Certain Extensions of Results of Siegel, Wilton and Hardy

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Abstract

Recently, Dixit et al. [22] established a very elegant generalization of Hardy's Theorem concerning the infinitude of zeros that the Riemann zeta function possesses at its critical line.

By introducing a general transformation formula for the theta function involving the Bessel and Modified Bessel functions of the first kind, we extend their result to a class of Dirichlet series satisfying Hecke's functional equation. In the process, we also find new generalizations of classical identities in Analytic number theory.

Contents

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1 Introduction and Main Results

Let $\zeta(s)$ denote the Riemann zeta function and define $\eta(s) := \pi^{-s/2}\Gamma(s/2)\,\zeta(s)$. A. Dixit, N. Robles, A. Roy and A. Zaharescu [23] proved the following theorem.

Theorem A: Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of non-zeros real numbers such that $\sum_{j=1}^{\infty}|c_j|<\infty$. Also, let $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers that attains its bounds. Then the function

$$F(s)=\sum_{j=1}^{\infty}c_j\,\eta\left(s+i\lambda_j\right):=\sum_{j=1}^{\infty}c_j\,\pi^{-\frac{s+i\lambda_j}{2}}\Gamma\left(\frac{s+i\lambda_j}{2}\right)\zeta\left(s+i\lambda_j\right)$$

has infinitely many zeros on the critical line $\mathrm{Re}(s)=\frac{1}{2}$.

Based on an integral representation of Jacobi's transformation formula due to Dixit [[19], p. 374, eq. (1.13)], A. Dixit, R. Kumar, B. Maji and A. Zaharescu [22] later generalized the aforementioned result and proved the more general theorem.

Theorem B: Let $(c_j)_{j\in\mathbb{N}}$ and $(\lambda_j)_{j\in\mathbb{N}}$ be as in Theorem A. Also, let $\mathscr{R}$ denote the region of the complex plane defined[1] by $\mathscr{R}:=\left\{z\in\mathbb{C}\,:\,|\mathrm{Re}(z)|<\sqrt{\frac{\pi}{2}},\,|\mathrm{Im}(z)|<\sqrt{\frac{\pi}{2}}\right\}$.

Then, for any $z\in\mathscr{R}$, the function

$$F_z(s)=\sum_{j=1}^{\infty}c_j\,\eta\left(s+i\lambda_j\right)\left\{{}_1F_1\left(\frac{1-(s+i\lambda_j)}{2};\,\frac{1}{2};\,\frac{z^2}{4}\right)+\,{}_1F_1\left(\frac{1-(\overline{s}-i\lambda_j)}{2};\,\frac{1}{2};\,\frac{\overline{z}^2}{4}\right)\right\}\tag{1.1}$$

has infinitely many zeros on the critical line $\mathrm{Re}(s)=\frac{1}{2}$.

The proofs of both theorems used a very elegant generalization of Hardy's method of studying the moments of the real function $\eta\left(\frac{1}{2}+it\right)$ [[32], [59], Chapter X]. One of the crucial steps in Hardy's proof is the transformation formula for Jacobi's theta function, usually defined as

$$\theta(x):=\sum_{n=-\infty}^{\infty}e^{-\pi n^2x}=2\psi(x)+1,\tag{1.2}$$

with $\psi(x)=\sum_{n=1}^{\infty}e^{-\pi n^2x}$. This transformation formula is:

$$x^{1/2}\left(1+2\psi(x)\right)=1+2\psi\left(\frac{1}{x}\right).\tag{1.3}$$

It was also established by Jacobi that, if we consider a two-variable generalization of $\psi(x)$ in the form,

$$\psi(x,z):=\sum_{n=1}^{\infty}e^{-\pi n^2x}\cos\left(\sqrt{\pi x}\,n\,z\right),\quad \mathrm{Re}(x)>0,\;\; z\in\mathbb{C},\tag{1.4}$$

[1] The region indicated in [22] is actually $|\mathrm{Re}(z)-\mathrm{Im}(z)|<\sqrt{\frac{\pi}{2}}-\sqrt{\frac{2}{\pi}}\,\mathrm{Re}(z)\,\mathrm{Im}(z)$, which contains $\mathscr{R}$, but in this paper we only focus on rectangular regions as the one indicated in the above statement.

the transformation formula takes place [[22], p. 312, eq. (2.6)]

$$\sqrt{x}\,(1+2\psi(x,z)) = e^{-z^2/4}\left(1+2\,\psi\left(\frac{1}{x},iz\right)\right). \tag{1.5}$$

Dixit [[19], p. 374, eq. (1.13)] realized that (1.5) could be achieved through an integral representation involving the Riemann zeta function, proving the elegant identity,

$$2x^{1/4}\psi(x,z) - x^{-1/4}e^{-z^2/4} = 2e^{-z^2/4}x^{-1/4}\,\psi\left(\frac{1}{x},\,iz\right) - x^{1/4} =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\zeta\left(\frac{1}{2}+it\right){}_1F_1\left(\frac{1}{4}+\frac{it}{2};\,\frac{1}{2};\,-\frac{z^2}{4}\right)x^{-\frac{it}{2}}\,dt. \tag{1.6}$$

This formula played a fundamental role in proving Theorem B above (see also the very interesting survey [20] for more integral representations of several modular-type transformation formulas). The purpose of this paper is to see how Theorem B can be extended to a class of Dirichlet series satisfying Hecke's functional equation.

Our method is an extension of the one employed in [22], but in order to use it we need to develop a general formulation of the integral identity (1.6).

Although we proceed in the set up of Chandrasekharan, Narasimhan and Berndt [5, 18], for the purpose of introducing the main results of this paper, we just give some examples.

For any $\alpha > 0$, define $r_\alpha(n)$ as the arithmetical function [[56], p. 481, eq. (1.6)]

$$\theta^{\alpha}(x) - 1 := \sum_{n=1}^{\infty} r_\alpha(n)\,e^{-\pi n x}. \tag{1.7}$$

For $\text{Re}(s)$ sufficiently large, we can consider formally the Dirichlet series

$$\zeta_\alpha(s) := \sum_{n=1}^{\infty}\frac{r_\alpha(n)}{n^s}, \tag{1.8}$$

and motivate its study through the transformation formula for $\theta(x)$ (1.3). The zeta function (1.8) seems to have been introduced in a paper by Lagarias and Rains [40] who studied, among several other things, how the distribution of its zeros could vary with the index α. Suzuki proved the analytic continuation and the functional equation for $\zeta_\alpha(s)$, with $0<\alpha<1$, and derived from it a formula similar to Selberg-Chowla's [50] involving its coefficients.

When $\alpha = k \in \mathbb{N}$, it is effortless to see that $r_\alpha(n)$ reduces to the arithmetical function counting the number of representations of n as a sum of k squares. Also, when $\alpha = 1$, $\zeta_1(s)$ reduces to $2\zeta(2s)$. Like $\zeta_k(s)$ and $\zeta(2s)$, $\zeta_\alpha(s)$ satisfies the functional equation

$$\eta_\alpha(s) := \pi^{-s}\Gamma(s)\,\zeta_\alpha(s) = \pi^{-\left(\frac{\alpha}{2}-s\right)}\Gamma\left(\frac{\alpha}{2}-s\right)\,\zeta_\alpha\left(\frac{\alpha}{2}-s\right) := \eta_\alpha\left(\frac{\alpha}{2}-s\right),$$

from which it is possible to conclude that $\eta_\alpha(s)$ is real on the critical line $\text{Re}(s) = \frac{\alpha}{4}$.

One of the first results of our paper is the extension of Theorem B above to $\zeta_\alpha(s)$. Its statement is as follows.

Theorem 1.1. *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_j |c_j|<\infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers attaining its bounds. Then, for any z satisfying the condition:*

$$z\in\mathscr{D}_{\alpha}:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<\sqrt{\frac{\pi\alpha}{2}},\,|Im(z)|<\sqrt{\frac{\pi\alpha}{2}}\right\}, \tag{1.9}$$

the function

$$F_{z,\alpha}(s):=\sum_{j=1}^{\infty}c_j\,\pi^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\,\zeta_{\alpha}(s+i\lambda_j)\,\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)+\,{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\frac{\alpha}{2};\,\frac{\overline{z}^2}{4}\right)\right\} \tag{1.10}$$

has infinitely many zeros on the critical line $Re(s)=\frac{\alpha}{4}$.

In particular, for $\alpha=1$ we can recover Theorem B above. For $\alpha=k\in\mathbb{N}$, our result generalizes a Theorem of Siegel [52] and Landau [41] (see also [33]), which says that the completed Dirichlet series

$$\eta_k(s):=\pi^{-s}\Gamma(s)\,\zeta_k(s)$$

possesses infinitely many zeros at its critical line[2] $\text{Re}(s)=\frac{k}{4}$.

One of the main ingredients in the proof Theorem 1.1 is a generalization of Dixit's integral formula (1.6): as it will be seen in Example 2.1 of the present paper, it is possible to prove the transformation formula

$$\begin{aligned}\sqrt{x}\,e^{z^2/8}\sum_{n=1}^{\infty}r_{\alpha}(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\,x}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi\,n\,x}\,z)-\left(\sqrt{\frac{\pi}{x}}\,\frac{z}{2}\right)^{\frac{\alpha}{2}-1}\frac{e^{-z^2/8}}{\Gamma\left(\frac{\alpha}{2}\right)\sqrt{x}}\\ =\frac{e^{-z^2/8}}{\sqrt{x}}\sum_{n=1}^{\infty}r_{\alpha}(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\frac{\pi n}{x}}\,I_{\frac{\alpha}{2}-1}\left(\sqrt{\frac{\pi n}{x}}\,z\right)-\left(\sqrt{\pi x}\,\frac{z}{2}\right)^{\frac{\alpha}{2}-1}\frac{\sqrt{x}\,e^{z^2/8}}{\Gamma\left(\frac{\alpha}{2}\right)}\\ =\left(\frac{\sqrt{\pi}z}{2}\right)^{\frac{\alpha}{2}-1}\frac{e^{z^2/8}}{2\pi\,\Gamma\left(\frac{\alpha}{2}\right)}\int_{-\infty}^{\infty}\pi^{-\frac{\alpha}{4}-it}\Gamma\left(\frac{\alpha}{4}+it\right)\zeta_{\alpha}\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}+it;\,\frac{\alpha}{2};\,-\frac{z^2}{4}\right)\,x^{-it}\,dt,\end{aligned} \tag{1.11}$$

where $J_{\nu}(z)$ and $I_{\nu}(z)$ denote the Bessel functions of the first kind. Formula (1.6) can be now recovered when $\alpha=1$.

Since the proof of (1.11) was achieved from a general point of view, identities akin to (1.11) are also valid to other Dirichlet series.

For example, a version of it holds for Epstein zeta functions attached to binary quadratic forms: if $Q(m,n)=Am^2+Bm\,n+Cn^2$ is a binary, integral and positive definite quadratic form, we can consider the Dirichlet series

$$\zeta\left(s,Q\right)=\sum_{m,n\neq0}\frac{1}{Q(m,n)^s},\quad \text{Re}(s)>1, \tag{1.12}$$

known as the Epstein zeta function attached to Q.

In 1934, Potter and Titchmarsh [46] extended Hardy's result to $\zeta(s,Q)$. Roughly one year later, Kober [37] furnished a considerable simplification of their proof, more similar in spirit to Hardy's own proof.

In this paper we are also able to extend the result given in Theorem B to a subclass of the zeta functions (1.12).

[2] In fact, Siegel's result is very sharp and provides detailed information about the zeros of $\zeta_k(s)$ on a fairly large strip of the complex plane.

Theorem 1.2. *Let $Q(x,y)=Ax^2+Bx\,y+Cy^2$ be a binary, integral and positive definite quadratic form and let $\Delta:=4AC-B^2$. Assume also that Q is reduced, this is, gcd$(A,\,B,\,C)=1$ and that $\sqrt{\Delta}\equiv 2 \mod 4$.*

Consider the Epstein zeta function attached to Q,

$$\zeta(s,\,Q)=\sum_{m,n\neq 0}\frac{1}{(Am^2+Bm\,n+Cn^2)^s},\quad Re(s)>1. \tag{1.13}$$

If $(c_j)_{j\in\mathbb{N}}$ is a sequence of non-zero real numbers such that $\sum_{j=1}^{\infty}|c_j|<\infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ is a bounded sequence of distinct real numbers attaining its bounds and z satisfies the condition:

$$z\in\mathscr{D}_Q:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<\frac{2\sqrt{\pi}}{\Delta^{3/4}},\;|Im(z)|<\frac{2\sqrt{\pi}}{\Delta^{3/4}}\right\}; \tag{1.14}$$

Then the function

$$F_{z,Q}(s)=\sum_{j=1}^{\infty}c_j\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\zeta(s+i\lambda_j,\,Q)\left\{{}_1F_1\left(1-s-i\lambda_j;\,1;\,\frac{z^2}{4}\right)+{}_1F_1\left(1-\overline{s}+i\lambda_j;\,1;\,\frac{\overline{z}^2}{4}\right)\right\}$$

has infinitely many zeros on the critical line $Re(s)=\frac{1}{2}$.

Another natural extension of Theorem B has to be concerned with Dirichlet L-functions. At the end of their paper, Dixit, Kumar, Maji and Zaharescu [22] state a character analogue of the identity (1.6) and write that *"it would be worthwhile (...) to find a character analogue of Theorem 2 [Theorem B above]".*

Under certain restrictions on even characters, we have managed to prove a character analogue of Theorem B. In fact, motivated by the study of our Theorem 1.1, we establish something more general than that.

In order to clarify our next statement, note that, in analogy with $\zeta_\alpha(s)$, we can define, just as Suzuki does [[56], p. 483, eq. (1.15)], a new Dirichlet series attached to the powers of the theta function:

$$\theta(x,\chi):=\sum_{n\in\mathbb{Z}}n^{\delta}\chi(n)\,e^{-\frac{\pi n^2 x}{q}},\quad \delta=\begin{cases}0 & \text{if } \chi(-1)=1\\ 1 & \text{if } \chi(-1)=-1.\end{cases} \tag{1.15}$$

However, in order to define $\theta^{\alpha}(x,\chi)$ for every $\alpha>0$, one would have to assure that χ is real and such that $\theta(x,\chi)>0$ for every $x>0$.

Whether this condition holds or not, it is always possible to take an arbitrary integer power of $\theta(x,\chi)$, which results in the following series:

$$\theta^k(x,\chi)=\sum_{n_1,...,n_k\in\mathbb{Z}}n_1^{\delta}\,\chi(n_1)\,...\,n_k^{\delta}\,\chi(n_k)\,e^{-\frac{\pi(n_1^2+...+n_k^2)\,x}{q}}.$$

Thus, in analogy to the Dirichlet series attached to the sum of $k-$squares, we study the zeros of arbitrary combinations of the Dirichlet series[3]

$$L_k\left(s,\chi\right):=\sum_{n_1,...,n_k\in\mathbb{Z}}\frac{n_1^{\delta}\,\chi(n_1)\,...\,n_k^{\delta}\,\chi(n_k)}{\left(n_1^2+...+n_k^2\right)^s},\quad \text{Re}(s)>\frac{k}{2}(1+\delta), \tag{1.16}$$

[3]Note that the zero term $(n_1,...,n_k)=(0,...,0)$ is not problematic in the expression of this Dirichlet series because the non-principal characters vanish at zero.

which is clearly attached to the powers of $\theta(x,\chi)$. Note that, if $k=1$, $L_1(s,\chi)=2\,L(2s-\delta,\chi)$. Also, $L_k\left(s,\chi\right)$ can be thought as a "character analogue" of $\zeta_k(s)$. See Lemma 1.3 below to see the functional equation satisfied by (1.16).

Using a class of transformation formulas related to $L_k(s,\chi)$, we will study the analytic continuation of (1.16) and establish the following Theorem.

Theorem 1.3. *Let χ be a primitive Dirichlet character modulo q and define $\delta=0$ if χ is even and $\delta=1$ if χ is odd.*

Moreover, if χ is even assume that $q\neq 0 \mod 4$.

For $L_k\left(s,\chi\right)$ given in (1.16), set

$$\eta_k\left(s,\chi\right):=\left(\frac{\pi}{q}\right)^{-s}\Gamma(s)\,L_k\left(s,\chi\right).$$

Assume that $(c_j)_{j\in\mathbb{N}}$ is a sequence of real numbers such that $\sum_j |c_j|<\infty$ and $\left(\lambda_j\right)_{j\in\mathbb{N}}$ is a bounded sequence of distinct real numbers attaining its bounds.

Moreover, assume that z satisfies the condition:

$$z\in\mathscr{D}_q:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<\sqrt{\frac{\pi}{2q}},\,|Im(z)|<\sqrt{\frac{\pi}{2q}}\right\}. \tag{1.17}$$

Then the function $F_{z,k,\chi}(s)$ given by

$$\sum_{j=1}^{\infty}c_j\,\eta_k\left(s+i\lambda_j,\chi\right)\,\left\{{}_1F_1\left(k\left(\frac{1}{2}+\delta\right)-s-i\lambda_j;\,k\left(\frac{1}{2}+\delta\right);\,\frac{z^2}{4}\right)+{}_1F_1\left(k\left(\frac{1}{2}+\delta\right)-\overline{s}+i\lambda_j;\,k\left(\frac{1}{2}+\delta\right);\,\frac{\overline{z}^2}{4}\right)\right\}$$

has infinitely many zeros on the critical line $Re(s)=\frac{k}{2}\left(\frac{1}{2}+\delta\right)$.

Finally, we consider an extension of Theorem B to Dirichlet series attached to cusp forms for the full modular group. Wilton [62] used Hardy's method to prove that, if p and q are two integers such that $p^2\equiv 1 \mod q$, then the twisted $L-$function,

$$L_{\tau}\left(s,\frac{p}{q}\right)=\sum_{n=1}^{\infty}\frac{\tau(n)\,e^{\frac{2\pi i p}{q}n}}{n^s}, \tag{1.18}$$

where $\tau(n)$ is Ramanujan's $\tau-$function, has infinitely many zeros on the line $\text{Re}(s)=6$. Since $\tau(n)$ defines the Fourier coefficients of a cusp form with weight $k=12$, it is natural that Wilton's proof extends to $L-$functions attached to different cusp forms. Very recently, Meher, Pujahari and Shankhadhar [43] employed Wilton's argument to prove that $L-$functions of cusp forms having half-integral weight with level $4N^2$ have infinitely many zeros on their critical lines.

Since any Dirichlet series of the form (1.18) satisfies Hecke's functional equation, our general formalism allows to prove the following extension of Wilton's result.

Theorem 1.4. *Let $f(\tau)$ be a cusp form of weight k for the full modular group with real Fourier coefficients $a_f(n)$. Consider the Dirichlet series:*

$$L_f(s,p/q)=\sum_{n=1}^{\infty}\frac{a_f(n)\,e^{\frac{2\pi ip}{q}n}}{n^s},\qquad p^2\equiv 1\mod q. \tag{1.19}$$

If $(c_j)_{j\in\mathbb{N}}$ is a sequence of non-zero real numbers such that $\sum_{j=1}^{\infty}|c_j|<\infty$, $(\lambda_j)_{j\in\mathbb{N}}$ is a bounded sequence of distinct real numbers attaining its bounds and z satisfies the condition:

$$z\in\mathscr{D}_q:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<2\sqrt{\frac{\pi}{q}},\ |Im(z)|<2\sqrt{\frac{\pi}{q}}\right\}; \tag{1.20}$$

Then the function

$$\sum_{j=1}^{\infty}c_j\left(\frac{2\pi}{q}\right)^{-s-i\lambda_j}\Gamma(s+i\lambda_j)\,L_f\left(s+i\lambda_j,\frac{p}{q}\right)\left\{{}_1F_1\left(k-s-i\lambda_j;\,k;\,\frac{z^2}{4}\right)+{}_1F_1\left(k-\overline{s}+i\lambda_j;\,k;\,\frac{\overline{z}^2}{4}\right)\right\} \tag{1.21}$$

has infinitely many zeros on the critical line $Re(s)=\frac{k}{2}$.

We would like to remark that it is possible to generalize Theorem B to even more Dirichlet series. Although we do not give the details in this paper, we have obtained similar results as the ones stated above in the following cases:

1. Although $L_k(s,\chi)$ is a nice analogue of the Dirichlet series $\zeta_k(s)$, there are other ways of constructing character analogues of Epstein zeta functions. One such way is due to H. M. Stark [[54, 55]] (see also [7] for more constructions), who introduced the character analogue of (1.12) in the form

$$L\left(s,\chi,Q\right):=\sum_{m,n\neq 0}\frac{\chi\left(Q(m,n)\right)}{Q(m,n)^s},\quad \mathrm{Re}(s)>1, \tag{1.22}$$

and studied its analytic properties when χ is primitive. One can prove an analogue of Theorem 1.2 for this Dirichlet series, under the additional assumption that $(q,\Delta)=1$ and Δ satisfying the same conditions as in Theorem 1.2.

2. Like in the case of (1.22), we could get similar results to different analogues of the Epstein zeta function such as

$$\zeta\left(s,Q,\chi\right):=\sum_{m,n\neq 0}\frac{\chi(m)\,\chi(n)}{Q(m,n)^s},\quad \mathrm{Re}(s)>1,$$

as well as

$$\sum_{m_1,\ldots,m_{2k}\neq 0}\frac{\chi\left(m_1^2+\ldots+m_{2k}^2\right)}{\left(m_1^2+\ldots+m_{2k}^2\right)^s},\quad \mathrm{Re}(s)>k.$$

3. There are also extensions of Theorem 1.4 to $L-$functions attached to cusp forms and twisted by primitive Dirichlet characters.

4. Instead of considering an Epstein zeta function in Theorem 1.2, one may prove prove a similar result for the Dirichlet series $\zeta(s)\,L(s,\chi)$, where χ is an odd and primitive Dirichlet character whose modulus is a perfect square.

This paper is organized as follows. In the next subsections we introduce some general definitions in the set up of Dirichlet series. We also give the functional equations of the main L-functions here studied, as well as some background on special functions.

Section 2 is devoted to prove a general analogue of the theta transformation formula, from which (1.6) is derived as a particular case. We also provide examples which will be helpful in establishing our main Theorems.

In the remaining sections of the paper we prove each particular Theorem.

1.1 Definitions

To give a generalization of (1.6) we need the following definition.

Definition 1.1. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ be two sequences of positive numbers strictly increasing to ∞ and $(a(n))_{n\in\mathbb{N}}$ and $(b(n))_{n\in\mathbb{N}}$ two sequences of complex numbers not identically zero. Consider the functions $\phi(s)$ and $\psi(s)$ representable as Dirichlet series

$$\phi(s)=\sum_{n=1}^{\infty}\frac{a(n)}{\lambda_n^s}\quad\text{and}\quad\psi(s)=\sum_{n=1}^{\infty}\frac{b(n)}{\mu_n^s}\tag{1.23}$$

with finite abscissas of absolute convergence σ_a and σ_b respectively. Let $\Delta(s)$ denote one of the following three gamma factors: $\Gamma(s)$, $\Gamma\left(\frac{s}{2}\right)$ and $\Gamma\left(\frac{s+1}{2}\right)$ and r be an arbitrary positive real number in the first case and 1 in the other two.

We say that $\phi(s)$ and $\psi(s)$ satisfy the functional equation

$$\Delta(s)\,\phi(s)=\Delta(r-s)\,\psi(r-s),\tag{1.24}$$

if there exists a meromorphic function $\chi(s)$ with the following properties:

1. $\chi(s)=\Delta(s)\,\phi(s)$ for $\text{Re}(s)>\sigma_a$ and $\chi(s)=\Delta(r-s)\,\psi(r-s)$ for $\text{Re}(s)<r-\sigma_b$;
2. $\lim_{|\text{Im}(s)|\to\infty}\chi(s)=0$ uniformly in every interval $-\infty<\sigma_1\leq\text{Re}(s)\leq\sigma_2<\infty$.
3. The singularities $\chi(s)$ are at most poles and are confined to some compact set.

We say that the pair of functions $(\phi,\,\psi)$ representable as Dirichlet series (1.23) satisfy **Hecke's functional equation** if they satisfy the conditions of the previous definition for $\Delta(s)=\Gamma(s)$. The particular case of (1.24) reads

$$\Gamma(s)\,\phi(s)=\Gamma(r-s)\,\psi(r-s),\quad r>0.\tag{1.25}$$

Similarly, we say that the pair of functions $(\phi,\,\psi)$ representable as (1.23) with finite abscissas of absolute convergence σ_a and σ_b satisfy **Bochner's functional equation** if they satisfy the functional equation

$$\Gamma\left(\frac{s+\delta}{2}\right)\phi(s)=\Gamma\left(\frac{1+\delta-s}{2}\right)\psi\left(1-s\right),\tag{1.26}$$

in the sense of Definition 1.1.

Before proceeding further, we shall denote $\sum_{m,n\neq 0}$ as the infinite sum over all integers m, n not simultaneously zero. The same notation is taken for multiple sums. Whenever we use the term "critical line" for a given Dirichlet series $\phi(s)$ satisfying Definition 1.1 and the functional equation (1.25) we will be referring to $\text{Re}(s) = \frac{r}{2}$, which is the line of symmetry.

The Dirichlet series considered in all of our Theorems have at most one simple pole as singularity. Therefore, it will be more convenient for the analysis of this paper to consider subclasses of the Dirichlet series defined above, with fewer singularities. We now give two definitions introducing these.

Definition 1.2. Let $\phi(s)$ be a Dirichlet series satisfying Definition 1.1 with $\Delta(s) = \Gamma(s)$. We say that $\phi(s)$ belongs to the class $\mathcal{A}$ if additionally:

1. $\phi(s)$ and $\psi(s)$ are analytic everywhere in $\mathbb{C}$ except for possible simple poles located at $s = r$ with residues ρ and $\rho^{\star}$ respectively.

Remark 1.1. By the functional equation (1.25) and the additional condition in the previous definition, $\phi(0) = -\rho^{\star}\Gamma(r)$, while $\psi(0) = -\rho\,\Gamma(r)$. In particular, if $\psi(s)$ is an entire Dirichlet series, then $\phi(0) = 0$. It is also clear that $\phi(s) \in \mathcal{A}$ if and only if $\psi(s) \in \mathcal{A}$. The purpose of introducing this class is to mimic as much as possible the class of Hecke Dirichlet series with a given signature, which is a subclass of $\mathcal{A}$. A similar definition was also considered [[3], p.221].

Remark 1.2. It is clear from the definition above and the functional equation (1.25) that $\phi(-n) = 0$ for every $n \in \mathbb{N}$.

Another class of Dirichlet series related to the previous one is given in the following definition:

Definition 1.3. Let $\phi(s)$ be a Dirichlet series satisfying Definition 1.1 with $\Delta(s) = \Gamma\left(\frac{s+\delta}{2}\right)$, $\delta \in \{0, 1\}$, $r = 1$ (Bochner class). We say that $\phi(s)$ belongs to the class $\mathcal{B}$ if additionally:

1. For $\delta = 0$, $\phi(s)$ and $\psi(s)$ are analytic everywhere in $\mathbb{C}$ except for possible simple poles located at $s = 1$ with residues ρ and $\rho^{\star}$ respectively.

2. For $\delta = 1$, $\phi(s)$ and $\psi(s)$ can be analytically continued as entire Dirichlet series.

Remark 1.3. Note that definition 1.3 mimics in some way the properties that the Dirichlet $L-$functions and the Riemann $\zeta-$function have. Note that $\phi(s) = \pi^{-s/2}\zeta(s)$ is a Dirichlet series belonging to the class $\mathcal{B}$ with $\delta = 0$.

By the functional equation for Bochner Dirichlet series (1.26), it is effortless to see that $\phi(0) = -\frac{\rho^{\star}}{2}\sqrt{\pi}$, while $\psi(0) = -\frac{\rho}{2}\sqrt{\pi}$. In particular, if $\psi(s)$ is entire then $\phi(0) = 0$.

Remark 1.4. If $\phi(s) \in \mathcal{B}$ with $\delta = 0$ and its residue at the simple pole $s = 1$ is ρ, then $\phi'(s) := \phi(2s) \in \mathcal{A}$ having a simple pole located at $s = 1/2$ with residue $\rho/2$. Moreover, $\phi'(s)$ satisfies Hecke's functional equation (1.25) with parameter $r = 1/2$.

If $\phi(s) \in \mathcal{B}$ with $\delta = 1$, then $\phi'(s) := \phi(2s - 1) \in \mathcal{A}$. Since $\phi(s)$ is entire when $\delta = 1$, $\phi'(s)$ will also be entire. Moreover, $\phi'(s)$ satisfies Hecke's functional equation (1.25) with parameter $r = 3/2$.

The functional equations (1.25) and (1.26) can be translated into arithmetical identities involving the sequences $a(n)$ and $b(n)$. From now on, if $\phi(s)$ satisfies Hecke's functional equation in the sense of Definition 1.1 and it is representable by the first Dirichlet series in (1.23), we shall use $\Phi(x)$ and $\Psi(x)$ to denote the generalized $\theta-$functions (1.2)

$$\Phi(x) := \sum_{n=1}^{\infty} a(n)\, e^{-\lambda_n x}, \quad \Psi(x) := \sum_{n=1}^{\infty} b(n)\, e^{-\mu_n x} \quad \text{Re}(x) > 0. \tag{1.27}$$

Following Bochner, for $\text{Re}(x) > 0$, let $P(x)$ denote the residual function

$$P(x) = \frac{1}{2\pi i} \int_{C} \Gamma(s)\, \phi(s)\, x^{-s} ds, \tag{1.28}$$

where C denotes a curve, or curves, encircling the singularities of $\chi(s)$ given in Definition 1.1. It was established in [14] that Hecke's functional equation for $\phi(s)$ and $\psi(s)$ (1.25) and the "modular relation"

$$\Phi(x) = x^{-r}\Psi\left(\frac{1}{x}\right) + P(x) \tag{1.29}$$

are equivalent.

It is interesting to note that the observation of this equivalence has its genesis in Riemann's revolutionary memoir [49], where one of the implications was proved for the first time.

Berndt [[4], [6]] established an expansion involving Modified Bessel function which is equivalent to (1.29). He proved that, if $x > 0$, then the curious formula takes place

$$\Gamma(s) \sum_{n=1}^{\infty} \frac{a(n)}{(\lambda_n + x^2)^s} = R(s, x) + 2x^{r-s} \sum_{n=1}^{\infty} b(n)\, \mu_n^{\frac{s-r}{2}}\, K_{s-r}\left(2\sqrt{\mu_n}\, x\right), \quad \text{Re}(s) > \sigma_a, \tag{1.30}$$

where $R(s, x)$ denotes the sum of residues of the function $\Gamma(w)\, \phi(w)\, \Gamma(s - w)\, x^{2w-2s}$ at the poles of $\Gamma(w)\, \phi(w)$.

Since we will generalize (1.29) in the next section (see Theorem 2.1 and Corollary 2.2 below), it will be natural to seek a generalization of (1.30), which will be given in Corollary 2.4.

1.2 Preliminary results

In several occasions throughout this paper, we shall need to estimate the asymptotic order of certain integrals involving the Dirichlet series $\phi(s)$ satisfying Definition 1.1. To justify most of the steps, we will often invoke the following version of Stirling's formula

$$\Gamma(\sigma + it) = (2\pi)^{\frac{1}{2}}\, t^{\sigma + it - \frac{1}{2}}\, e^{-\frac{\pi t}{2} - it + \frac{i\pi}{2}(\sigma - \frac{1}{2})} \left(1 + \frac{1}{12(\sigma + it)} + O\left(\frac{1}{t^2}\right)\right), \tag{1.31}$$

as $t \to \infty$, uniformly for $-\infty < \sigma_1 \leq \sigma \leq \sigma_2 < \infty$. A similar formula can be written for $t < 0$ as t tends to $-\infty$ by using the fact that $\Gamma(\overline{s}) = \overline{\Gamma(s)}$. Of course, a direct consequence of this exact version is

$$|\Gamma(\sigma + it)| = (2\pi)^{\frac{1}{2}}\, |t|^{\sigma - \frac{1}{2}}\, e^{-\frac{\pi}{2}|t|} \left(1 + O\left(\frac{1}{|t|}\right)\right), \quad |t| \to \infty. \tag{1.32}$$

To estimate the order of $\phi(s)$ at the line $\text{Re}(s) = \sigma$ we shall need a version of the classical Phragmén-Lindelöf theorem given in [[58], p.180, 5.65] (see also [[48], p. 11] for details). Adapted to our current needs, this version states that, if $\phi(s)$ satisfies Hecke's functional equation (1.25) and has abscissa of absolute convergence σ_a, then, as $|t| \to \infty$,

$$\phi(\sigma+it) = O\left(|t|^{\sigma_a+\delta-\sigma}\right), \quad r-\sigma_a-\delta \leq \sigma \leq \sigma_a+\delta. \tag{1.33}$$

The next few lemmas are mainly concerned with the analytic continuation of the Dirichlet series studied in this paper.

We start with $\zeta_\alpha(s)$. The theta function $\vartheta_3(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$ is a modular form of weight $\frac{1}{2}$ with a multiplier system with respect to the theta group Γ_θ (see [[40], pp. 15-16]). Therefore, for every $\alpha > 0$, $\vartheta_3^\alpha(\tau)$ is a modular form of weight $\alpha/2$ with a multiplier system on the same group.

