}$, $\widetilde{R}$, and $\widetilde{C}$ will stand for the blow-ups of X, R, and C. Next, recall that, whenever two vector fields commute, then the flow of one of them will preserve the foliation associated with the other. This simple remark hints at a basic property of Halphen vector fields. Indeed, since X and C commute up to the Radial vector field, the flow of X "tends" to preserve the projection of the foliation arising from C along the orbits of R. To make this remark accurate, we first note that the space of orbits of R is naturally identified with the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{CP}^2$ though, on $\pi^{-1}(0)$, $\widetilde{X}$ vanishes identically and $\widetilde{C}$ has poles. However, the restrictions $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ and $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$ to $\pi^{-1}(0)$ of the foliations $\widetilde{\mathcal{F}}_X$ and $\widetilde{\mathcal{F}}_C$ have a specific property of "mutual transversality" which is reminiscent from the previous observation on commuting vector fields. This can be stated as follows:

Definition 6.3. Two (singular) foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be mutually transverse if they are (regular and) transverse away from an algebraic curve C which, in addition, is invariant by both $\mathcal{F}_1$ and $\mathcal{F}_2$. In particular, the curve C contains all singular points of $\mathcal{F}_1$ and of $\mathcal{F}_2$.

Keeping in mind that $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$ is nothing but a pencil of projective lines, the "mutual transversality" condition makes it easy to work out the structure of $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ directly on $\pi^{-1}(0) \simeq \mathbb{CP}^2$. Namely, we have:

- Generically, $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ leaves exactly 3 projective lines C_1, C_2 and C_3 invariant. These projective lines belong to the pencil $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$ and they intersect mutually at a radial singularity in $\pi^{-1}(0)$ (the base locus of the pencil) which is given in homogeneous coordinates by $[1, 1, 1]$. Also, the eigenvalues of $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ at $[1, 1, 1]$ are 1 and 1 (radial singularity).
- In fact, $[1, 1, 1]$ is a radial singularity for the foliation in the 3-dimensional space. In other words, the eigenvalue of $\widetilde{\mathcal{F}}_X$ at $[1, 1, 1]$ associated to the direction transverse to the exceptional divisor is again 1.
- Away from the invariant projective lines C_1, C_2, and C_3, the foliation $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ is transverse to the remaining projective lines in the pencil $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$.

Next, since X is homogeneous, the dynamics of the foliation $\widetilde{\mathcal{F}}_X$ on $\widetilde{\mathbb{C}}^3$ can basically be recovered from the dynamics of the *core foliation* $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ on $\pi^{-1}(0) \simeq \mathbb{CP}^2$. We will return to this point later.

In view of the preceding, let us first focus on the core foliation $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$. Note that both $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ and the pencil $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$, the latter viewed as foliation, share the singular point $[1, 1, 1] \in \pi^{-1}(0)$. Consider the (2-dimensional) blow-up of $\pi^{-1}(0) \simeq \mathbb{CP}^2$ at $[1, 1, 1]$. The resulting surface is the Hirzebruch surface F_1, the $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ with a section of self-intersection -1. On F_1, the foliation (pencil) $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$ becomes the standard fibration $\mathrm{P} : F_1 \to \mathbb{CP}^1$. In turn, the transform $\mathcal{F}_{X,F_1}$ of the foliation $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ on F_1 is regular on a neighborhood of the -1-rational curve of F_1 (identified with the exceptional divisor $\pi^{-1}(0)$ of the blow-up of $\mathbb{CP}^2$). Also, there are 3 fibers of P that are invariant by $\mathcal{F}_{X,F_1}$ and these fibers will still be denoted by C_1, C_2, and C_3 by evident reasons. Away from these three fibers, $\mathcal{F}_{X,F_1}$ is regular and transverse to the fibration induced by P on the open manifold $F_1 \setminus \{C_1, C_2, C_3\}$.

The dynamics of $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ can naturally be read off the dynamics of $\mathcal{F}_{X,F_1}$ which, in turn, is essentially described by the *holonomy representation*. In fact, the restriction of $\mathcal{F}_{X,F_1}$ to the open surface $(F_1 \setminus \{C_1, C_2, C_3\})$ is transverse to the restriction to $(F_1 \setminus \{C_1, C_2, C_3\})$ of the fibration $\mathrm{P} : F_1 \to \mathbb{CP}^1$. Since the fibers of P are compact, Ehresmann's observation ensures that the restriction of P to the leaves of $\mathcal{F}_{X,F_1}$ yields a covering map from the leaf in question to $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$, where z_1, z_2, z_3 are in natural correspondence with the invariant fibers C_1, C_2, C_3. The dynamics of $\mathcal{F}_{X,F_1}$ is therefore essentially encoded in the holonomy representation, namely: the homomorphism ρ from the fundamental group of $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$ to the group of automorphisms of the fiber of P arising from parallel transport along leaves of $\mathcal{F}_{X,F_1}$.

