

A Deleting Derivations Algorithm for Quantum Nilpotent Algebras at Roots of Unity

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Abstract

This paper extends an algorithm and canonical embedding in [Cau03a] to a large class of quantum algebras. It applies to iterated Ore extensions over a field satisfying some suitable assumptions which cover those of Cauchon's original setting but also allows for roots of unity. The extended algorithm constructs a quantum affine space A' from the original quantum algebra A via a series of change of variables within the division ring of fractions $\text{Frac}(A)$. The canonical embedding takes a completely prime ideal $P \triangleleft A$ to a completely prime ideal $Q \triangleleft A'$ such that when A is a PI algebra, $\text{PI-deg}(A/P) = \text{PI-deg}(A'/Q)$. When the quantum parameter is a root of unity, combining our construction with results from [BLR22] allows us to state an explicit formula for the PI degree of completely prime quotient algebras. This paper ends with a method to construct a maximum dimensional irreducible representation of A/P given a suitable irreducible representation of A'/Q when A is PI.

In [Cau03a], Cauchon developed an algorithm for a large class of quantum algebras originally named CGL extensions after Cauchon, Goodearl and Letzter who first proved a set of unifying results on the algebras in this class, and later renamed quantum nilpotent algebras [LLR06]. These are algebras which can be written in the form of an iterated Ore extension over a field with certain conditions placed on the defining automorphisms and skew-derivations. In essence, the algorithm takes a prime quotient of a suitable CGL extension and, via a series of changes of indeterminates within its division ring of fractions, obtains a prime quotient of a quantum affine space which shares the same division ring of fractions as the original algebra. Properties from the resulting quotiented quantum affine space can then be pulled back to the original quotient algebra. When the prime ideal is invariant under a rational action of a torus, the algorithm combined with Goodearl and Letzter's \mathcal{H} -stratification theory was used to prove the *quantum Gel'fand Kirillov Conjecture* on prime quotients of a large class of generic quantum algebras, such as quantum euclidean spaces, quantum symplectic spaces, quantum matrices, and quantum Weyl algebras [Cau03a, Théorème 6.1.1]. Cauchon's algorithm has also been applied by many authors to gain a better understanding of the structure of quantum algebras in the generic setting. Such works include the aforementioned [Cau03b] and [GL98] as well as [Cas14, GY14, GLL19, GY16, LLN19].

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Cauchon's algorithm cannot be applied to such algebras at roots of unity, however, as the homomorphism defining the algorithm is not compatible with roots of unity. Several authors have recreated the main results of [Cau03a] for quantum algebras at roots of unity satisfying slightly different conditions ([Hay08] and [LM11]). However, the explicit change of variables in the original algorithm, that was essential in the applications named above, was not recreated in either paper. This paper addresses this by extending Cauchon's deleting derivations algorithm, including the canonical embedding to track completely prime ideals, to include the root of unity case by utilising an adapted version, provided in [Hay08, Section 3], of the homomorphism which lies at the heart of the procedure.

The main motivation for this paper is the fact that many quantum algebras become *polynomial identity (PI) algebras* when taken at roots of unity and in this case the *PI degree* is a useful invariant for deducing various properties of the algebra (see, for example, [BY17], [CPWZ16], and [CPWZ15]). The PI degree also plays a role in the representation theory of prime affine PI algebras, giving an upper bound on the dimension of irreducible representations [BG02, Theorem I.13.5].

The extended algorithm defined in Section 2 of this paper takes a suitable iterated Ore extension A and a completely prime ideal $P \in \text{C.Spec}(A)$ and constructs a quantum affine space A' and a canonical embedding $\psi : \text{C.Spec}(A) \rightarrow \text{C.Spec}(A')$ such that $\text{Frac}(A/P) = \text{Frac}(A'/\psi(P))$. Furthermore, if A is PI then, so too is A' and thus $\text{PI-deg}(A/P) = \text{PI-deg}(A'/\psi(P))$. When $A'/\psi(P)$ is itself a quantum affine space then the PI degree is determined by the rank and values of the invariant factors of its commutation matrix (see for instance [BLR22, Lemma 2.4]). With this in mind, we focus on those ideals P_w which give rise to quantum affine spaces $A'/\psi(P_w)$. For such ideals, the PI degree of A/P_w is then computed in Theorem 4.4. We end with a method to pass certain maximum-dimensional irreducible representations of any $A'/\psi(P)$ back through the deleting derivations algorithm to obtain maximum irreducible representations of A/P (Section 5) and illustrate our result by constructing an explicit irreducible representation of maximal dimension for a completely prime quotient of $U_q^+(\mathfrak{so}_5)$ at roots of unity.