By definition, $r_\alpha(n)$ are the Fourier coefficients of the expansion of $\vartheta_3(\tau)$ at the cusp $i\infty$, this is:

$$\vartheta_3^\alpha(ix) := \theta^\alpha(x) = 1 + \sum_{n=1}^\infty r_\alpha(n)\, e^{-\pi n\, x}, \quad x>0. \tag{1.34}$$

It is helpful to see how these coefficients show up: these are computable by using the expansion

$$\begin{aligned}\vartheta_3^\alpha(ix) = \theta^\alpha(x) &= \left(1+2\sum_{n=1}^\infty e^{-\pi n^2 x}\right)^\alpha = \sum_{j=0}^\infty \left(\begin{array}{c}\alpha\\ j\end{array}\right) 2^j \left(\sum_{n=1}^\infty e^{-\pi n^2 x}\right)^j \\ &= 1+\sum_{j=1}^\infty \left(\begin{array}{c}\alpha\\ j\end{array}\right) 2^j \sum_{n_1,\dots,n_j=1}^\infty e^{-\pi\left(n_1^2+\dots+n_j^2\right)x}.\end{aligned} \tag{1.35}$$

Defining a new summation variable, say $m = n_1^2+...+n_j^2$, we have that the number of $(n_i)_{1\leq i\leq j}$ decomposing n in this way is at most m. Therefore, the sum over j is finite (in fact, a polynomial of degree m in α), so we may define $r_\alpha(m)$ as

$$\sum_{j=1}^\infty \left(\begin{array}{c}\alpha\\ j\end{array}\right) 2^j \sum_{n_1,\dots,n_j=1}^\infty e^{-\pi\left(n_1^2+\dots+n_j^2\right)x} := \sum_{m=1}^\infty r_\alpha(m)\, e^{-\pi m x}. \tag{1.36}$$

The order of growth of $r_\alpha(n)$ as $n\to\infty$ is determined by classical estimates due to Petersson and Lehner [40, 42, 56] for the Fourier coefficients of arbitrary modular forms of positive real weight with multiplier systems. These estimates show that $r_\alpha(n)$ grows polynomially, more precisely in the form:

$$r_\alpha(n) \ll_\alpha \begin{cases} n^{\alpha/2-1} & \alpha>4 \\ n^{\alpha/2-1}\log(n) & \alpha=4 \\ n^{\alpha/4} & 0<\alpha<4. \end{cases} \tag{1.37}$$

See also [[40], p. 18, Theorem 3.3.] for estimates whose constants are independent of α. These bounds determine the existence of a finite abscissa of absolute convergence for $\zeta_\alpha(s)$ given in (1.8).

Our first lemma concerns the analytic continuation of (1.8), which should resemble the analytic continuation of $\zeta_k(s)$. For a proof see [[40], p. 11, Theorem 2.1.].

Lemma 1.1. *Let $r_\alpha(n)$ be the sequence defined by (1.34). Consider the Dirichlet series:*

$$\zeta_\alpha(s)=\sum_{n=1}^{\infty}\frac{r_\alpha(n)}{n^s},\qquad Re(s)>\sigma_\alpha:=\begin{cases}\frac{\alpha}{2} & \alpha\geq 4\\ 1+\frac{\alpha}{4} & 0<\alpha<4\end{cases}.$$

Then $\zeta_\alpha(s)$ can be analytically continued to the entire complex plane as a meromorphic function with a simple pole located at $s=\frac{\alpha}{2}$ with residue $Res_{s=\alpha/2}\zeta_\alpha(s)=\frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)}$.

Moreover, it satisfies Hecke's functional equation:

$$\pi^{-s}\Gamma(s)\,\zeta_\alpha(s)=\pi^{-\left(\frac{\alpha}{2}-s\right)}\Gamma\left(\frac{\alpha}{2}-s\right)\,\zeta_\alpha\left(\frac{\alpha}{2}-s\right). \tag{1.38}$$

Our next lemma is due to Paul Epstein [27,28] and gives the analytic continuation and the functional equation for the Epstein zeta function attached to any positive quadratic form. Before stating it, let $\mathbf{x}$ denote a vector in $\mathbb{R}^n$ and $Q(\mathbf{x})$ a positive definite quadratic form in n variables. Write $Q(\mathbf{x})$ in the matrix form:

$$Q(\mathbf{x})=\frac{1}{2}\mathbf{x}^T\,A\,\mathbf{x}, \tag{1.39}$$

where A is a symmetric $n\times n$ matrix of real numbers. Define the discriminant of Q to be

$$D(Q):=\det(A),$$

and the adjoint quadratic form $Q^{\dagger}(\mathbf{x})$ by

$$Q^{\dagger}(\mathbf{x})=\frac{1}{2}\mathbf{x}^T\,A^{\dagger}\,\mathbf{x}, \tag{1.40}$$

with $A^{\dagger}$ denoting the adjoint matrix of A. The Epstein zeta functions attached to Q are defined by the series:

$$\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right):=\sum_{\mathbf{m}\in\mathbb{Z}^n,\,\mathbf{m}+\mathbf{g}\neq\mathbf{0}}\frac{\exp\left(2\pi i\,\mathbf{m}\cdot\mathbf{h}\right)}{Q\left(\mathbf{m}+\mathbf{g}\right)^s},\quad \text{Re}(s)>\frac{n}{2}, \tag{1.41}$$

where the $\mathbf{g}$ and $\mathbf{h}$ are vectors in $\mathbb{R}^n$. The following lemma gives the analytic properties of (1.41) (for an even more detailed statement, see [[51], p. 54, Theorem 3]).

Lemma 1.2. *The Epstein zeta function $\zeta\left(s,\mathbf{g},\mathbf{h},\,Q\right)$ has an analytic continuation into the entire complex plane as:*

1. *An entire function if $\mathbf{h}\notin\mathbb{Z}^n$;*

2. *A meromorphic function with a simple pole at $s=\frac{n}{2}$ if $\mathbf{h}\in\mathbb{Z}^n$. The residue that $\zeta\left(s,\mathbf{g},\mathbf{h},\,Q\right)$ possesses at $s=n/2$ is given by $(2\pi)^{n/2}\Gamma\left(\frac{n}{2}\right)/\sqrt{D(Q)}$.*

Moreover, $\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)$ satisfies Hecke's functional equation:

$$\left(\frac{2\pi}{D(Q)^{1/n}}\right)^{-s}\Gamma(s)\,\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)=e^{-2\pi i\mathbf{g}\cdot\mathbf{h}}\left(\frac{2\pi}{D(Q^{\dagger})^{1/n}}\right)^{-\left(\frac{n}{2}-s\right)}\Gamma\left(\frac{n}{2}-s\right)\,\zeta\left(\frac{n}{2}-s,\mathbf{h},-\mathbf{g},Q^{\dagger}\right). \tag{1.42}$$

The previous functional equation will be very useful in studying our next result which, although making its first appearance here as a lemma, will be proven in subsection 5.2. It concerns the analytic continuation of the series (1.16).

Lemma 1.3. *Let χ be a primitive Dirichlet character modulo q and set $\delta=0$ if χ is even and $\delta=1$ if χ is odd.*

If $L_k(s,\chi)$ is the Dirichlet series given by

$$L_k(s,\chi):=\sum_{n_1,\dots,n_k\in\mathbb{Z}}\frac{n_1^{\delta}\chi(n_1)\dots n_k^{\delta}\chi(n_k)}{\left(n_1^2+\dots+n_k^2\right)^s},\quad Re(s)>\frac{k}{2}(1+\delta), \tag{1.43}$$

then $L_k(s,\chi)$ can be analytically continued as an entire function and it satisfies the functional equation:

$$\left(\frac{\pi}{q}\right)^{-s}\Gamma(s)\,L_k(s,\chi)=\frac{(-i)^{\delta k}G^k(\chi)}{q^{k/2}}\left(\frac{\pi}{q}\right)^{-\left(k(\frac{1}{2}+\delta)-s\right)}\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)L_k\left(k\left(\frac{1}{2}+\delta\right)-s,\,\overline{\chi}\right), \tag{1.44}$$

where $G(\chi)$ denotes the Gauss sum

$$G(\chi):=\sum_{r=0}^{q}\chi(r)\,e^{2\pi i n\,r/q}. \tag{1.45}$$

We will prove this result on subsection 5.2 because we could not track any reference containing it, namely when $\delta=1$. Of course, when $k=1$, (1.44) easily reduces to the functional equation for Dirichlet L-functions and for the case $\delta=0$ it actually comes from a more general result of Berndt [7]. For $\delta=1$, however, (1.44) seems to be absent in the literature.

To conclude our set of lemmas, let $f(\tau)$ be a cusp form of weight $k\geq 12$ for $\mathrm{SL}(2,\mathbb{Z})$ on the upper half plane $\mathbb{H}:=\{\tau\in\mathbb{C}\,:\,\mathrm{Im}(\tau)>0\}$ with Fourier expansion given by

$$f(\tau)=\sum_{n=1}^{\infty}a_f(n)\,e^{2\pi i n\tau}. \tag{1.46}$$

Then, for all $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z})$ and $\tau\in\mathbb{H}$, $f(\tau)$ satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k\,f(\tau),\qquad \lim_{\mathrm{Im}(\tau)\to\infty}f(\tau)=0. \tag{1.47}$$

The following lemma, due to Wilton [62], gives the analytic continuation of a twisted Dirichlet series attached to $f(\tau)$. We quote Wilton's result in the same form as exposed in Jutila's text [[35], p. 14, Lemma 1.2.].

Lemma 1.4. *Let $f(\tau)$ be a holomorphic cusp form of weight k for the full modular group and let $a_f(n)$ be its Fourier coefficients. Also, assume that p,q are integers such that $(p,q)=1$ and consider the Dirichlet series*

$$L_f(s,p/q):=\sum_{n=1}^{\infty}\frac{a_f(n)\,e^{\frac{2\pi i p}{q}n}}{n^s},\qquad Re(s)>\frac{k+1}{2}. \tag{1.48}$$

Then $L_f(s,p/q)$ can be continued analytically as an entire function satisfying Hecke's functional equation

$$\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)\,L_f\left(s,\frac{p}{q}\right)=(-1)^{k/2}\left(\frac{2\pi}{q}\right)^{-(k-s)}\Gamma(k-s)\,L_f\left(k-s,\,-\overline{p}/q\right), \tag{1.49}$$

where $\overline{p}$ is such that $p\,\overline{p}\equiv 1\mod q$.

Throughout this paper we will also require the asymptotic formulas for the Bessel functions. It can be seen in [[61], pp. 199-202] that $J_{\nu}(z)$, $I_{\nu}(z)$ and $K_{\nu}(z)$ satisfy the asymptotic expansions:

$$J_{\nu}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \left\{\cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \sum_{n=0}^{\infty} \frac{(-1)^n (\nu, 2n)}{(2z)^{2n}} - \sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \sum_{n=0}^{\infty} \frac{(-1)^n(\nu, 2n+1)}{(2z)^{2n+1}}\right\}, \tag{1.50}$$

$$I_{\nu}(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(-1)^n (\nu, n)}{(2z)^n} + \frac{e^{-z\pm(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(\nu, n)}{(2z)^n}, \tag{1.51}$$

$$K_{\nu}(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{n=0}^{\infty} \frac{(\nu, n)}{(2z)^n}. \tag{1.52}$$

Formulas (1.50) and (1.52) are valid in the limit $|z| \to \infty$, $|\arg(z)| < \pi$. The asymptotic expansion (1.51) is taken with the plus sign if $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$ and with minus sign if $-\frac{3\pi}{2} < \arg(z) < \frac{\pi}{2}$ (see also [[44], pp. 223, 249 and 252, relations 10.7.3, 10.25.3, 10.30.4, 10.30.5]). Here, (ν, n) stands for the Hankel symbol:

$$(\nu, n) := \frac{(-1)^n \cos(\pi\nu)}{\pi\, n!} \Gamma\left(\frac{1}{2} + \nu + n\right) \Gamma\left(\frac{1}{2} - \nu + n\right). \tag{1.53}$$

In the sequel, we will employ at some points well-known integral representations for the Bessel functions given above, as well as to Kummer's function. For a matter of clarity in our exposition, we will invoke them solely when they are needed.

2 A general theta transformation formula involving Bessel functions

To prove all the theorems given at the introduction, we need to develop a new summation formula which generalizes Jacobi's formula (1.5) with respect to the parameter r in the functional equation (1.25).

In the same way that the "modular relation" (1.29) acts as a generalization of (1.3), in this section we will try to seek which kind of modular transformation generalizes (1.5).

We will restrict ourselves to Dirichlet series belonging to the class $\mathcal{A}$ (see definition 1.2 above) although the study given in this section also works for Dirichlet series with a more general distribution of singularities. We will need to find a way to study a summation formula for

$$\sum_{n=1}^{\infty} a(n)\, \lambda_n^{\frac{1-r}{2}}\, e^{-\alpha\lambda_n}\, J_{r-1}(\beta\sqrt{\lambda_n}), \quad r > 0, \tag{2.1}$$

where $a(n)$ and λ_n are given in definition 1.1. Here, r is the parameter appearing in Hecke's functional equation (1.25).

This study employs an integral representation, known as Hankel's formula (see [[16], p. 8, eq. (15)], [[15], p.155, eq.(3.10.3.2)] and also [[1], pp. 221-222] for a brief proof),

$$\int_0^{\infty} t^{s-1} e^{-p^2t^2} J_{\nu}(at)\, dt = \frac{\Gamma\left(\frac{s+\nu}{2}\right) a^{\nu}}{2^{\nu+1} p^{s+\nu} \Gamma(\nu+1)}\, {}_1F_1\left(\frac{s+\nu}{2};\, \nu+1;\, -\frac{a^2}{4p^2}\right), \tag{2.2}$$

valid for $\mathrm{Re}(s) > -\mathrm{Re}(\nu)$, $\mathrm{Re}(p) > 0$, $|\arg(a)| < \pi$.

Clearly, for $\nu = r-1$ and $\mathrm{Re}(s) > 1-r$, $\mathrm{Re}(\alpha) > 0$ and $|\arg(\beta)| < \pi$, the following particular case of the previous formula holds

$$\int_0^\infty t^{s-1}e^{-\alpha t}J_{r-1}(\beta\sqrt{t})\,dt = \frac{\Gamma\left(s+\frac{r-1}{2}\right)\beta^{r-1}}{2^{r-1}\alpha^{s+\frac{r-1}{2}}\Gamma(r)}\,{}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right). \tag{2.3}$$

This suggests the inversion formula, now valid for $\mathrm{Re}(\alpha) > 0$, $\beta \in \mathbb{C}$ and σ, $x > 0$,

$$e^{-\alpha x}J_{r-1}(\beta\sqrt{x}) = \frac{1}{\Gamma(r)}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(s+\frac{r-1}{2}\right){}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)(\alpha x)^{-s}ds, \tag{2.4}$$

which might be proved by invoking directly the power series expansion of Kummer's function and using the power series for the Bessel function $J_\nu(z)$.

To represent the series given in (2.1), we need an asymptotic formula for the confluent hypergeometric function on the right-hand side of (2.4) which is valid when $|s| \to \infty$. Following the reasoning in [[19], p. 379], recall that the Whittaker function $M_{\lambda,\mu}(z)$ has the asymptotic formula [[29], Vol. , p.274], [[44], p. 341, eq. (13.21.1)]

$$M_{\lambda,\mu}(z) = \frac{z^{1/4}}{\sqrt{\pi}}\lambda^{-\mu-\frac{1}{4}}\Gamma(2\mu+1)\cos\left(2\sqrt{\lambda z}-\frac{\pi}{4}-\mu\pi\right)+O\left(|\lambda|^{-\mu-\frac{3}{4}}\right)$$

as $|\lambda| \to \infty$ and z such that $|\arg(\lambda z)| < 2\pi$. Furthermore, by its definition,

$$M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-z/2}\,{}_1F_1\left(\mu-\lambda+\frac{1}{2};\,2\mu+1;\,z\right), \tag{2.5}$$

we see that, once we replace λ by $\frac{1}{2}-s$, μ by $\frac{r-1}{2}$ and z by $-\frac{\beta^2}{4\alpha}$,

$${}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right) = \frac{e^{-\frac{\beta^2}{8\alpha}}\Gamma(r)}{\sqrt{\pi}}\left(\frac{\beta^2}{4\alpha}\left(s-\frac{1}{2}\right)\right)^{\frac{1}{4}-\frac{r}{2}}\cos\left(\frac{\beta}{\alpha}\sqrt{s-\frac{1}{2}}+\frac{\pi}{4}-\pi\frac{r}{2}\right)+O\left(|s|^{-\frac{r}{2}-\frac{1}{4}}\right), \tag{2.6}$$

as $|s| \to \infty$ and $\left|\arg\left(\frac{\beta^2}{4\alpha}\left(s-\frac{1}{2}\right)\right)\right| < 2\pi$.

For the purposes of this paper, it will be enough to invoke a simpler bound of the form,

$$\left|{}_1F_1\left(\sigma+it+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)\right| \leq A\,|t|^{\frac{1}{4}-\frac{r}{2}}\,e^{B\sqrt{|t|}}+O\left(|t|^{-\frac{r}{2}-\frac{1}{4}}\right),\quad |t|\to\infty, \tag{2.7}$$

where A and B are positive constants depending only on α, β and r and $c \leq \sigma \leq d$.

Since we will use the Mellin inverse representation (2.4), the confluent hypergeometric function lying in its kernel will have to satisfy a transformation formula compatible with Hecke's functional equation (1.25). The following formula due to Kummer [[1], p. 191, eq. (4.1.11)] will be useful

$${}_1F_1(a;\,c;\,x) = e^x\,{}_1F_1(c-a;\,c;\,-x). \tag{2.8}$$

2.1 General identities for Dirichlet series

We are now ready to prove the following Theorem, which has independent interest because it is an identity concerning general Dirichlet series.

Theorem 2.1. *Let $\phi(s)$ and $\psi(s)$ be the pair of Dirichlet series satisfying the functional equation (1.25). Furthermore, assume that $\phi(s)\in\mathcal{A}$ and let us denote by ρ the residue that $\phi(s)$ possesses at its pole $s=r$.*

Then, for $\mathrm{Re}(\alpha)>0$ and $\beta\in\mathbb{C}$, the following transformation formula holds

$$\sum_{n=1}^{\infty}a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\alpha\lambda_n}\,J_{r-1}(\beta\sqrt{\lambda_n})=\frac{\phi(0)\beta^{r-1}}{2^{r-1}\Gamma(r)}+\frac{\beta^{r-1}\rho}{2^{r-1}\alpha^{r}}\,e^{-\frac{\beta^2}{4\alpha}}+\frac{e^{-\frac{\beta^2}{4\alpha}}}{\alpha}\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{1-r}{2}}\,e^{-\mu_n/\alpha}I_{r-1}\left(\frac{\beta\sqrt{\mu_n}}{\alpha}\right). \quad (2.9)$$

Remark 2.1. The series on the left-hand side is obviously convergent under the conditions established for the sequences $a(n)$ and λ_n in definition 1.1. The convergence of the series on the right-hand side comes from the Hankel expansion for the Modified Bessel function (1.51), which gives $|I_{\nu}(z)|\ll_{\nu}e^{|\mathrm{Re}(z)|}/\sqrt{2\pi|z|}$ as $|z|\to\infty$.

Proof. Pick $\mu>\max\left\{\sigma_a-\frac{r}{2}+\frac{1}{2},\sigma_b-\frac{r}{2}+\frac{1}{2},\frac{r+1}{2},\frac{3r}{2}-\frac{1}{2}\right\}$. By (2.6) and (2.4), and arguing by absolute convergence, we can write the series (2.1) as

$$\sum_{n=1}^{\infty}a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\alpha\lambda_n}\,J_{r-1}(\beta\sqrt{\lambda_n})=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\phi\left(s+\frac{r-1}{2}\right)\frac{\Gamma\left(s+\frac{r-1}{2}\right)\beta^{r-1}}{2^{r-1}\alpha^{s+\frac{r-1}{2}}\Gamma(r)}\,{}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)ds. \quad (2.10)$$

We will now integrate along a positively oriented rectangular contour $\mathcal{R}$ containing the vertices $\mu\pm iT$ and $r-\mu\pm iT$ for $T>0$. By the choice of μ, we know that the line $\mathrm{Re}(s)=r-\mu$ is located at the left of the line $\mathrm{Re}(s)=\mu$. By using the residue theorem, we have

$$\frac{1}{2\pi i}\int_{\mu-iT}^{\mu+iT}\phi\left(s+\frac{r-1}{2}\right)\frac{\Gamma\left(s+\frac{r-1}{2}\right)\beta^{r-1}}{2^{r-1}\alpha^{s+\frac{r-1}{2}}\Gamma(r)}\,{}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)ds=$$

$$=\frac{1}{2\pi i}\left[\int_{r-\mu+iT}^{\mu+iT}+\int_{r-\mu-iT}^{r-\mu+iT}+\int_{\mu-iT}^{r-\mu-iT}\right]\phi\left(s+\frac{r-1}{2}\right)\frac{\Gamma\left(s+\frac{r-1}{2}\right)\beta^{r-1}}{2^{r-1}\alpha^{s+\frac{r-1}{2}}\Gamma(r)}\,{}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)ds$$

$$+\sum_{\rho\in\mathcal{R}}\mathrm{Res}_{s=\rho}\left\{\phi\left(s+\frac{r-1}{2}\right)\frac{\Gamma\left(s+\frac{r-1}{2}\right)\beta^{r-1}}{2^{r-1}\alpha^{s+\frac{r-1}{2}}\Gamma(r)}\,{}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)\right\}. \quad (2.11)$$

Clearly, by Stirling's formula (1.32), the convex estimates (1.33) and (2.6), the integrals over the horizontal segments $[r-\mu\pm iT,\,\mu\pm iT]$ tend to zero as $T\to\infty$. Also, we know that $\phi(s)$ has a simple pole located at $s=r$ and, by Remark 1.2, has zeros located at the negative integers.

Since ${}_1F_1\left(z;\,r;\,-\frac{\beta^2}{4\alpha}\right)$ is an entire function in the variable z, the only poles that we need to take into account in the sum above are located at the points $\rho=\frac{1-r}{2}$ and $\rho=\frac{r+1}{2}$. Denoting the integrand in (2.10) by $\mathcal{I}_{\alpha,\beta}(s)$, we see that the residues at these poles are respectively

$$\mathrm{Res}_{s=\frac{r+1}{2}}\left\{\mathcal{I}_{\alpha,\beta}(s)\right\}=\frac{\beta^{r-1}\rho}{2^{r-1}\alpha^{r}}\,e^{-\frac{\beta^2}{4\alpha}},\quad \mathrm{Res}_{s=\frac{1-r}{2}}\left\{\mathcal{I}_{\alpha,\beta}(s)\right\}=\frac{\phi(0)\beta^{r-1}}{2^{r-1}\Gamma(r)}. \quad (2.12)$$

Letting $T \to \infty$ in (2.11), we now need to evaluate the integral

$$\frac{1}{\Gamma(r)}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}}\frac{1}{2\pi i}\int_{r-\mu-i\infty}^{r-\mu+i\infty}\Gamma\left(s+\frac{r-1}{2}\right)\phi\left(s+\frac{r-1}{2}\right){}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)\alpha^{-s}\,ds,$$

which we do by appealing to the functional satisfied by $\phi(s)$ (1.25) and to Kummer's formula (2.8),

$${}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)=e^{-\frac{\beta^2}{4\alpha}}\,{}_1F_1\left(\frac{r+1}{2}-s;\,r;\,\frac{\beta^2}{4\alpha}\right). \tag{2.13}$$

Performing such transformations and changing the variable back to the region of absolute convergence of $\psi(s)$ (which is possible due to the fact that we have chosen $\mu > \sigma_b - \frac{r}{2}+\frac{1}{2}$), we get

$$\begin{aligned}
&\frac{1}{\Gamma(r)}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}}\frac{1}{2\pi i}\int_{r-\mu-i\infty}^{r-\mu+i\infty}\Gamma\left(s+\frac{r-1}{2}\right)\phi\left(s+\frac{r-1}{2}\right){}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)\alpha^{-s}\,ds\\
&=\frac{e^{-\frac{\beta^2}{4\alpha}}}{2\pi i\,\Gamma(r)}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}}\int_{r-\mu-i\infty}^{r-\mu+i\infty}\Gamma\left(\frac{r+1}{2}-s\right)\psi\left(\frac{r+1}{2}-s\right){}_1F_1\left(\frac{r+1}{2}-s;\,r;\,\frac{\beta^2}{4\alpha}\right)\alpha^{-s}\,ds\\
&=\frac{e^{-\frac{\beta^2}{4\alpha}}}{2\pi i\,\Gamma(r)}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}}\alpha^{-r}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(s+\frac{1-r}{2}\right)\psi\left(s+\frac{1-r}{2}\right){}_1F_1\left(s+\frac{1-r}{2};\,r;\,\frac{\beta^2}{4\alpha}\right)\alpha^{s}\,ds.
\end{aligned} \tag{2.14}$$

We now use the expansion of $\psi(s)$ as a Dirichlet series (1.23) together with the power series expansion for the confluent hypergeometric function. Due to absolute convergence of both series, we can write the right-hand side of (2.14) as

$$\begin{aligned}
&e^{-\frac{\beta^2}{4\alpha}}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}}\alpha^{-r}\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{r-1}{2}}\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\Gamma\left(s+\frac{1-r}{2}\right)}{\Gamma(r)}\,{}_1F_1\left(s+\frac{1-r}{2};\,r;\,\frac{\beta^2}{4\alpha}\right)(\alpha/\mu_n)^s ds\\
&=e^{-\frac{\beta^2}{4\alpha}}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}}\alpha^{-r}\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{r-1}{2}}\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\sum_{k=0}^{\infty}\frac{\Gamma\left(s+\frac{1-r}{2}+k\right)}{\Gamma(r+k)\,k!}\frac{\beta^{2k}}{4^k\alpha^k}(\alpha/\mu_n)^s ds\\
&=e^{-\frac{\beta^2}{4\alpha}}\left(\frac{\beta}{2}\right)^{r-1}\alpha^{-r}\sum_{n=1}^{\infty}b(n)\sum_{k=0}^{\infty}\frac{1}{\Gamma(r+k)\,k!}\left(\frac{\beta^2\mu_n}{4\alpha^2}\right)^k\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,(\alpha/\mu_n)^z dz\\
&=e^{-\frac{\beta^2}{4\alpha}}\left(\frac{\beta}{2}\right)^{r-1}\alpha^{-r}\sum_{n=1}^{\infty}b(n)e^{-\mu_n/\alpha}\sum_{k=0}^{\infty}\frac{1}{\Gamma(r+k)\,k!}\left(\frac{\beta\sqrt{\mu_n}}{2\alpha}\right)^{2k}.
\end{aligned} \tag{2.15}$$

Recalling the definition of the modified Bessel function of the first kind [[57], p. 233, eq. (9.28)],

$$\left(\frac{x}{2}\right)^{-\nu}I_\nu(x)=\sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m},$$

we see immediately that the latter infinite series in (2.15) is represented by the modified Bessel function of the first kind. This proves (2.9). □

Since the class $\mathcal{B}$ can be included in the class $\mathcal{A}$, i.e., if $\phi(s)\in\mathcal{B}$ then $\phi(2s-\delta)\in\mathcal{A}$ (see Remark 1.4), we can deduce the following formula obtained in [[48], Lemma 3.1.].

Corollary 2.1. *Let $\phi(s)$ be a Dirichlet series representable as (1.23) and belonging to the class $\mathcal{B}$ with $\delta = 0$. Then, for any $x > 0$ and $Re(\alpha) > 0$, $\beta \in \mathbb{C}$, the following identity holds*

$$\sum_{n=1}^{\infty} a(n)\, e^{-\alpha\lambda_n^2 x^2}\, \cos(\beta\lambda_n x) = \phi(0) + \sqrt{\frac{\pi}{\alpha}}\,\frac{\rho}{2\,x}\, e^{-\frac{\beta^2}{4\alpha}} + \frac{e^{-\frac{\beta^2}{4\alpha}}}{\sqrt{\alpha}\, x} \sum_{n=1}^{\infty} b(n)\, e^{-\frac{\mu_n^2}{\alpha x^2}} \cosh\left(\frac{\beta\,\mu_n}{\alpha x}\right). \tag{2.16}$$

Moreover, if $\phi(s)$ belongs to the class $\mathcal{B}$ with $\delta = 1$, the analogous formula takes place

$$\sum_{n=1}^{\infty} a(n)\, e^{-\alpha\lambda_n^2 x^2}\, \sin(\beta\lambda_n x) = \frac{e^{-\frac{\beta^2}{4\alpha}}}{\sqrt{\alpha x}} \sum_{n=1}^{\infty} b(n)\, e^{-\frac{\mu_n^2}{\alpha\, x^2}}\, \sinh\left(\frac{\beta\mu_n}{\alpha\, x}\right). \tag{2.17}$$

Proof. Let $\phi'(s) = \phi(2s - \delta)$, $\delta \in \{0, 1\}$. Then $\phi'(s)$ satisfies Hecke's functional equation

$$\Gamma(s)\, \phi'(s) = \Gamma\left(\frac{1}{2} + \delta - s\right)\, \psi'\left(\frac{1}{2} + \delta - s\right) \tag{2.18}$$

and belongs to the class $\mathcal{A}$, with (at most) a simple pole located at $s = \frac{1}{2}$ (if $\delta = 0$) with residue $\rho/2$ (see Remark 1.4 above).

In general, if $\phi(s)$ and $\psi(s) \in \mathcal{B}$ are representable as (1.23), $\phi'(s)$ and $\psi'(s)$ can be written in the form

$$\phi'(s) = \sum_{n=1}^{\infty} \frac{a(n)\,\lambda_n^{\delta}}{\lambda_n^{2s}}, \quad \psi'(s) = \sum_{n=1}^{\infty} \frac{b(n)\,\mu_n^{\delta}}{\mu_n^{2s}}, \quad \mathrm{Re}(s) > \sigma_a/2.$$

From (2.18), we can apply the summation formula (2.9) replacing there $a(n)$ by $a(n)\,\lambda_n^{\delta}$, λ_n by λ_n^2, $b(n)$ by $b(n)\,\mu_n^{\delta}$, μ_n by μ_n^2 and finally ρ by $\rho/2$: this gives

$$\sum_{n=1}^{\infty} a(n)\,\lambda_n^{\frac{1}{2}}\, e^{-\alpha\lambda_n^2}\, J_{\delta-\frac{1}{2}}(\beta\,\lambda_n) = \sqrt{\frac{2}{\pi\beta}} \left\{\phi(0) + \frac{\rho}{2}\sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\beta^2}{4\alpha}}\right\}(1-\delta) + \frac{e^{-\frac{\beta^2}{4\alpha}}}{\alpha} \sum_{n=1}^{\infty} b(n)\,\mu_n^{\frac{1}{2}}\, e^{-\mu_n^2/\alpha}\, I_{\delta-\frac{1}{2}}\left(\frac{\beta\mu_n}{\alpha}\right). \tag{2.19}$$

Using now the particular values for the Bessel functions [[57], p.248],

$$J_{\delta-\frac{1}{2}}(x) = \begin{cases} \sqrt{\frac{2}{\pi x}}\,\cos(x) & \text{if } \delta = 0 \\ \sqrt{\frac{2}{\pi x}}\,\sin(x) & \text{if } \delta = 1 \end{cases}, \quad I_{\delta-\frac{1}{2}}(x) = \begin{cases} \sqrt{\frac{2}{\pi x}}\,\cosh(x) & \text{if } \delta = 0 \\ \sqrt{\frac{2}{\pi x}}\,\sinh(x) & \text{if } \delta = 1 \end{cases}, \tag{2.20}$$

we obtain immediately the identities (2.16) and (2.17) after replacing α and β in (2.19) respectively by αx^2 and βx, $x > 0$. □

Remark 2.2. In particular, since $\phi(s) := \pi^{-s/2}\zeta(s) \in \mathcal{B}$ by Remark 1.3, an application of the identity (2.16) gives Jacobi's formula (1.5). This seems to be the first indication that the general formula (2.9) will prove to be the rightful analogue of (1.5).

Identities (2.16) and (2.17) were employed in [[48], Theorem 3.1.] to derive a generalization of the Selberg-Chowla formula for Dirichlet series in the class $\mathcal{B}$. Based on the "theta-like" features of the series appearing in (2.16) and (2.17), we now give a definition which introduces a new general analogue of the theta function.

Definition 2.1. Let $\mathrm{Re}(x) > 0$, $y \in \mathbb{C}$ and assume that $\phi(s)$ is a Dirichlet series satisfying Hecke's functional equation (1.25). We define the generalized $\theta-$function as the following series

$$\Phi(x,y) = 2^{r-1}\Gamma(r)\, y^{1-r} \sum_{n=1}^{\infty} a(n)\, \lambda_n^{\frac{1-r}{2}}\, e^{-\lambda_n x}\, J_{r-1}\left(\sqrt{\lambda_n}\, y\right). \tag{2.21}$$

Our next corollary, which actually consists in rewriting (2.9) in a compact form, gives a transformation formula for $\Phi(x,y)$. Moreover, just like Dixit's formula (1.6), we are able to compare this modular relation with an integral involving $\phi(s)$ and the confluent hypergeometric function.