Let $\pi_1(\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\})$ denote the fundamental group of $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$. Since $\mathcal{F}_{X,F_1}$ is holomorphic, the image of the holonomy representation ρ is contained in the group of holomorphic diffeomorphisms of $\mathbb{CP}^1$ which can be identified with $\mathrm{PSL}\,(2, \mathbb{C})$. The *holonomy group* Γ of $\mathcal{F}_{X,F_1}$ is the image of $\pi_1(\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\})$ by ρ, i.e. $\Gamma \subset \mathrm{PSL}\,(2, \mathbb{C})$ is defined by $\Gamma = \rho[\pi_1(\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\})]$.

Next, for each $i = 1, 2, 3$, let $\xi_i \in \mathrm{PSL}\,(2, \mathbb{C})$ be the holonomy map obtained by lifting a small loop around $z_i \in \mathbb{CP}^1$ in the leaves of $\mathcal{F}_{X,F_1}$. The Möebious transformations ξ_1, ξ_2, ξ_3 clearly generate the holonomy group Γ and satisfy the relation $\xi_1\, \xi_2\, \xi_3 = \mathrm{id}$. With the evident identifications, the dynamics of Γ on $\mathbb{CP}^1$ also accounts for the global dynamics of the foliation $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ on $\mathbb{CP}^2$.

All of the preceding considerations apply to every Halphen vector field in the family defined by (8), regardless of whether or not they are semicomplete. To detect semicomplete Halphen vector fields in the family (8), we proceed as follows. First, notice that the singularities of $\widetilde{\mathcal{F}}_X$ and of $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ do coincide. Naturally there is the point $[1, 1, 1]$ lying at the intersection of all the lines in the pencil $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$. Moreover, around $[1, 1, 1]$, the foliation $\widetilde{\mathcal{F}}_X$ is conjugate to the Radial vector field in dimension 3.

To describe the structure of the remaining singular points, for $i = 1, 2, 3$, let $m_i = (\alpha_1 + \alpha_2 + \alpha_3 - 2)/\alpha_i$ provided that $\alpha_i \neq 0$, and set $m_i = \infty$ otherwise. The remaining singular points of $\widetilde{\mathcal{F}}_X$ are contained in the lines C_1, C_2, C_3 and are as follows.

(1) If $m_i \neq \infty$. Then, aside from $[1, 1, 1]$, $\widetilde{\mathcal{F}}_X$ possesses exactly two singular points p_i and q_i in the line C_i. The eigenvalues of $\widetilde{\mathcal{F}}_X$ at p_i are $-1, 1, -m_i$ while at q_i the eigenvalues are $-1, 1, m_i$. In both cases, the eigenvalues are ordered so that the first eigenvalue corresponds to a direction transverse to the exceptional divisor, the second eigenvalue is associated with the direction of C_i and the third eigenvalue is associated with a direction transverse to C_i and contained in the exceptional divisor.

(2) If $m_i = \infty$. Then, aside from $[1, 1, 1]$, $\widetilde{\mathcal{F}}_X$ possesses a unique singular point $p_i = q_i$ in C_i. At this singular point, the eigenvalues are $-1, 0, -1$ with the same ordering used in the above item.

When $m_i = \infty$, the holonomy map ξ_i is a parabolic map in $\mathrm{PSL}\,(2, \mathbb{C})$ since $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ has a (2-dimensional) saddle-node singularity at $p_i = q_i$ with strong invariant manifold transverse to C_i. Next, we have:

Proposition 6.4. *Assume that X is semicomplete and that $m_i \neq \infty$. Then m_i is an integer (which can be assumed positive up to reversing the roles of p_i and q_i). Moreover the holonomy map $\xi_i : \mathbb{CP}(1) \to \mathbb{CP}(1)$ is periodic of period m_i.*

Proof. Again, up to renaming p_i and q_i, the singular point q_i of $\widetilde{\mathcal{F}}_X$ lies in the Siegel domain and the eigenvalues of the mentioned foliation at the singular point in question fulfill the conditions 1., 2., 3. and 4. of Theorem 1 in [49] (or, equivalently, Theorem 2.19 in [42]). More precisely

(1) q_i is an isolated singular point for $\widetilde{\mathcal{F}}_X$.

(2) q_i is a singularity of Siegel type for $\widetilde{\mathcal{F}}_X$, i.e., the origin of $\mathbb{C}$ belongs to the convex hull determined by the eigenvalues.

(3) no eigenvalue at q_i is equal to zero. Furthermore, there is a straight line through $0 \in \mathbb{C}$ in the complex plane separating one of the eigenvalue from the remaining ones;

(4) up to a change of coordinates, the coordinate planes are invariant for $\widetilde{\mathcal{F}}_X$.

The eigenvalue that can be "separated" from the others is -1 (i.e. the eigenvalue associated with direction transverse to the exceptional divisor). In order to prove the statement, it suffices to prove that the holonomy map of $\widetilde{\mathcal{F}}_X$ associated with the separatrix tangent to the eigenspace transverse to the exceptional divisor is trivial, i.e. coincides with the identity map. In fact, taking this assertion for granted, the holonomy map of $\widetilde{\mathcal{F}}_X$ arising from the separatrix in question is conjugate to the holonomy map arising from the x-axis in the foliation defined by

$$-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + m_i z\frac{\partial}{\partial z}$$

since both holonomy maps coincide with the identity. Now either Theorem 1 in [49] or Theorem 2.19 in [42] ensures that these foliations are analytically equivalent or, in other words, that $\widetilde{\mathcal{F}}_X$ is linearizable around q_i.