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1 Quantum Nilpotent Algebras at roots of unity

Take \mathbb{K} to be an arbitrary field and R a \mathbb{K} -algebra. One can form an Ore extension $R[x, \sigma, \delta]$ in one indeterminate x by defining a \mathbb{K} -automorphism σ and a (left) σ -derivation δ (also called a *skew-derivation*) such that for any $a, b \in R$ we have $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$. These maps define the commutation rules for x with elements of R : given $r \in R$ we have $xr = \sigma(r)x + \delta(r)$. Ore extensions satisfy a universal mapping property ([GW04, Proposition 2.4]) and thus are unique up to isomorphism ([GW04, Corollary 2.5]). Iterating this process with N indeterminates gives an iterated Ore extension $R[X_1, \sigma_1, \delta_1] \dots [X_N, \sigma_N, \delta_N]$. Note that this is a noetherian domain by the Skew Hilbert Basis Theorem [BG02, Theorem I.1.13]. The exact form of R and the properties of the pairs (σ_i, δ_i) are what will define quantum nilpotent algebras at roots of unity and distinguish them from CGL extensions.

If all the skew-derivations in the iterated Ore extension above are the 0 map, all the automorphisms act by scalar multiplication on each indeterminate, i.e. $\sigma_i(X_j) = \lambda_{i,j}X_j$ for all $1 \leq j < i \leq N$ and some $\lambda_{i,j} \in \mathbb{K}^*$, and R is a field \mathbb{K} then the iterated Ore extension would describe a quantum affine space, which we can write as follows:

$$\mathbb{K}[X_1, \sigma_1] \dots [X_N, \sigma_N] = \mathbb{K}_\Lambda[X_1, \dots, X_N] =: \mathcal{O}_{q^\Lambda}(\mathbb{K}^N)$$

where $\Lambda := (\lambda_{i,j})_{i,j} \in M_N(\mathbb{K}^*)$ with $\lambda_{j,i} = \lambda_{i,j}^{-1}$ for all $1 \leq j \leq i \leq N$.

The following terms are needed to define our class of algebras of interest.

Definition 1.1. Let R be a \mathbb{K} -algebra and $R[x, \sigma, \delta]$ be an Ore extension.

1. We call the pair (σ, δ) , and indeed the Ore extension, *q-skew* if the automorphism and skew-derivation satisfy the relation $\delta \circ \sigma = q\sigma \circ \delta$, for some $1 \neq q \in \mathbb{K}^*$.
2. The skew-derivation δ is *locally nilpotent* if, for every $0 \neq a \in R$, there exists an integer $n_a \geq 0$ such that $\delta^{n_a}(a) = 0$ and $\delta^m(a) \neq 0$ for any $m < n_a$. We define such an n_a as the *δ -nilpotence index of a* .
3. [Hay08] We say that the σ -derivation δ *extends to a higher q-skew σ -derivation (h.q-s. σ -d.)* on R if there is a sequence $\{d_n\}_{n=0}^\infty$ (denoted simply by $\{d_n\}$ when it is obvious which subscript indexes the sequence) of \mathbb{K} -linear operators on R such that:

- d_0 is the identity;
- $d_1 = \delta$;
- $d_n(rs) = \sum_{i=0}^n \sigma^{n-i} d_i(r) d_{n-i}(s)$ for all $r, s \in R$ and all n ,
- $d_n \circ \sigma = q^n \sigma \circ d_n$ for all n .

A (h.q-s. σ -d.) is *locally nilpotent* if, for all $0 \neq r \in R$, there exists an integer $n_r \geq 0$ such that $d_n(r) = 0$ for all $n \geq n_r$, and $d_m(r) \neq 0$ for any $m < n_r$. In this case we call n_r the *d -nilpotence index of r* . A h.q-s. σ -d is called *iterative* if $d_n d_m = \binom{n+m}{m}_q d_{n+m}$ for all n, m .