Corollary 2.2. *Let* $Re(x) > 0$, $y \in \mathbb{C}$ *and assume that* $\phi(s) \in \mathcal{A}$. *Let us consider the generalized Theta functions:*

$$\Phi(x,y) := 2^{r-1}\Gamma(r)\, y^{1-r} \sum_{n=1}^{\infty} a(n)\, \lambda_n^{\frac{1-r}{2}}\, e^{-\lambda_n x}\, J_{r-1}\left(\sqrt{\lambda_n}\, y\right) \tag{2.22}$$

and

$$\Psi\left(x,y\right) := 2^{r-1}\Gamma(r)\, y^{1-r} \sum_{n=1}^{\infty} b(n)\, \mu_n^{\frac{1-r}{2}}\, e^{-\mu_n x}\, J_{r-1}\left(\sqrt{\mu_n}\, y\right). \tag{2.23}$$

Then $\Phi(x,y)$ *(resp.* $\Psi(x,y)$*) has the following properties:*

1. *It is entire in* $y \in \mathbb{C}$ *and analytic in* $Re(x) > 0$*;*
2. *It satisfies the transformation formula*

$$\Phi(x,y) = \phi(0) + \frac{\rho}{x^r}\Gamma(r)\, e^{-\frac{y^2}{4x}} + \frac{e^{-\frac{y^2}{4x}}}{x^r}\,\Psi\left(\frac{1}{x},\, \frac{iy}{x}\right). \tag{2.24}$$

Moreover, (2.24) can be written in terms of an integral involving $\Gamma(s)\,\phi(s)\,{}_1F_1(s;\, r;\, -y^2/4x)$ *at the critical line of* $\phi(s)$*, this is,*

$$x^{r/2}\Phi\left(x,y\right) - \frac{\rho\Gamma(r)}{x^{r/2}}\, e^{-\frac{y^2}{4x}} = e^{-\frac{y^2}{4x}}x^{-r/2}\Psi\left(\frac{1}{x},\, i\,\frac{y}{x}\right) + \phi(0)\,x^{r/2}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma\left(\frac{r}{2}+it\right)\,\phi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\, r;\, -\frac{y^2}{4x}\right)x^{-it}\,dt. \tag{2.25}$$

Proof. We first show that $\Phi(x,y)$ satisfies item 1., i.e., that is entire in $y \in \mathbb{C}$ and analytic in the half-plane $\mathrm{Re}(x) > 0$. We divide this proof in two cases: $r > \frac{1}{2}$ or $0 < r \leq \frac{1}{2}$. For the first case, if $\mathrm{Re}(x) = \sigma \geq \sigma_0 > 0$ and y belongs to a bounded subset of $\mathbb{C}$, contained in $|y| \leq M$ say, then the series defining $\Phi(x,y)$ is absolutely and uniformly convergent, since, for $r > \frac{1}{2}$,

$$\begin{aligned}
|\Phi(x,y)| &\leq \Gamma\left(r\right)\sum_{n=1}^{\infty}|a(n)|e^{-\lambda_n\sigma_0}\left|\left(\frac{\sqrt{\lambda_n}\,y}{2}\right)^{1-r} J_{r-1}\left(\sqrt{\lambda_n}\,y\right)\right| \\
&\leq \frac{2\Gamma(r)}{\sqrt{\pi}\Gamma\left(r-\frac{1}{2}\right)}\sum_{n=1}^{\infty}|a(n)|e^{-\lambda_n\sigma_0}\int_0^1\left(1-t^2\right)^{r-\frac{3}{2}}\left|\cos\left(\sqrt{\lambda_n}\,y\,t\right)\right|dt \\
&\leq \sum_{n=1}^{\infty}|a(n)|\,\exp\left(-\lambda_n\sigma_0+\sqrt{\lambda_n}\,M\right) < \infty,
\end{aligned}$$

where in the second inequality we have used the well-known Fourier representation (due to Poisson) [[44], p. 224, eq. (10.9.4)]

$$\left(\frac{z}{2}\right)^{-\nu} J_{\nu}(z)=\frac{2}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{0}^{1}\left(1-t^{2}\right)^{\nu-\frac{1}{2}}\cos(zt)\,dt,\qquad \mathrm{Re}(\nu)>-\frac{1}{2},\ z\in\mathbb{C}, \tag{2.26}$$

as well as the fact that the Dirichlet series $\phi(s)$ converges in some half-plane.

Note that (2.26) can only be invoked for $J_{r-1}(z)$ if $r>\frac{1}{2}$. To address the case where $0<r\leq\frac{1}{2}$, we just bound trivially $\Phi(x,y)$, which gives

$$\begin{aligned}|\Phi(x,y)| &\leq \Gamma(r)\sum_{n=1}^{\infty}|a(n)|e^{-\lambda_n\sigma_0}\left|\left(\frac{\sqrt{\lambda_n}\,y}{2}\right)^{1-r}J_{r-1}\left(\sqrt{\lambda_n}\,y\right)\right| \\ &\leq \Gamma(r)\sum_{n=1}^{\infty}|a(n)|e^{-\lambda_n\sigma_0}\left(\frac{\sqrt{\lambda_n}\,|y|}{2}\right)^{1-r}I_{r-1}\left(\sqrt{\lambda_n}\,|y|\right)<\infty,\end{aligned}$$

where we have used the well-known bound $|J_{\nu}(z)|\leq I_{\nu}(|z|)$, valid for $\nu\in\mathbb{R}$. The finitude of the last series comes from the well-known asymptotic estimate (1.51) for $I_{\nu}(x)$, $x>0$ [[44], p. 252, eq. (10.30(ii))], which gives

$$\left(\frac{\sqrt{\lambda_n}\,|y|}{2}\right)^{1-r}I_{r-1}\left(\sqrt{\lambda_n}\,|y|\right)\ll\exp\left(\sqrt{\lambda_n}\,M\right)\lambda_n^{\frac{1}{4}-\frac{r}{2}}|y|^{\frac{1}{2}-r},\quad n\to\infty. \tag{2.27}$$

Therefore, for each fixed x in the half-plane $\mathrm{Re}(x)>0$, the function $\Phi(x,\cdot)$ is entire and for each fixed $y\in\mathbb{C}$ the function $\Phi(\cdot,y)$ is holomorphic in the right half-plane $\mathrm{Re}(x)>0$. This proves item 1. above.

Now, we prove the integral representation (2.25), which immediately gives (2.24). We start our proof with the integral on the right-hand side of (2.25), transforming it into a contour integral along the critical line $\mathrm{Re}(s)=\frac{r}{2}$,

$$\begin{aligned}&\frac{y^{r-1}x^{-r/2}}{2^{r}\pi\,\Gamma(r)}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\phi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{y^2}{4x}\right)x^{-it}\,dt= \\ &=\frac{y^{r-1}}{2^{r-1}\Gamma(r)}\frac{1}{2\pi i}\int_{\frac{r}{2}-i\infty}^{\frac{r}{2}+i\infty}\Gamma(s)\,\phi(s)\,{}_1F_1\left(s;\,r;\,-\frac{y^2}{4x}\right)x^{-s}\,ds= \\ &=\frac{y^{r-1}}{2^{r-1}\Gamma(r)}\frac{e^{-\frac{y^2}{4x}}}{2\pi i}\int_{\frac{r}{2}-i\infty}^{\frac{r}{2}+i\infty}\Gamma(s)\,\phi(s)\,{}_1F_1\left(r-s;\,r;\,\frac{y^2}{4x}\right)x^{-s}\,ds,\end{aligned}$$

where we have used Kummer's formula (2.13). We now change the line of integration to $\mathrm{Re}(s)=\sigma_a+\delta$, for some positive δ. Doing so we integrate over a (positively oriented) rectangular contour whose vertices are $r/2\pm iT$ and $\sigma_a+\delta\pm iT$, where T is a (sufficiently large) positive real number. The integrals along the horizontal segments vanish as $T\to\infty$ due to Stirling's formula (1.32) and (2.6).

Doing this shift, due to the simple pole of $\phi(s)$ located at $s=r$, we find the residue

$$\mathrm{Res}_{s=r}\left\{\Gamma(s)\,\phi(s)\,{}_1F_1\left(r-s;\,r;\,\frac{y^2}{4x}\right)x^{-s}\right\}=\Gamma(r)\rho\,x^{-r}.$$

Henceforth, since $\phi(s)$ converges absolutely for $\mathrm{Re}(s)>\sigma_a$, we have by absolute convergence

$$\frac{y^{r-1}}{2^{r-1}\,\Gamma(r)}\,\frac{e^{-\frac{y^2}{4x}}}{2\pi i}\int_{\frac{r}{2}-i\infty}^{\frac{r}{2}+i\infty}\Gamma\left(s\right)\,\phi\left(s\right)\,{}_1F_1\left(r-s;\,r;\,\frac{y^2}{4x}\right)x^{-s}\,ds$$

$$=\frac{y^{r-1}}{2^{r-1}\,\Gamma(r)}\,\frac{e^{-\frac{y^2}{4x}}}{2\pi i}\int_{\sigma_a+\delta-i\infty}^{\sigma_a+\delta+i\infty}\Gamma\left(s\right)\,\phi\left(s\right)\,{}_1F_1\left(r-s;\,r;\,\frac{y^2}{4x}\right)x^{-s}\,ds-\frac{y^{r-1}\rho}{2^{r-1}x^{r}}\,e^{-\frac{y^2}{4x}}$$

$$=\frac{y^{r-1}}{2^{r-1}\,\Gamma(r)}\,\frac{e^{-\frac{y^2}{4x}}}{2\pi i}\sum_{n=1}^{\infty}a(n)\int_{\sigma_a+\delta-i\infty}^{\sigma_a+\delta+i\infty}\Gamma\left(s\right)\,{}_1F_1\left(r-s;\,r;\,\frac{y^2}{4x}\right)(x\lambda_n)^{-s}\,ds-\frac{y^{r-1}\rho}{2^{r-1}x^{r}}\,e^{-\frac{y^2}{4x}}$$

$$=\sum_{n=1}^{\infty}a(n)\,e^{-\lambda_n x}J_{r-1}\left(\sqrt{\lambda_n}\,y\right)-\frac{y^{r-1}\rho}{2^{r-1}x^{r}}\,e^{-\frac{y^2}{4x}},$$

where in the last step we have used the integral representation (2.4). This proves

$$\frac{y^{r-1}x^{-r/2}}{2^{r}\pi\,\Gamma(r)}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\phi\left(\frac{r}{2}+it\right)\,{}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{y^2}{4x}\right)x^{-it}\,dt$$

$$=\sum_{n=1}^{\infty}a(n)\,e^{-\lambda_n x}J_{r-1}\left(\sqrt{\lambda_n}\,y\right)-\frac{y^{r-1}\rho}{2^{r-1}x^{r}}\,e^{-\frac{y^2}{4x}}. \tag{2.28}$$

After multiplying both sides of (2.28) by the factor $\Gamma(r)\,2^{r-1}y^{1-r}x^{r/2}$ and appealing to the definition of the generalized theta function (2.22), we derive

$$x^{r/2}\Phi\left(x,y\right)-\frac{\rho\Gamma(r)}{x^{r/2}}\,e^{-\frac{y^2}{4x}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\phi\left(\frac{r}{2}+it\right)\,{}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{y^2}{4x}\right)x^{-it}\,dt. \tag{2.29}$$

Now look at the integral on the right-hand side of (2.29): by the functional equation for $\phi(s)$ (1.25) and Kummer's formula (2.8), we can rewrite it as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\phi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{y^2}{4x}\right)x^{-it}\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\psi\left(\frac{r}{2}+it\right)\,{}_1F_1\left(\frac{r}{2}-it;\,r;\,-\frac{y^2}{4x}\right)x^{it}\,dt$$

$$=\frac{e^{-\frac{y^2}{4x}}}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\psi\left(\frac{r}{2}+it\right)\,{}_1F_1\left(\frac{r}{2}+it;\,r;\,\frac{y^2}{4x}\right)x^{it}\,dt. \tag{2.30}$$

But the last integral in (2.30) is actually the same as the one on the right side of (2.29) if we replace there x by $1/x$, y by iy/x and $\phi(s)$ by $\psi(s)$: hence (2.29) implies

$$x^{-r/2}\Psi\left(\frac{1}{x},\,i\,\frac{y}{x}\right)-\rho^{\star}\Gamma(r)\,x^{r/2}\,e^{\frac{y^2}{4x}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\psi\left(\frac{r}{2}+it\right)\,{}_1F_1\left(\frac{r}{2}+it;\,r;\,\frac{y^2}{4x}\right)x^{it}\,dt, \tag{2.31}$$

where $\rho^{\star}$ is the residue that $\psi(s)$ possesses at its simple pole at $s=r$. From the simple properties of the class $\mathcal{A}$ (see Remark 1.1 above), we know that $\rho^{\star}\Gamma(r)=-\phi(0)$, which gives, after using (2.31), (2.30) and (2.29),

$$e^{-\frac{y^2}{4x}}x^{-r/2}\Psi\left(\frac{1}{x},\,i\,\frac{y}{x}\right)+\phi(0)\,x^{r/2}=\frac{e^{-\frac{y^2}{4x}}}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\psi\left(\frac{r}{2}+it\right)\,{}_1F_1\left(\frac{r}{2}+it;\,r;\,\frac{y^2}{4x}\right)x^{it}\,dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\phi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{y^2}{4x}\right)x^{-it}\,dt=x^{r/2}\Phi\left(x,y\right)-\frac{\rho\Gamma(r)}{x^{r/2}}\,e^{-\frac{y^2}{4x}},$$

completing our proof. □

Since the previous "modular" relation (2.24) is an analogue of Bochner's "modular" relation (1.29), in the next corollary we note that (1.29) can indeed be obtained from (2.9).

Corollary 2.3. *Assume that $\phi(s)\in\mathcal{A}$: then for any Re(x) > 0, Bochner's formula (1.29) takes place*

$$\sum_{n=1}^{\infty}a(n)\,e^{-\lambda_n x}=\phi(0)+\rho\Gamma(r)\,x^{-r}+x^{-r}\sum_{n=1}^{\infty}b(n)\,e^{-\mu_n/x}. \tag{2.32}$$

Proof. The transformation formula (2.24) implies, for Re$(x)>0$ and $y\in\mathbb{C}$,

$$\Gamma(r)\,2^{r-1}y^{1-r}\sum_{n=1}^{\infty}a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\lambda_n x}\,J_{r-1}(\sqrt{\lambda_n}\,y)=\phi(0)+\frac{\rho}{x^r}\Gamma(r)\,e^{-\frac{y^2}{4x}}+$$
$$+\frac{e^{-\frac{y^2}{4x}}}{x}\,\Gamma(r)\,2^{r-1}y^{1-r}\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{1-r}{2}}\,e^{-\frac{\mu_n}{x}}\,I_{r-1}\left(\frac{\sqrt{\mu_n}y}{x}\right).$$

We have seen above that the series on both sides converge absolutely and uniformly with respect to y contained in any bounded subset of $\mathbb{C}$. Therefore, we can take $y\to0^{+}$ and interchange the orders of limit and summation on both sides. From the limiting relations for the Bessel functions [[44], p. 223, eq. (10.7.3)],

$$\lim_{y\to0}y^{-\nu}J_{\nu}(y)=\frac{2^{-\nu}}{\Gamma(\nu+1)},\quad\lim_{y\to0}y^{-\nu}I_{\nu}(y)=\frac{2^{-\nu}}{\Gamma(\nu+1)}, \tag{2.33}$$

(2.32) is immediately obtained. □

By the equivalence between the identities (1.29), (1.30) and Hecke's functional equation, the fact that (2.24) generalizes (1.29) puts now an interesting question: is it possible to find a generalization of (1.30) in this context? The following corollary furnishes a positive answer to this question.

Corollary 2.4. *Let $\phi(s)$ and $\psi(s)$ be two Dirichlet series satisfying Definition 1.2 and let σ_a be the abscissa of absolute convergence of $\phi(s)$. Furthermore, let x, y be two positive real numbers and s be any complex number such that Re$(s)>\sigma_a$.*

Then the following identity takes place

$$\frac{x^{s-r}}{2}\,\frac{\Gamma(s)\,y^{r-1}}{\Gamma(r)}\sum_{n=1}^{\infty}\frac{a(n)}{(\lambda_n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,r;\,\frac{4y^2\lambda_n}{(\lambda_n+x^2+y^2)^2}\right)$$
$$=\frac{\rho\,y^{r-1}\,x^{r-s}\,\Gamma(s-r)}{2}+\frac{x^{s-r}y^{r-1}\phi(0)\,\Gamma(s)}{2\Gamma(r)\,(x^2+y^2)^s}+\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{s+1}{2}-r}\,J_{r-1}(2\sqrt{\mu_n}\,y)\,K_{s-r}\left(2\sqrt{\mu_n}\,x\right). \tag{2.34}$$

Moreover, if x and y are two positive numbers such that $x>y$ and Re$(s)>\sigma_a$, then the analogous identity holds:

$$\frac{x^{s-r}}{2}\,\frac{\Gamma(s)\,y^{r-1}}{\Gamma(r)}\sum_{n=1}^{\infty}\frac{a(n)}{(\lambda_n+x^2-y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,r;\,-\frac{4y^2\lambda_n}{(\lambda_n+x^2-y^2)^2}\right)$$
$$=\frac{\rho\,y^{r-1}x^{r-s}\,\Gamma(s-r)}{2}+\frac{x^{s-r}y^{r-1}\phi(0)\,\Gamma(s)}{2\Gamma(r)\,(x^2-y^2)^s}+\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{s+1}{2}-r}\,I_{r-1}(2\sqrt{\mu_n}\,y)\,K_{s-r}\left(2\sqrt{\mu_n}\,x\right). \tag{2.35}$$

Proof. Since $I_\nu(x) = i^{-\nu} J_\nu(ix)$ for $x > 0$, it is simple to see that (2.35) can be derived from (2.34) once we assure the convergence of both sides of (2.35). Indeed, the series on the right-hand side of (2.35) converges absolutely for every $s \in \mathbb{C}$ because of the fact that $x > y$ and the asymptotic formulas (1.51) and (1.52) [[61], p. 199-203],

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \left(1 + O\left(\frac{1}{x}\right)\right), \quad I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \left(1 + O\left(\frac{1}{x}\right)\right), \quad x \to \infty. \tag{2.36}$$

To justify the convergence of the series on the left-hand side of (2.35), note that, as $n \to \infty$,

$$\frac{1}{(\lambda_n + x^2 - y^2)^s}\, {}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}; r; -\frac{4y^2\lambda_n}{(\lambda_n + x^2 - y^2)^2}\right) \sim \frac{1}{(\lambda_n + x^2 - y^2)^s}$$

since the hypergeometric function factor tends to 1 as $\lambda_n \to \infty$. Thus, if $\mathrm{Re}(s) > \sigma_a$, the series on the left of (2.35) converges absolutely. Similar comments can be made about the series on both sides of (2.34) without the additional hypothesis that $x > y$.

From the previous comments, we just need to prove the identity (2.34). To that end, we look first at the infinite series on its right-hand side and invoke the well-known representation for the Modified Bessel function of the second kind [[31], eq. 3.471.9, p. 368],

$$\int_0^\infty x^{s-1} e^{-\beta x}\, e^{-\frac{\gamma}{x}}\, dx = 2\left(\frac{\gamma}{\beta}\right)^{s/2} K_s\left(2\sqrt{\beta\,\gamma}\right), \quad \mathrm{Re}(\beta),\, \mathrm{Re}(\gamma) > 0. \tag{2.37}$$

Using (2.37) on the infinite series appearing on the right side of (2.34), we obtain

$$\sum_{n=1}^\infty b(n)\, \mu_n^{\frac{s+1-2r}{2}}\, J_{r-1}(2\sqrt{\mu_n}\, y)\, K_{s-r}\left(2\sqrt{\mu_n}\, x\right) = \frac{x^{s-r}}{2} \int_0^\infty t^{s-r-1} e^{-x^2 t} \sum_{n=1}^\infty b(n)\, \mu_n^{\frac{1-r}{2}}\, e^{-\frac{\mu_n}{t}}\, J_{r-1}\left(2\sqrt{\mu_n}\, y\right) dt,$$

where interchanging the orders of summation and integration is possible due to absolute convergence. Now we invoke the transformation formula (2.9) with the roles of $\phi(s)$ and $\psi(s)$ being reversed and we get

$$\sum_{n=1}^\infty b(n)\, \mu_n^{\frac{s+1-2r}{2}}\, J_{r-1}(2\sqrt{\mu_n}\, y)\, K_{s-r}\left(2\sqrt{\mu_n}\, x\right) = \frac{x^{s-r}}{2} \int_0^\infty t^{s-r-1} e^{-x^2 t} \cdot \left\{\frac{\psi(0) y^{r-1}}{\Gamma(r)} + y^{r-1} \rho^\star\, t^r\, e^{-y^2 t}\right\} dt +$$

$$+ \frac{x^{s-r}}{2} \int_0^\infty t^{s-r-1} e^{-x^2 t} \cdot \left\{t\, e^{-y^2 t} \sum_{n=1}^\infty a(n)\, \lambda_n^{\frac{1-r}{2}}\, e^{-\lambda_n t}\, I_{r-1}\left(2y\, t\, \sqrt{\lambda_n}\right)\right\} dt. \tag{2.38}$$

The computation of the first integral is simple and it is equal to:

$$\frac{x^{s-r}}{2} \int_0^\infty t^{s-r-1} e^{-x^2 t} \left\{\frac{\psi(0) y^{r-1}}{\Gamma(r)} + y^{r-1} \rho^\star\, t^r\, e^{-y^2 t}\right\} dt = \frac{\psi(0)\, y^{r-1}\, x^{r-s}\, \Gamma(s-r)}{2\Gamma(r)} + \frac{x^{s-r} y^{r-1} \rho^\star\, \Gamma(s)}{2\,(x^2+y^2)^s}$$

$$= -\frac{\rho\, y^{r-1}\, x^{r-s}\, \Gamma(s-r)}{2} - \frac{x^{s-r} y^{r-1} \phi(0)\, \Gamma(s)}{2\Gamma(r)\, (x^2+y^2)^s}, \tag{2.39}$$

where we have used the simple properties of the class $\mathcal{A}$, namely that $\psi(0) = -\rho\, \Gamma(r)$ and $\rho^\star = -\phi(0)/\Gamma(r)$ (see Remark 1.1). To evaluate the second integral, we use absolute convergence (assured by item 1. of Corollary 2.2 above) and the integral given in [[15], p. 191, eq. 3.13.2.1],

$$\int_0^\infty x^{z-1}\, e^{-Ax} I_\nu(Bx)\, dx = A^{-z-\nu} \left(\frac{B}{2}\right)^\nu \frac{\Gamma(z+\nu)}{\Gamma(\nu+1)}\, {}_2F_1\left(\frac{z+\nu}{2}, \frac{z+\nu+1}{2}; \nu+1; \frac{B^2}{A^2}\right), \tag{2.40}$$

valid for $\text{Re}(A) > |\text{Re}(B)|$ and $\text{Re}(z) > -\text{Re}(\nu)$.

Invoking (2.40) on the last integral in (2.38), we obtain

$$\int_0^{\infty} t^{s-r}e^{-x^2t}\,e^{-y^2t}\sum_{n=1}^{\infty}a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\lambda_n t}\,I_{r-1}\left(2y\,t\,\sqrt{\lambda_n}\right)\,dt = \sum_{n=1}^{\infty}a(n)\,\lambda_n^{\frac{1-r}{2}}\int_0^{\infty}t^{s-r}\,e^{-\left(x^2+y^2+\lambda_n\right)t}\,I_{r-1}\left(2yt\,\sqrt{\lambda_n}\right)\,dt$$
$$= \frac{\Gamma(s)\,y^{r-1}}{\Gamma(r)}\sum_{n=1}^{\infty}\frac{a(n)}{(\lambda_n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,r;\,\frac{4y^2\lambda_n}{(\lambda_n+x^2+y^2)^2}\right), \tag{2.41}$$

which is legitimate through (2.40) because $\text{Re}(s) > \sigma_a \geq \frac{r}{2} > 0$ by hypothesis and $x^2+y^2+\lambda_n > y^2+\lambda_n > 2y\sqrt{\lambda_n}$. Combining (2.41) with (2.38) and (2.39) proves our formula (2.34).

□

Remark 2.3. A formula similar to (2.35) involving the product $I_{\nu}(y)\,K_{\nu}(z)$ was found by Berndt, Dixit, Kim and Zaharescu in [11] and later generalized to a class of Dirichlet series by three of the authors and Gupta in [[9], Theorem 11.1].

Restricted to the case where the Dirichlet series is attached to the sum of $k-$squares, $\zeta_k(s)$, their identity reads [[11], p. 315, Theorem 1.6.]

$$\sum_{n=1}^{\infty}r_k(n)\,I_{\nu}\left(\pi\sqrt{n}\left(\sqrt{\alpha}-\sqrt{\beta}\right)\right)\,K_{\nu}\left(\pi\sqrt{n}\left(\sqrt{\alpha}+\sqrt{\beta}\right)\right)$$
$$= -\frac{1}{2\nu}\left(\frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right)^{\nu} + \frac{\Gamma\left(\frac{k}{2}+\nu\right)}{\pi^{k/2}2^{k-1}\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{r_k(n)}{\sqrt{n+\alpha}\,\sqrt{n+\beta}}\left(\frac{\sqrt{n+\alpha}-\sqrt{n+\beta}}{\sqrt{n+\alpha}+\sqrt{n+\beta}}\right)^{\nu}\times$$
$$\times\left(\frac{1}{\sqrt{n+\alpha}}+\frac{1}{\sqrt{n+\beta}}\right)^{k-2}{}_2F_1\left(1-\frac{k}{2}+\nu,\,1-\frac{k}{2};\,\nu+1;\,\left(\frac{\sqrt{n+\alpha}-\sqrt{n+\beta}}{\sqrt{n+\alpha}+\sqrt{n+\beta}}\right)^2\right), \tag{2.42}$$

where $\text{Re}(\sqrt{\alpha}) \geq \text{Re}(\sqrt{\beta}) > 0$ and $\text{Re}(\nu) > 0$. Compare formula (2.42) with (2.62) below for $\alpha = k$.

Our identity (2.35) and its consequences (c.f. (2.62) below) are somewhat akin to (2.42) but they seem to be independent because in (2.42) it is required that the indices of the Bessel functions are the same. In our formula (2.35) we have a free complex parameter s, although the index of the modified Bessel function $I_{\nu}(x)$ is fixed by the functional equation, being equal to $r-1$. In the case of (2.42) the indices of the Bessel functions are kept equal but they are independent of the functional equation for $\zeta_k(s)$, this is, they do not depend on k.

The formula obtained by Berndt, Dixit, Kim and Zaharescu (2.42) can be used to extend known formulas due to Dixon and Ferrar [24–26]. For example, by using (2.42), they were able to establish the beautiful identity (see [[11], p. 329, Corollary 4.6.])

$$\frac{\beta^{\nu/2}\Gamma\left(\nu+\frac{k}{2}\right)}{2\pi^{\nu+\frac{k}{2}}}\sum_{n=0}^{\infty}\frac{r_k(n)}{(n+\beta)^{\nu+\frac{k}{2}}} = \sum_{n=0}^{\infty}r_k(n)\,n^{\nu/2}K_{\nu}\left(2\pi\sqrt{n\beta}\right), \tag{2.43}$$

which is valid for any positive integer $k > 1$ and $\text{Re}(\sqrt{\beta})$, $\text{Re}(\nu) > 0$. On the other hand, (2.43) can be established immediately from Berndt's general Bessel expansion (1.30) because $\zeta_k(s)$ satisfies Hecke's functional equation.

In a general context, we now prove that (2.34) implies (1.30) in the same way that the similar formula (2.42) gives (2.43).

Corollary 2.5. *Let $Re(s) > \sigma_a$ and $\phi(s)$ be a Dirichlet series belonging to the class $\mathcal{A}$. Then, for every $x > 0$, the following Bessel expansion takes place*

$$\sum_{n=1}^{\infty} \frac{a(n)}{(\lambda_n + x^2)^s} = \frac{\rho\,\Gamma(r)\,x^{2r-2s}\,\Gamma(s-r)}{\Gamma(s)} + x^{-2s}\phi(0) + \frac{2x^{r-s}}{\Gamma(s)} \sum_{m=1}^{\infty} b(m)\,\mu_m^{\frac{s-r}{2}}\,K_{s-r}\left(2\sqrt{\mu_m}\,x\right). \tag{2.44}$$

Proof. Multiply both sides of (2.34) by $\Gamma(r)\,y^{1-r}$: we obtain

$$\Gamma(r)\,y^{1-r} \sum_{m=1}^{\infty} b(m)\,\mu_m^{\frac{s+1-2r}{2}}\,J_{r-1}(2\sqrt{\mu_m}\,y)\,K_{s-r}\left(2\sqrt{\mu_m}\,x\right) + \frac{\rho\,\Gamma(r)\,x^{r-s}\,\Gamma(s-r)}{2} + \frac{x^{s-r}\phi(0)\,\Gamma(s)}{2\,(x^2+y^2)^s}$$
$$= \frac{x^{s-r}}{2}\,\Gamma(s) \sum_{n=1}^{\infty} \frac{a(n)}{(\lambda_n + x^2 + y^2)^s}\,{}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}; r; \frac{4y^2\lambda_n}{(\lambda_n + x^2+y^2)^2}\right).$$

We have argued at the beginning of the proof of corollary 2.4 that both series above converge absolutely and uniformly with respect to y for $0 \leq y < M$. Thus, letting $y \to 0^+$, using the first limiting relation in (2.33) and recalling that ${}_2F_1(a, b;\,c;\,0) = 1$, we obtain the identity

$$\sum_{n=1}^{\infty} \frac{a(n)}{(\lambda_n + x^2)^s} = \frac{\rho\,\Gamma(r)\,x^{2r-2s}\,\Gamma(s-r)}{\Gamma(s)} + x^{-2s}\phi(0) + \frac{2x^{r-s}}{\Gamma(s)} \sum_{m=1}^{\infty} b(m)\,\mu_m^{\frac{s-r}{2}}\,K_{s-r}\left(2\sqrt{\mu_m}\,x\right),$$

which is Berndt's formula (1.30) restricted to the class $\mathcal{A}$. □

Since (2.44) can be thought as a generalization of Watson's formula [[60], p. 299, eq. (4)], we now see that Corollary 2.4 also gives another analogue of Watson's formula for the class $\mathcal{B}$. [4]

Corollary 2.6. *Let $Q(x,y) = Ax^2+Bxy+Cy^2$ be a positive definite and real quadratic form, i.e., $\Delta := 4AC-B^2 > 0$ and $A > 0$. Also, define $k := \sqrt{\Delta}/2A$.*

If $\phi(s)$ is a Dirichlet series belonging to the class $\mathcal{B}$ and s is a complex number such that $Re(s) > \frac{\sigma_a}{2}$, then the following formulas take place:

$$-2\phi(0)\,C^{-s} + \sum_{n=1}^{\infty} \frac{a(n)}{(A\lambda_n^2 + B\,\lambda_n + C)^s} + \sum_{n=1}^{\infty} \frac{a(n)}{(A\lambda_n^2 - B\,\lambda_n + C)^s} = \rho\sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,A^{-s}k^{1-2s} +$$
$$+ \frac{4k^{\frac{1}{2}-s}A^{-s}}{\Gamma(s)} \sum_{n=1}^{\infty} b(n)\mu_n^{s-\frac{1}{2}}\,\cos\left(\frac{B\mu_n}{A}\right) K_{s-\frac{1}{2}}\left(2k\,\mu_n\right), \quad \delta = 0, \tag{2.45}$$

$$\sum_{n=1}^{\infty} \frac{a(n)}{(A\lambda_n^2 + B\,\lambda_n + C)^s} - \sum_{n=1}^{\infty} \frac{a(n)}{(A\lambda_n^2 + B\,\lambda_n + C)^s} = -\frac{4k^{\frac{1}{2}-s}A^{-s}}{\Gamma(s)} \sum_{n=1}^{\infty} b(n)\,\mu_n^{s-\frac{1}{2}}\,\sin\left(\frac{B\mu_n}{A}\right)\,K_{s-\frac{1}{2}}\left(2k\,\mu_n\right), \quad \delta = 1. \tag{2.46}$$

[4]Corollary 2.6 is essentially given in [[48], Example 5.1., eq. (5.2), (5.3)], although in this reference there are some additional assumptions on the arithmetical function $a(n)$.

Proof. Starting with (2.45), we use the fact that if $\phi(s) \in \mathcal{B}$ with $\delta = 0$, then $\phi(2s) \in \mathcal{A}$ with $r = \frac{1}{2}$. Thus, to obtain (2.45), we start by using (2.34) with $\phi(s)$ being replaced by $\phi(2s)$ and $r = \frac{1}{2}$.