It remains to prove that the holonomy map associated with the separatrix S of $\widetilde{\mathcal{F}}_X$ tangent to the eigenspace transverse to the exceptional divisor coincides with the identity. For this, note that the restriction of $\widetilde{X}$ to S is given, in local coordinates, by $-z^2\partial/\partial z$. Being X semicomplete, there follows that the local holonomy map of $\widetilde{\mathcal{F}}$ arising from a small loop in S encircling q_i must agree with the identity and the result follows, c.f. [14].

In particular, the foliation $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ is also linearizable around p_i. It follows that $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ possesses a separatrix transverse to C_i and that the holonomy map arising from this separatrix is locally conjugate to a rotation of angle $2\pi/m_i$. Because $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ is transverse to a fibration, this local holonomy map is, in fact, the restriction of a global Möebius transformation $\xi_i \in \mathrm{PSL}\,(2, \mathbb{C})$ which, therefore, must verify $\xi_i^{m_i} = \mathrm{id}$. □

Remark 6.5. Concerning the preceding proposition, it is worth pointing out that A. Guillot provides a direct argument valid for Halphen vector fields in his paper [17].

Also, the results in [49] and [42], as well as other similar previously known results, were the object of a far reaching generalization recently obtained by F. Chaves in [10].

As an immediate consequence, we have the following

Proposition 6.6. *If X is semicomplete, then the holonomy group $\Gamma \subset \mathrm{PSL}\,(2, \mathbb{C})$ describing the global dynamics of $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ is given by*

$$\Gamma =< \xi_1,\, \xi_2,\, \xi_3 :\; \xi_1^{m_1} = \xi_2^{m_2} = \xi_3^{m_3} = \xi_1\xi_2\xi_3 = \mathrm{id} > \,.$$

In other words, Γ is a triangular group.

In Proposition 6.6, when $m_i = \infty$, the condition $\xi_i^{\infty} = \mathrm{id}$ must be interpreted as simply saying that ξ_i is parabolic. In the sequel, we shall also use the convention that $1/m_i = 0$ provided that $m_i = \infty$. In order to obtain semicomplete Halphen vector fields with complicate dynamics, we assume also that

$$m = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1\,. \tag{10}$$

The effect of inequality (10) is just to rule out finitely many cases where the group Γ is "elementary", either finite or conjugate to a subgroup of the affine group of $\mathbb{C}$. Assuming m_1, m_2, m_3 fixed and as in (10), the resulting triangular group Γ satisfy all of the following conditions:

- The group Γ is unique (up to conjugation).
- Γ is discrete and non-elementary.
- Γ leaves a real projective line in $\mathbb{CP}^1$ invariant so that Γ is actually a non-elementary Fuchsian group (i.e. Γ can also be viewed as a subgroup of $\mathrm{PSL}\,(2, \mathbb{R})$).
- The limit set $\Lambda(\Gamma)$ of Γ coincides with the invariant circle S^1. In particular, Γ acts properly discontinuously on each connected component of $\mathbb{CP}^1 \setminus \Lambda(\Gamma)$.

As is well known, the dynamics of Γ on its limit set $\Lambda(\Gamma) = S^1$ is very non-trivial: the dynamics has all orbits are dense and it is ergodic with respect to the Lebesgue measure. Also stationary measures are unique (and hard to understand in detail). Clearly, these issues are directly reflected in the saturated of $\Lambda(\Gamma)$ by the foliation $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$ whose dynamics is hence very non-trivial as well.

It is also convenient to say a few words on the actual dynamics of $\widetilde{\mathcal{F}}_X$ on $\widetilde{\mathbb{C}}^3$ rather than limiting ourselves to its core foliation $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$. To describe this dynamics, we can follow essentially the same ideas used to describe the foliation $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$. Beginning with the pencil $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$, we define a family of surfaces in $\widetilde{\mathbb{C}}^3$ by considering the preimage of each line in $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$ by the canonical projection $\Pi : \widetilde{\mathbb{C}}^3 \to \pi^{-1}(0) \simeq \mathbb{CP}^2$. More precisely, for every projective line D in the pencil $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$, $\Pi^{-1}(D)$ is the line bundle over $\mathbb{CP}^1$ whose Chern class equals -1. Alternatively, by adding a "section at infinity", $\Pi^{-1}(D)$ can naturally be compactified into the Hirzebruch surface F_1. In other words, up to adding a "plane at infinity" to $\widetilde{\mathbb{C}}^3$, we obtain a family of F_1 surfaces parameterized by the lines in the pencil $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$. Now, if we remove the three Hirzebruch surfaces sitting on the top of the lines in $\widetilde{\mathcal{F}}_{C|\pi^{-1}(0)}$ *that are invariant under* $\widetilde{\mathcal{F}}_{X|\pi^{-1}(0)}$, it is straightforward to conclude that $\widetilde{\mathcal{F}}_X$ is *transverse to the fibration* by F_1-surfaces over $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$. Thus, once again we obtain a representation $\overline{\rho}$ from the fundamental group of $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$ in the group $\mathrm{Aut}\,(F_1)$ of holomorphic diffeomorphisms of F_1. Let $\overline{\Gamma}$ be the image of $\overline{\rho}$, i.e. the holonomy group of $\widetilde{\mathcal{F}}_X$. Clearly, $\overline{\rho}$ is generated by the maps Ξ_i obtained by lifting a small loop around z_i, $i = 1, 2, 3$. The maps Ξ_i can explicitly be computed. Fix a surface F_1 equipped with coordinates (x, w) where x is projective coordinate on the projective line $\Pi(F_1)$ and w is an affine coordinate on the fibers of F_1 that equals zero in the intersection with the exceptional divisor. Then we have