Here $\binom{n+m}{m}_q$ is the q -Gaussian binomial coefficient: a polynomial in q over \mathbb{Z} with properties similar to those of regular binomial coefficients. More precisely, for all $n \geq i \geq 0$, we have $\binom{n}{i}_q := \frac{(n)!_q}{(i)!_q (n-i)!_q}$, where $(i)_q = \frac{q^i - 1}{q - 1}$ and $(i)!_q = (1)_q \cdots (i-1)_q (i)_q$ with the convention that $(0)!_q = 1$.

The higher skew-derivations are what allowed Haynal to adapt Cauchon's *effacement des derivations* homomorphism to a broader homomorphism which allows for roots of unity. For practical use, [Hay08, Theorem 2.8] gives a sufficient condition on a q -skew Ore extension $R[x; \sigma, \delta]$ for the σ -derivation δ to extend to a h.q-s. σ -d. for any $1 \neq q \in \mathbb{K}^*$. This is used to check that the specific examples we are interested in do, indeed, satisfy this condition.

The algebras that we study in this article satisfy the following:

Hypothesis 1.2. Let $A = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_N; \sigma_N, \delta_N]$, where the σ_i are \mathbb{K} -algebra automorphisms, and the δ_i are σ_i -derivations on the relevant subalgebras of A . Denote these subalgebras by $A_{i-1} := \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_{i-1}; \sigma_{i-1}, \delta_{i-1}]$ so that $A_0 := \mathbb{K}$ and $A_N := A$. We place the following conditions on A :

H1.2.1: $\sigma_i(X_l) = \lambda_{i,l} X_l$ for all $l < i$ and $2 \leq i \leq N$, where $\lambda_{i,l} \in \mathbb{K}^*$.

H1.2.2: $\Lambda := (\lambda_{i,j}) \in M_N(\mathbb{K}^*)$ is a multiplicatively antisymmetric matrix. That is, $\lambda_{i,l} = \lambda_{l,i}^{-1}$ for all $1 \leq i, l \leq N$.

H1.2.3: For $2 \leq i \leq N$ there exists some $1 \neq q_i \in \mathbb{K}^*$ such that $\delta_i \circ \sigma_i = q_i \sigma_i \circ \delta_i$, i.e. (σ_i, δ_i) is q_i -skew.

H1.2.4: For all $2 \leq i \leq N$ each δ_i extends to a locally nilpotent, iterative h.q-i-s. σ_i -d., $\{d_{i,n}\}_{n=0}^\infty$, on A_{i-1} , and $\sigma_l \circ d_{i,n} = \lambda_{l,i}^n d_{i,n} \circ \sigma_l$ on A_{i-1} for all $n \geq 0$ and $i+1 \leq l \leq N$.

Algebras satisfying Hypothesis 1.2 are called *quantum nilpotent algebras* and include quantum Schubert cell algebras, quantised Weyl algebras, quantised coordinate rings of affine, symplectic, and euclidean spaces, and quantum matrices.

2 Deleting Derivations Algorithm

The main algorithm in this section will remove each skew-derivation from an algebra A satisfying Hypothesis 1.2 via an iterative change of indeterminates within the division ring of fractions of A , until we obtain a quantum affine space. This is developed in much the same way as in [Cau03a, Section 3.2] but applies the homomorphism defined in [Hay08, Proposition 3.4]. For this application we need the ability to reorder the extensions of A . This is achieved in the following lemma.

Lemma 2.1. *Let*

$$\begin{aligned} A &= A_{j-1}[X_j; \sigma_j, \delta_j][X_{j+1}; \sigma_{j+1}] \dots [X_N; \sigma_N], \\ \hat{A} &= A_{j-1}[X_j; \sigma_j, \delta_j][X_{j+1}^{\pm 1}; \sigma_{j+1}] \dots [X_N^{\pm 1}; \sigma_N], \end{aligned}$$

where the automorphisms and skew-derivations satisfy properties H1.2.1–H1.2.3 and $\delta_j \neq 0$.

(I) *Then*

$$\begin{aligned} A &= A_{j-1}[X_{j+1}; \sigma_{j+1}^*] \dots [X_N; \sigma_N^*][X_j; \sigma_j', \delta_j'], \\ \hat{A} &= A_{j-1}[X_{j+1}^{\pm 1}; \sigma_{j+1}^*] \dots [X_N^{\pm 1}; \sigma_N^*][X_j; \sigma_j', \delta_j'], \end{aligned}$$

where

- (i) $\sigma_i^*|_{A_{j-1}} = \sigma_i|_{A_{j-1}}$ for all $j+1 \leq i \leq N$ and $\sigma_i^*(X_l) = \sigma_i(X_l) = \lambda_{i,l}X_l$ for all $j+1 \leq l < i$;
- (ii) $\sigma_j'|_{A_{j-1}} = \sigma_j$ and $\delta_j'|_{A_{j-1}} = \delta_j$;
- (iii) $\sigma_j'(X_l) = \lambda_{j,l}X_l = \lambda_{l,j}^{-1}X_l$ and $\delta_j'(X_l) = 0$ for all $j+1 \leq l \leq N$.