Appealing to the well-known case for Gauss Hypergeometric function [[47], vol. 3, p. 461, eq. (7.3.1.106)],

$$ {}_2F_1\left(a, a+\frac{1}{2}; \frac{1}{2}; z\right) = \frac{1}{2}\left[\left(1+\sqrt{z}\right)^{-2a} + \left(1-\sqrt{z}\right)^{-2a}\right], \tag{2.47} $$

the left-hand side of (2.34) can be simplified to

$$ \frac{x^{s-\frac{1}{2}}}{2}\,\Gamma(s)\sum_{n=1}^{\infty}\frac{a(n)}{\left(\lambda_n^2+x^2+y^2\right)^s}\,{}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}; \frac{1}{2}; \frac{4y^2\lambda_n^2}{\left(\lambda_n^2+x^2+y^2\right)^2}\right) $$

$$ \frac{x^{s-\frac{1}{2}}}{4}\,\Gamma(s)\sum_{n=1}^{\infty}a(n)\left\{\left(\lambda_n^2+2y\,\lambda_n+x^2+y^2\right)^{-s} + \left(\lambda_n^2-2y\,\lambda_n+x^2+y^2\right)^{-s}\right\}, \ \text{Re}(s) > \frac{\sigma_a}{2}. \tag{2.48} $$

On the other hand, for every $s \in \mathbb{C}$, we can write the right-hand side of (2.34) as follows:

$$ \sum_{m=1}^{\infty} b(m)\,\mu_m^{\frac{s-\frac{1}{2}}{2}}\cos\left(2\mu_m y\right)\,K_{s-\frac{1}{2}}\left(2\,\mu_m\,x\right) + \frac{\sqrt{\pi}\,\rho\,x^{\frac{1}{2}-s}\,\Gamma(s-\frac{1}{2})}{4} + \frac{x^{s-\frac{1}{2}}\,\phi(0)\,\Gamma(s)}{2\,\left(x^2+y^2\right)^s}, \tag{2.49} $$

where we have used the particular case $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\cos(x)$ (2.20) and the fact that $\phi(2s)$ has residue $\rho/2$ at $s = 1/2$ (see Remark 1.4).

Comparing (2.48) with (2.49) gives the formula:

$$ -\frac{2\phi(0)}{\left(x^2+y^2\right)^s} + \sum_{n=1} a(n)\left\{\left(\lambda_n^2+2y\,\lambda_n+x^2+y^2\right)^{-s} + \left(\lambda_n^2-2y\,\lambda_n+x^2+y^2\right)^{-s}\right\} = $$

$$ = \frac{\sqrt{\pi}\,\rho\,x^{1-2s}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} + \frac{4x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{m=1}^{\infty} b(m)\,\mu_m^{\frac{s-\frac{1}{2}}{2}}\cos\left(2\mu_m y\right)\,K_{s-\frac{1}{2}}\left(2\,\mu_m\,x\right). $$

Multiplying both sides of the previous equality by A^{-s} and defining $y = \frac{B}{2A}$ and $x = \frac{\sqrt{\Delta}}{2A} := k$, we get immediately (2.45).

The proof of (2.46) is analogous: in this case, note that $\phi(2s-1) \in \mathcal{A}$ with $r = \frac{3}{2}$ (see Remark 1.4), so we just need to use (2.34) with $r = \frac{3}{2}$. Appealing to the formula [[47], vol. 3, p. 461, eq. (7.3.1.107)],

$$ {}_2F_1\left(a, a+\frac{1}{2}; \frac{3}{2}; z\right) = \frac{1}{2(2a-1)\sqrt{z}}\left\{\left(1-\sqrt{z}\right)^{1-2a} - \left(1+\sqrt{z}\right)^{1-2a}\right\}, $$

we see after some straightforward simplifications that the left-hand side of (2.34) is reduced to

$$ \frac{x^{s-\frac{3}{2}}}{4\sqrt{\pi y}}\,\Gamma(s-1)\sum_{n=1}^{\infty}a(n)\left\{\frac{1}{\left(\lambda_n^2-2y\lambda_n+x^2+y^2\right)^{s-1}} - \frac{1}{\left(\lambda_n^2+2y\lambda_n+x^2+y^2\right)^{s-1}}\right\}, \tag{2.50} $$

for $\text{Re}(s) > \frac{\sigma_a}{2}$.

In what concerns the right-hand side of (2.34), note that when $\delta = 1$ in the class $\mathcal{B}$, we have $\rho = \phi(0) = 0$, so we just need to simplify the series involving the Bessel functions.

For every $s \in \mathbb{C}$, this series can be written as

$$\sum_{n=1}^{\infty} b(n)\,\mu_n^{s-1}\,J_{\frac{1}{2}}(2\mu_n\,y)\,K_{s-\frac{3}{2}}(2\mu_n\,x) = \frac{1}{\sqrt{\pi y}}\sum_{n=1}^{\infty} b(n)\,\mu_n^{s-\frac{3}{2}}\,\sin(2\mu_n\,y)\,K_{s-\frac{3}{2}}(2\mu_n\,x), \tag{2.51}$$

where we have used $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\,\sin(x)$ (2.20). Comparing both sides (2.50) and (2.51) and replacing s by $s+1$ and defining $y=\frac{B}{2A}$ and $x=\frac{\sqrt{\Delta}}{2A}:=k$, we obtain (2.46). □

Remark 2.4. The previous corollary is somewhat remindful of the classical Selberg-Chowla formula for the Epstein zeta function [2, 50]. Being a generalization of Watson's formula, it is expected that (2.34) can be used to derive new analogues of Guinand and Koshliakov's formulas [10, 12, 13, 30, 39, 45, 53], as well as new extensions of the classical Epstein zeta function. This study will be given elsewhere and for the purposes of the present paper it is enough to work with the formula (2.25).

2.2 Some Useful Examples

In their paper [[22], p. 312] Dixit, Kumar, Maji and Zaharescu use a variant of Jacobi's $\psi-$function of the form

$$\psi(x,z):=\sum_{n=1}^{\infty} e^{-\pi n^2 x}\cos(\sqrt{\pi\,x}\,nz),\quad \text{Re}(x)>0,\quad z\in\mathbb{C}. \tag{2.52}$$

The parameter z in this consideration is exactly the same parameter that appears in the statement of their Theorem (see eq. (1.1) above). So, in order to extend their result, we need to convert the generalized theta function (2.22) into the more symmetric version (2.52) depending on a new parameter z.

This is done in the following definition:

Definition 2.2. Let $\phi(s)$ be any Dirichlet series belonging to the class $\mathcal{A}$ in the sense of Definition 1.2. Then, for $\text{Re}(x)>0$ and $z\in\mathbb{C}$, we define the generalized Jacobi's $\psi-$function attached to $\phi(s)$ as follows:

$$\psi_{\phi}(x,z):=2^{r-1}\,\Gamma(r)\left(\sqrt{x}\,z\right)^{1-r}\sum_{n=1}^{\infty}a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\lambda_n x}\,J_{r-1}\left(\sqrt{\lambda_n x}\,z\right)=\Phi\left(x,\sqrt{x}\,z\right), \tag{2.53}$$

where $\Phi(x,y)$ is given in (2.22). Analogously, one may also define:

$$\tilde{\psi}_{\phi}(x,z):=2^{r-1}\Gamma(r)\,\left(\sqrt{x}\,z\right)^{1-r}\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{1-r}{2}}\,e^{-\mu_n x}\,J_{r-1}\left(\sqrt{\mu_n x}\,z\right)=\Psi\left(x,\sqrt{x}\,z\right). \tag{2.54}$$

With this new parameter $z\in\mathbb{C}$, we note that the transformation formula (2.24) can be rewritten as

$$\begin{aligned}\psi_{\phi}(x,z)&:=2^{r-1}\,\Gamma(r)\,x^{\frac{1-r}{2}}\,z^{1-r}\sum_{n=1}^{\infty}a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\lambda_n x}\,J_{r-1}\left(\sqrt{\lambda_n x}\,z\right)\\ &=\phi(0)+\frac{\rho}{x^r}\Gamma(r)\,e^{-\frac{z^2}{4}}+2^{r-1}\Gamma(r)\,z^{1-r}\,x^{-\frac{r+1}{2}}\,e^{-z^2/4}\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{1-r}{2}}\,e^{-\frac{\mu_n}{x}}\,I_{r-1}\left(\sqrt{\frac{\mu_n}{x}}\,z\right):=\\ &:=\phi(0)+\frac{\rho\Gamma(r)}{x^r}\,e^{-\frac{z^2}{4}}+\frac{e^{-\frac{z^2}{4}}}{x^r}\,\tilde{\psi}_{\phi}\left(\frac{1}{x},iz\right).\end{aligned} \tag{2.55}$$

Also, identity (2.25) takes the new form:

$$x^{r/2}\psi_{\phi}(x,z)-\frac{\rho\Gamma(r)}{x^{r/2}}\,e^{-\frac{z^2}{4}}=e^{-\frac{z^2}{4}}\,x^{-r/2}\,\tilde{\psi}_{\phi}\left(\frac{1}{x},iz\right)+\phi(0)\,x^{r/2}=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\phi\left(\frac{r}{2}+it\right)\,{}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{z^2}{4}\right)\,x^{-it}\,dt. \tag{2.56}$$

which is valid for every $z\in\mathbb{C}$ and $\text{Re}(x)>0$.

Example 2.1. Let $\alpha>0$ and $r_{\alpha}(n)$ be defined as the coefficients of the expansion (1.34). Recall the Dirichlet series attached to it:

$$\zeta_{\alpha}(s)=\sum_{n=1}^{\infty}\frac{r_{\alpha}(n)}{n^s},\quad \text{Re}(s)>\sigma_{\alpha}:=\begin{cases}\alpha/2 & \text{if } \alpha\geq 4\\ 1+\frac{\alpha}{4} & \text{if } 0<\alpha<4\end{cases}. \tag{2.57}$$

We know by Lemma 1.1 that $\phi(s)=\pi^{-s}\zeta_{\alpha}(s)$ satisfies Hecke's functional equation (1.25) with parameter $r=\frac{\alpha}{2}$ and belongs to the class $\mathcal{A}$.

For this particular example, the analogue of Jacobi's $\psi-$function (2.53) is

$$\psi_{\alpha}(x,z):=2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi x}\,z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty}r_{\alpha}(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n x}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi\,n\,x}\,z),\ \text{Re}(x)>0,\quad z\in\mathbb{C} \tag{2.58}$$

Note that we can recover the classical Jacobi's function (2.52) from (2.58). Using the particular cases

$$J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\,\cos(x),\quad r_1(n)=\begin{cases}2 & \text{if } n \text{ is a square}\\ 0 & \text{otherwise}\end{cases},$$

we obtain from (2.58) and (2.20)

$$\psi_1(x,z):=\sqrt{2\pi z}\,(\pi x)^{1/4}\sum_{n=1}^{\infty}\sqrt{n}\,e^{-\pi n^2 x}\,J_{-\frac{1}{2}}(\sqrt{\pi\,x}\,n\,z)=2\sum_{n=1}^{\infty}e^{-\pi n^2 x}\,\cos\left(\sqrt{\pi x}\,n\,z\right):=2\,\psi(x,z).$$

From the transformation formula (2.55), we see that the following identity takes place

$$\psi_{\alpha}(x,z)=-1+\frac{e^{-z^2/4}}{x^{\alpha/2}}+\frac{e^{-z^2/4}}{x^{\alpha/2}}\,\psi_{\alpha}\left(\frac{1}{x},\,iz\right). \tag{2.59}$$

Note again that, when $\alpha=1$, we recover (1.5).

Furthermore, we can write (2.59) in the suitable integral form (2.56). Note that the critical line for $\phi(s):=\pi^{-s}\zeta_{\alpha}(s)$ is the line $\text{Re}(s)=\frac{\alpha}{4}$, so that (2.56) yields the representation

$$x^{\frac{\alpha}{4}}\psi_{\alpha}(x,z)-x^{-\alpha/4}\,e^{-\frac{z^2}{4}}=e^{-\frac{z^2}{4}}x^{-\alpha/4}\,\psi_{\alpha}\left(\frac{1}{x},\,i\,z\right)-x^{\alpha/4}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_{\alpha}\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}+it;\,\frac{\alpha}{2};\,-\frac{z^2}{4}\right)\,x^{-it}\,dt, \tag{2.60}$$

where

$$\eta_{\alpha}(s)=\pi^{-s}\,\Gamma(s)\,\zeta_{\alpha}(s). \tag{2.61}$$

Representation (2.60) will be of great use in proving Theorem 1.1 below.

From Corollary 2.4 we can derive the curious formula

$$\frac{1}{(x^2+y^2)^s}+\sum_{n=1}^{\infty}\frac{r_{\alpha}(n)}{(n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,\frac{\alpha}{2};\,\frac{4n\,y^2}{(n+x^2+y^2)^2}\right)$$
$$=\frac{\pi^{\frac{\alpha}{2}}\,\Gamma(s-\frac{\alpha}{2})\,x^{\alpha-2s}}{\Gamma(s)}+\frac{2\,\Gamma\left(\frac{\alpha}{2}\right)\,\pi^{s+1-\frac{\alpha}{2}}x^{\frac{\alpha}{2}-s}y^{1-\frac{\alpha}{2}}}{\Gamma(s)}\sum_{n=1}^{\infty}r_{\alpha}(n)\,n^{\frac{s+1-\alpha}{2}}\,J_{\frac{\alpha}{2}-1}(2\pi\sqrt{n}y)\,K_{s-\frac{\alpha}{2}}\left(2\pi\sqrt{n}\,x\right),\qquad(2.62)$$

valid for every positive pair of positive numbers x and y and s such $\text{Re}(s)>\sigma_{\alpha}$, with σ_{α} being given by (2.57).

When $\alpha=1$ we obtain an extension of Watson's formula due to Kober [[38], p.614]: since $r_1(n)=2$ only when n is a perfect square, we can use the steps leading to the proof of Corollary 2.6 to obtain the formula

$$\sum_{n\in\mathbb{Z}}\frac{1}{(n^2+2y\,n+x^2+y^2)^s}=\frac{\sqrt{\pi}x^{1-2s}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)}+\frac{4\pi^s}{x^{s-\frac{1}{2}}\,\Gamma(s)}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\,\cos(2\pi n\,y)\,K_{s-\frac{1}{2}}\left(2\pi\,n\,x\right),\qquad(2.63)$$

which is valid for every $\text{Re}(s)>\frac{1}{2}$ and positive x and y. By letting $y=0$ in (2.63), one can deduce Watson's formula [60].

Example 2.2. Let Q be a real, binary and positive definite quadratic form and let $\Delta:=4AC-B^2$ be its discriminant. The Epstein zeta function attached to Q (1.12) satisfies Hecke's functional equation

$$\eta_Q(s):=\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-s}\Gamma(s)\,\zeta(s,Q)=\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-(1-s)}\Gamma(1-s)\,\zeta(1-s,Q):=\eta_Q(1-s).$$

Thus, we can apply the previous formalism to the pair of Dirichlet series

$$\phi(s)=\psi(s)=\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-s}\sum_{n=1}^{\infty}\frac{r_Q(n)}{n^s},\quad\text{Re}(s)>1,\qquad(2.64)$$

where $r_Q(n)$ denotes the representation number of n by Q. Since $\zeta(s,Q)$ has a simple pole at $s=1$ with residue $2\pi/\sqrt{\Delta}$, $\phi(s)$ has a simple pole at $s=1$ with residue 1. Moreover, $\phi(0)=-1$.

From (2.53), we can write the analogue of Jacobi's $\psi-$function (2.53) as follows:

$$\psi_Q(x,z)=\sum_{n=1}^{\infty}r_Q(n)\,e^{-\frac{2\pi nx}{\sqrt{\Delta}}}\,J_0\left(\sqrt{\frac{2\pi nx}{\Delta^{1/2}}}\,z\right),\quad\text{Re}(x)>0,\;\;z\in\mathbb{C}.\qquad(2.65)$$

Using (2.24), the transformation formula for (2.65) reads

$$\sqrt{x}\,\psi_Q(x,z)-\frac{e^{-\frac{z^2}{4}}}{\sqrt{x}}=\frac{e^{-\frac{z^2}{4}}}{\sqrt{x}}\,\psi_Q\left(\frac{1}{x},\,iz\right)-\sqrt{x}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_Q\left(\frac{1}{2}+it\right)\,{}_1F_1\left(\frac{1}{2}+it;\,1;\,-\frac{z^2}{4}\right)x^{-it}\,dt,\qquad(2.66)$$

where

$$\eta_Q(s):=\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-s}\Gamma(s)\,\zeta(s,Q).\qquad(2.67)$$

Like the previous formula (2.60), (2.66) will be of great importance in the proof of Theorem 1.2.

Further identities can be obtained. From a straightforward application of Corollary 2.4 we can derive the formula,

$$\frac{1}{(x^2+y^2)^s}+\sum_{n=1}^{\infty}\frac{r_Q(n)}{(n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\ \frac{s+1}{2};\ 1;\ \frac{4ny^2}{(n+x^2+y^2)^2}\right)$$
$$=\frac{2\pi}{\sqrt{\Delta}}\cdot\frac{x^{2-2s}}{s-1}+\frac{2x^{1-s}}{\Gamma(s)}\left(\frac{2\pi}{\sqrt{\Delta}}\right)^s\sum_{n=1}^{\infty}r_Q(n)\,n^{\frac{s-1}{2}}\,J_0\left(4\pi\sqrt{\frac{n}{\Delta}}\,y\right)\,K_{s-1}\left(4\pi\sqrt{\frac{n}{\Delta}}\,x\right),$$

which seems to be novel.

Example 2.3. Let χ be a primitive Dirichlet character modulo q. If χ is even, we define $\delta=0$ and if χ is odd, we take $\delta=1$. In this example we consider the Dirichlet series

$$L_k\left(s,\chi\right):=\sum_{n_1,\ldots,n_k\in\mathbb{Z}}\frac{n_1^{\delta}\,\chi(n_1)\ldots n_k^{\delta}\,\chi(n_k)}{\left(n_1^2+\ldots+n_k^2\right)^s},\quad \text{Re}(s)>\frac{k}{2}(1+\delta).$$

First, let us note that $L_k\left(s,\chi\right)$ can be rewritten as

$$L_k\left(s,\chi\right):=\sum_{n_1,\ldots,n_k\in\mathbb{Z}}\frac{n_1^{\delta}\,\chi(n_1)\ldots n_k^{\delta}\,\chi(n_k)}{\left(n_1^2+\ldots+n_k^2\right)^s}=\sum_{n=1}^{\infty}\frac{r_{k,\chi}(n)}{n^s},\qquad \text{Re}(s)>\frac{k}{2}(1+\delta),$$

where $r_{k,\chi}(n)$ is the character analogue of $r_k(n)$,

$$r_{k,\chi}(n):=\sum_{n_1^2+\ldots+n_k^2=n}n_1^{\delta}\chi(n_1)\ \cdot\ldots\ \cdot n_k^{\delta}\,\chi(n_k), \tag{2.68}$$

where the sum runs over any decomposition of n as a sum of k squares.

It will be shown in Subsection 5.2 of this paper that $L_k(s,\chi)$ can be analytically continued to an entire function satisfying Hecke's functional equation

$$\left(\frac{\pi}{q}\right)^{-s}\Gamma(s)\,L_k\left(s,\chi\right)=\frac{(-i)^{\delta k}G^k(\chi)}{q^{k/2}}\,\left(\frac{\pi}{q}\right)^{-\left(k(\frac{1}{2}+\delta)-s\right)}\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)\,L_k\left(k\left(\frac{1}{2}+\delta\right)-s;\,\overline{\chi}\right). \tag{2.69}$$

Therefore, we can apply the previous formulas to the pair of Dirichlet series

$$\phi(s):=\left(\frac{\pi}{q}\right)^{-s}\sum_{n=1}^{\infty}\frac{r_{k,\chi}(n)}{n^s},\quad \psi(s):=\frac{(-i)^{\delta k}G^k(\chi)}{q^{k/2}}\,\left(\frac{\pi}{q}\right)^{-s}\sum_{n=1}^{\infty}\frac{r_{k,\overline{\chi}}(n)}{n^s}. \tag{2.70}$$

From the definition (2.53), we can consider the analogue of Jacobi's $\psi-$function attached to $L_k(s,\chi)$,

$$\psi_{\chi,k}(x,z):=2^{\frac{k}{2}+k\delta-1}\,\Gamma\left(\frac{k}{2}+k\delta\right)\left(\sqrt{\frac{\pi}{q}}x\,z\right)^{1-\frac{k}{2}-k\delta}\sum_{n=1}^{\infty}r_{k,\chi}(n)\,n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\,e^{-\frac{\pi n}{q}\,x}\,J_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi}{q}n\,x\,z}\right). \tag{2.71}$$

When $k=1$, (2.71) reduces to the character analogue of Jacobi's $\psi-$function given at the end of the paper [[22], p. 321] and in [[21], Theorem 1.3.]. Indeed, since $r_{1,\chi}(n)=2\sqrt{n}^{\delta}\chi(\sqrt{n})$ only if n is a perfect square, we see that (2.71) gives

$$\psi_{\chi,1}(x,z):=2^{\frac{1}{2}+\delta}\,\Gamma\left(\frac{1}{2}+\delta\right)\left(\sqrt{\frac{\pi}{q}}x\,z\right)^{\frac{1}{2}-\delta}\sum_{n=1}^{\infty}\chi(n)\,\sqrt{n}\,e^{-\frac{\pi n^2}{q}\,x}\,J_{\delta-\frac{1}{2}}\left(\sqrt{\frac{\pi}{q}}x\,n\,z\right)$$
$$=\begin{cases}2\,\sum_{n=1}^{\infty}\chi(n)\,e^{-\frac{\pi n^2}{q}\,x}\,\cos\left(\sqrt{\frac{\pi}{q}}x\,n\,z\right) & \delta=0\\ \frac{2}{z}\sqrt{\frac{q}{\pi x}}\,\sum_{n=1}^{\infty}\chi(n)\,e^{-\frac{\pi n^2}{q}\,x}\,\sin\left(\sqrt{\frac{\pi}{q}}x\,n\,z\right) & \delta=1.\end{cases}$$

Since $\phi(0) = \rho = 0$ for $\phi(s)$ given by (2.70), we can deduce from our general formulas,

$$\sqrt{x}\sum_{n=1}^{\infty} r_{k,\chi}(n)\, n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\, e^{-\frac{\pi n}{q}\,x}\, J_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi}{q}n\,x}\,z\right) = \frac{(-i)^{\delta k}G^{k}(\chi)}{q^{k/2}}\,\frac{e^{-z^2/4}}{\sqrt{x}}\sum_{n=1}^{\infty} r_{k,\overline{\chi}}(n)\, n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\, e^{-\frac{\pi n}{q\,x}}\, I_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi n}{qx}}\,z\right)$$

which, by virtue of relation (2.56), can be rewritten as

$$x^{\frac{k}{4}+\frac{k\delta}{2}}\,\psi_{\chi,k}\left(x,z\right) = \frac{(-i)^{\delta k}G^{k}(\chi)\,e^{-\frac{z^2}{4}}}{q^{k/2}}\,x^{-\frac{k}{4}-\frac{k}{2}\delta}\,\psi_{\overline{\chi},k}\left(\frac{1}{x},iz\right) =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_{k}\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)\cdot {}_1F_1\left(\frac{k}{4}+\frac{k\delta}{2}+it;\,\frac{k}{2}+k\delta;\,-\frac{z^2}{4}\right)x^{-it}\,dt, \tag{2.72}$$

where

$$\eta_{k}(s,\chi) := \left(\frac{\pi}{q}\right)^{-s}\Gamma(s)\,L_{k}\left(s,\chi\right). \tag{2.73}$$

From Corollary 2.4, we may derive a character analogue of (2.62), namely:

$$\frac{q^{k/2}i^{\delta k}}{G^{k}(\chi)}\sum_{n=1}^{\infty}\frac{r_{k,\chi}(n)}{(n+x^2+y^2)^{s}}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,\frac{k}{2}+k\delta;\,\frac{4y^2n}{(n+x^2+y^2)^2}\right) =$$

$$= \frac{2\,\Gamma\left(\frac{k}{2}+k\delta\right)}{\Gamma(s)\,y^{\frac{k}{2}+k\delta-1}\,x^{s-\frac{k}{2}-k\delta}}\left(\frac{\pi}{q}\right)^{1+s-\frac{k}{2}-k\delta}\sum_{n=1}^{\infty} r_{k,\overline{\chi}}(n)\,n^{\frac{s+1-k}{2}-k\delta}\,J_{\frac{k}{2}+k\delta-1}\left(\frac{2\pi\sqrt{n}}{q}y\right)K_{s-\frac{k}{2}-k\delta}\left(\frac{2\pi\sqrt{n}}{q}\,x\right). \tag{2.74}$$

If $k=1$ and $\delta=0$, we can also appeal to the particular case (2.47) to obtain the character analogue of Watson's formula, valid for $\mathrm{Re}(s) > \frac{1}{2}$,

$$\sum_{n\in\mathbb{Z}}\frac{\chi(n)}{(n^2+2yn+x^2+y^2)^{s}} = \frac{4\,x^{\frac{1}{2}-s}\,G(\chi)}{\Gamma(s)\sqrt{\pi}}\left(\frac{\pi}{q}\right)^{\frac{1}{2}+s}\sum_{n=1}^{\infty}\overline{\chi}(n)\,n^{s-\frac{1}{2}}\,\cos\left(\frac{2\pi n}{q}y\right)K_{s-\frac{1}{2}}\left(\frac{2\pi n}{q}x\right).$$

By letting $y=0$ above, we rederive a formula due to Berndt, Dixit and Sohn [[10], p. 56, eq. (2.9.)].

When $k=1$ and $\delta=1$, we see that $L_1\left(s,\chi\right) = 2\,L(2s-1,\chi)$, with χ being an odd Dirichlet character. In this case, (2.74) implies the identity:

$$\sum_{n\in\mathbb{Z}}\frac{\chi(n)}{(n^2+2yn+x^2+y^2)^{s}} = -\frac{4i\,x^{\frac{1}{2}-s}\,G(\chi)}{\Gamma(s)\sqrt{\pi}}\left(\frac{\pi}{q}\right)^{s+\frac{1}{2}}\sum_{n=1}^{\infty}\overline{\chi}(n)\,n^{s-\frac{1}{2}}\sin\left(\frac{2\pi n}{q}y\right)K_{s-\frac{1}{2}}\left(\frac{2\pi n}{q}x\right),$$

valid for $\mathrm{Re}(s) > \frac{1}{2}$. This is another analogue of Watson's formula.

Example 2.4. Let $f(\tau)$ be a holomorphic cusp form with weight $k \geq 12$ for the full modular group and $L_f(s)$ the associated $L-$function,

$$L_{f}(s) = \sum_{n=1}^{\infty}\frac{a_f(n)}{n^{s}},\quad \mathrm{Re}(s) > \frac{k+1}{2}. \tag{2.75}$$

From Lemma 1.4 we know $L_f(s)$ can be analytically continued as an entire function with functional equation

$$(2\pi)^{-s}\Gamma(s)\,L_f(s) = (-1)^{k/2}\,(2\pi)^{-(k-s)}\,\Gamma(k-s)\,L_f(k-s). \tag{2.76}$$

From the previous results we can conceive the analogue of Jacobi's $\psi-$function (2.53) in the following form

$$\psi_f(x,z) := (k-1)!\left(\sqrt{\frac{\pi x}{2}}\,z\right)^{1-k}\sum_{n=1}^{\infty} a_f(n) n^{\frac{1-k}{2}}\,e^{-2\pi n\,x}\,J_{k-1}\left(\sqrt{2\pi n\,x}\,z\right),\ \ \text{Re}(x)>0,\ \ z\in\mathbb{C}. \tag{2.77}$$

Moreover, we can establish the formula

$$x^{\frac{k}{2}}\psi_f(x,z) = (-1)^{k/2}\,e^{-\frac{z^2}{4}}x^{-k/2}\,\psi_f\left(\frac{1}{x},\,iz\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_f\left(\frac{k}{2}+it\right)\,{}_1F_1\left(\frac{k}{2}+it;\,k;\,-\frac{z^2}{4}\right)\,x^{-it}dt, \tag{2.78}$$

where

$$\eta_f(s) := (2\pi)^{-s}\Gamma(s)\,L_f(s). \tag{2.79}$$

Also, for $\text{Re}(s) > \frac{k+1}{2}$ and $x, y > 0$, the following particular case of Corollary 2.4 holds:

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{a_f(n)}{(n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,k;\,\frac{4n\,y^2}{(n+x^2+y^2)^2}\right)\\ &= \frac{2\,(2\pi)^{s+1-k}\,(k-1)!\,(-1)^{k/2}}{\Gamma(s)\,y^{k-1}x^{s-k}}\sum_{n=1}^{\infty} a_f(n)\,n^{\frac{s+1}{2}-k}\,J_{k-1}(4\pi\sqrt{n}\,y)\,K_{s-k}\left(4\pi\sqrt{n}\,x\right).\end{aligned} \tag{2.80}$$

For example, when $L_f(s)$ is the Dirichlet series associated with Ramanujan's $\tau-$function,

$$L_{\tau}(s) = \sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}, \quad \text{Re}(s) > \frac{13}{2},$$

we obtain from (2.80)

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\tau(n)}{(n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,12;\,\frac{4ny^2}{(n+x^2+y^2)^2}\right)\\ &= \frac{2\times 11!\times(2\pi)^{s-11}}{\Gamma(s)\,y^{11}x^{s-12}}\sum_{n=1}^{\infty}\tau(n)\,n^{\frac{s-23}{2}}\,J_{11}(4\pi\sqrt{n}\,y)\,K_{s-12}\left(4\pi\sqrt{n}\,x\right),\ \ x,y>0,\ \ \text{Re}(s)>\frac{13}{2},\end{aligned}$$

which appears to be new (see [[9], eq. (13.1)] for a companion formula).

Example 2.5. Let $f(\tau)$ be a holomorphic cusp form of weight k for the full modular group and $a_f(n)$ its Fourier coefficients. Also, assume that p, q are integers such that $(p,q)=1$ and consider the Dirichlet series

$$L_f(s,p/q) := \sum_{n=1}^{\infty}\frac{a_f(n)\,e^{\frac{2\pi ip}{q}n}}{n^s}, \quad \text{Re}(s) > \frac{k+1}{2}.$$

We know from Lemma 1.4 that $L_f(s,p/q)$ is an entire function and satisfies the functional equation:

$$\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)\,L_f\left(s,\,\frac{p}{q}\right) = (-1)^{k/2}\left(\frac{2\pi}{q}\right)^{-(k-s)}\Gamma(k-s)\,L_f\left(k-s,\,-\overline{p}/q\right),$$

where $\overline{p}$ is such that $p\,\overline{p}\equiv 1 \mod q$.

From this, we can create the analogue of Jacobi's $\psi-$function (2.53) as follows

$$\psi_{f,p/q}(x,z) := (k-1)!\,\left(\sqrt{\frac{\pi x}{2q}}\,z\right)^{1-k}\sum_{n=1}^{\infty}a_f(n)\,e^{\frac{2\pi ip}{q}n}\,n^{\frac{1-k}{2}}\,e^{-\frac{2\pi n}{q}\,x}\,J_{k-1}\left(\sqrt{\frac{2\pi n}{q}x}\,z\right), \tag{2.81}$$

defined for $\text{Re}(x)>0$ and $z\in\mathbb{C}$.

Clearly, the transformation formula for $\psi_{f,p/q}(x,z)$ (2.55) is explicitly given by

$$\sum_{n=1}^{\infty}a_f(n)\,e^{\frac{2\pi ip}{q}n}\,n^{\frac{1-k}{2}}\,e^{-\frac{2\pi n}{q}\,x}\,J_{k-1}\left(\sqrt{\frac{2\pi n}{q}x}\,z\right)=\frac{(-1)^{k/2}\,e^{-z^2/4}}{x}\sum_{n=1}^{\infty}a_f(n)\,e^{-\frac{2\pi i\overline{p}}{q}n}\,n^{\frac{1-k}{2}}\,e^{-\frac{2\pi n}{qx}}\,I_{k-1}\left(\sqrt{\frac{2\pi n}{qx}}\,z\right), \tag{2.82}$$

and it admits the integral representation,

$$x^{k/2}\psi_{f,p/q}(x,z)=\frac{e^{-z^2/4}(-1)^{k/2}}{x^{k/2}}\,\psi_{f,-\frac{\overline{p}}{q}}\left(\frac{1}{x},iz\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_f\left(\frac{k}{2}+it,\frac{p}{q}\right)\,{}_1F_1\left(\frac{k}{2}+it;\,k;\,-\frac{z^2}{4}\right)\,x^{-it}dt, \tag{2.83}$$

where

$$\eta_f\left(s,\frac{p}{q}\right):=\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)\,L_f\left(s,\frac{p}{q}\right). \tag{2.84}$$

3 Zeros of combinations attached to $\zeta_\alpha(s)$

3.1 Dirichlet series attached to powers of the Theta functions

At the core of Hardy's proof of his Theorem is the fact that Jacobi's $\theta-$function satisfies

$$\lim_{\omega\to\frac{\pi}{4}^-}\frac{d^n}{d\omega^n}\,\theta\left(e^{2i\omega}\right)=0\quad\forall n\in\mathbb{N}_0. \tag{3.1}$$

If we replace the traditional role of $\theta(x)$ in (3.1) by the analogue $\psi_\alpha(x,z)$, we will see below that the study of the limit (3.1) can be made through the evaluation of

$$\lim_{\delta\to0^+}\psi_\alpha(i+\delta,z).$$

However, to inspect $\psi_\alpha(i+\delta,z)$ is the same as to study the series

$$2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty}(-1)^n\,r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\,\delta}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi\,n\,(i+\delta)}\,z) \tag{3.2}$$

as $\delta\to0^+$. For $\alpha=1$, this study was done in [[22], p. 314] and it represents an easier case because $r_1(n)=2$ if n is a perfect square and it is zero otherwise. Indeed, the expression (3.2) reduces to

$$\psi_1(i+\delta,z)=2\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2\delta}\,\cos\left(\sqrt{\pi(i+\delta)}\,n\,z\right),$$

which can be evaluated simply as (once we separate the previous series for n even and n odd)

$$\psi_1(i+\delta,z)=2\,\psi_1\left(4\delta,\sqrt{\frac{i+\delta}{\delta}}\,z\right)-\psi_1\left(\delta,\sqrt{\frac{i+\delta}{\delta}}\,z\right).$$

Thus, the study of the limit $\delta\to 0^+$ is complete once we apply the transformation formula (2.59) for $\psi_1(x,z)$!