$$\Xi_i(x, w) = (\xi_i(x), \sqrt{\xi'(x)}\, w) \tag{11}$$

c.f. [44]. Keeping in mind that the dynamics of $\widetilde{\mathcal{F}}_X$ on $\widetilde{\mathbb{C}}^3$ and the dynamics of $\mathcal{F}_X$ on $\mathbb{C}^3$ can be identified, what precedes can be summarized as follows (the slight abuse of language should not really lead to any misunderstanding):

Proposition 6.7. *The dynamics of $\mathcal{F}_X$ on $\mathbb{C}^3$ is essentially equivalent to the dynamics of the group $\overline{\Gamma} = \langle \Xi_1, \Xi_2, \Xi_3 \rangle$ on F_1. In particular, the (-1)-section of F_1 is invariant by $\overline{\Gamma}$ and the restriction of the action of $\overline{\Gamma}$ to this section is nothing but the action of the triangular group Γ on $\mathbb{CP}^1$.*

By now, we have provided a description of the (rather non-trivial) dynamics of Halphen vector fields such that the quantities m_i are integers satisfying the condition in (10) and the reader is referred to [17] for additional information. However, strictly speaking, we still *do not know* whether or not Halphen vector fields satisfying the conditions in question are, indeed, semicomplete. In fact, Proposition 6.4 provides only necessary conditions for the vector field to be semicomplete. Hence, there remains the problem of checking that these conditions are also *sufficient.*

Curiously enough the fact that the corresponding Halphen vector fields are semicomplete is basically included in Halphen original papers [20], [21]. Halphen begins his Note by pointing out that, if ϕ is a solution of a Halphen vector field, then so is

$$\widetilde{\phi} = \frac{1}{(ct+d)^2}\, \phi\left(\frac{at+b}{ct+d}\right) - \frac{c}{ct+d}\,, \tag{12}$$

for every a, b, c, $d \in \mathbb{C}$ with $ad - bc \neq 0$. From this he concludes that all solutions can be described out of a single "known" solution. He then goes on to obtain a particular solutions by skillfully manipulating theta functions. In this sense, the converse to Proposition 6.4 can be derived from his work.

Yet, Guillot [17] provides a different proof of the semicomplete nature of Halphen vector fields satisfying the conditions in Proposition 6.4. Guillot's argument dispenses with the remarkable identities satisfied by theta functions and, perhaps more importantly, lends itself well to deep generalizations. We will close this paragraph by sketching this argument.

First, it is convenient to recall the basic notions of *translation, affine, and projective structures* on Riemann surfaces since they play a role in the discussion below.

Definition 6.8. Let S be a Riemann surface along with a covering $\{(B_i, \varphi_i)\}$ by local coordinates. The covering $\{(B_i, \varphi_i)\}$ is said to define a translation structure (resp. affine structure, projective structure) on S if and only if the changes of coordinates $\varphi_i \circ \varphi_j^{-1} : \varphi_j(B_i \cap B_j) \to \varphi_i(B_i \cap B_j)$ are restrictions of translations of $\mathbb{C}$ (resp. affine maps, Möebius transformations).

In particular, if S is endowed with a nowhere zero holomorphic vector field X, then the covering whose local coordinates are the inverse maps of the (local) solutions of X endows S with a *translation structure*. This simple remark will be useful below.

Also, a translation structure (resp. affine structure, projective structure) gives rise to a *monodromy homomorphism* ρ from the fundamental group of S to the group of translations of $\mathbb{C}$ (resp. affine maps, Möebius transformations). Following [17], [19], denote by S_ρ the covering space of S associated with the kernel of ρ. On S_ρ, we can define a *developing map* $\mathcal{D}_\rho : S_\rho \to \mathbb{C}$ (resp. $\mathbb{C}$, $\mathbb{CP}^1$). In fact, S_ρ is the *smallest covering of S on which a developing map is well defined.* This developing map will be called *the monodromy-developing map* of the corresponding structure. Naturally, all developing maps are well defined up to composition with an element of the corresponding group (translation, affine map, or Möebius transformations).

Remark 6.9. The preceding offers us yet another equivalent way to define semicomplete vector fields on a Riemann surface, and thus in general since a vector field will be semicomplete if and only if its restriction to each leaf of its associated foliation is semicomplete. Namely, the vector field X on the Riemann surface S is semicomplete if and only if the monodromy-developing map of the corresponding translation structure is *injective.* We are now ready to explain Guillot's argument.