(II) (σ_j', δ_j') is q_j -skew.

(III) *Suppose that property H1.2.4 is also satisfied. Then δ_j' extends to a locally nilpotent, iterative h.q.j-s. σ_j' -d., $\{d'_{j,n}\}_{n=0}^\infty$, on $A_{j-1}\langle X_{j+1}^{\pm 1}, X_{j+2}^{\pm 1}, \dots, X_N^{\pm 1} \rangle$, where the $d'_{j,n}$ coincide with the $d_{j,n}$ on A_{j-1} and, for all $j+1 \leq l \leq N$ and $n \geq 1$, $d'_{j,n}(X_l) = 0$. Moreover, $\{d'_{j,n}\}_{n=0}^\infty$ restricts to a h.q.j-s. σ_j' -d. on $A_{j-1}\langle X_{j+1}, X_{j+2}, \dots, X_N \rangle$ which is also locally nilpotent and iterative.*

Proof. This is an inductive corollary to [Hay08, Lemma 4.1] and the details are left to the reader. \square

In the results that follow, we abuse notation slightly and use the same notation for maps defined on isomorphic algebras. We do this in the case where the action of the map on the generators of the algebra does not change, even though the algebras do. For example, suppose we have two isomorphic iterated Ore extensions

$$\mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cong \mathbb{K}[x_1][x_2; \bar{\sigma}_2, \bar{\delta}_2],$$

where $\sigma_2(X_1) = \lambda X_1$ and $\bar{\sigma}_2(x_1) = \lambda x_1$ for the same $\lambda \in \mathbb{K}^*$. In this case, we simply denote $\bar{\sigma}_2$ by σ_2 (similarly for $\bar{\delta}_2$) and write $\mathbb{K}[x_1][x_2; \sigma_2, \delta_2]$.

We may also abuse notation in a similar way for restrictions of maps to isomorphic subalgebras.

We now describe the deleting derivations algorithm for algebras satisfying Hypothesis 1.2.

2.1 The Algorithm

For each $j \in \llbracket 2, N+1 \rrbracket$ we define a sequence $(X_1^{(j)}, \dots, X_N^{(j)})$ of elements of $F := \text{Frac}(A)$ and we set $A^{(j)} := \mathbb{K}\langle X_1^{(j)}, \dots, X_N^{(j)} \rangle$ to be the subalgebra of F generated by these elements. For $j = N+1$ we set $(X_1^{(N+1)}, \dots, X_N^{(N+1)}) := (X_1, \dots, X_N)$ so that $A^{(N+1)} = A$. For some fixed $j \in \llbracket 2, N \rrbracket$ we assume that the algebra $A^{(j+1)}$ satisfies the following hypothesis, first setting $(x_1, \dots, x_N) := (X_1^{(j+1)}, \dots, X_N^{(j+1)})$ for ease of notation.

Hypothesis 2.2.

H2.2.1: $A^{(j+1)} \cong \mathbb{K}[X_1] \dots [X_j; \sigma_j, \delta_j][X_{j+1}; \sigma_{j+1}^{(j+1)}] \dots [X_N; \sigma_N^{(j+1)}]$ by an isomorphism sending x_i to X_i for all $i \in \llbracket 1, N \rrbracket$.

H2.2.2: For each $i \in \llbracket j+1, N \rrbracket$, the map $\sigma_i^{(j+1)}$ is an automorphism such that $\sigma_i^{(j+1)}(X_l) = \lambda_{i,l} X_l$ for all $l \in \llbracket 1, i-1 \rrbracket$. Furthermore, we have $\sigma_i^{(j+1)} \circ d_{l,n} = \lambda_{i,l}^n d_{l,n} \circ \sigma_i^{(j+1)}$ for all $l \in \llbracket 2, j \rrbracket$ and $n \geq 0$.