By looking at (3.2), one can see that it may be difficult to employ the previous trick (i.e., to separate the sum for even and odd n) for general $\alpha>0$, and so we need to have more information about the coefficients $(-1)^n r_\alpha(n)$ and the Dirichlet series attached to them. This study will be developed through a sequence of important lemmas.

These lemmas concern the powers of certain variants of Jacobi's theta function. Like in the case of $\vartheta_3(\tau)$ (1.35), we can consider arbitrary powers of the remaining thetanulls,

$$\vartheta_2(\tau)=2\sum_{n=0}^{\infty}e^{\pi i\tau\left(n+\frac{1}{2}\right)^2},\quad \vartheta_4(\tau)=1+2\sum_{n=1}^{\infty}(-1)^n e^{\pi i\tau n^2},\quad \text{Im}(\tau)>0, \tag{3.3}$$

and study their expansions at $i\infty$ by taking

$$\theta_2(x):=\vartheta_2(ix)=2\sum_{n=0}^{\infty}e^{-\pi\left(n+\frac{1}{2}\right)^2x},\quad \theta_4(x):=\vartheta_4(ix)=1+2\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2 x}.$$

It follows from Jacobi's formula (1.5) that $\theta_2(x)$ and $\theta_4(x)$ are connected via the transformation formula

$$\theta_2\left(\frac{1}{x}\right)=\sqrt{x}\,\theta_4\left(x\right). \tag{3.4}$$

Arithmetical functions akin to $r_\alpha(n)$ arise from the study of the powers of $\theta_2(x)$ and $\theta_4(x)$. For example, modifying slightly the computations in (1.35) and (1.36), we can define the Fourier coefficients of $\theta_2^\alpha(x)$ as follows:

$$\begin{aligned}\vartheta_2^\alpha(ix):=\theta_2^\alpha(x)&=\left(2\sum_{n=0}^{\infty}e^{-\pi x\left(n+\frac{1}{2}\right)^2}\right)^\alpha=\left(2\,e^{-\frac{\pi x}{4}}+2\sum_{n=1}^{\infty}e^{-\pi x\left(n+\frac{1}{2}\right)^2}\right)^\alpha\\ &=2^\alpha e^{-\frac{\pi\alpha x}{4}}\left(1+\sum_{n=1}^{\infty}e^{-\pi x\left(n^2+n\right)}\right)^\alpha=2^\alpha\,e^{-\frac{\pi\alpha x}{4}}\sum_{j=0}^{\infty}\binom{\alpha}{j}\left(\sum_{n=1}^{\infty}e^{-\pi x\left(n^2+n\right)}\right)^j\\ &:=\sum_{m=0}^{\infty}\tilde{r}_\alpha(m)\,e^{-\pi\left(m+\frac{\alpha}{4}\right)x}\end{aligned} \tag{3.5}$$

where we have taken the new variable of summation as $m:=n_1(n_1+1)+...+n_j(n_j+1)$. Analogously to $r_\alpha(n)$, it is possible to show that the coefficients $\tilde{r}_\alpha(n)$ grow polynomially[5] with n. Note also that $\tilde{r}_\alpha(0)=2^\alpha$ by construction.

Henceforth, it is meaningful to introduce a Dirichlet series attached to the coefficients $\tilde{r}_\alpha(n)$, this is, for some finite $\tilde{\sigma}_\alpha$, we define

$$\tilde{\zeta}_\alpha(s):=\sum_{n=0}^{\infty}\frac{\tilde{r}_\alpha(n)}{\left(n+\frac{\alpha}{4}\right)^s},\qquad \text{Re}(s)>\tilde{\sigma}_\alpha. \tag{3.6}$$

[5] We do not necessarily need a bound of the form (1.37) to proceed with our reasoning. It suffices to check that the argument by Lagarias and Rains [[40], p. 18, Theorem 3.3.] works for $\vartheta_2(\tau)$. See Lemma 3.1 below.

We can also introduce a Dirichlet series attached to any positive power of $\theta_4(x)$. Proceed once more as in (1.35):

$$\begin{aligned}
\vartheta_4^{\alpha}(ix) := \theta_4^{\alpha}(x) &= \left(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2 x}\right)^{\alpha} \\
&= 1 + \sum_{j=1}^{\infty}\begin{pmatrix}\alpha \\ j\end{pmatrix} 2^j \left(\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2 x}\right)^j \\
&= 1 + \sum_{j=1}^{\infty}\begin{pmatrix}\alpha \\ j\end{pmatrix} 2^j \sum_{n_1,\dots,n_j=1}^{\infty} (-1)^{n_1+n_2+\dots+n_j}\, e^{-\pi(n_1^2+\dots+n_j^2)\,x} \\
&= 1 + \sum_{j=1}^{\infty}\begin{pmatrix}\alpha \\ j\end{pmatrix} 2^j \sum_{n_1,\dots,n_j=1}^{\infty} (-1)^{n_1^2+n_2^2+\dots+n_j^2}\, e^{-\pi(n_1^2+\dots+n_j^2)\,x}.
\end{aligned}$$

If we now define a new variable of summation $m = n_1^2 + \dots + n_j^2$, then the finite sum over j is exactly the same as we have obtained in defining (1.36) but with an extra factor containing $(-1)^m$. Following (1.36), we see that

$$\begin{aligned}
\theta_4^{\alpha}(x) &= 1 + \sum_{j=1}^{\infty}\begin{pmatrix}\alpha \\ j\end{pmatrix} 2^j \sum_{n_1,\dots,n_j=1}^{\infty} (-1)^{n_1^2+n_2^2+\dots+n_j^2}\, e^{-\pi(n_1^2+\dots+n_j^2)\,x} \\
&= 1 + \sum_{m=1}^{\infty}(-1)^m\, r_{\alpha}(m)\, e^{-\pi m x}.
\end{aligned} \tag{3.7}$$

The computations above show that $(-1)^n r_{\alpha}(n)$ are the coefficients of the Fourier expansion of $\vartheta_4^{\alpha}(\tau)$ at the cusp $i\infty$.

The Dirichlet series attached to these coefficients is then:

$$\zeta_{\alpha}^{\star}(s) := \sum_{n=1}^{\infty}\frac{(-1)^n\, r_{\alpha}(n)}{n^s}, \quad \mathrm{Re}(s) > \sigma_{\alpha} := \begin{cases}\alpha/2 & \text{if } \alpha \geq 4 \\ 1 + \frac{\alpha}{4} & \text{if } 0 < \alpha < 4.\end{cases} \tag{3.8}$$

In the following subsection, we connect the Dirichlet series (3.6) and (3.8) through a functional equation and prove a new summation formula for their coefficients.

3.2 Summation formulas

To establish the connection between the Dirichlet series $\zeta_{\alpha}^{\star}(s)$ and $\tilde{\zeta}_{\alpha}(s)$, we need to assure that the latter is actually a Dirichlet series, i.e., that $\tilde{\sigma}_{\alpha}$ in (3.6) is a finite number. The next lemma gives an estimate for $\tilde{r}_{\alpha}(n)$ similar in spirit to the bounds found in [40].

Lemma 3.1. *Let $\alpha > 0$ and $\tilde{r}_{\alpha}(n)$ be defined by (3.5). Then, for any $n \in \mathbb{N}$, $\tilde{r}_{\alpha}(n)$ satisfies the estimate*

$$|\tilde{r}_{\alpha}(n)| < 540\,(2n)^{\alpha/2}. \tag{3.9}$$

Therefore, the Dirichlet series defined by (3.6) converges absolutely in the right half-plane $Re(s) > \tilde{\sigma}_{\alpha} := \frac{\alpha}{2} + 1$.

Proof. Our proof is nothing but a simple adaptation of [[40], Theorem 3.3., p. 19, eq. (3.12)]. Write $\varphi_2(q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}$ for $|q|<1$: invoking Cauchy's integral formula, the following bound takes place

$$|\tilde{r}_{\alpha}(n)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} \varphi_2^{\alpha}\left(Re^{i\phi}\right) R^{-n}e^{-in\phi}\,d\phi\right| \leq R^{-n} \max_{\phi\in[-\pi,\pi]}\left|\varphi_2^{\alpha}\left(Re^{i\phi}\right)\right|$$
$$= R^{-n}\left(\max_{\phi\in[-\pi,\pi]}\left|\varphi_2\left(Re^{i\phi}\right)\right|\right)^{\alpha}, \quad 0<R<1,$$

since $\alpha>0$ by hypothesis. Clearly, for every $\phi\in[-\pi,\pi]$,

$$\left|\varphi_2\left(Re^{i\phi}\right)\right| \leq \varphi_2(R),$$

which immediately gives

$$|\tilde{r}_{\alpha}(n)| \leq R^{-n}\,\varphi_2^{\alpha}(R), \tag{3.10}$$

for any choice of $R\in(0,1)$. For each n, take $R:=R(n)=e^{-\pi A/n}$ for a fixed $A>1$. Then (3.10) yields

$$|\tilde{r}_{\alpha}(n)| \leq e^{\pi A}\,\varphi_2^{\alpha}\left(e^{-\pi A/n}\right) = e^{\pi A}\,\theta_2^{\alpha}\left(\frac{A}{n}\right) = e^{\pi A}\left(\frac{n}{A}\right)^{\alpha/2}\theta_4^{\alpha}\left(\frac{n}{A}\right)$$
$$= \frac{e^{\pi A}}{A^{\alpha/2}}\left\{1+2\sum_{k=1}^{\infty}(-1)^k e^{-\pi k^2\frac{n}{A}}\right\}^{\alpha} n^{\alpha/2} \leq \frac{e^{\pi A}\,n^{\alpha/2}}{A^{\alpha/2}}\left\{1+\frac{2}{e^{\pi n/A}-1}\right\}^{\alpha}$$
$$\leq \frac{e^{\pi A}}{A^{\alpha/2}}\left\{1+\frac{2}{e^{\pi/A}-1}\right\}^{\alpha} n^{\alpha/2}. \tag{3.11}$$

where we have used (3.4) together with elementary bounds on the series defining $\theta_4(x)$. The desired estimate (3.9) now follows once we take $A=2$ in (3.11). □

Our next lemma gives a functional equation for the zeta functions $\zeta_{\alpha}^{\star}(s)$ and $\tilde{\zeta}_{\alpha}(s)$.

Lemma 3.2. *Let $r_{\alpha}(n)$ and $\tilde{r}_{\alpha}(n)$ be given by (1.36) and (3.5) respectively.*

Then the Dirichlet series defined by (3.8) can be analytically continued to an entire function satisfying Hecke's functional equation

$$\pi^{-s}\Gamma(s)\,\zeta_{\alpha}^{\star}(s) = \pi^{-\left(\frac{\alpha}{2}-s\right)}\Gamma\left(\frac{\alpha}{2}-s\right)\tilde{\zeta}_{\alpha}\left(\frac{\alpha}{2}-s\right), \tag{3.12}$$

where $\tilde{\zeta}_{\alpha}(s)$ is given by (3.6).

Proof. We have deduced above that $(-1)^n r_{\alpha}(n)$ are the coefficients of the Fourier expansion of $\vartheta_4^{\alpha}(\tau)$ at the cusp $i\infty$. It is clear from this that the Mellin transform holds

$$\pi^{-s}\Gamma(s)\,\zeta_{\alpha}^{\star}(s) = \int_0^{\infty} x^{s-1}\left\{\theta_4^{\alpha}(x)-1\right\}dx, \quad \mathrm{Re}(s)>\sigma_{\alpha},$$

where σ_{α} is explicitly given by (3.8). By following Riemann's paper [49, 59], we study the functional equation for $\zeta_{\alpha}^{\star}(s)$: indeed, for $\mathrm{Re}(s)>\sigma_{\alpha}$,

$$\pi^{-s}\Gamma(s)\,\zeta_{\alpha}^{\star}(s) = \int_0^{\infty} x^{s-1}\left\{\theta_4^{\alpha}(x)-1\right\}dx = \int_0^{1} x^{s-1}\left\{\theta_4^{\alpha}(x)-1\right\}dx + \int_1^{\infty} x^{s-1}\left\{\theta_4^{\alpha}(x)-1\right\}dx. \tag{3.13}$$

Using a particular case of Jacobi's formula (1.5) given in (3.4),

$$\theta_4^\alpha(1/x) - 1 = x^{\alpha/2}\,\theta_2^\alpha(x) - 1,$$

we see that the first integral on the right of (3.13) can be written as:

$$\int_0^1 x^{s-1}\left\{\theta_4^\alpha(x) - 1\right\} dx = \int_1^\infty x^{-s-1}\left\{\theta_4^\alpha\left(1/x\right) - 1\right\} dx = \int_1^\infty x^{\frac{\alpha}{2}-s-1}\theta_2^\alpha(x)\, dx - \frac{1}{s}. \tag{3.14}$$

Combining (3.13) and (3.14) leads to the representation

$$\pi^{-s}\Gamma\left(s\right)\zeta_\alpha^\star(s) = \int_1^\infty \left[x^{s-1}\left\{\theta_4^\alpha(x) - 1\right\} + x^{\frac{\alpha}{2}-s-1}\,\theta_2^\alpha(x)\right] dx - \frac{1}{s}. \tag{3.15}$$

Since the integral on the right-hand side of (3.15) converges absolutely for every $s \in \mathbb{C}$, it represents an entire function of $s \in \mathbb{C}$. Hence, the previous representation gives the analytic continuation of $\zeta_\alpha^\star(s)$ as an entire function. Furthermore, since $\Gamma(s)$ has a simple pole at $s = 0$, it is clear to see that $\zeta_\alpha^\star(0) = -1$.

We can do the same kind of computations with $\tilde{\zeta}_\alpha(s)$. We find that, for $\mathrm{Re}(s) > \tilde{\sigma}_\alpha := \frac{\alpha}{2} + 1$ (see (3.6) above)

$$\begin{aligned}
\pi^{-s}\Gamma(s)\,\tilde{\zeta}_\alpha(s) &= \int_0^\infty x^{s-1}\theta_2^\alpha(x)\, dx = \int_0^1 x^{s-1}\theta_2^\alpha(x)\, dx + \int_1^\infty x^{s-1}\theta_2^\alpha(x)\, dx \\
&= \int_1^\infty x^{-s-1}\theta_2^\alpha(1/x)\, dx + \int_1^\infty x^{s-1}\,\theta_2^\alpha(x)\, dx \\
&= \int_1^\infty x^{-s-1}\left(x^{\alpha/2}\left(\theta_4^\alpha(x) - 1\right) + x^{\alpha/2}\right) dx + \int_1^\infty x^{s-1}\theta_2^\alpha(x)\, dx \\
&= \int_1^\infty \left(x^{\frac{\alpha}{2}-s-1}\left(\theta_4^\alpha(x) - 1\right) + x^{s-1}\theta_2^\alpha(x)\right) dx + \int_1^\infty x^{-s-1+\frac{\alpha}{2}}\, dx \\
&= \int_1^\infty \left(x^{\frac{\alpha}{2}-s-1}\left(\theta_4^\alpha(x) - 1\right) + x^{s-1}\theta_2^\alpha(x)\right) dx + \frac{1}{s-\frac{\alpha}{2}}.
\end{aligned} \tag{3.16}$$

From this we can see that $\tilde{\zeta}_\alpha(s)$ can be continued to the complex plane as a meromorphic function having a simple pole at $s = \frac{\alpha}{2}$ with residue $\pi^{\alpha/2}/\Gamma(\alpha/2)$. Furthermore, the right-hand sides of (3.15) and (3.16) can be turned into one another under the reflection $s \longleftrightarrow \frac{\alpha}{2} - s$. This implies the functional equation (3.12) and completes the proof. $\square$

Since the functional equation (3.12) holds, we might expect that the summation formula (2.9) must hold as well. The next lemma gives precisely this.

Lemma 3.3. *Let $r_\alpha(n)$ be the coefficients of the series expansion of $\theta^\alpha(x)-1$ and $\tilde{r}_\alpha(n)$ be defined by (3.5). Then for Re$(x)>0$ and any $y\in\mathbb{C}$, the following identity holds*

$$\sum_{n=1}^{\infty}(-1)^n\, r_\alpha(n)\, n^{\frac{1}{2}-\frac{\alpha}{4}}\, e^{-\pi n x}\, J_{\frac{\alpha}{2}-1}(\sqrt{\pi\, n}\, y) = -\frac{y^{\frac{\alpha}{2}-1}\pi^{\frac{\alpha}{4}-\frac{1}{2}}}{2^{\frac{\alpha}{2}-1}\Gamma(\alpha/2)}+$$
$$+\frac{e^{-\frac{y^2}{4x}}}{x}\sum_{n=0}^{\infty}\tilde{r}_\alpha(n)\,\left(n+\frac{\alpha}{4}\right)^{\frac{1}{2}-\frac{\alpha}{4}}\, e^{-\frac{\pi}{x}\left(n+\frac{\alpha}{4}\right)}I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}\,y}{x}\right). \tag{3.17}$$

Proof. By the previous result, we have that the pair of Dirichlet series

$$\phi(s)=\pi^{-s}\sum_{n=1}^{\infty}\frac{(-1)^n r_\alpha(n)}{n^s},\quad \psi(s)=\pi^{-s}\sum_{n=0}^{\infty}\frac{\tilde{r}_\alpha(n)}{\left(n+\frac{\alpha}{4}\right)^s},$$

satisfies Hecke's functional equation (1.25). Thus, by an application of our summation formulas (2.9) or (2.24) to this pair, we just need to take the simple substitutions $r=\frac{\alpha}{2}$, $\rho=0$ (because $\phi(s)$ is entire by the previous lemma) and $\phi(0)=-1$, obtaining (3.17). □

3.3 The behavior of $\psi_\alpha(x,z)$

Using the previous Lemma 3.3 we are now able to show that $\psi_\alpha(x,z)$ has the same kind of behavior as the classical Jacobi's theta function when x approaches the imaginary axis. We now establish one of the most important results of our paper.

Lemma 3.4. *Let $\alpha>0$ and $r_\alpha(n)$ be defined as the coefficients of the Fourier expansion of $\theta^\alpha(x)$ (1.34). For Re$(x)>0$ and $z\in\mathbb{C}$, let $\psi_\alpha(x,z)$ denote the analogue of Jacobi's $\psi-$function (2.58),*

$$\psi_\alpha(x,z)=2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi x}\,z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty}r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\,x}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi\,n\,x}\,z). \tag{3.18}$$

Then, for any z satisfying the condition:

$$z\in\mathscr{D}_\alpha:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<\sqrt{\frac{\pi\alpha}{2}},\,|Im(z)|<\sqrt{\frac{\pi\alpha}{2}}\right\}, \tag{3.19}$$

and every $m\in\mathbb{N}_0$, one has that:

$$\frac{d^m}{d\omega^m}\left(1+\psi_\alpha\left(e^{2i\omega},z\right)\right)\to 0,\quad as\quad \omega\to\frac{\pi}{4}^{-}. \tag{3.20}$$

Proof. Note that $e^{2i\omega}\to i$ as $\omega\to\frac{\pi}{4}^{-}$ along a circular path where $\text{Re}(e^{2i\omega}),\ \text{Im}(e^{2i\omega})>0$. Thus, we can write $e^{2i\omega}$ as $i+\delta$, where $\delta\to0$ along any path in the region $|\arg(\delta)|<\frac{\pi}{2}$. With this change of variable and Faà di Bruno's formula, we can write

$$\frac{d^m}{d\omega^m}\psi_\alpha\left(e^{2i\omega},z\right)=(2i)^m\sum_{b_1,\dots,b_m}\frac{m!}{b_1!\cdot\ldots\cdot b_m!}\prod_{j=1}^{m}\left(\frac{i+\delta}{j!}\right)^{b_j}\frac{d^{b_1+\ldots+b_m}}{d\delta^{b_1+\ldots+b_m}}\psi_\alpha\left(i+\delta,z\right), \tag{3.21}$$

where the sum is over all nonnegative integers $b_1,\dots,b_m$ such that $b_1+2b_2+\ldots+mb_m=m$.

Therefore, to prove (3.20), we shall evaluate

$$\psi_\alpha(i+\delta,\,z) = 2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\,\sum_{n=1}^{\infty} r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\,(i+\delta)}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi\,n\,(i+\delta)}\,z)$$
$$= 2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\,\sum_{n=1}^{\infty}(-1)^n\,r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\,\delta}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi\,n\,(i+\delta)}\,z).$$

Using the previous formula (3.17) with $x=\delta$ and $y=\sqrt{i+\delta}\,z$, we obtain

$$1+\psi_\alpha(i+\delta,z) = 2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\,\frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\delta}\,\sum_{n=0}^{\infty}\tilde{r}_\alpha(n)\,\left(n+\frac{\alpha}{4}\right)^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\frac{\pi}{\delta}\,(n+\frac{\alpha}{4})}I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})\,(i+\delta)}\,z}{\delta}\right) \tag{3.22}$$

Note that (3.20) is established once we prove that any derivative (with respect to δ) of the right-hand side of (3.22) tends to zero as $\delta\to 0$ along any path in the region $|\arg(\delta)|<\frac{\pi}{2}$. We will prove first that (3.20) holds for $m=0$ and the remaining cases will follow. In order to check this, we use the fact that $\psi_\alpha(x,z)$ is analytic as a function of both x in $\mathrm{Re}(x)>0$ and $z\in\mathbb{C}$ [6], so it suffices to bound each term of the series as $\delta\to 0^+$.

Like in Corollary 2.2, we divide the proof in two cases: $\alpha>1$ or $0<\alpha\leq 1$. For the first case, we use the integral representation for the modified Bessel function [[44], p.252, eq. (10.32.2)], connected to the Poisson integral (2.26)

$$\left(\frac{z}{2}\right)^{-\nu}I_\nu(z) = \frac{1}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}}\,e^{zt}dt,\quad \mathrm{Re}(\nu)>-\frac{1}{2},\quad z\in\mathbb{C}, \tag{3.23}$$

from which one can immediately obtain the bound (with real $\nu>-\frac{1}{2}$ and $z\in\mathbb{C}$)

$$\left|\left(\frac{z}{2}\right)^{-\nu}I_\nu(z)\right| \leq \frac{e^{|\mathrm{Re}(z)|}}{\Gamma(\nu+1)}. \tag{3.24}$$

Using (3.24), Lemma 3.1 and the fact that $\tilde{r}_\alpha(0)=2^\alpha$, we can bound $|1+\psi_\alpha(i+\delta,z)|$ simply as follows:

$$|1+\psi_\alpha(i+\delta,z)| \leq \delta^{-\frac{\alpha}{2}}\,e^{-\frac{\mathrm{Re}\left((i+\delta)\,z^2\right)}{4\delta}}\,\sum_{n=0}^{\infty}|\tilde{r}_\alpha(n)|\,e^{-\frac{\pi}{\delta}\,(n+\frac{\alpha}{4})}\,\exp\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}}{\delta}\,\left|\mathrm{Re}\left(\sqrt{i+\delta}\,z\right)\right|\right)$$
$$< 2^\alpha\delta^{-\frac{\alpha}{2}}\,e^{-\frac{\mathrm{Re}\left((i+\delta)\,z^2\right)}{4\delta}}\,e^{-\frac{\pi\alpha}{4\delta}}\,\exp\left(\frac{\sqrt{\pi\alpha}}{2\delta}\,\left|\mathrm{Re}\left(\sqrt{i+\delta}\,z\right)\right|\right)+$$
$$+\,540\,\delta^{-\frac{\alpha}{2}}\,e^{-\frac{\mathrm{Re}\left((i+\delta)\,z^2\right)}{4\delta}}\,\sum_{n=1}^{\infty}(2n)^{\alpha/2}\,e^{-\frac{\pi}{\delta}\,(n+\frac{\alpha}{4})}\,\exp\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}}{\delta}\,\left|\mathrm{Re}\left(\sqrt{i+\delta}\,z\right)\right|\right).$$

Thus, to prove (3.20) it is sufficient to show that, for every fixed $n\in\mathbb{N}_0$ and $z\in\mathscr{D}_\alpha$ (see condition (3.19)),

$$\lim_{\delta\to 0^+}\,\delta^{-\frac{\alpha}{2}}\,\exp\left(-\frac{\pi}{\delta}\,\left(n+\frac{\alpha}{4}\right)+\frac{\sqrt{\pi(n+\frac{\alpha}{4})}}{\delta}\,\left|\mathrm{Re}\left(\sqrt{i+\delta}\,z\right)\right|-\frac{\mathrm{Re}\left((i+\delta)\,z^2\right)}{4\delta}\right)=0. \tag{3.25}$$

Using the fact that $\mathrm{Re}\left(\sqrt{i+\delta}\,z\right)\to\mathrm{Re}(\sqrt{i}\,z)=\frac{1}{\sqrt{2}}\left(\mathrm{Re}(z)-\mathrm{Im}(z)\right)$ as $\delta\to 0^+$ and $\mathrm{Re}\left((i+\delta)\,z^2\right)\to -2\,\mathrm{Re}(z)\,\mathrm{Im}(z)$ also in this limit, we get

$$\lim_{\delta\to 0^+}\,\delta^{-\frac{\alpha}{2}}\,e^{-\frac{\mathrm{Re}\left((i+\delta)\,z^2\right)}{4\delta}}\,e^{-\frac{\pi}{\delta}\,(n+\frac{\alpha}{4})}\,\exp\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}}{\delta}\,\left|\mathrm{Re}\left(\sqrt{i+\delta}\,z\right)\right|\right)$$
$$=\lim_{\delta\to 0^+}\,\delta^{-\frac{\alpha}{2}}\,\exp\left[-\frac{\pi}{\delta}\left(\left(n+\frac{\alpha}{4}\right)-\frac{|\mathrm{Re}(z)-\mathrm{Im}(z)|}{\sqrt{2\pi}}\sqrt{n+\frac{\alpha}{4}}-\frac{\mathrm{Re}(z)\,\mathrm{Im}(z)}{2\pi}\right)\right]. \tag{3.26}$$

[6]The analyticity of $\psi_\alpha(x,z)$ is, of course, a direct consequence of the more general result given in Corollary 2.2.

Note that the term in the exponential is a quadratic polynomial of variable $X = \sqrt{n+\frac{\alpha}{4}}$, with the polynomial being explicitly given by

$$P(X) = X^2 - \frac{|\mathrm{Re}(z) - \mathrm{Im}(z)|}{\sqrt{2\pi}} X - \frac{\mathrm{Re}(z)\,\mathrm{Im}(z)}{2\pi}. \tag{3.27}$$

We know that, for a given $X_0 \in \mathbb{R}_{>0}$, $P(X_0) > 0$ if $X_0 > \sup\{y \in \mathbb{R} : P(y) = 0\}$. It is an easy task to compute the zeros of the polynomial $P(X)$, so the previous condition on X_0 reduces to:

$$P(X_0) > 0 \ \text{ if } \ \frac{|\mathrm{Re}(z) - \mathrm{Im}(z)|}{2\sqrt{2\pi}} + \frac{|\mathrm{Re}(z) + \mathrm{Im}(z)|}{2\sqrt{2\pi}} < X_0. \tag{3.28}$$

Since the polynomial in the exponential is defined for $X = \sqrt{n+\frac{\alpha}{4}}$, $n \in \mathbb{N}_0$, a sufficient condition to assure that $P\left(\sqrt{n+\frac{\alpha}{4}}\right) > 0$ for every $n \in \mathbb{N}_0$ is that z must satisfy

$$|\mathrm{Re}(z) - \mathrm{Im}(z)| + |\mathrm{Re}(z) + \mathrm{Im}(z)| < \sqrt{2\pi\alpha}, \tag{3.29}$$

which actually is the case if $z \in \mathscr{D}_\alpha = \left\{z \in \mathbb{C} : |\mathrm{Re}(z)| < \sqrt{\frac{\pi\alpha}{2}},\ |\mathrm{Im}(z)| < \sqrt{\frac{\pi\alpha}{2}}\right\}$. Therefore, under this hypothesis one sees that the limit in (3.26) goes to zero as $\delta \to 0^+$. Hence,

$$1 + \psi_\alpha\left(e^{2i\omega}, z\right) \to 0 \quad \text{as} \quad \omega \to \frac{\pi}{4}^- \quad \text{and} \quad z \in \mathscr{D}_\alpha.$$

Finally, since $\exp(-A/\delta)$, $A > 0$, tends to zero faster than any power δ^N as $\delta \to 0^+$, any derivative (with respect to δ) of $\psi_\alpha(i+\delta, z)$ will go to zero in this limit, proving (3.20) for every $m \geq 1$. This concludes the proof of the lemma for the case where $\alpha > 1$.

Assume now that $0 < \alpha \leq 1$: then the same argument must follow but we cannot invoke the integral representation (3.23) for $I_{\frac{\alpha}{2}-1}(z)$. Just like in (3.25) we need to evaluate the limit

$$\lim_{\delta \to 0^+} \left| 2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right) \left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}} \frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\delta}\, e^{-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right)} I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})\,(i+\delta)}\,z}{\delta}\right)\right| \tag{3.30}$$

for every fixed $n \in \mathbb{N}_0$. This can be done by appealing to the well-known Hankel expansion of the modified Bessel function (see [[44], p. 255, eq. (10.40.5)] and (1.51) above),

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \frac{\cos(\pi\nu)}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+\nu+n\right)\,\Gamma\left(\frac{1}{2}-\nu+n\right)}{(2z)^n n!} + \frac{e^{-z+(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}} \frac{\cos(\pi\nu)}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma\left(\frac{1}{2}+\nu+n\right)\,\Gamma\left(\frac{1}{2}-\nu+n\right)}{(2z)^n n!}, \tag{3.31}$$

which is valid for $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$, $|z| \to \infty$. Alternatively, in the range $-\frac{3}{2}\pi < \arg(z) < \frac{\pi}{2}$, $|z| \to \infty$, we have the expansion

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \frac{\cos(\pi\nu)}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+\nu+n\right)\Gamma\left(\frac{1}{2}-\nu+n\right)}{(2z)^n\, n!} + \frac{e^{-z-(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}} \frac{\cos(\pi\nu)}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma\left(\frac{1}{2}+\nu+n\right)\,\Gamma\left(\frac{1}{2}-\nu+n\right)}{(2z)^n n!}. \tag{3.32}$$

In any case, from (3.31) and (3.32) and real ν, one can obtain the simple bound

$$|I_\nu(z)| \leq 2\left(\frac{e^{\mathrm{Re}(z)}}{\sqrt{2\pi|z|}} + \frac{e^{-\mathrm{Re}(z)}}{\sqrt{2\pi|z|}}\right) \leq \sqrt{\frac{8}{\pi|z|}}\,\exp\left(|\mathrm{Re}(z)|\right) \tag{3.33}$$

which becomes valid for every $|z| > M$, where M is sufficiently large.

We now apply (3.33) to (3.30) and we obtain, for $\delta > 0$ sufficiently small, some fixed $n \in \mathbb{N}_0$ and every $z \in \mathscr{D}_\alpha$,

$$\left|2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\delta}\,e^{-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right)}I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})\,(i+\delta)}\,z}{\delta}\right)\right|$$

$$< \frac{\pi^{-\frac{\alpha+1}{4}}\,2^{\frac{\alpha}{2}+1}\,\Gamma\left(\frac{\alpha}{2}\right)\,|z|^{\frac{1-\alpha}{2}}}{\left(n+\frac{\alpha}{4}\right)^{1/4}\sqrt{\delta}}\,\exp\left(-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right)+\frac{\sqrt{\pi(n+\frac{\alpha}{4})}}{\delta}\left|\mathrm{Re}\left(\sqrt{i+\delta}\,z\right)\right|-\frac{\mathrm{Re}((i+\delta)z^2)}{4\delta}\right).$$

Note that there is no problem if z is arbitrarily close to the origin because we are supposing that $0 < \alpha \leq 1$. Thus, to prove that (3.30) tends to zero, we just follow the same logic as before, because the term in the exponential is exactly the same as in the previous case (3.26) and it involves the quadratic polynomial (3.27). Thus, the condition $z \in \mathscr{D}_\alpha$ needs to be kept in this range of α and this completes the proof of (3.20). □

3.4 Some additional lemmas

With Lemma 3.4 completely established, we are ready to prove Theorem 1.1. Before doing this we provide two additional lemmas, one of which being an easy consequence of Lemma 3.4.

Lemma 3.5. *Let $h : \mathbb{C} \longmapsto \mathbb{C}$ be an analytic function. Then, for every $z \in \mathscr{D}_\alpha$ (3.19) and arbitrary $m \in \mathbb{N}_0$, we have*

$$\lim_{\omega\to\frac{\pi}{4}^-}\frac{d^m}{d\omega^m}\left\{h(\omega)\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\} = -2\,h^{(m)}\left(\frac{\pi}{4}\right)\sinh\left(\frac{z^2}{8}\right). \tag{3.34}$$

Proof. Applying the Leibniz formula for the derivative, we obtain

$$\frac{d^m}{d\omega^m}\left\{h(\omega)\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\} = \sum_{k=0}^{m}\binom{m}{k}\frac{d^{m-k}}{d\omega^{m-k}}h(\omega)\,\frac{d^k}{d\omega^k}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right). \tag{3.35}$$

Thus, letting $\omega \to \frac{\pi}{4}^-$, we see from lemma 3.4 that any derivative of order $k \geq 1$ of $\psi_\alpha\left(e^{2i\omega},z\right)$ tends to zero as $\omega \to \frac{\pi}{4}^-$. Thus, the only surviving term is when $k=0$, which immediately gives (3.34). □

Since our proof will follow very closely the argument in [22], we will also require Kronecker's lemma [34].