Sketch of Guillot's proof that Halphen vector fields as in Proposition 6.4 are semicomplete. We might start by recalling that the vector fields R and C generate the Lie algebra of the affine group $\mathrm{Aff}\,(\mathbb{C}, 0)$. Of course a similar remark applies to their blow-ups $\widetilde{R}$ and $\widetilde{C}$. Then we consider the Zariski open subset W of $\widetilde{\mathbb{C}}^3$ given as the complement of $\pi^{-1}(0)$ and of the 3 invariant Hirzebruch surfaces. In the setting of Proposition 6.7, W is a U-bundle over $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$, where $U \subset F_1$ is the Zariski open set defined as the complement of the two rational sections of F_1. In particular, U is in a natural correspondence with an orbit of $\mathrm{Aff}\,(\mathbb{C}, 0)$. Finally, $\widetilde{\mathcal{F}}_X$ is transverse to the fibers of the fibration $W \to \mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$ and admits a global holonomy group determined by Proposition 6.7.

It suffices to show that the restriction of $\widetilde{X}$ to W is semicomplete. Guillot basic observation is that $\widetilde{X}$ induces a natural projective structure on $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$ viewed as the base of the U-bundle W. This deserves a few comments. Small discs $B \subset \mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$ can be identified with discs on the leaves L of $\widetilde{\mathcal{F}}_X$ by means of the fiber bundle structure. Next, each leaf L of the restriction of $\widetilde{\mathcal{F}}_X$ to W is endowed with a translation structure induced by $\widetilde{X}$. These transverse structures vary with the leaf but its *underlining projective structure does not.* In fact, taking into account that a fiber U of the U-bundle W is identified with an orbit of Aff ($\mathbb{C}$) - and thus parameterized by the flows of R and of C, Equation (12) can be interpreted as an identity involving the flows of R, C, and X (or of their blow-ups which amounts to the same). With this interpretation, it becomes clear that the time taken by $\widetilde{X}$ to move between two fixed fibers U_1 and U_2 along leaves L and L' are related by a Möebius transformation. Thus the covering of the base $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$ obtained by taking the inverses of the local solutions of X, as above, over all possible leaves of $\widetilde{\mathcal{F}}_X$ defines a projective structure on $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$.

Next, consider the monodromy-developing map $\mathcal{D}_\rho$ for the projective structure $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$. It is straightforward to check that *representatives* for this developing-map can be obtained by simply considering the monodromy-developing maps associated with the *translation structures induced by* $\widetilde{X}$ on the leaves of $\widetilde{\mathcal{F}}_X$ (or more accurately of the restriction of $\widetilde{\mathcal{F}}_X$ to W). In view of Remark 6.9, there follows that X is semicomplete if and only if $\mathcal{D}_\rho$ is injective. Guillot's then "compute" the projective structure in question by means of the Schwarzian operator so as to show that $\mathcal{D}_\rho$ is essentially Schwarz triangular functions and the proof follows. □

Remark 6.10. In fairness, we should note that the material covered in this paragraph is essentially the first part of Guillot's paper [17]. The content of [17] also includes realizing semicomplete Halphen vector fields as actual *complete* vector fields on complex manifolds as well as several important applications to the study of SL (2$\mathbb{C}$) actions and homogeneous spaces.

Let us close this paragraph with a couple of questions about dynamics of semicomplete vector fields, the first one being kind of inevitable.

Problem 1. Are there semicomplete vector fields with complicated dynamics which genuinely different from the dynamics obtained by means of Halphen vector fields?

Another interesting question which may or may not have a saying in the above problem concerns geodesic flows on semisimple Lie groups. These geodesic flows have already been considered in works by S. Dumitrescu and by Elshafei-Ferreira-Reis, see [11], [13] and their references. Given a (semisimple) Lie group G and a left-invariant holomorphic metric $\langle \, . \, \rangle$ on G, the complex geodesic flow on G can be expressed by a quadratic vector field defined on the Lie algebra of G by means of the Euler-Arnold formalism. This yields a particular, yet large and with geometric nature, class of quadratic vector fields. Referring to vector fields in this class as *Euler-Arnold* vector fields, their dynamics is definitely worth study. Thus we can formulate the following special case of the preceding question which, however, holds interest in its own:

Problem 2. Are there semicomplete Euler-Arnold vector fields exhibiting complicated dynamical behavior?

6.2. **Local aspects of semicomplete vector fields and applications.** Partly, the interest of semicomplete vector fields comes from the fact that they provide local obstructions for a germ of vector field be realized as singularity of a complete one. In the sequel, we will talk about *germs of semicomplete vector fields* or about *semicomplete singularities* as synonymous.

From the basic properties discussed at the beginning of this section, it follows that semicomplete vector fields can be viewed as a "local counterpart" of complete ones. In fact, a singularity that *is not semicomplete* cannot be realized by a complete vector field. In particular, it cannot be realized by a globally defined holomorphic vector field on a compact manifold. The understanding of semicomplete singularities is therefore useful to the description of holomorphic vector fields (globally) defined on compact manifolds.