This allows us to write

$$A^{(j+1)} = \mathbb{K}\langle x_1, \dots, x_N \rangle \quad (1)$$

$$= \mathbb{K}[x_1] \dots [x_j; \sigma_j, \delta_j][x_{j+1}; \sigma_{j+1}^{(j+1)}] \dots [x_N; \sigma_N^{(j+1)}] \quad (2)$$

where, for $i \in \llbracket 2, j \rrbracket$, σ_i and δ_i satisfy Hypothesis 1.2 and δ_i extends to a locally nilpotent, iterative h.q.i-s.σ_i-d., $\{d_{i,n}\}_{n=0}^\infty$, on A_{i-1} .

We define a new sequence of elements in F , $(y_1, \dots, y_N) := (X_1^{(j)}, \dots, X_N^{(j)})$:

$$y_l = \begin{cases} x_l & l \geq j; \\ \sum_{n=0}^\infty q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} d_{j,n} \circ \sigma_j^{-n}(x_l) x_j^{-n} & l < j. \end{cases} \quad (3)$$

Note that the sum stated above is finite, since the sequence $\{d_{j,n}\}_{n=0}^\infty$ is locally nilpotent. With this we define $A^{(j)} := \mathbb{K}\langle y_1, \dots, y_N \rangle$.

Theorem 2.3. *Let A be as in Hypothesis 1.2, with $A^{(j+1)}$ defined as above and satisfying Hypothesis 2.2. Then we have the following:*

(I) $A^{(j)} \cong \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \dots [X_{j-1}; \sigma_{j-1}, \delta_{j-1}][X_j; \sigma_j^{(j)}] \dots [X_N; \sigma_N^{(j)}]$ by an isomorphism which sends y_l to X_l for all $l \in \llbracket 1, N \rrbracket$.

(II) For all $i \in \llbracket j, N \rrbracket$, the $\sigma_i^{(j)}$ are automorphisms satisfying

(i) $\sigma_i^{(j)}(X_l) = \lambda_{i,l} X_l$ for all $l \in \llbracket 1, i-1 \rrbracket$;

(ii) $\sigma_i^{(j)} \circ d_{l,n} = \lambda_{i,l}^n d_{l,n} \circ \sigma_i^{(j)}$ for all $n \geq 0$ and all $l \in \llbracket 2, j-1 \rrbracket$.

(III) Let $S_j := \{x_j^n \mid n \geq 0\} = \{y_j^n \mid n \geq 0\}$. This is a multiplicatively closed set of regular elements satisfying the two sided Ore-condition in $A^{(j+1)}$ and $A^{(j)}$ and, furthermore, $A^{(j)} S_j^{-1} = A^{(j+1)} S_j^{-1}$.

Proof. By Hypothesis 2.2, and the discussion thereafter, we can write $A^{(j+1)}$ as

$$A^{(j+1)} = \mathbb{K}[x_1] \dots [x_j; \sigma_j, \delta_j][x_{j+1}; \sigma_{j+1}^{(j+1)}] \dots [x_N; \sigma_N^{(j+1)}].$$

Define

$$A_{j-1}^{(j+1)} := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \dots [x_{j-1}; \sigma_{j-1}, \delta_{j-1}].$$

If $\delta_j = 0$ then $A^{(j)} = A^{(j+1)}$ and the result is trivial. So, assuming that $\delta_j \neq 0$ and applying Lemma 2.1 to $A^{(j+1)} = A_{j-1}^{(j+1)}[x_j; \sigma_j, \delta_j][x_{j+1}; \sigma_{j+1}^{(j+1)}] \dots [x_N; \sigma_N^{(j+1)}]$ gives

$$A^{(j+1)} = A_{j-1}^{(j+1)}[x_{j+1}; \sigma_{j+1}^{(j+1)*}] \dots [x_N; \sigma_N^{(j+1)*}][x_j; \sigma'_j, \delta'_j], \quad (4)$$

where

- (a) $\sigma_i^{(j+1)*}|_{A_{j-1}^{(j+1)}} = \sigma_i^{(j+1)}|_{A_{j-1}^{(j+1)}}$ for all $i \in \llbracket j+1, N \rrbracket$, and $\sigma_i^{(j+1)*}(x_l) = \sigma_i^{(j+1)}(x_l) = \lambda_{i,l}x_l$ for all $l \in \llbracket j+1, i-1 \rrbracket$;
- (b) $\sigma'_j|_{A_{j-1}^{(j+1)}} = \sigma_j$ and $\delta'_j|_{A_{j-1}^{(j+1)}} = \delta_j$;
- (c) $\sigma'_j(x_l) = \lambda_{j,l}x_l = \lambda_{l,j}^{-1}x_l$ and $\delta'_j(x_l) = 0$ for all $l \in \llbracket j+1, N \rrbracket$.