Lemma 3.6. *If $x \notin \mathbb{Q}$, then the sequence of fractional parts $(\{n\,x\})_{n\in\mathbb{N}}$ is dense in the interval* $(0,1)$.

3.5 Proof of Theorem 1.1

The proof that follows uses the previous lemmas and integral representations, but the main ideas are due to Hardy and to a variation of his argument given by Dixit et al. in the papers [22, 23].

We start by using the integral representation given in Example 2.1 (2.60), replacing there x by $e^{2i\omega}$, $-\frac{\pi}{4} < \omega < \frac{\pi}{4}$. We obtain the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}+it;\,\frac{\alpha}{2};\,-\frac{z^2}{4}\right)\,e^{2\omega t}\,dt = e^{\frac{i\omega\alpha}{2}}\,\psi_\alpha\left(e^{2i\omega},z\right)-e^{-\frac{i\omega\alpha}{2}}\,e^{-z^2/4}.$$

We can use now Kummer's formula (2.8) on the integrand to obtain

$$\frac{e^{-z^2/8}}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt=e^{\frac{i\omega\alpha}{2}}\,e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)-e^{-\frac{i\omega\alpha}{2}}e^{-z^2/8}. \tag{3.36}$$

Now if we add and subtract the term $e^{-z^2/8}\,e^{i\omega\alpha/2}$ on the right-hand side of the previous equality, we rewrite (3.36) in the form

$$\frac{e^{-z^2/8}}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt=-2\,e^{-z^2/8}\,\cos\left(\frac{\omega\alpha}{2}\right)+e^{\frac{i\omega\alpha}{2}}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right). \tag{3.37}$$

After having an identity like (3.37), the structure of our proof follows [22] almost line by line. Since we want to study the zeros of an infinite combination of $\eta_\alpha(s+i\lambda_j)\,{}_1F_1(s+i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4})$, we change the variable to get

$$\begin{aligned}
&\frac{e^{-z^2/8}}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+i(t+\lambda_j)\right)\,{}_1F_1\left(\frac{\alpha}{4}-i(t+\lambda_j);\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt=\\
&=\frac{e^{-z^2/8}e^{-2\omega\lambda_j}}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt=\\
&=e^{-2\omega\lambda_j}\left\{-2\,e^{-z^2/8}\,\cos\left(\frac{\omega\alpha}{2}\right)+e^{\frac{i\omega\alpha}{2}}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}=\\
&=-\,e^{-z^2/8}\,e^{-2\omega\lambda_j+i\,\frac{\omega\alpha}{2}}-\,e^{-z^2/8}\,e^{-2\omega\lambda_j-i\,\frac{\omega\alpha}{2}}+e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right).
\end{aligned} \tag{3.38}$$

Proceeding now as in Hardy's own proof [32], let us take the $p-$th derivative of both sides of (3.38) with respect to ω: since the integral on the left of (3.38) converges absolutely and uniformly with respect of $\omega\in(-\pi/4,\,\pi/4)$, one can apply the Leibniz rule and deduce the equality:

$$\begin{aligned}
&\frac{2^p\,e^{-z^2/8}}{2\pi}\int_{-\infty}^{\infty}t^p\eta_\alpha\left(\frac{\alpha}{4}+i\,(t+\lambda_j)\right)\,{}_1F_1\left(\frac{\alpha}{4}-i\,(t+\lambda_j);\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt=\\
&=-\,e^{-z^2/8}\left(-2\lambda_j+i\frac{\alpha}{2}\right)^p e^{-2\omega\lambda_j+i\,\frac{\omega\alpha}{2}}-\,e^{-z^2/8}\left(-2\lambda_j-i\frac{\alpha}{2}\right)^p e^{-2\omega\lambda_j-i\,\frac{\omega\alpha}{2}}+\\
&\qquad+\frac{d^p}{d\omega^p}\left\{e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}.
\end{aligned} \tag{3.39}$$

We now consider the change of variables,

$$\frac{i\alpha}{2}-2\lambda_j:=r_j\,e^{i\theta_j},\;\;0<\theta_j<\frac{\pi}{2}, \tag{3.40}$$

and, using these new coordinates, write (3.39) in a more suitable form

$$\begin{aligned}
&\int_{-\infty}^{\infty}t^p\eta_\alpha\left(\frac{\alpha}{4}+i\,(t+\lambda_j)\right)\,{}_1F_1\left(\frac{\alpha}{4}-i\,(t+\lambda_j);\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt=\\
&=-4\pi\,\left(\frac{r_j}{2}\right)^p\,e^{-2\omega\lambda_j}\,\cos\left(p\,\theta_j+\frac{\omega\alpha}{2}\right)+\frac{2\pi}{2^p}\,e^{z^2/8}\,\frac{d^p}{d\omega^p}\left\{e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}.
\end{aligned} \tag{3.41}$$

To apply Hardy's method, we need to have a real integrand on the left-hand side of (3.41) in order to count the changes of its sign. This is done if we sum another integral

$$\int_{-\infty}^{\infty} t^p \eta_\alpha\left(\frac{\alpha}{4}+i\left(t+\lambda_j\right)\right)\,{}_1F_1\left(\frac{\alpha}{4}+i\left(t+\lambda_j\right);\,\frac{\alpha}{2};\,\frac{\overline{z}^2}{4}\right)\,e^{2\omega t}\,dt,$$

whose integrand is the complex conjugate of the integrand in (3.41).

Taking the real part in both sides of (3.41), multiplying by $c_j \in \ell^1$ and summing over j, we can formally derive:

$$\int_{-\infty}^{\infty} t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\,e^{2\omega t}\,dt=-\frac{8\pi}{2^p}\sum_{j=1}^{\infty}c_j\,r_j^p e^{-2\omega\lambda_j}\,\left[\cos\left(p\,\theta_j+\frac{\omega\alpha}{2}\right)\right]+$$
$$+\frac{4\pi}{2^p}\,\mathrm{Re}\left(e^{z^2/8}\,\frac{d^p}{d\omega^p}\left\{\sum_{j=1}^{\infty}c_j e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}\right),\tag{3.42}$$

where $F_{z,\alpha}(s)$ is the function given by (1.10),

$$F_{z,\alpha}(s)=\sum_{j=1}^{\infty}c_j\,\eta_\alpha\left(s+i\lambda_j\right)\,\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)+\,{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\frac{\alpha}{2};\,\frac{\overline{z}^2}{4}\right)\right\},$$

whose zeros we will now study.

But before this, we justify (3.42) by following a similar reasoning to that given in [[22], p. 316]. By Stirling's formula and convex estimates for $\zeta_\alpha(s)$ (obtained from the general Phragmén-Lindelöf principle (1.33)), we know that

$$\left|\eta_\alpha\left(\frac{\alpha}{4}+it\right)\right|\ll_\alpha |t|^{A(\alpha)}\,e^{-\frac{\pi}{2}|t|},\quad |t|\to\infty.$$

This, together with the asymptotic estimate for Kummer's function (2.6),

$$\left|{}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right|\ll_{\alpha,z}\,|t|^{\frac{1}{4}-\frac{\alpha}{4}}\exp\left(|z|\sqrt{|t|}\right),\quad |t|\to\infty,$$

yields

$$\sum_{j=1}^{\infty}\left|c_j\,\eta_\alpha\left(\frac{\alpha}{4}+i\left(t+\lambda_j\right)\right)\mathrm{Re}\left(\,{}_1F_1\left(\frac{\alpha}{4}-i\left(t+\lambda_j\right);\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right)\right|\ll_{\alpha,z}C_\lambda\,|t|^{B(\alpha)}\,e^{-\frac{\pi}{2}|t|+|z|\sqrt{|t|}}\sum_{j=1}^{\infty}|c_j|<\infty\tag{3.43}$$

where we have used the fact that $(\lambda_j)_{j\in\mathbb{N}}$ is a bounded sequence and $(c_j)_{j\in\mathbb{N}}\in\ell^1$. The term C_λ only stands for a positive constant which depends on the bounds of the sequence $(\lambda_j)_{j\in\mathbb{N}}$. This observation now allows the interchange of the orders of integration and summation in the process leading to the left-hand side of (3.42).

The very same reasoning can be used to show the procedure

$$\sum_{j=1}^{\infty}c_j\,\frac{d^p}{d\omega^p}\left\{e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}=\frac{d^p}{d\omega^p}\left\{\sum_{j=1}^{\infty}c_j e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\},$$

used on the right-hand side of (3.42).

The strategy now is to let $\omega \to \frac{\pi}{4}^-$ on both sides of (3.42). The previous lemmas are indispensable to study the behavior of the right-hand side under this limit. Since $(c_j)_{j\in\mathbb{N}} \in \ell^1$, it is clear that the function

$$h_\alpha(\omega) := \sum_{j=1}^{\infty} c_j e^{\frac{i\omega\alpha}{2} - 2\omega\lambda_j}$$

is analytic for every $\omega \in \mathbb{C}$. Hence, according to Lemma 3.5,

$$\lim_{\omega\to\frac{\pi}{4}^-} \frac{d^p}{d\omega^p}\left\{\sum_{j=1}^{\infty} c_j e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\} = \lim_{\omega\to\frac{\pi}{4}^-} \frac{d^p}{d\omega^p}\left\{h_\alpha(\omega)\,\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}$$

$$= -2\sinh\left(\frac{z^2}{8}\right)\sum_{j=1}^{\infty} c_j\left(\frac{i\alpha}{2}-2\lambda_j\right)^p e^{\frac{i\pi\alpha}{8}-\frac{\pi}{2}\lambda_j} = -2\sinh\left(\frac{z^2}{8}\right)\sum_{j=1}^{\infty} c_j\, r_j^p\, e^{-\frac{\pi}{2}\lambda_j}\, e^{i\left(\frac{\pi\alpha}{8}+p\,\theta_j\right)}, \qquad (3.44)$$

where in the last step we have just considered the coordinates (3.40).

Returning to the identity (3.42), we get from (3.44),

$$\lim_{\omega\to\frac{\pi}{4}^-}\int_{-\infty}^{\infty} t^p F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{2\omega t}\,dt = -\frac{8\pi}{2^p}\sum_{j=1}^{\infty} c_j r_j^p\, e^{-\frac{\pi}{2}\lambda_j}\left[\cos\left(p\,\theta_j+\frac{\pi}{8}\alpha\right)+\text{Re}\left(e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\,e^{i\left(\frac{\pi\alpha}{8}+p\,\theta_j\right)}\right)\right]$$

$$= -\frac{8\pi}{2^p}\sum_{j=1}^{\infty} c_j r_j^p e^{-\frac{\pi}{2}\lambda_j}\left[\cos\left(p\,\theta_j+\frac{\pi}{8}\alpha\right)\left(1+\text{Re}\left(e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\right)-\sin\left(p\theta_j+\frac{\pi}{8}\alpha\right)\,\text{Im}\left(e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\right]. \qquad (3.45)$$

Now, if we once more follow [22] and set

$$u_z := 1+\text{Re}\left(e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right), \quad v_z := \text{Im}\left(e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right), \qquad (3.46)$$

we are able to write the right-hand side of (3.45) in the approriate form

$$-\frac{8\pi}{2^p}\,w_z\sum_{j=1}^{\infty} c_j r_j^p\, e^{-\frac{\pi}{2}\lambda_j}\,\cos\left(p\theta_j+\frac{\pi\alpha}{8}+\beta_z\right), \text{ for } w_z=\sqrt{u_z^2+v_z^2},\ \ \beta_z := \arctan\left(\frac{v_z}{u_z}\right). \qquad (3.47)$$

Since all the parameters in the previous expression are fixed either by the Dirichlet series (α) or by the sequence taken in the statement $(\theta_j)_{j\in\mathbb{N}}$, the only free parameter is the integer p, which is the number of times we have differentiated both sides of (3.38). Therefore, the previous expression can be thought of as a sequence of real numbers, say $(s_p)_{p\in\mathbb{N}}$.

Under only some minor modifications of the argument given in [[22], pp. 317-321], we now show that the sequence $(s_p)_{p\in\mathbb{N}}$ defined by (3.47) changes its sign for infinitely many values of p.

Since $(\lambda_j)_{j\in\mathbb{N}}$ attains its bounds and is made of distinct elements, we know that there is some positive integer M such that

$$|\lambda_M| = \max_{j\in\mathbb{N}}\{|\lambda_j|\}, \quad \lambda_j \neq \lambda_M \text{ for } j\neq M \implies r_j < r_M \text{ for } j\neq M, \qquad (3.48)$$

where we have used the change of coordinates (3.40). Observe now that we can write the series in (3.47) as

$$-\frac{8\pi}{2^p}\, w_z\, c_M r_M^p\, e^{-\frac{\pi}{2}\lambda_M}\, \cos\left(p\theta_M + \frac{\pi\alpha}{8} + \beta_z\right) \left\{1 + E(X,z) + H(X,z)\right\}, \tag{3.49}$$

where

$$E(X,z) := \sum_{j\neq M,\, j\leq X} \frac{c_j}{c_M}\, e^{-\frac{\pi}{2}(\lambda_j-\lambda_M)} \left(\frac{r_j}{r_M}\right)^p \frac{\cos\left(p\theta_j + \frac{\pi\alpha}{8} + \beta_z\right)}{\cos\left(p\theta_M + \frac{\pi\alpha}{8} + \beta_z\right)}$$

and

$$H(X,z) := \sum_{j\neq M,\, j> X} \frac{c_j}{c_M}\, e^{-\frac{\pi}{2}(\lambda_j-\lambda_M)} \left(\frac{r_j}{r_M}\right)^p \frac{\cos\left(p\theta_j + \frac{\pi\alpha}{8} + \beta_z\right)}{\cos\left(p\theta_M + \frac{\pi\alpha}{8} + \beta_z\right)},$$

where X is some large parameter that will be chosen later.

We will now see that, as a function of p, the sign of $\cos\left(p\theta_M + \frac{\pi\alpha}{8} + \beta_z\right)$ will prevail in (3.49), making the sign of the integral on the left of (3.45) to change infinitely often.

We divide the proof of this fact in two items: in the first item, we show it for $\frac{\theta_M}{2\pi} \notin \mathbb{Q}$ and in the second we argue that the case $\frac{\theta_M}{2\pi} \in \mathbb{Q}$ ultimately reduces to the former.

1. Assume that $\frac{\theta_M}{2\pi} \notin \mathbb{Q}$: by Kronecker's lemma (see Lemma 3.6), $\left(\left\{n\,\frac{\theta_M}{2\pi}\right\}\right)_{n\in\mathbb{N}}$ is a dense subset of $(0,1)$. Hence,

$$\left(\cos\left(n\,\theta_M + \frac{\pi\alpha}{8} + \beta_z\right)\right)_{n\in\mathbb{N}} = \left(\cos\left(2\pi\left\{\frac{n\,\theta_M}{2\pi}\right\} + \frac{\pi\alpha}{8} + \beta_z\right)\right)_{n\in\mathbb{N}}$$

is a dense subset of $[-1,1]$ due to continuity and surjectivity of the function $f(X) = \cos(X+a)$. Therefore, we can find two sequences of integers $(q_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ such that

$$\cos\left(2\pi\left\{\frac{q_n\,\theta_M}{2\pi}\right\} + \frac{\pi\alpha}{8} + \beta_z\right) \to \frac{1}{2}, \quad \cos\left(2\pi\left\{\frac{r_n\,\theta_M}{2\pi}\right\} + \frac{\pi\alpha}{8} + \beta_z\right) \to -\frac{1}{2}, \quad \text{as } n\to\infty. \tag{3.50}$$

Thus, for a sufficiently large N_0 and $p \in (q_n)_{n\geq N_0} \cup (r_n)_{n\geq N_0}$, we have the inequality

$$\left|\cos\left(p\theta_M + \frac{\pi\alpha}{8} + \beta_z\right)\right| \geq \frac{1}{3}.$$

Since $(\lambda_k)_{k\in\mathbb{N}}$ is bounded and attains its bounds, under this choice of N_0 we have the bound for $H(X,z)$,

$$|H(X,z)| \leq \frac{3}{|c_M|}\, e^{\frac{\pi}{2}\max_{k,\ell}|\lambda_k-\lambda_\ell|} \sum_{j\neq M,\, j>X} |c_j| < \frac{1}{3},$$

once we take $X \geq X_0$ large enough. With the same choice of X, we bound $E(X,z)$ as follows

$$|E(X,z)| \leq \frac{3\mu_X^p}{|c_M|}\, e^{\frac{\pi}{2}\max_{k,\ell}|\lambda_k-\lambda_\ell|} \sum_{j\neq M,\, j\leq X} |c_j|,$$

where

$$\mu_X = \max_{j\leq X}\left\{\frac{r_j}{r_M}\right\}.$$

By (3.48), we know that $\mu_X < 1$ and so, for N_0 sufficiently large and $p \geq \max\{q_{N_0}, r_{N_0}\}$,

$$|E(X,z)| < \frac{1}{3}.$$

Therefore, for our choice of N_0 and X, $1+E(X,z)+H(X,z)>\frac{1}{3}$, and so, for $p\in (q_n)_{n\geq N_0}\cup (r_n)_{n\geq N_0}$ and from (3.45), (3.49), the limit

$$\lim_{\omega\to\frac{\pi}{4}^-}\int_{-\infty}^{\infty} t^p F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{2\omega t}\,dt$$

must have the same sign (as a sequence depending on $p\in (q_n)_{n\in\mathbb{N}}\cup (r_n)_{n\in\mathbb{N}})$ as the sequence

$$\left(s_p'\right)_{p\in (q_n)\cup(r_n)_{n\in\mathbb{N}}} := -\frac{8\pi}{2^p}\, w_z\, c_M r_M^p\, e^{-\frac{\pi}{2}\lambda_M}\, \cos\left(p\theta_M+\frac{\pi\alpha}{8}+\beta_z\right),$$

whose sign changes infinitely often by the construction (3.50).

2. We now deal with the case where $\frac{\theta_M}{2\pi}\in\mathbb{Q}$ making the same remark as in [[22], pp. 320-321]. Note that, for a fixed small $\delta>0$, Theorem 1.1 is equivalent to the statement that $F_{z,\alpha}(s+i\delta)$ has infinitely many zeros at the line $\text{Re}(s)=\frac{\alpha}{4}$. However, throughout the course of the previous argument we just need to replace the elements of the sequence $(\lambda_j)_{j\in\mathbb{N}}$ by $\lambda_j^\star:=\lambda_j+\delta$. But now in this new analysis we need to rewrite (3.40) as $\frac{i\alpha}{2}-2\lambda_j^\star=r_j^\star\, e^{i\theta_j^\star}$. Since we have the freedom of choosing δ, we may take it in such a way that $\frac{\theta_M^\star}{2\pi}\notin\mathbb{Q}$, due to density of the irrational numbers in the real line. Therefore, if we prove Theorem 1.1 under the hypothesis $\frac{\theta_M}{2\pi}\notin\mathbb{Q}$, we can easily see that the same strategy will work for $\frac{\theta_M}{2\pi}\in\mathbb{Q}$.

We are ready to finish the proof of Theorem 1.1. By contradiction, let us assume that $F_{z,\alpha}(s)$ has only a finite number of zeros located on the line $\text{Re}(s)=\frac{\alpha}{4}$.

Under this hypothesis, there exists some $T_0>0$ such that one of the following situations takes place:

1. $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)>0$ for $|t|>T_0$;
2. $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)<0$ for $|t|>T_0$;
3. $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)>0$ for $t>T_0$ and $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)<0$ for $t<-T_0$;
4. $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)<0$ for $t>T_0$ and $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)>0$ for $t<-T_0$.

But the second contradiction hypothesis easily reduces to the first if we carry out the proof with $(c_j)_{j\in\mathbb{N}}$ replaced by $(-c_j)_{j\in\mathbb{N}}$. Analogously, the fourth hypothesis reduces to the third. Henceforth, for the sake of completing the proof, we just need to deal with cases 1. and 3. above.

By hypotheses 1. and 3., we know that $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)>0$ for $t>T_0$: this implies the inequality,

$$\lim_{\omega\to\frac{\pi}{4}^-}\int_{T_0}^{T} t^p F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{2\omega t}\,dt \leq \lim_{\omega\to\frac{\pi}{4}^-}\int_{T_0}^{\infty} t^p F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{2\omega t}\,dt := L_{z,p}(T_0), \tag{3.51}$$

where $L_{z,p}(T_0)$ exists and is finite, according to (3.45). Therefore, for every $T\geq T_0$, the following inequality holds

$$\int_{T_0}^{T} t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t}\,dt \leq L_{z,p}\left(T_0\right). \tag{3.52}$$

But since $T \geq T_0$ is arbitrary, (3.52) implies

$$\int_{T_0}^{\infty} t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t}\, dt \leq L_{z,p}\left(T_0\right). \tag{3.53}$$

Using (3.53) and one of the contradiction hypotheses 1. or 3., we now prove

$$\int_{-\infty}^{\infty} t^p\, \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|\, e^{\frac{\pi}{2}t}\, dt < \infty. \tag{3.54}$$

In fact, since the hypotheses 1. and 3. imply that $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) = \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|$ for $t > T_0$ and $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) = \pm\left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|$ for[7] $t < -T_0$,

$$\begin{aligned}
\int_{-\infty}^{\infty} t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t}dt &= \int_{0}^{\infty}\left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\, e^{-\frac{\pi}{2}t}\right\}\, t^p dt \\
&= \int_{0}^{T_0}\left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\, e^{-\frac{\pi}{2}t}\right\}\, t^p dt \\
&+ \int_{T_0}^{\infty}\left\{\left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|\, e^{\frac{\pi}{2}t} \pm (-1)^p\, \left|F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\right|\, e^{-\frac{\pi}{2}t}\right\}\, t^p dt.
\end{aligned} \tag{3.55}$$

If $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) > 0$ for $t < -T_0$ (condition 1. above) we take p as an even integer and if $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) < 0$ for $t < -T_0$ (condition 3. above) we assume p to be odd. In any case, the second integral in (3.55) reduces to

$$\int_{T_0}^{\infty}\left\{\left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|\, e^{\frac{\pi}{2}t} + \left|F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\right|\, e^{-\frac{\pi}{2}t}\right\}\, t^p dt = \int_{-\infty}^{\infty}|t|^p\left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|\, e^{\frac{\pi}{2}t}dt - \int_{-T_0}^{T_0}|t|^p\, \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|\, e^{\frac{\pi}{2}t}dt, \tag{3.56}$$

and so, combining (3.55) and (3.56),

$$\begin{aligned}
\int_{-\infty}^{\infty}|t|^p\left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|\, e^{\frac{\pi}{2}t}dt &= \int_{-\infty}^{\infty} t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t}dt + \int_{-T_0}^{T_0}|t|^p\, \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|\, e^{\frac{\pi}{2}t}dt \\
&- \int_{0}^{T_0}\left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\, e^{-\frac{\pi}{2}t}\right\}\, t^p dt < \infty,
\end{aligned} \tag{3.57}$$

establishing (3.54).

We are now free to use the dominated convergence Theorem: by (3.45), (3.49) and (3.54),

$$\begin{aligned}
&\int_{0}^{\infty}\left\{t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t} + (-1)^p\, t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\, e^{-\frac{\pi}{2}t}\right\}\, t^p\, dt \\
&= -\frac{8\pi}{2^p}\, w_z\, c_M r_M^p\, e^{\frac{\pi}{2}\lambda_M}\, \cos\left(p\theta_M + \frac{\pi\alpha}{8} + \beta_z\right)\, \left\{1 + E(X,z) + H(X,z)\right\}.
\end{aligned} \tag{3.58}$$

[7]here, the + sign is under hypothesis 1. while the − sign is under hypothesis 3.

But we have already seen that there are infinitely many integers $p \in (q_n)_{n\geq N_0} \cup (r_n)_{n\geq N_0}$ for which the right-hand side of (3.58) is negative, this is, there are infinitely many p such that

$$\int_{T_0}^{\infty} \left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\, e^{-\frac{\pi}{2}t}\right\} t^p dt < -\int_{0}^{T_0} \left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\, e^{-\frac{\pi}{2}t}\right\} t^p dt <$$

$$< T_0^p \int_{0}^{T_0} \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\, e^{-\frac{\pi}{2}t}\right| dt \leq K(T_0)\, T_0^p, \tag{3.59}$$

where K only depends on T_0 and not on p. By our hypotheses (1. or 3.) over $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$ (and for the choice of p depending whether 1. or 3. takes place), we know that there exists some $\epsilon := \epsilon(T_0) > 0$ such that

$$F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\, e^{-\frac{\pi}{2}t} = \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|\, e^{\frac{\pi}{2}t} + \left|F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\right|\, e^{-\frac{\pi}{2}t}$$
$$\geq \epsilon(T_0), \quad \forall t \in [2T_0,\, 2T_0+1]\,.$$

This proves the lower bound:

$$\int_{T_0}^{\infty} \left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\, e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\, e^{-\frac{\pi}{2}t}\right\} t^p dt >$$

$$> \int_{2T_0}^{2T_0+1} \left\{\left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|\, e^{\frac{\pi}{2}t} + \left|F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\right|\, e^{-\frac{\pi}{2}t}\right\} t^p dt \geq \epsilon\,(T_0)\, (2T_0)^p\,. \tag{3.60}$$

Comparing (3.59) with (3.60), we conclude that the inequality

$$2^p \leq \frac{K(T_0)}{\epsilon(T_0)}$$

must hold for infinitely many values of $p \in (p_n)_{n\geq N_0} \cup (q_n)_{n\geq N_0}$. This is absurd because we can take p as large as desired. ■

4 Zeros of combinations attached to $\zeta(s,Q)$

In this section we apply all the previous work to prove a version of Theorem 1.1 where $\zeta_\alpha(s)$ is replaced by the Epstein zeta function attached to a binary, integral and positive definite quadratic form. To adapt the work done in the proof of Theorem 1.1, as well as to apply the Summation formula (2.9), we need some lemmas concerning twists of the Epstein zeta function by an additive character $e^{2\pi inp/q}$, $p/q \in \mathbb{Q}$.

The first lemma given in the next subsection is essentially given in [[17], Theorems 2 and 3] and was used by Jutila to derive exponential sums attached to Epstein zeta functions in [36].

However, to match the notation introduced in our paper and to be easier to apply the next result to our summation formula, we will give a brief proof of it.

4.1 Exponential sums attached to Quadratic forms

Lemma 4.1. *Let $(p,\,q)=1$ and $Q(m,\,n)=Am^2+Bm\,n+Cn^2$ be a binary, positive definite and integral quadratic form with discriminant $\Delta:=4AC-B^2$. Consider the periodic Epstein zeta function,*

$$\zeta\left(s,\,Q,\,\frac{p}{q}\right):=\sum_{m,n\neq 0}\frac{\exp\left(-\frac{2\pi ip}{q}\left(Am^2+Bm\,n+Cn^2\right)\right)}{\left(Am^2+Bm\,n+Cn^2\right)^s},\quad Re(s)>1. \tag{4.1}$$

Then $\zeta\left(s,\,Q,\,\frac{p}{q}\right)$ can be analytically continued (at most) as a meromorphic function with a simple pole located at $s=1$, whose residue is

$$Res_{s=1}\zeta\left(s,\,Q,\,\frac{p}{q}\right)=\frac{2\pi}{q^2\sqrt{\Delta}}\sum_{k_1,k_2=0}^{q-1}e^{-\frac{2\pi ip}{q}Q(k_1,k_2)} \tag{4.2}$$

(if the previous value is zero, $\zeta\left(s,\,Q,\,\frac{p}{q}\right)$ is entire).

Moreover, it satisfies the functional equation

$$\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-s}\Gamma(s)\,\zeta\left(s,\,Q,\,\frac{p}{q}\right)=\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-(1-s)}\Gamma\left(1-s\right)\tilde{\zeta}\left(1-s,\,Q,\,\frac{p}{q}\right), \tag{4.3}$$

where $\tilde{\zeta}(s;\,Q;\,p/q)$ represents the Dirichlet series

$$\tilde{\zeta}\left(s,\,Q,\,\frac{p}{q}\right):=\sum_{m,n\neq 0}\frac{b_Q(m,n,p/q)}{Q(m,n)^s},\quad Re(s)>1, \tag{4.4}$$

with

$$b_Q(m,n,p/q):=\frac{1}{q}\sum_{k_1,k_2=0}^{q-1}e^{-\frac{2\pi ip}{q}Q(k_1,k_2)}\,e^{\frac{2\pi i}{q}(mk_1+nk_2)}. \tag{4.5}$$

Proof. For $\text{Re}(s)>1$, we start by writing (4.1) as

$$\begin{aligned}\zeta\left(s,\,Q,\,\frac{p}{q}\right)&:=\sum_{m,n\neq 0}\frac{e^{-2\pi i\,\frac{p}{q}\,Q(m,n)}}{Q(m,n)^s}=\sum_{k_1,k_2=0}^{q-1}e^{-2\pi i\,\frac{p}{q}Q(k_1,\,k_2)}\sum_{\ell_1,\ell_2\neq 0}\frac{1}{Q\left(\ell_1q+k_1,\,\ell_2q+k_2\right)^s}\\&=q^{-2s}\sum_{k_1,k_2=0}^{q-1}e^{-2\pi i\,\frac{p}{q}Q(k_1,\,k_2)}\sum_{\ell_1,\ell_2\neq 0}\frac{1}{Q\left(\ell_1+\frac{k_1}{q},\,\ell_2+\frac{k_2}{q}\right)^s}.\end{aligned} \tag{4.6}$$

Note that the infinite series with respect to ℓ_1,ℓ_2 is well defined because the denominator cannot be zero. For $\text{Re}(s)>1$, it represents a particular example of the Epstein zeta function (1.41)

$$\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right):=\sum_{\mathbf{m}\in\mathbb{Z}^n,\,\mathbf{m}+\mathbf{g}\neq\mathbf{0}}\frac{\exp\left(2\pi i\,\mathbf{m}\cdot\mathbf{h}\right)}{Q\left(\mathbf{m}+\mathbf{g}\right)^s},\quad \text{Re}(s)>\frac{n}{2} \tag{4.7}$$

when Q is binary and $\mathbf{h}=(0,0)$. We know from Lemma 1.2 above that, for $\mathbf{h}\notin\mathbb{Z}^n$, $\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)$ can be analytically continued as an entire function. If, however, $\mathbf{h}\in\mathbb{Z}^n$, $\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)$ has an analytic continuation into the complex plane as a meromorphic function possessing a simple pole located at $s=n/2$ with residue

$$\text{Res}_{s=\frac{n}{2}}\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)=\frac{(2\pi)^{n/2}}{\sqrt{D(Q)}}\,\Gamma\left(\frac{n}{2}\right). \tag{4.8}$$

For $n=2$, we know that $D(Q)$ reduces to the discriminant $\Delta := 4AC - B^2$ of the binary quadratic form. Hence, for $\text{Re}(s)>1$, we can reduce (4.6) to

$$\zeta\left(s,\,Q,\,\frac{p}{q}\right) = q^{-2s}\sum_{k_1,k_2=0}^{q-1} e^{-2\pi i\,\frac{p}{q}Q(k_1,\,k_2)}\,\zeta\left(s,\,\frac{\mathbf{k}}{q},\,\mathbf{0},\,Q\right), \tag{4.9}$$

which proves that $\zeta\left(s,\,Q,\,\frac{p}{q}\right)$ has an analytic continuation as a meromorphic function with a simple pole located at $s=1$ with residue

$$\text{Res}_{s=1}\zeta\left(s,\,Q,\,\frac{p}{q}\right) = \frac{2\pi}{q^2\sqrt{\Delta}}\sum_{k_1,k_2=0}^{q-1} e^{-\frac{2\pi i p}{q}Q(k_1,k_2)}.$$

This proves the first part of the lemma. To prove the second part, we use the functional equation for the Epstein zeta function (1.42)

$$\left(\frac{2\pi}{D(Q)^{1/n}}\right)^{-s}\Gamma(s)\,\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right) = \exp\left(-2\pi i\,\mathbf{g}\cdot\mathbf{h}\right)\left(\frac{2\pi}{D(Q^{\dagger})^{1/n}}\right)^{s-\frac{n}{2}}\Gamma\left(\frac{n}{2}-s\right)\zeta\left(\frac{n}{2}-s,\mathbf{h},\,-\mathbf{g},\,Q^{\dagger}\right), \tag{4.10}$$

where $Q^{\dagger}$ is the adjoint quadratic form (1.40). Since $\mathbf{h}=(0,0)$ and Q is binary, $\zeta\left(s,\mathbf{0},-\frac{\mathbf{k}}{q},\,Q^{\dagger}\right)=\zeta\left(s,\mathbf{0},\frac{\mathbf{k}}{q},\,Q\right)$, so by (4.6) and the previous functional equation (4.10) with $n=2$ and $D(Q)=\Delta$, we obtain

$$\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-s}\Gamma(s)\,\zeta\left(s,\,Q,\,\frac{p}{q}\right) = \left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-(1-s)}\Gamma\left(1-s\right)\frac{1}{q}\sum_{k_1,k_2=0}^{q-1} e^{-2\pi i\,\frac{p}{q}Q(k_1,\,k_2)}\,\zeta\left(1-s,\,\mathbf{0},\,\frac{\mathbf{k}}{q},\,Q\right). \tag{4.11}$$

Now, for $\text{Re}(s)>1$, it is simple to see that

$$\frac{1}{q}\sum_{k_1,k_2=0}^{q-1} e^{-2\pi i\,\frac{p}{q}Q(k_1,\,k_2)}\,\zeta\left(s,\,\frac{\mathbf{k}}{q},\,\mathbf{0},\,Q\right) = \frac{1}{q}\sum_{k_1,k_2=0}^{q-1} e^{-2\pi i\,\frac{p}{q}Q(k_1,\,k_2)}\sum_{m,n\neq 0}\frac{e^{\frac{2\pi i}{q}(m\,k_1+n\,k_2)}}{Q\left(m,n\right)^s} =$$

$$= \sum_{m,n\neq 0}\frac{1}{Q(m,n)^s}\,\frac{1}{q}\sum_{k_1,k_2=0}^{q-1}\exp\left(-2\pi i\,\frac{p}{q}Q\left(k_1,\,k_2\right)+\frac{2\pi i}{q}\left(m\,k_1+n\,k_2\right)\right) = \sum_{m,n\neq 0}\frac{b_Q(m,n,p/q)}{Q(m,n)^s},$$

proving (4.3). □

Remark 4.1. Note that both $\zeta\left(s,\,Q,\,\frac{p}{q}\right)$ and $\tilde{\zeta}\left(s,\,Q,\,\frac{p}{q}\right)$ can be written as Dirichlet series over one variable of summation. It is not hard to see that, if $r_Q(n)$ denotes the number of representations of n by Q, then

$$\zeta\left(s,\,Q,\,\frac{p}{q}\right) = \sum_{n=1}^{\infty}\frac{r_Q(n)\,\exp\left(-\frac{2\pi i\,p}{q}n\right)}{n^s},\quad \text{Re}(s)>1 \tag{4.12}$$

and

$$\tilde{\zeta}\left(s,\,Q,\,\frac{p}{q}\right) := \sum_{n=1}^{\infty}\frac{\tilde{b}_Q(n,p/q)}{n^s},\quad \text{Re}(s)>1, \tag{4.13}$$

where

$$\tilde{b}_Q(n,p/q) := \frac{1}{q}\sum_{Q(\alpha,\beta)=n}\sum_{k_1,k_2=0}^{q-1}\exp\left(-\frac{2\pi i p}{q}Q(k_1,k_2)+\frac{2\pi i}{q}(\alpha k_1+\beta k_2)\right). \tag{4.14}$$

In the definition above we are summing over pairs of integers (α,β) such that $Q(\alpha,\beta)=n$. Note that $|\tilde{b}_Q(n,p/q)|\leq r_Q(n)$, so the absolute convergence of the Dirichlet series on the right-hand side of (4.13) is assured for $\text{Re}(s)>1$.