To better explain this issue, it is convenient to center the discussion around a rather concrete and well known question due to E. Ghys that can be formulated in terms of semicomplete vector fields as follows: *let X be a semicomplete holomorphic vector field on* $(\mathbb{C}^n, 0)$ *with isolated* *singular points. Is it true that $J^2X(0) \neq 0$, i.e. must the second jet of X at the singular point be different from zero?*

Ghys' original motivation seems to be related to problems about bounds for the dimension of automorphism group of compact complex manifolds. To be more precise, consider a compact complex manifold M and denote by $\mathrm{Aut}\,(M)$ the group of holomorphic diffeomorphisms of M. It is well known that $\mathrm{Aut}\,(M)$ is a finite dimensional complex Lie group whose Lie algebra can be identified with $\mathfrak{X}\,(M)$, the space of all holomorphic vector fields defined on M. A too naïve question, would be to wonder if the dimension of $\mathrm{Aut}\,(M)$ can be bounded by a function of the dimension of M. It turns out, however, that the dimension of the automorphism group of the Hirzebruch surface F_n is $n+5$ provided that $n \geq 1$. In particular, already in the case of compact surfaces, the dimension of $\mathrm{Aut}\,(M)$ can be arbitrarily large. However, analogous questions can be raised to better effect for specific classes of manifolds. For example, the following question was communicated to us by J.V. Pereira who attributes it to Hwuang and Mok: among projective manifolds of dimension n with Picard group isomorphic to $\mathbb{Z}$, is it $\mathbb{CP}^n$ the one whose automorphism group has the largest dimension?

As a matter of fact, Ghys' question is part of a general principle with vaguely stated as follows: semicomplete singularities cannot be "too degenerate". Here it is convenient to explain how *limiting the extent to which a semicomplete singularity can be degenerate* becomes a useful tool to deal with the previous questions. Consider a n-dimensional compact complex manifold M and let $\mathrm{Aut}\,(M)$ and $\mathfrak{X}\,(M)$ be as above. Fix a point $p \in M$ and let $k \in \mathbb{N}$ be given. Finally, let $\mathfrak{X}_p^k(M)$ stand for the set of holomorphic vector fields with vanishing k-jet at p and denote by $J_p^k(M)$ the space of k-jets at p. The natural mappings

$$\mathfrak{X}_p^k\,(M) \to \mathfrak{X}\,(M) \to J_p^k(M)\,,$$

give rise to a short exact sequence so that we have

$$\dim \mathfrak{X}\,(M) \leq \dim \mathfrak{X}_p^k\,(M) + \dim J_p^k(M)\,.$$

The dimensions of the jet spaces $J_p^k(M)$ are explicitly given in terms of k and of $n = \dim\,(M)$. In particular, if for some $p \in M$ and $k \in \mathbb{N}$, we can obtain bounds for $\dim \mathfrak{X}_p^k\,(M)$ in terms of $\dim\,(M)$ then bounds for $\dim\,(\mathrm{Aut}\,(M))$ follow immediately. For example, suppose that we happen to know that for a certain class of compact manifolds every singularity of a globally defined holomorphic vector field is necessarily isolated. Then, assuming Ghys conjecture holds, it follows that $\dim \mathfrak{X}_p^3\,(M) = 0$ and therefore the dimension of $\mathrm{Aut}\,(M)$ would be bounded by $(n^3 + 3n^2 + 2n)/2$. Of course, in general, non-isolated singularities also appear so that it is convenient to be able to handle them as well.

Aside from introducing the notion of semicomplete singularity, the content of [40] can fairly be summarized by the following theorem:

Theorem 6.11. [40] *Let X be a holomorphic semicomplete vector field on $(\mathbb{C}^2, 0)$. If the origin is an isolated singular point for X, then $J_0^2X \neq 0$.*

The proof of Theorem 6.11 relies on Camacho-Sad theorem on the existence of separatrices for foliations on $(\mathbb{C}^2, 0)$. Indeed, since the singular set of X is reduced to the origin, the restriction of X to any analytic invariant curve going through the origin cannot vanish identically. Furthermore, this restriction is still a semicomplete vector field. Considering then the restriction of X to a separatrix, whose existence is ensured by Camacho-Sad theorem, the problem becomes essentially reduced to the one-dimensional situation (whether or not the separatrix is smooth). The resulting (one-dimensional) problem is settled in the same paper by direct methods.

The question on whether or not Ghys conjecture holds for semicomplete vector fields in higher dimensions is hence natural. The first deep investigations involving semicomplete vector fields in higher dimensions were conducted by A. Guillot in [16], and [17] (here "higher" means ≥ 3). The mentioned papers by Guillot contain, in particular, numerous examples of quadratic semicomplete vector fields exhibiting a wide range of geometric and dynamical behaviors. Among these examples, we have already discussed the case of Halphen vector fields that have complicated dynamics and no (non-trivial) holomorphic/meromorphic first integral (cf. Section 6.1). Moreover, Guillot's work also make clear that in dimensions ≥ 3, an exhaustive classification of all semicomplete vector fields with zero linear part - paralleling the list provided in [14] - is unlikely to exist or, at least, it would be too long to be truly useful.

This is therefore a good moment to elaborate on the difficulties in extending to $(\mathbb{C}^3, 0)$ the general classification results in dimension 2 of [40], [14], not to mention the more general results of [19] encompassing also meromorphic vector fields. Indeed, whether or not obtaining these generalizations is a tall order, it certainly seems useful to explicitly list some of the new difficulties arising in dimensions greater than 2. Aside from the existence of *core dynamics*, that is a general difficulty already emphasized in this work, the following issues are worth mentioning.

1. The basic approach to Ghys conjecture stemming from [40] consists of finding a separatrix. Namely, the following holds: let X be a semicomplete (holomorphic) vector field on $(\mathbb{C}^n, 0)$ with an isolated singularity at the origin. If X possesses a separatrix, then $J_0^2 X \neq 0$. However, as previously seen, Gomez-Mont and Luengo [15] have proved that separatrices do not exist in general for germs of 1-dimensional foliations on $(\mathbb{C}^n, 0)$, $n \geq 3$ (cf. Section 5).
2. The examples provided in [15], however, are not semicomplete so that it is conceivable that all semicomplete vector field possesses a separatrix. While this seems to suggest that Ghys conjecture may be proved by showing that semicomplete vector fields do have separatrices, a direct approach to the latter question does not seem feasible.
3. A more promising point of view regarding item 2 above consists of noticing that the detailed classification of semicomplete vector fields in dimension 2, as developed in [14] or in [19], dispenses with Camacho-Sad theorem. In fact, these deeper analysis yield directly the classification. Hence the existence of separatrices for *semicomplete vector fields* on $(\mathbb{C}^2, 0)$ becomes a *corollary*, as opposed to a statement needed a priori.
4. In dimension 2, a fundamental ingredient permeating virtually all works on singularities of vector fields or foliations is the resolution theorem of Seidenberg [54]. Since resolutions theorems for 1-dimensional foliations have been established in the past few years, c.f. Section 4, this initial difficulty has now been overcome. In fact, as far as semicomplete vector fields are concerned, a totally faithful analogue of Seidenberg's theorem is available in dimension 3 as will be seen below.
5. Difficulties, however, are not limited to reduction of singularities procedures nor to the phenomenon of core dynamics. For example, assume our objective is to establish Ghys conjecture by means of proving the existence of separatrices (in which case the role played by core dynamics is significantly reduced, c.f. Section 5.2). Assume, in addition, that we are given a holomorphic foliation admitting a simple reduction of singularities. In dimension 3, the existence of saddle-node singularities appearing in the resolution procedure cannot easily be ruled out. In particular, codimension 2 saddle-nodes (i.e. with two eigenvalues equal to zero) may appear and these singularities are still poorly understood.

The remainder of this paper is to complement the above list with further comments and results, some of them proposing simpler approaches that can be effective pending on the specific application targeted.

Concerning items 1 and 2, some partial results have been proved in [46]. In fact, recall that Theorem 5.8 states that in the case we are given two holomorphic vector fields X and Y yielding a representation of a Lie algebra of dimension 2 and not everywhere parallel, then they possess a common separatrix. By elaborating on this theorem, the following weaker version of Ghys conjecture in dimension 3 was proven in [46]:

Theorem 6.12. [46] *Consider a compact complex manifold M of dimension 3 and assume that the dimension of $\mathrm{Aut}\,(M)$ is at least 2. Let Z be an element of $\mathfrak{X}\,(M)$ and suppose that $p \in M$ is an isolated singularity of Z. Then*

$$J^2(Z)\,(p) \neq 0\,,$$

i.e. the second jet of Z at the point p is different from zero.

The reader will note that, as far as estimates on the dimension of automorphism groups are concerned, Theorem 6.12 is as effective as an affirmative solution to Ghys conjecture in dimension 3 in the sense that if the additional assumption needed for Theorem 6.12 is not verified, then the dimension of $\mathrm{Aut}\,(M)$ is at most 1.

One of the advantages of looking for bounds for the dimension of $\mathrm{Aut}\,(M)$ by means of local considerations is that the results obtained are essentially valid for open manifolds as well. For example, studying finite dimensional Lie group actions on Stein manifolds is an active topic in several complex variables whose roots lie in a classical work of Suzuki [55]. In this direction, our techniques yield:

Theorem 6.13. [46] *Let N denote a Stein manifold of dimension 3 and consider a finite dimensional Lie algebra $\mathfrak{G}$ embedded in $\mathfrak{X}_{\mathrm{comp}}(N)$ (the space of complete holomorphic vector fields on N). Assume that the dimension of $\mathfrak{G}$ is at least 2. If Z is an element of $\mathfrak{G} \subseteq \mathfrak{X}\,(M)$ possessing an isolated singular point $p \in N$, then the linear part of Z at p cannot vanish, i.e. p is a non-degenerate singularity of Z.*

We may point out that the automorphism group of a Stein manifold is not a finite dimensional Lie group in general. Indeed, even $\mathbb{C}^2$ has an infinite dimensional group of automorphisms with hardly any non-trivial structure of Lie group. This difficulty is avoided in the statement of Theorem 6.13 by the assumption that, from the beginning, we are dealing with some finite dimensional Lie algebra: owing to Lie theorem, such Lie algebra can always be integrated to yield a (complete) action of the corresponding Lie group. This part of the statement actually holds for arbitrary complex manifolds of dimension 3 (whether or not they are compact or Stein) and parallels Theorem 6.12 in the sense that the second jet of a vector field $Z \in \mathfrak{G}$ will never be zero at an isolated singular point. The role played by the Stein condition is to ensure that the *first jet*, rather than the second one, is different from zero at isolated singular points.