From the previous lemma, it is natural to apply the summation formula given in Theorem 2.1 not only to $\zeta(s,Q)$ (as we have done in Example 2.2) but also to $\zeta(s,Q,p/q)$. The next result gives one particular case of our summation formula which will be crucial in the proof of Theorem 1.2.

Lemma 4.2. *Let $Q(m,n) = Am^2 + Bm\,n + Cn^2$ be an integral and positive definite quadratic form and $\Delta := 4AC - B^2$ be its discriminant. Then, for every $Re(x) > 0$, $y \in \mathbb{C}$ and $(p,q) = 1$, the following summation formula takes place*

$$1 + \sum_{n=1}^{\infty} r_Q(n)\, exp\left(-\frac{2\pi i p}{q} n\right)\, e^{-\frac{2\pi n\, x}{q\sqrt{\Delta}}}\, J_0\left(\frac{\sqrt{2\pi n}\, y}{\sqrt{q}\Delta^{1/4}}\right)$$
$$= \frac{e^{-\frac{y^2}{4x}}}{qx} \sum_{k_1,k_2=0}^{q-1} e^{-\frac{2\pi i p}{q}Q(k_1,k_2)} + \frac{e^{-\frac{y^2}{4x}}}{x} \sum_{n=1}^{\infty} \tilde{b}_Q(n,p/q)\, e^{-\frac{2\pi n}{q\sqrt{\Delta}\, x}}\, I_0\left(\frac{\sqrt{2\pi n}\, y}{\sqrt{q}\,\Delta^{1/4}\, x}\right). \tag{4.15}$$

where $\tilde{b}_Q(n,p/q)$ is explicitly given by (4.14).

Proof. Since the functional equation (4.3) takes place and the analytic continuation of $\zeta(s,\, Q,\, p/q)$ satisfies all the properties of the class $\mathcal{A}$, we can invoke (2.9) for the pair of Dirichlet series:

$$\phi(s) = \left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-s} \sum_{n=1}^{\infty} \frac{\exp\left(-\frac{2\pi i\, p}{q} n\right)\, r_Q(n)}{n^s}, \quad \text{Re}(s) > 1$$

and

$$\psi(s) = \left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-s} \sum_{n=1}^{\infty} \frac{\tilde{b}_Q(n,p/q)}{n^s}, \quad \text{Re}(s) > 1.$$

Under this template, from (4.2) we know that the residue of $\phi(s)$ is $\rho = q^{-1} \sum_{k_1,k_2=0}^{q-1} e^{-\frac{2\pi i p}{q}Q(k_1,k_2)}$. To compute $\phi(0)$, it suffices to know $\zeta(0,Q,p/q)$. To this end, we use (4.11): multiplying both sides of this equation by s and letting $s \to 0$, we obtain

$$\zeta\left(0,\, Q,\, \frac{p}{q}\right) = \lim_{s\to 0} s\, \left(\frac{2\pi q}{\sqrt{\Delta}}\right)^{s-1} \Gamma\left(1-s\right) \frac{1}{q} \sum_{k_1,k_2=0}^{q-1} e^{-2\pi i\, \frac{p}{q} Q(k_1,\, k_2)}\, \zeta\left(1-s,\, \mathbf{0},\, \frac{\mathbf{k}}{q},\, Q\right)$$
$$= \lim_{s\to 0} s\, \left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{s-1} \Gamma\left(1-s\right) \frac{1}{q} \left\{\zeta\left(1-s,\, \mathbf{0},\, \mathbf{0},\, Q\right) + \sum_{k_1,k_2 \neq 0}^{q-1} e^{-2\pi i\, \frac{p}{q} Q(k_1,\, k_2)}\, \zeta\left(1-s,\, \mathbf{0},\, \frac{\mathbf{k}}{q},\, Q\right)\right\}.$$

If $k_1 \neq 0$ or $k_2 \neq 0$, then $\mathbf{k}/q \notin \mathbb{Z}^2$, and so each Dirichlet series on the second sum $\zeta(1-s, \mathbf{0}, \mathbf{k}/q, Q)$ is an entire function by Lemma 1.2. Therefore, the only singularity comes from $\zeta(1-s, \mathbf{0}, \mathbf{0}, Q)$ at $s = 0$, giving

$$\zeta\left(0,\, Q,\, \frac{p}{q}\right) = -\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-1} \frac{1}{q} \lim_{s\to 1} \left(s-1\right) \zeta\left(s,\, \mathbf{0},\, \mathbf{0},\, Q\right) = -1, \tag{4.16}$$

where we have used (4.8). An application of the summation formula (2.9) yields (4.15) immediately. □

4.2 The behavior of $\psi_Q(x,z)$

Using the previous lemma, we can now study the behavior of the analogue of Jacobi's $\psi-$function attached to $\zeta(s,Q)$. The following lemma essentially says the same as Lemma 3.4 in the present situation.

Lemma 4.3. *Let $Q(m,n)=Am^2+Bm\,n+Cn^2$ be a binary, integral and positive definite quadratic form and let $\Delta:=4AC-B^2$. Assume also that Q is reduced, this is, gcd$(A,\,B,\,C)=1$ and that $\sqrt{\Delta}\equiv 2 \mod 4$. Consider the generalized Jacobi's $\psi-$function attached to Q (given in Example 2.2),*

$$\psi_Q(x,z)=\sum_{n=1}^{\infty}r_Q(n)\,e^{-\frac{2\pi nx}{\sqrt{\Delta}}}\,J_0\left(\sqrt{\frac{2\pi nx}{\Delta^{1/2}}}\,z\right),\quad Re(x)>0,\ \ z\in\mathbb{C}. \tag{4.17}$$

Then, for every z satisfying the condition:

$$z\in\mathscr{D}_Q:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<\frac{2\sqrt{\pi}}{\Delta^{3/4}},\ |Im(z)|<\frac{2\sqrt{\pi}}{\Delta^{3/4}}\right\}, \tag{4.18}$$

every $m\in\mathbb{N}_0$ and every analytic function $h:\mathbb{C}\longmapsto\mathbb{C}$, one has the asymptotic formula

$$\lim_{\omega\to\frac{\pi}{4}^-}\frac{d^m}{d\omega^m}\left\{h(\omega)\,\left(e^{-z^2/8}+e^{z^2/8}\,\psi_Q\left(e^{2i\omega},z\right)\right)\right\}=-2\,h^{(m)}\left(\frac{\pi}{4}\right)\,\sinh\left(\frac{z^2}{8}\right). \tag{4.19}$$

Proof. We proceed as in the proof of lemma 3.4. By Lemma 3.5 and Corollary 2.2, it suffices to compute $\psi_Q(i+\delta,\,z)$ as $\delta\to0^+$. Note that

$$1+\psi_Q(i+\delta,z)=1+\sum_{n=1}^{\infty}r_Q(n)\,e^{-\frac{2\pi ni}{\sqrt{\Delta}}}\,e^{-\frac{2\pi n\delta}{\sqrt{\Delta}}}\,J_0\left(\sqrt{\frac{2\pi n(i+\delta)}{\Delta^{1/2}}}\,z\right).$$

Applying the previous summation formula (4.15) with $p=1$, $q=\sqrt{\Delta}$, $x=\sqrt{\Delta}\,\delta$ and $y=\sqrt{i+\delta}\,z\,\Delta^{1/4}$, we obtain

$$1\,+\,\psi_Q(i+\delta,z)=\frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\Delta\,\delta}\sum_{k_1,k_2=0}^{\sqrt{\Delta}-1}e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)}+\frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\sqrt{\Delta}\,\delta}\sum_{n=1}^{\infty}\tilde{b}_Q\left(n,1/\sqrt{\Delta}\right)\,e^{-\frac{2\pi n}{\Delta^{3/2}\delta}}\,I_0\left(\frac{\sqrt{2\pi n(i+\delta)}}{\Delta^{3/4}}\,\frac{z}{\delta}\right) \tag{4.20}$$

To have the conclusion (4.19), we need to show that the right-hand side of (4.20) tends to zero exponentially fast as $\delta\to0^+$. To do this, one needs to get rid of the residual term involving δ^{-1}.

We now show that, under the conditions above, the residual term (4.2) is zero: by hypothesis, we know that $\sqrt{\Delta}$ is an integer and that $\sqrt{\Delta}^2+B^2=4AC$. Since the sum of two odd numbers is not divisible by 4, this automatically shows that B and $\sqrt{\Delta}$ must be even.

Since $(A,\,B,\,C)=1$ and B is even, at least A or C is odd and without any loss of generality suppose that A is odd. We now evaluate the Gauss sum (4.2), restricting ourselves to the sum with respect to k_1. We will see that this sum vanishes if $\sqrt{\Delta}\equiv 2 \mod 4$. Indeed

$$\begin{aligned}\sum_{k_1,k_2=0}^{\sqrt{\Delta}-1}e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)}&=\sum_{k_2=0}^{\sqrt{\Delta}}\left\{\sum_{k_1=0}^{\sqrt{\Delta}/2-1}e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)}+\sum_{k_1=\sqrt{\Delta}/2}^{\sqrt{\Delta}-1}e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)}\right\}\\&=\sum_{k_2=0}^{\sqrt{\Delta}}\left\{\sum_{k_1=0}^{\sqrt{\Delta}/2-1}e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)}+\sum_{k_1=0}^{\sqrt{\Delta}/2-1}e^{-\frac{2\pi i}{\sqrt{\Delta}}Q\left(k_1+\frac{\sqrt{\Delta}}{2},k_2\right)}\right\}.\end{aligned}$$

However, one easily checks that

$$Q\left(k_1+\frac{\sqrt{\Delta}}{2},\,k_2\right)=Q\left(k_1,k_2\right)+\sqrt{\Delta}\left(\frac{A}{4}\sqrt{\Delta}+A\,k_1+\frac{B}{2}k_2\right), \tag{4.21}$$

and so,

$$\exp\left(-\frac{2\pi i}{\sqrt{\Delta}}Q\left(k_1+\frac{\sqrt{\Delta}}{2},\,k_2\right)\right)=\exp\left(-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)-2\pi i\left(\frac{A}{4}\sqrt{\Delta}+A\,k_1+\frac{B}{2}k_2\right)\right)$$
$$=\exp\left(-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)\right)\cdot\exp\left(-\frac{\pi i}{2}A\sqrt{\Delta}\right)=-\exp\left(-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)\right),$$

because A is odd and $\sqrt{\Delta}\equiv 2 \mod 4$. Therefore,

$$\sum_{k_1,k_2=0}^{\sqrt{\Delta}-1} e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)}=0,$$

so that our summation formula is reduced to

$$1+\tilde{\psi}_Q(i+\delta,z)=\frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\sqrt{\Delta}\,\delta}\sum_{n=1}^{\infty}\tilde{b}_Q\left(n,1/\sqrt{\Delta}\right)\,e^{-\frac{2\pi n}{\Delta^{3/2}\delta}}\,I_0\left(\frac{\sqrt{2\pi n(i+\delta)}}{\Delta^{3/4}}\,\frac{z}{\delta}\right). \tag{4.22}$$

Now it is a matter of following the proof of Lemma 3.4: we need to show that

$$\lim_{\delta\to0^+}\left|\frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\delta}\,e^{-\frac{2\pi n}{\Delta^{3/2}\delta}}\,I_0\left(\frac{\sqrt{2\pi n(i+\delta)}}{\Delta^{3/4}}\,\frac{z}{\delta}\right)\right|=0$$

for any fixed $n\in\mathbb{N}$. To do it, we just need to invoke the simple bound (3.24) and to recall that $\mathrm{Re}\left(\sqrt{i+\delta}\,z\right)\to\mathrm{Re}(\sqrt{i}\,z)=\frac{1}{\sqrt{2}}\left(\mathrm{Re}(z)-\mathrm{Im}(z)\right)$ and $\mathrm{Re}\left((i+\delta)\,z^2\right)\to-2\,\mathrm{Re}(z)\,\mathrm{Im}(z)$ as $\delta\to0^+$, so that the term in the previous equation is bounded by

$$\lim_{\delta\to0^+}\frac{1}{\delta}\exp\left(-\frac{2\pi}{\Delta^{3/2}\delta}\left(n-\frac{\Delta^{3/4}}{2\sqrt{\pi}}\left|\mathrm{Re}(z)-\mathrm{Im}(z)\right|\sqrt{n}-\frac{\mathrm{Re}(z)\,\mathrm{Im}(z)\,\Delta^{3/2}}{4\pi}\right)\right).$$

As before, the expression in the exponential is a quadratic polynomial with variable $X=\sqrt{n}$ of the form

$$P(X)=X^2-\frac{\Delta^{3/4}}{2\sqrt{\pi}}\left|\mathrm{Re}(z)-\mathrm{Im}(z)\right|\,X-\frac{\mathrm{Re}(z)\,\mathrm{Im}(z)\,\Delta^{3/2}}{4\pi}.$$

After computing the zero of $P(X)$ with highest value, we know that a sufficient condition for the positivity of $P(X)$ at a point $X=X_0$ is that

$$\frac{\Delta^{3/4}}{4\sqrt{\pi}}\left[\left|\mathrm{Re}(z)-\mathrm{Im}(z)\right|+\left|\mathrm{Re}(z)+\mathrm{Im}(z)\right|\right]<X_0. \tag{4.23}$$

Thus, we want (4.23) to hold for every $X_0=\sqrt{n}$ and every $n\in\mathbb{N}$. This is satisfied if:

$$\left|\mathrm{Re}(z)-\mathrm{Im}(z)\right|+\left|\mathrm{Re}(z)+\mathrm{Im}(z)\right|<\frac{4\sqrt{\pi}}{\Delta^{3/4}}, \tag{4.24}$$

which is the case whenever $z\in\mathscr{D}_Q=\left\{z\in\mathbb{C}\,:\,|\mathrm{Re}(z)|<\frac{2\sqrt{\pi}}{\Delta^{3/4}},\ |\mathrm{Im}(z)|<\frac{2\sqrt{\pi}}{\Delta^{3/4}}\right\}$. Now it is just a matter of following the conclusion of Lemmas 3.4 and 3.5 to deduce (4.19).

□

Remark 4.2. Note that the condition (4.24) is indicated to assure that (4.23) holds for every $X_0 = \sqrt{n}$, $n \in \mathbb{N}$. But this condition can be relaxed and replaced by

$$|\text{Re}(z) - \text{Im}(z)| + |\text{Re}(z) + \text{Im}(z)| < \frac{4\sqrt{\pi\, m_Q}}{\Delta^{3/4}}, \tag{4.25}$$

where m_Q is the least integer for which $\tilde{b}_Q\left(m_Q, 1/\sqrt{\Delta}\right) \neq 0$. For example, when $Q(m,n) = Q_0(m,n) := m^2 + n^2$, $\Delta = 4$, the condition (4.18) does not reduce to (1.9) for $\alpha = 2$. This is because

$$\begin{aligned}\tilde{b}_{Q_0}\left(1, \frac{1}{2}\right) &:= \frac{1}{2} \sum_{\alpha^2+\beta^2=1} \sum_{k_1,k_2=0}^{1} \exp\left(-\pi i\, \left(k_1^2 + k_2^2\right) + \pi i\left(\alpha k_1 + \beta k_2\right)\right) \\ &= 0,\end{aligned}$$

while $\tilde{b}_{Q_0}(2, 1/2) \neq 0$. Thus, $m_{Q_0} = 2$, and then (4.25) now reduces to (1.9). It is a nice exercise of notation and arrangement of variables to check that (4.22) reduces to (3.17) when $Q(m,n) = m^2 + n^2$.

4.3 Proof of Theorem 1.2

Now, the proof of our Theorem 1.2 is almost the same (modulo different computations) as the proof of Theorem 1.1. Therefore, we will brief and just indicate the main steps. Start with integral representation (2.66) given in Example 2.2, replacing there $x = e^{2i\omega}$, with $-\frac{\pi}{4} < \omega < \frac{\pi}{4}$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \eta_Q\left(\frac{1}{2} + it\right) {}_1F_1\left(\frac{1}{2} + it;\, 1;\, -\frac{z^2}{4}\right) e^{2\omega t}\, dt = e^{i\omega}\, \psi_Q\left(e^{2i\omega}, z\right) - e^{-i\omega}\, e^{-z^2/4}.$$

From Kummer's formula (2.8) and adding and subtracting the term $e^{-z^2/8}\, e^{i\omega}$, we obtain

$$\frac{e^{-z^2/8}}{2\pi} \int_{-\infty}^{\infty} \eta_Q\left(\frac{1}{2} + it\right) {}_1F_1\left(\frac{1}{2} - it;\, 1;\, -\frac{z^2}{4}\right) e^{2\omega t}\, dt = -2\, e^{-z^2/8} \cos(\omega) + e^{i\omega}\left(e^{-z^2/8} + e^{z^2/8}\, \psi_Q\left(e^{2i\omega}, z\right)\right), \tag{4.26}$$

which is analogous to (3.37) for $\alpha = 2$. Therefore, we can use the same computations as in the previous section: changing the variable t to $t + \lambda_j$, differentiating p times on both sides and appealing to the notation (3.40), we deduce

$$\begin{aligned}&\int_{-\infty}^{\infty} t^p \eta_Q\left(\frac{1}{4} + i\left(t + \lambda_j\right)\right) {}_1F_1\left(\frac{1}{2} - i\left(t + \lambda_j\right);\, 1;\, \frac{z^2}{4}\right) e^{2\omega t}\, dt \\ &= -4\pi \left(\frac{r_j}{2}\right)^p e^{-2\omega\lambda_j} \cos\left(p\,\theta_j + \omega\right) + \frac{2\pi}{2^p}\, e^{z^2/8} \frac{d^p}{d\omega^p}\left\{e^{i\omega - 2\omega\lambda_j}\left(e^{-z^2/8} + e^{z^2/8}\, \psi_Q\left(e^{2i\omega}, z\right)\right)\right\}.\end{aligned}$$

Continuing to argue in the same manner, we can fully justify the equality (c.f. (3.42) above)

$$\begin{aligned}\int_{-\infty}^{\infty} t^p\, F_{z,Q}\left(\frac{1}{2} + it\right) e^{2\omega t}\, dt = &-\frac{8\pi}{2^p} \sum_{j=1}^{\infty} c_j\, r_j^p e^{-2\omega\lambda_j} \left[\cos\left(p\,\theta_j + \omega\right)\right] + \\ &+\frac{4\pi}{2^p} \text{Re}\left(e^{z^2/8} \frac{d^p}{d\omega^p}\left\{\sum_{j=1}^{\infty} c_j e^{i\omega - 2\omega\lambda_j}\left(e^{-z^2/8} + e^{z^2/8}\, \psi_Q\left(e^{2i\omega}, z\right)\right)\right\}\right),\end{aligned} \tag{4.27}$$

where

$$F_{z,Q}(s)=\sum_{j=1}^{\infty}c_j\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\,\zeta(s+i\lambda_j,\,Q)\,\left\{{}_1F_1\left(1-s-i\lambda_j;\,1;\,\frac{z^2}{4}\right)+\,{}_1F_1\left(1-\overline{s}+i\lambda_j;\,1;\,\frac{\overline{z}^2}{4}\right)\right\}.$$

By letting $\omega\to\frac{\pi}{4}^{-}$, we see from Lemma 4.3 above (namely, relation (4.19)) that the right-hand side of (4.27) can be simplified to:

$$-\frac{8\pi}{2^p}\sum_{j=1}^{\infty}c_jr_j^pe^{-\frac{\pi}{2}\lambda_j}\left[\cos\left(p\,\theta_j+\frac{\pi}{4}\right)\left(1+\text{Re}\left(e^{z^2/8}\,\sinh\left(\frac{z^2}{8}\right)\right)\right)-\sin\left(p\theta_j+\frac{\pi}{4}\right)\,\text{Im}\left(e^{z^2/8}\,\sinh\left(\frac{z^2}{8}\right)\right)\right],$$

which is exactly the same as the right-hand side of (3.45) with $\alpha=2$. Therefore, from this point onward, the proof follows exactly the same principles as the proof of Theorem 1.1. ■

5 Zeros of combinations attached to $L_k(s,\chi)$

Similarly to the previous sections, in order to study the zeros of $L_k(s,\chi)$ we need a Lemma which gives the analytic continuation of a periodic Dirichlet series containing information about the behavior of $\psi_{k,\chi}(x,z)$ (2.71).

Before proving such an important lemma, we first need an auxiliary result, which will be the next lemma.

Although the proof of our next result is quite standard and it is a known result for $\delta=0$ (see Lemma 1.2 above) we prove it for the case where $\delta=1$ because this might be instructive to some readers. Although straightforward, we could not track any reference containing explicitly its statement.

5.1 Exponential sums attached to Dirichlet characters

Lemma 5.1. *Let $\delta\in\{0,1\}$ and $0<a_i<1$ for every $i\in\{1,...,k\}$. Consider the Dirichlet series*

$$\varphi_{k,\delta}(s;a_1,...,a_k):=\sum_{n_1,...,n_k\in\mathbb{Z}}\frac{(n_1+a_1)^{\delta}\cdot...\cdot(n_k+a_k)^{\delta}}{\left((n_1+a_1)^2+...+(n_k+a_k)^2\right)^s},\quad Re(s)>\frac{k}{2}\left(1+\delta\right). \tag{5.1}$$

Then $\varphi_{k,\delta}(s;\,a_1,...,a_k)$ has the following properties:

1. *If $\delta=0$, it can be continued as a meromorphic function with a simple pole located at $s=\frac{k}{2}$ with residue $Res_{s=k/2}\,\varphi_{k,0}(s;\,a_1,...,a_k)=\frac{\pi^{k/2}}{\Gamma(k/2)}$.*
2. *If $\delta=1$, it can be continued as an entire function.*

Moreover, it satisfies the functional equation:

$$\pi^{-s}\Gamma(s)\,\varphi_{k,\delta}(s;\,a_1,...,a_k)=(-i)^{\delta k}\,\pi^{-\left(k\left(\frac{1}{2}+\delta\right)-s\right)}\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)\,\tilde{\varphi}_{k,\delta}\left(k\left(\frac{1}{2}+\delta\right)-s;\,a_1,...,a_k\right), \tag{5.2}$$

where, for $Re(s)>\frac{k}{2}(1+\delta)$, $\tilde{\varphi}_{k,\delta}(s;\,a_1,...,a_k)$ can be written as the Dirichlet series

$$\tilde{\varphi}_{k,\delta}\left(s;\,a_1,...,a_k\right)=\sum_{n_1,...,n_k\neq0}\frac{n_1^{\delta}\cdot...\cdot n_k^{\delta}\,e^{2\pi i(n_1,...,n_k)\cdot(a_1,...,a_k)}}{\left(n_1^2+...+n_k^2\right)^s}. \tag{5.3}$$

Furthermore, $\tilde{\varphi}_{k,\delta}\left(s;\,a_1,...,a_k\right)$ can be analytically continued as an entire function no matter the value of δ.

Proof. Note that the condition $0 < a_i < 1$ implies that the multiple series (5.1) is well-defined for any $(n_1, ..., n_k) \in \mathbb{Z}^k$. Note also that for $\delta = 0$ this result is already known because the Dirichlet series is reduced (5.1) to the Epstein zeta function (1.41), this is

$$\varphi_{k,0}(s; a_1, ..., a_k) = \zeta\left(s, \mathbf{a}, \mathbf{0}, \mathbf{I}_k\right),$$

where $\mathbf{a} = (a_1, ..., a_k)$ and $\mathbf{I}_k$ denotes the diagonal quadratic form $n_1^2 + ... + n_k^2$. The pole of $\varphi_{k,0}(s; a_1, ..., a_k)$ comes from the pole of the Epstein zeta function at $s = k/2$. Furthermore, we have

$$\tilde{\varphi}_{k,0}\left(s;\, a_1, ..., a_k\right) = \zeta\left(s,\, \mathbf{0},\, \mathbf{a},\, \mathbf{I}_k\right).$$

Since $(a_1, ..., a_k) \notin \mathbb{Z}^k$, by the analytic continuation of Epstein's zeta function we see that $\tilde{\varphi}_{k,0}\left(s;\, a_1, ..., a_k\right)$ must be entire. The functional equation (5.2) is an immediate corollary of (1.42).

We now focus on the case where $\delta = 1$. We start by writing (5.1) as a Mellin transform

$$\pi^{-s}\Gamma(s)\,\varphi_{k,1}(s; a_1, ..., a_k) = \int_0^{\infty} x^{s-1} \prod_{j=1}^{k} \sum_{n_j \in \mathbb{Z}} (n_j + a_j)\; e^{-\pi (n_j + a_j)^2 x}\, dx, \quad \text{Re}(s) > k, \tag{5.4}$$

so that the proof of (5.2) relies on applying Poisson's summation formula to each factor. This is standard and we obtain for each $j \in \{1, ..., k\}$,

$$\sum_{n_j \in \mathbb{Z}} (n_j + a_j)\, e^{-\pi(n_j + a_j)^2 x} = -\frac{i}{x^{3/2}} \sum_{n_j \in \mathbb{Z}} n_j\, \exp\left(-\frac{\pi}{x} n_j^2 + 2\pi i\, n_j\, a_j\right). \tag{5.5}$$

Hence, breaking the integral on (5.4) into $(0, 1)$ and $(1, \infty)$ and applying (5.5) on the first integral, we get the representation:

$$\pi^{-s}\Gamma(s)\,\varphi_{k,1}(s; a_1, ..., a_k) = \int_1^{\infty} x^{s-1} \sum_{n_1, ..., n_k \in \mathbb{Z}} (n_1 + a_1) \cdot ... \cdot (n_k + a_k)\, \exp\left(-\pi\left((n_1 + a_1)^2 + ... + (n_k + a_k)^2\right) x\right)\, dx$$

$$+ \int_1^{\infty} x^{\frac{3}{2}k - s - 1} (-i)^k \sum_{n_1, ..., n_k \in \mathbb{Z}} n_1 \cdot ... \cdot n_k \cdot \exp\left(-\pi x \left(n_1^2 + ... + n_k^2\right) + 2\pi i\, (n_1, ..., n_k) \cdot (a_1, ..., a_k)\right)\, dx,$$

which is invariant once we replace s by $\frac{3k}{2} - s$ and $\varphi_{k,1}(s; a_1, ..., a_k)$ by $\tilde{\varphi}_{k,1}\left(s;\, a_1, ..., a_k\right)$.

Moreover, the uniform and absolute convergence of the integrals given in the equality above assure that $\varphi_{k,1}(s; a_1, ..., a_k)$ and $\tilde{\varphi}_{k,1}\left(s;\, a_1, ..., a_k\right)$ are entire functions of s.

□

The next lemma has a role similar to Lemmas 3.2 and 4.1 of the previous sections.

Lemma 5.2. *Let χ be a Dirichlet character modulo q and p be an integer such that $(p, q) = 1$. If χ is even, define $\delta = 0$. Otherwise, if χ is odd, set $\delta = 1$.*

Consider the Dirichlet series

$$L_k(s, \chi, p/q) := \sum_{n_1, ..., n_k \neq 0} \frac{n_1^{\delta}\, \chi(n_1) \ldots n_k^{\delta}\, \chi(n_k)}{\left(n_1^2 + ... + n_k^2\right)^s}\, e^{-\frac{i\pi p}{q}\left(n_1^2 + ... + n_k^2\right)}, \quad Re(s) > \frac{k}{2}(1 + \delta). \tag{5.6}$$

Then $L_k\left(s, \chi, p/q\right)$ has the following properties:

1. *If χ is even, it can be analytically continued as a meromorphic function with at most one simple pole located at $s = k/2$ with residue given by*

$$Res_{s=k/2}L_k(s,\chi,p/q) = \frac{\pi^{k/2}}{\Gamma(k/2)(2q)^k}\left(\sum_{r=1}^{2q-1}\chi(r)\,e^{-\frac{\pi i p}{q}r^2}\right)^k. \tag{5.7}$$

In particular, if p, q are odd integers, $L_k(s,\chi,p/q)$ is entire.

2. *If χ is odd, it can be analytically continued as an entire function.*

Moreover, it satisfies the functional equation

$$\left(\frac{\pi}{2q}\right)^{-s}\Gamma(s)\,L_k(s,\chi,p/q) = (-i)^{\delta k}\left(\frac{\pi}{2q}\right)^{-\left(k\left(\frac{1}{2}+\delta\right)-s\right)}\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)\tilde{L}_k\left(k\left(\frac{1}{2}+\delta\right)-s,\chi,p/q\right), \tag{5.8}$$

where $\tilde{L}_k(s,\chi,p/q)$ is representable by the series

$$\tilde{L}_k(s,\chi,p/q) = \sum_{n_1,\ldots,n_k\neq 0}\frac{b_\chi(n_1,\ldots,n_k;\,p/q)\,\cdot\,n_1^\delta\cdot\ldots\cdot n_k^\delta}{\left(n_1^2+\ldots+n_k^2\right)^s}, \quad Re(s) > \frac{k}{2}(1+\delta), \tag{5.9}$$

with

$$b_\chi(n_1,\ldots,n_k;\,p/q) := (2q)^{-\frac{k}{2}}\sum_{r_1,\ldots,r_k=0}^{2q-1}\chi(r_1)\cdot\ldots\cdot\chi(r_k)\,\exp\left(-\frac{i\pi p}{q}\left(r_1^2+\ldots+r_k^2\right)+2\pi i\,(n_1,\ldots,n_k)\cdot\left(\frac{r_1}{2q},\ldots,\frac{r_k}{2q}\right)\right). \tag{5.10}$$

Furthermore, (5.9) can be continued as an entire function no matter the parity of χ.

Proof. Since the arithmetical function $\chi(n)\,e^{-\frac{i\pi p}{q}n^2}$ has period $2q$, we may decompose the series (5.6) into residue classes modulo $2q$, which gives:

$$L_k(s,\chi,p/q) := \sum_{n_1,\ldots,n_k\neq 0}\frac{n_1^\delta\,\chi(n_1)\ldots n_k^\delta\,\chi(n_k)}{\left(n_1^2+\ldots+n_k^2\right)^s}\,e^{-\frac{i\pi p}{q}\left(n_1^2+\ldots+n_k^2\right)} =$$

$$= (2q)^{k\delta-2s}\sum_{r_1,\ldots,r_k=1}^{2q-1}\chi(r_1)\cdot\ldots\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+\ldots+r_k^2\right)}\sum_{m_1,\ldots,m_k\in\mathbb{Z}}\frac{\left(m_1+\frac{r_1}{2q}\right)^\delta\cdot\ldots\cdot\left(m_k+\frac{r_k}{2q}\right)^\delta}{\left(\left(m_1+\frac{r_1}{2q}\right)^2+\ldots+\left(m_k+\frac{r_k}{2q}\right)^2\right)^s}$$

$$= (2q)^{k\delta-2s}\sum_{r_1,\ldots,r_k=1}^{2q-1}\chi(r_1)\cdot\ldots\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+\ldots+r_k^2\right)}\varphi_{k,\delta}\left(s;\,\frac{r_1}{2q},\ldots,\frac{r_k}{2q}\right), \quad \text{Re}(s) > \frac{k}{2}(1+\delta), \tag{5.11}$$

where $\varphi_{k,\delta}(s;\,a_1,\ldots,a_k)$ is given by (5.1). We now apply the previous lemma: if $\delta = 1$, i.e., if χ is an odd Dirichlet character, we have that $L_k(s,\chi,p/q)$ is entire because $\varphi_{k,1}(s;\,a_1,\ldots,a_k)$ is entire.