To close this paper, let us go back to resolution theorems as discussed in Section 4 and further sharpen the results under the *additional assumption of having semicomplete vector fields.* As already mentioned, resolution theorems always play a central role in singularity theory and sharp resolution statements exist for 1-dimensional foliations in dimension 3. Yet, given their importance, it is convenient to have available the simplest possible resolution statements in every circumstance. In particular, it is natural to wonder if semicomplete singularities or other special classes of singular points allow for simpler resolution theorems facilitating a more detailed analysis of their structures.

The possibility of having simpler resolution theorems valid for semicomplete singularities was also considered in [50] whose initial motivation was, in fact, to obtain a resolution theorem for *semicomplete singularities* that would faithfully parallel Seidenberg theorem for foliations on $(\mathbb{C}^2, 0)$. This type of statement is useful to approach problems such as Ghys conjecture or to investigate compact complex manifolds of dimension 3 equipped with holomorphic vector fields. In this setting, Theorem 6.14 below is proved in [50].

Theorem 6.14. [50] *Let X be a semicomplete vector field defined on a neighborhood of the origin of $\mathbb{C}^3$ and denote by $\mathcal{F}$ the holomorphic foliation associated with X. Then one of the following holds:*

(1) *The linear part of X at the origin is nilpotent (non-zero).*
(2) *There exists a finite sequence of (standard) blow-ups along with transformed foliations*

$$\mathcal{F} = \mathcal{F}_0 \xleftarrow{\pi_1} \mathcal{F}_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} \mathcal{F}_r$$

such that all of the singular points of $\mathcal{F}_r$ are elementary. Moreover, each blow-up map π_i is centered in the singular set of the corresponding foliation $\mathcal{F}_{i-1}$. In other words, the foliation $\mathcal{F}$ can be resolved by means of standard blow-ups.

Let us emphasize that item 1 in Theorem 6.14 involves the linear part of the vector field X rather than the linear part of the associated foliation $\mathcal{F}$. In fact, it is the linear part of X that has to be (nilpotent) *non-zero from the outset*. Furthermore, this property is "universal" in the sense that it does not depend on any sequence of blow-ups carried out. It is indeed also independent of blow-downs if X is defined on a manifold. More precisely, let X be a semicomplete vector field defined on a complex manifold M of dimension 3. Suppose that $p \in M$ is a nilpotent singularity of X which cannot be reduced in the sense of item 2 of Theorem 6.14. Then the existence of this nilpotent singular point for X is invariant by sequences of blow-ups/blow-downs of M. In particular, we can choose a "minimal model" for M and the corresponding transform of X will still have non-zero nilpotent linear part at the corresponding point.

Along similar lines, the following also deserves to be singled out.

Corollary 6.15. [50] *Let X be a semicomplete vector field defined on a neighborhood of $(0, 0, 0) \in \mathbb{C}^3$ and assume that the linear part of X at the origin is equal to zero. Then item (2) of Theorem 6.14 holds.*

Accurate normal forms for persistent nilpotent singular points were provided in the same paper, c.f. Theorem 4.5. However, not all of them need to be semicomplete or, indeed, realized as singularity of a complete flow. Taking into account the global setting of complete vector fields, it is natural to wonder if there is, indeed, complete vector fields inducing a foliation with singular points that cannot be resolved by standard blow-ups as in item (2) of Theorem 6.14. As a matter of fact, these singularities *do exist* and an explicit example is provided by the polynomial vector field

$$Z = x^2 \frac{\partial}{\partial x} + xz \frac{\partial}{\partial y} + (y - xz) \frac{\partial}{\partial z} .$$

Although Z is not complete on $\mathbb{C}^3$, it can be extended to a complete vector field defined on a suitable *open manifold*. In particular, the point corresponding to the origin of the above coordinates (x, y, z) constitutes a nilpotent singular point of Z that cannot be resolved by means of standard blow-ups with centers in the singular set.

Finally, we might emphasize that the example above involves a complete vector field defined on an open manifold. We might then ask if this phenomenon still occurs in the far more restrictive context of compact manifolds of dimension 3. Since in the compact case the completeness condition becomes automatic, we are simply asking whether or not there is a compact complex manifold of dimension 3 equipped with a (global) holomorphic vector field X which exhibits a singular point that cannot be resolved by means of standard blow-ups. This time, the answer turns out to be negative as the following holds:

Corollary 6.16. *Let $\mathcal{F}$ be the foliation associated with a vector field X globally defined on some compact complex manifold M of dimension 3. Then every singular point of $\mathcal{F}$ can be resolved by a sequence of standard blow-ups.*

In closing, let us just point out that both Corollary 6.15 and Corollary 6.16 are strictly speaking by-products of the methods used to prove Theorem 6.14 rather than formal consequences of the statement of this theorem.

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