On the other hand, if χ is even, then $L_k(s,\chi,p/q)$ must have a simple pole at $s = k/2$ with residue given explicitly given by (5.7), due to item 1. of Lemma 5.1.

In particular, if χ is even and p, q are odd integers,

$$\sum_{r_1,\ldots,r_k=1}^{2q-1}\chi(r_1)\cdot\ldots\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+\ldots+r_k^2\right)} = \left(\sum_{r=1}^{2q-1}\chi(r)\,e^{-\frac{\pi i p}{q}r^2}\right)$$

$$= \left(\sum_{r=1}^{q-1}\chi(r)\,e^{-\frac{\pi i p}{q}r^2}+\sum_{r=q+1}^{2q-1}\chi(r)\,e^{-\frac{\pi i p}{q}r^2}\right)^k = \left(\sum_{r=1}^{q-1}\chi(r)\,e^{-\frac{\pi i p}{q}r^2}-\sum_{r=1}^{q-1}\chi(r)\,e^{-\frac{\pi i p}{q}r^2}\right)^k = 0,$$

which shows that, under these conditions, $L_k(s,\chi,p/q)$ must be entire.

The functional equation (5.8) now follows from (5.3): invoking (5.11) and (5.3), we see that

$$\left(\frac{\pi}{2q}\right)^{-s}\Gamma(s)\,L_k(s,\chi,p/q)=\frac{(-i)^{\delta k}}{(2q)^{k/2}}\left(\frac{\pi}{2q}\right)^{-\left(k\left(\frac{1}{2}+\delta\right)-s\right)}\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)\times$$
$$\times\sum_{r_1,...,r_k=1}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,\times e^{-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)}\,\tilde{\varphi}_{k,\delta}\left(k\left(\frac{1}{2}+\delta\right)-s;\,\frac{r_1}{2q},...,\frac{r_k}{2q}\right).$$

But for $\text{Re}(s)>\frac{k}{2}(1+\delta)$,

$$\frac{1}{(2q)^{k/2}}\sum_{r_1,...,r_k=1}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)}\,\tilde{\varphi}_{k,\delta}\left(s;\,\frac{r_1}{2q},...,\frac{r_k}{2q}\right)=$$
$$=\sum_{n_1,...,n_k\neq0}\frac{n_1^{\delta}\cdot...\cdot n_k^{\delta}}{\left(n_1^2+...+n_k^2\right)^s}\,(2q)^{-\frac{k}{2}}\sum_{r_1,...,r_k=0}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,\exp\left(-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)+2\pi i\,(n_1,...,n_k)\cdot\left(\frac{r_1}{2q},...,\frac{r_k}{2q}\right)\right)$$
$$=\sum_{n_1,...,n_k\neq0}\frac{b_{\chi}(n_1,...,n_k;\,p/q)\,n_1^{\delta}\cdot...\cdot n_k^{\delta}}{\left(n_1^2+...+n_k^2\right)^s}=\tilde{L}_k\left(s,\chi,p/q\right).$$

To conclude, we just need to show that $\tilde{L}_k(s,\chi,p/q)$ is an entire function: this comes from the previous expression of $\tilde{L}_k(s,\chi,p/q)$ as the sum

$$\frac{1}{(2q)^{k/2}}\sum_{r_1,...,r_k=1}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)}\,\tilde{\varphi}_{k,\delta}\left(s;\,\frac{r_1}{2q},...,\frac{r_k}{2q}\right).$$

Since $0<r_j/(2q)<1$, we know that $\tilde{\varphi}_{k,\delta}\left(s;\,\frac{r_1}{2q},...,\frac{r_k}{2q}\right)$ is always entire (no matter if δ is 0 or 1) and so $\tilde{L}_k(s,\chi,p/q)$ must be entire as well.

□

Remark 5.1. Note that we can write the multiple series defining (5.6) and (5.9) as Dirichlet series involving a single variable of summation. In fact, we may write (5.6) as the Dirichlet series

$$L_k(s,\chi,p/q)=\sum_{n=1}^{\infty}\frac{r_{k,\chi}(n)\,e^{-\frac{i\pi p}{q}n}}{n^s},\quad\text{Re}(s)>\frac{k}{2}\left(1+\delta\right),\tag{5.12}$$

where $r_{k,\chi}(n)$ is explicitly given by (2.68). We can also write (5.9) as

$$\tilde{L}_k\left(s,\chi,p/q\right)=\sum_{n=1}^{\infty}\frac{\tilde{b}_{k,\chi}(n,\,p/q)}{n^s},\quad\text{Re}(s)>\frac{k}{2}(1+\delta),\tag{5.13}$$

with $\tilde{b}_{k,\chi}(n,p/q)$ being

$$(2q)^{-\frac{k}{2}}\sum_{n_1^2+...+n_k^2=n}n_1^{\delta}\cdot...\cdot n_k^{\delta}\sum_{r_1,...,r_k=0}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,\exp\left(-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)+2\pi i\,(n_1,...,n_k)\cdot\left(\frac{r_1}{2q},...,\frac{r_k}{2q}\right)\right).\tag{5.14}$$

Remark 5.2. Note that we did not use the primitivity of the character χ in the proof of Lemma 5.2.

In the next subsection we will see that, under this additional hypothesis and by setting $p=0$ in (5.8), we can recover the functional equation (1.43) given at the beginning of the paper.

5.2 Proof of Lemma 1.3

Since χ is a primitive character, we know that the Gauss sum splits in the form

$$G(n,\overline{\chi}) := \sum_{j=1}^{q-1} \overline{\chi}(j)\, e^{2\pi i n j/q} = \chi(n)\, G(\overline{\chi}), \tag{5.15}$$

where $G(\chi) := G(1,\chi)$.

By the representation as Dirichlet series (5.6) and (1.43), we clearly have that $L_k(s,\chi) = L_k(s,\chi,0)$. Since (5.7) is zero when $p=0$, we conclude that $L_k(s,\chi)$ is entire. To prove the functional equation (1.44), we look at (5.8) and see

$$\left(\frac{\pi}{2q}\right)^{-s} \Gamma(s)\, L_k(s,\chi) = (-i)^{\delta k} \left(\frac{\pi}{2q}\right)^{-\left(k\left(\frac{1}{2}+\delta\right)-s\right)} \Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)\, \tilde{L}_k\left(k\left(\frac{1}{2}+\delta\right)-s,\chi,0\right). \tag{5.16}$$

Thus, it remains to simplify the expression for $\tilde{L}_k(s,\chi,p/q)$, (5.9), when $p=0$: by (5.10), we know that

$$\tilde{L}_k(s,\chi,0) = (2q)^{-\frac{k}{2}} \sum_{n_1,\dots,n_k\neq 0} \sum_{r_1,\dots,r_k=0}^{2q-1} \chi(r_1)\cdot\ldots\cdot\chi(r_k)\, \exp\left(2\pi i\, (n_1,\dots,n_k)\cdot\left(\frac{r_1}{2q},\dots,\frac{r_k}{2q}\right)\right) \frac{n_1^{\delta}\cdot\ldots\cdot n_k^{\delta}}{\left(n_1^2+\ldots+n_k^2\right)^{s}}, \tag{5.17}$$

for $\mathrm{Re}(s) > \frac{k}{2}(1+\delta)$. Now, for every pair (n_j, r_j), the resulting Gauss sum is:

$$\begin{aligned}
\sum_{r_j=0}^{2q-1} \chi(r_j)\, \exp\left(2\pi i\, n_j\, \frac{r_j}{2q}\right) &= \sum_{r_j=1}^{q-1} \chi(r_j)\, \exp\left(2\pi i\, n_j\, \frac{r_j}{2q}\right) + \sum_{j=1}^{q-1} \chi(r_j)\, \exp\left(2\pi i\, n_j\, \frac{r_j+q}{2q}\right) \\
&= (1+(-1)^{n_j}) \sum_{r_j=1}^{q-1} \chi(r_j)\, \exp\left(2\pi i\, n_j\, \frac{r_j}{2q}\right) = \begin{cases} 2\sum_{r_j=1}^{q-1} \chi(r_j)\, \exp\left(2\pi i\, m_j\, \frac{r_j}{q}\right) = 2\, G(m_j,\chi) & n_j = 2m_j \\ 0 & n_j \ \text{odd}. \end{cases}
\end{aligned} \tag{5.18}$$

Therefore, the sum in (5.17) over $(n_1,\dots,n_k)$ does not vanish only if each n_j is an even integer: writing $n_j = 2m_j$, we derive from (5.17) and (5.18),

$$\begin{aligned}
\tilde{L}_k(s,\chi,0) &= (2q)^{-\frac{k}{2}} 2^{-2s+k(\delta+1)} \sum_{m_1,\dots,m_k\neq 0} \frac{G(m_1,\chi)\, m_1^{\delta}\cdot\ldots\cdot G(m_k,\chi)\, m_k^{\delta}}{\left(m_1^2+\ldots+m_k^2\right)^{s}} \\
&= \frac{G^k(\chi)}{q^{k/2}}\, 2^{-2s+k\delta+\frac{k}{2}} \sum_{m_1,\dots,m_k\neq 0} \frac{\overline{\chi}(m_1)m_1^{\delta}\cdot\ldots\cdot\overline{\chi}(m_k)m_k^{\delta}}{\left(m_1^2+\ldots+m_k^2\right)^{s}} \\
&= \frac{G^k(\chi)}{q^{k/2}}\, 2^{-2s+k\delta+\frac{k}{2}}\, L_k(s,\overline{\chi}),
\end{aligned} \tag{5.19}$$

where in the second step we have used the fact that χ is primitive (5.15). Joining (5.19) and (5.16) gives (1.44). ■

5.3 The behavior of $\psi_{\chi,k}(x,z)$

We finally prove that the analogue of Jacobi's $\psi-$function given at (2.71) has the same properties as the previous ones. We start with a summation formula involving $r_{k,\chi}(n)$.

Lemma 5.3. *Let χ be a primitive Dirichlet character modulo q and let $r_{k,\chi}(n)$ and $\tilde{b}_{k,\chi}(n,\, p/q)$ be the arithmetical functions given in (2.68) and (5.14).*

Assume further that $q \neq 0 \mod 4$ when χ is even.

Then, for every $Re(x)>0$ and $y \in \mathbb{C}$, the following transformation formula takes place

$$\sum_{n=1}^{\infty} r_{k,\chi}(n)\, e^{-\frac{i\pi}{q}n}\, n^{\frac{1}{2}-\frac{k}{2}\left(\frac{1}{2}+\delta\right)}\, e^{-\frac{\pi}{2q}n\,x}\, J_{k\left(\frac{1}{2}+\delta\right)-1}\left(\sqrt{\frac{\pi n}{2q}}\,y\right)$$
$$=\frac{e^{-\frac{y^2}{4x}}}{x}\,\sum_{n=1}^{\infty}(-i)^{\delta k}\tilde{b}_{k,\chi}\,(n,\,1/q)\;n^{\frac{1}{2}-\frac{k}{2}\left(\frac{1}{2}+\delta\right)}\,e^{-\frac{\pi n}{2q\,x}}\,I_{k\left(\frac{1}{2}+\delta\right)-1}\left(\sqrt{\frac{\pi n}{2q}}\,\frac{y}{x}\right). \tag{5.20}$$

Proof. Apply the summation formula (2.9) or (2.24) to the pair of Dirichlet series (which satisfy Hecke's functional equation and belong to the class $\mathcal{A}$ by Lemma 5.2):

$$\phi(s)=\left(\frac{\pi}{2q}\right)^{-s}\,L_k\left(s,\chi,1/q\right):=\left(\frac{\pi}{2q}\right)^{-s}\sum_{n=1}^{\infty}\frac{r_{k,\chi}(n)\,e^{-\frac{i\pi}{q}n}}{n^s},\quad \mathrm{Re}(s)>\frac{k}{2}(1+\delta) \tag{5.21}$$

and

$$\psi(s)=\left(\frac{\pi}{2q}\right)^{-s}\,(-i)^{\delta k}\,\tilde{L}_k\left(s,\chi,1/q\right)=\left(\frac{\pi}{2q}\right)^{-s}\sum_{n=1}^{\infty}\frac{(-i)^{\delta k}\,\tilde{b}_{k,\chi}(n,\,1/q)}{n^s},\quad \mathrm{Re}(s)>\frac{k}{2}(1+\delta). \tag{5.22}$$

This requires to take the simple substitutions $\lambda_n=\frac{\pi n}{2q}$, $\mu_n=\frac{\pi n}{2q}$, $a(n)=r_{k,\chi}(n)\,e^{-\frac{i\pi p}{q}n}$, $b(n)=(-i)^{\delta k}\tilde{b}_{k,\chi}(n,1/q)$ and $r=k\left(\frac{1}{2}+\delta\right)$ in (2.9).

We first study the possible poles that $\phi(s)$ may have: if χ is odd, it follows lemma 5.2 that $\phi(s)$ is entire.

Assume now that χ is even. Since there are no primitive characters (mod q) such that $q\equiv 2 \mod 4$, our imposition that $q\neq 0$ mod 4 when χ is even actually implies that q must be odd. By item 1. of Lemma 5.2, we know that $\phi(s)$ must be an entire function under this assumption.

Thus, $\rho=0$ regardless of χ being even or odd. In order to apply (2.9), we need to find also $\phi(0)$. Since $\phi(s)\in\mathcal{A}$ and $\psi(s)$ given by (5.21) is an entire function, we know by Remark 1.1 that $\phi(0)$ must be zero.

Finally, replacing $\rho=\phi(0)=0$ in (2.9), we are able to derive (5.20). □

We now prove the analogue of Lemmas 3.4 and 4.3 of the previous sections.

Lemma 5.4. *Let χ be a primitive Dirichlet character modulo q and $r_{k,\chi}(n)$ be defined by (2.68). For $Re(x)>0$ and $z\in\mathbb{C}$, consider the analogue of Jacobi's $\psi-$function (2.71):*

$$\psi_{\chi,k}(x,z):=2^{\frac{k}{2}+k\delta-1}\,\Gamma\left(\frac{k}{2}+k\delta\right)\left(\sqrt{\frac{\pi}{q}x}\,z\right)^{1-\frac{k}{2}-k\delta}\sum_{n=1}^{\infty}r_{k,\chi}(n)\,n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\,e^{-\frac{\pi n}{q}\,x}\,J_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi}{q}n\,x}\,z\right).$$

Furthermore, let $h:\mathbb{C}\longmapsto\mathbb{C}$ be an analytic function and assume that z satisfies the condition:

$$z\in\mathscr{D}_{\chi}:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<\sqrt{\frac{\pi}{2q}},\;|Im(z)|<\sqrt{\frac{\pi}{2q}}\right\}. \tag{5.23}$$

Then:

1. if χ is even and $q \neq 0 \mod 4$ or

2. if χ is odd with arbitrary modulus,

the asymptotic behavior takes place:

$$\lim_{\omega\to\frac{\pi}{4}^-} \frac{d^m}{d\omega^m}\left\{h(\omega)\,\psi_{\chi,k}\left(e^{2i\omega},\, z\right)\right\}=0 \ \ \forall m\in\mathbb{N}_0. \tag{5.24}$$

Proof. Following the proofs of Lemmas 3.4 and 3.5 (in particular, see (3.21) and (3.35)) (and changing the $\delta-$notation there to ϵ), we just need to consider the behavior of $\psi_{\chi,k}(i+\epsilon,z)$ as $\epsilon\to0^+$. From (2.71), the expression of $\psi_{\chi,k}(i+\epsilon,z)$ is

$$\begin{aligned}
&2^{\frac{k}{2}+k\delta-1}\,\Gamma\left(\frac{k}{2}+k\delta\right)\left(\sqrt{\frac{\pi}{q}(i+\epsilon)}\,z\right)^{1-\frac{k}{2}-k\delta}\sum_{n=1}^{\infty} r_{k,\chi}(n)\,n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\,e^{-\frac{\pi n}{q}(i+\epsilon)}\,J_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi}{q}n(i+\epsilon)}\,z\right)\\
&=2^{\frac{k}{2}+k\delta-1}\,\Gamma\left(\frac{k}{2}+k\delta\right)\left(\sqrt{\frac{\pi}{q}(i+\epsilon)}\,z\right)^{1-\frac{k}{2}-k\delta}\sum_{n=1}^{\infty} r_{k,\chi}(n)\,n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\,e^{-\frac{\pi n}{q}i}\,e^{-\frac{\pi n}{q}\epsilon}\,J_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi}{q}n(i+\epsilon)}\,z\right).
\end{aligned} \tag{5.25}$$

We may apply the summation formula (5.20) to the right-hand side of (5.25) with x in (5.20) being replaced by 2ϵ and y by $\sqrt{2(i+\epsilon)}\,z$. This gives the transformation:

$$\begin{aligned}
\psi_{\chi,k}(i+\epsilon,z)=&2^{\frac{k}{2}+k\delta-1}\,\Gamma\left(\frac{k}{2}+k\delta\right)\left(\sqrt{\frac{\pi}{q}(i+\epsilon)}\,z\right)^{1-\frac{k}{2}-k\delta}\frac{e^{-\frac{(i+\epsilon)z^2}{4\epsilon}}}{2\epsilon}\times\\
&\times\sum_{n=1}^{\infty}(-i)^{\delta k}\tilde{b}_{k,\chi}\left(n,\,1/q\right)\,n^{\frac{1}{2}-\frac{k}{2}\left(\frac{1}{2}+\delta\right)}\,e^{-\frac{\pi n}{4q\epsilon}}\,I_{k\left(\frac{1}{2}+\delta\right)-1}\left(\sqrt{\frac{\pi n(i+\epsilon)}{q}}\,\frac{z}{2\epsilon}\right).
\end{aligned} \tag{5.26}$$

If we bound $I_\nu(z)$ in the same way as before (see (3.24)), a sufficient condition ensuring that (5.26) tends to zero as $\epsilon\to0^+$ is

$$\lim_{\epsilon\to0^+}\epsilon^{-\frac{k}{2}-k\delta}\exp\left(-\frac{\pi n}{4q\epsilon}+\frac{1}{2\epsilon}\sqrt{\frac{\pi n}{q}}\frac{|\mathrm{Re}(z)-\mathrm{Im}(z)|}{\sqrt{2}}+\frac{\mathrm{Re}(z)\,\mathrm{Im}(z)}{2\epsilon}\right)=0.$$

But by examining the quadratic polynomial (with variable $\sqrt{n}$),

$$P(X)=X^2-X\sqrt{\frac{2q}{\pi}}\,|\mathrm{Re}(z)-\mathrm{Im}(z)|-\frac{2q\,\mathrm{Re}(z)\,\mathrm{Im}(z)}{\pi},$$

one can see that $P(\sqrt{n})>0$ for every $n\in\mathbb{N}$ if:

$$|\mathrm{Re}(z)-\mathrm{Im}(z)|+|\mathrm{Re}(z)+\mathrm{Im}(z)|<\sqrt{\frac{2\pi}{q}}, \tag{5.27}$$

which is satisfied whenever z satisfies the condition (5.23). □

Remark 5.3. In contrast with the statement of Theorem 1.1, the domain in (5.23) does not depend on k. Like in Remark 4.2, enlarging this domain requires to get more information about the arithmetical function $\tilde{b}_{k,\chi}\left(n,\,1/q\right)$. As in (4.25), (5.27) can be replaced by:

$$|\mathrm{Re}(z)-\mathrm{Im}(z)|+|\mathrm{Re}(z)+\mathrm{Im}(z)|<\sqrt{\frac{2\pi\,m_{\chi,k}}{q}},$$

where $m_{\chi,k}$ is the least integer for which $\tilde{b}_{k,\chi}\left(n,\,1/q\right)\neq0$.

5.4 Proof of Theorem 1.3

Since the Dirichlet series $L_k(s,\chi)$ is entire, the argument in the proof of Theorem 1.1 becomes easier, because we do not need to count the sign changes of the residual terms like in the expression (3.45).

We shall see how the proof in this case goes: start with the integral representation (2.72) obtained in Example 2.3 and replace there x by $e^{2i\omega}$, $-\frac{\pi}{4}<\omega<\frac{\pi}{4}$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)\,{}_1F_1\left(\frac{k}{4}+\frac{k\delta}{2}+it;\,\frac{k}{2}+k\delta;\,-\frac{z^2}{4}\right)e^{2\omega t}\,dt=e^{i\omega k\left(\frac{1}{2}+\delta\right)}\,\psi_{\chi,k}\left(e^{2i\omega},z\right).$$

From Kummer's formula, we can rewrite the previous equality in the symmetric form:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)\,{}_1F_1\left(\frac{k}{4}+\frac{k\delta}{2}-it;\,\frac{k}{2}+k\delta;\,\frac{z^2}{4}\right)e^{2\omega t}\,dt=e^{i\omega k\left(\frac{1}{2}+\delta\right)}\,e^{z^2/4}\,\psi_{\chi,k}\left(e^{2i\omega},z\right), \qquad (5.28)$$

which is somewhat remindful of the previous steps given in (3.36) and (4.26). In the present case, however, we are in a much easier situation because there are no residual terms on the right-hand side of (5.28).

After changing the variable $t\to t+\lambda_j$ and differentiate over ω a number p of times, we are able to get:

$$\int_{-\infty}^{\infty}t^p\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+i\left(t+\lambda_j\right),\chi\right)\,{}_1F_1\left(\frac{k}{4}+\frac{k\delta}{2}-i(t+\lambda_j);\,\frac{k}{2}+k\delta;\,\frac{z^2}{4}\right)e^{2\omega t}\,dt$$
$$=\frac{2\pi}{2^p}\,e^{z^2/4}\,\frac{d^p}{d\omega^p}\left\{e^{i\omega k\left(\frac{1}{2}+\delta\right)-2\omega\lambda_j}\,\psi_{\chi,k}\left(e^{2i\omega},z\right)\right\}. \qquad (5.29)$$

Now, we make an important remark: in order to apply Hardy's method, we need the integrand on the left of (5.29) to be a real function of t.

In the previous two cases, it was enough to take the real part of the hypergeometric factor to assure this. In this case, however, we note that the function

$$\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)$$

is not real-valued. This can be easily seen from the asymmetry in the functional equation (1.44),

$$\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)=\left(\frac{(-i)^{\delta}G(\chi)}{\sqrt{q}}\right)^k\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}-it,\overline{\chi}\right). \qquad (5.30)$$

However, since $|G(\chi)/\sqrt{q}|=1$, we can write $(-i)^{\delta}\,G(\chi)/\sqrt{q}:=e^{i\gamma}$ for some real γ and then set:

$$\tilde{\eta}_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right):=e^{-i\gamma k/2}\,\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right). \qquad (5.31)$$

It is now clear from (5.30) that $\tilde{\eta}_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)$ is real-valued and its zeros are exactly the same as the ones of $\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)$. Thus, we may multiply both sides of (5.29) by $e^{-i\gamma k/2}$ and have:

$$\int_{-\infty}^{\infty}t^p\,\tilde{\eta}_k\left(\frac{k}{4}+\frac{k\delta}{2}+i\left(t+\lambda_j\right),\chi\right)\,{}_1F_1\left(\frac{k}{4}+\frac{k\delta}{2}-i(t+\lambda_j);\,\frac{k}{2}+k\delta;\,\frac{z^2}{4}\right)e^{2\omega t}\,dt$$
$$=\frac{2\pi e^{-i\gamma k/2}}{2^p}\,e^{z^2/4}\,\frac{d^p}{d\omega^p}\left\{e^{i\omega k\left(\frac{1}{2}+\delta\right)-2\omega\lambda_j}\,\psi_{\chi,k}\left(e^{2i\omega},z\right)\right\}.$$

We take the real part of the hypergeometric factor, multiply by $c_j \in \ell^1$ and sum over j: with the same kind of justifications as in the proof of Theorem 1.1, we can deduce

$$\int_{-\infty}^{\infty} t^p \tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) e^{2\omega t}\, dt = \frac{4\pi}{2^p} \operatorname{Re}\left[e^{-i\gamma k/2+z^2/4} \frac{d^p}{d\omega^p}\left\{\sum_{j=1}^{\infty} c_j\, e^{i\omega k\left(\frac{1}{2}+\delta\right)-2\omega\lambda_j} \cdot \psi_{\chi,k}\left(e^{2i\omega}, z\right)\right\}\right], \quad (5.32)$$

where

$$\tilde{F}_{z,k,\chi}(s) := e^{-i\gamma k/2}\, F_{z,k,\chi}(s),$$

with $F_{z,k,\chi}(s)$ being

$$\sum_{j=1}^{\infty} c_j\, \eta_k\left(s+i\lambda_j,\chi\right)\left\{{}_1F_1\left(k\left(\frac{1}{2}+\delta\right)-s-i\lambda_j;\, k\left(\frac{1}{2}+\delta\right);\, \frac{z^2}{4}\right)+{}_1F_1\left(k\left(\frac{1}{2}+\delta\right)-\overline{s}+i\lambda_j;\, k\left(\frac{1}{2}+\delta\right);\, \frac{\overline{z}^2}{4}\right)\right\}.$$

If we show that the real-valued function $\tilde{F}_{z,k,\chi}(s)$ has infinitely many zeros, we are done. This can be proved similarly as in Theorem 1.1: taking the limit $\omega \to \frac{\pi}{4}^-$ on both sides and using (5.24), we deduce that

$$\lim_{\omega\to\frac{\pi}{4}^-} \int_{-\infty}^{\infty} t^p \tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) e^{2\omega t}\, dt = 0. \quad (5.33)$$

If we now assume that the integrand has only a finite amount of zeros, we can use our previous reasoning (see (3.55), (3.56) and (3.57) above) to show that

$$\int_{-\infty}^{\infty} t^p \left|\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+i\left(t+\lambda_j\right)\right)\right| e^{\frac{\pi}{2}t}\, dt < \infty$$

and so, by (5.33) and the dominated convergence theorem

$$\int_{-\infty}^{\infty} t^p \tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) e^{\frac{\pi}{2}t}\, dt = 0.$$

Mimicking (3.59) and (3.60), we find that

$$\begin{aligned}
&\int_{T_0}^{\infty}\left\{\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) e^{\frac{\pi}{2}t}+(-1)^p\,\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt \\
&= -\int_{0}^{T_0}\left\{\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) e^{\frac{\pi}{2}t}+(-1)^p\,\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt \leq K(T_0)\, T_0^p \quad (5.34)
\end{aligned}$$

where T_0 is a (sufficiently large) parameter such that $\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) \neq 0$ when $t > T_0$. Literally as in (3.60), we may deduce the lower bound:

$$\int_{T_0}^{\infty}\left\{\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) e^{\frac{\pi}{2}t}+(-1)^p\,\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt \geq \epsilon(T_0)\,(2T_0)^p. \quad (5.35)$$

Comparing (5.34) and (5.35) we conclude that

$$2^p \leq \frac{K(T_0)}{\epsilon(T_0)}$$

must hold for infinitely many values of p, which is absurd. ■

Remark 5.4. As it is clear from the proof, we only need the condition that the sequence $(\lambda_j)_{j\in\mathbb{N}}$ is bounded, but we have not used the fact that it attains its bounds. Thus, we may relax this latter condition in the statement of Theorem 1.3. Similar comments can be made for Theorem 1.4

6 Zeros of combinations attached to $L_f(s, p/q)$

In this final section we extend Wilton's result by proving Theorem 1.4. We will be very brief because, unlike in previous sections, we do not need to develop "exponential summation formulas" for the Dirichlet series under study. The basic Dirichlet series in this section is:

$$L_f(s, p/q) := \sum_{n=1}^{\infty} \frac{a_f(n)\, e^{\frac{2\pi i p}{q}n}}{n^s}, \quad \text{Re}(s) > \frac{k+1}{2}, \quad (p,q) = 1, \tag{6.1}$$

where $a_f(n)$ are the Fourier coefficients of the cusp form $f(\tau)$ and are assumed to be real. From the functional equation for $L_f(s, p/q)$ (see (1.49) and (2.84) above):

$$\eta_f\left(s, \frac{p}{q}\right) = (-1)^{k/2}\, \eta_f\left(k-s, -\frac{\overline{p}}{q}\right),$$

one can check that $\eta_f(k/2+it, p/q)$ is only real (or purely imaginary) when $p^2 \equiv 1 \mod q$.

Therefore, from now on, we will assume that $p^2 \equiv 1 \mod q$. Since $L_f(s, p/q)$ is entire, we expect that the proof of Theorem 1.4 will have exactly the same nature as the proofs of theorem 1.3 and so we shall omit it.

The only purpose of this section is to show that the analogue of Jacobi's $\psi-$function for $L_f(s, p/q)$, (2.81), vanishes exponentially fast when $x \to i$ through the half circle $|x| = 1$, $\text{Re}(x) > 0$. This is the last result of our paper. After showing it, Theorem 1.4 is automatically proved once we follow the lines of the previous sections.

Lemma 6.1. *Let $f(\tau)$ be a cusp form of weight k for the full modular group. Consider the Dirichlet series:*

$$L_f(s, p/q) = \sum_{n=1}^{\infty} \frac{a_f(n)\, e^{\frac{2\pi i p}{q}n}}{n^s}, \quad (p,q) = 1, \ \ p^2 \equiv 1 \mod q$$

and its analogue of Jacobi's $\psi-$function,

$$\psi_{f,p/q}(x,z) := (k-1)!\left(\sqrt{\frac{\pi x}{2q}}\, z\right)^{1-k} \sum_{n=1}^{\infty} a_f(n)\, e^{\frac{2\pi i p}{q}n}\, n^{\frac{1-k}{2}}\, e^{-\frac{2\pi n}{q}x}\, J_{k-1}\left(\sqrt{\frac{2\pi n}{q}x}\, z\right),$$

defined for Re$(x) > 0$ and $z \in \mathbb{C}$.

Suppose further that z belongs to the region:

$$z \in \mathscr{D}_q := \left\{ z \in \mathbb{C} \, : \, |Re(z)| < 2\sqrt{\frac{\pi}{q}}, \,\, |Im(z)| < 2\sqrt{\frac{\pi}{q}} \right\}. \tag{6.2}$$

Then for every z satisfying (6.2), any $m \in \mathbb{N}_0$, and any analytic function $h : \mathbb{C} \longmapsto \mathbb{C}$, the following relation takes place:

$$\lim_{\omega \to \frac{\pi}{4}^{-}} \frac{d^m}{d\omega^m} \left\{ h(\omega) \, \psi_{f,p/q} \left(e^{2i\omega}, z \right) \right\} = 0. \tag{6.3}$$

Proof. The proof is instantaneous because we already have the transformation (2.82) in the right template. Indeed,

$$\psi_{f,p/q}(i+\delta, z) = (k-1)! \, 2^{k-1} (\sqrt{i+\delta}\, z)^{1-k} \left(\frac{2\pi}{q} \right)^{\frac{1-k}{2}} \sum_{n=1}^{\infty} a_f(n) \, e^{\frac{2\pi i \, (p-1)}{q} n} \, n^{\frac{1-k}{2}} \, e^{-\frac{2\pi n}{q}\delta} \, J_{k-1}\left(\sqrt{\frac{2\pi n}{q}} \, (i+\delta) \, z \right).$$

Applying (2.82) and replacing there x by δ and z by $\frac{\sqrt{i+\delta}}{\sqrt{\delta}} \, z$ and p by $p-1$, we get:

$$\psi_{f,p/q}(i+\delta, z) = (k-1)! \, 2^{k-1} (\sqrt{i+\delta}\, z)^{1-k} \left(\frac{2\pi}{q} \right)^{\frac{1-k}{2}} \frac{e^{-\frac{(i+\delta) z^2}{4\delta}}}{\delta} (-1)^{k/2} \times$$
$$\times \sum_{n=1}^{\infty} a_f(n) \, e^{-\frac{2\pi i \, R \, n}{q}} \, n^{\frac{1-k}{2}} \, e^{-\frac{2\pi n}{q\delta}} \, I_{k-1}\left(\sqrt{\frac{2\pi n \, (i+\delta)}{q}} \, \frac{z}{\delta} \right),$$

where R is such $R\,(p-1) \equiv 1 \mod q$. From the bound (3.24) and the work done in Lemmas 3.4 and 3.5, the proof of (6.3) reduces once more to show that

$$\lim_{\delta \to 0^+} \delta^{-1} \exp\left(-\frac{2\pi n}{q\delta} + \sqrt{\frac{2\pi n}{q}} \, \frac{|\mathrm{Re}(z) - \mathrm{Im}(z)|}{\sqrt{2}\delta} + \frac{\mathrm{Re}(z) \, \mathrm{Im}(z)}{2\delta} \right) = 0.$$

But a sufficient condition for this to hold is that z satisfies (6.2). □

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