In particular, δ'_j extends to a h.q_j-s.σ_j'-d., $\{d'_{j,n}\}_{n=0}^\infty$, on $A_{j-1}^{(j+1)}\langle x_{j+1}^{\pm 1}, \dots, x_N^{\pm 1} \rangle$ where

$$d'_{j,n}|_{A_{j-1}^{(j+1)}} = d_{j,n} \quad (\forall n \geq 0), \quad (5)$$

$$d'_{j,n}(x_l) = 0 \quad (\forall l \in \llbracket j+1, N \rrbracket \text{ and } n \geq 1). \quad (6)$$

Define

$$\widehat{A^{(j+1)}} := A_{j-1}^{(j+1)}[x_{j+1}; \sigma_{j+1}^{(j+1)*}] \dots [x_N; \sigma_N^{(j+1)*}]$$

so that equation (4) becomes

$$A^{(j+1)} = \widehat{A^{(j+1)}}[x_j; \sigma'_j, \delta'_j].$$

Applying [Hay08, Theorem 3.7] to $A^{(j+1)}$ yields the isomorphism,

$$\begin{aligned} f : \widehat{A^{(j+1)}}[x_j^{\pm 1}; \sigma'_j] &\longrightarrow \widehat{A^{(j+1)}}[x_j; \sigma'_j, \delta'_j]S_j^{-1} \\ \widehat{A^{(j+1)}} \ni a &\longmapsto f(a) = \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} d'_{j,n} \circ (\sigma'_j)^{-n}(a) x_j^{-n} \\ x_j &\longmapsto x_j. \end{aligned}$$

Note that $f(x_l) = x_l$ for all $l \in \llbracket j+1, N \rrbracket$ since, by (6), $d'_{j,n}(x_l) = 0$ for all $n \geq 1$. If $l \in \llbracket 1, j-1 \rrbracket$, then x_l and $(\sigma'_j)^{-n}(x_l) \in A_{j-1}^{(j+1)}$. Thus, by (5) and (b) above, we can replace σ'_j and $d'_{j,n}$ in $f(x_l)$ with σ_j and $d_{j,n}$ to obtain

$$f(x_l) = \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} d_{j,n} \circ \sigma_j^{-n}(x_l) x_j^{-n} = y_l,$$

as defined in (3). Therefore, for any $l \in \llbracket 1, N \rrbracket$ we see that

$$f(x_l) = \begin{cases} x_l & l \geq j; \\ y_l & l < j. \end{cases}$$

Hence the isomorphism f takes x_l to y_l for all $l \in \llbracket 1, N \rrbracket$.

Using [Hay08, Theorem 3.7] we see that restricting f to $\widehat{A^{(j+1)}}[x_j; \sigma'_j]$ gives the deleting derivation homomorphism as defined in [Hay08, Proposition 3.4]. Therefore

$$\mathrm{Im}(f) \cong \widehat{A^{(j+1)}}[x_j; \sigma'_j], \quad (7)$$

where $\mathrm{Im}(f) = f(\widehat{A^{(j+1)}}[x_j; \sigma'_j])$ is the subalgebra of $\widehat{A^{(j+1)}}[x_j^{\pm 1}; \sigma'_j, \delta'_j]$ generated by x_j and $f(\widehat{A^{(j+1)}})$. Since $f(\widehat{A^{(j+1)}})$ is generated by \mathbb{K} and $\{y_l \mid l \neq j\}$, and since $x_j = y_j$, this simply tells us that

$$\mathrm{Im}(f) = \mathbb{K}\langle y_1, \dots, y_N \rangle = A^{(j)}.$$

Using (7) we see that

$$A^{(j)} = \mathrm{Im}(f) \cong \mathbb{K}[x_1] \dots [x_{j-1}; \sigma_{j-1}, \delta_{j-1}][x_{j+1}; \sigma_{j+1}^{(j+1)*}] \dots [x_N; \sigma_N^{(j+1)*}][x_j; \sigma'_j],$$

and therefore,

$$A^{(j)} = \mathbb{K}[y_1] \dots [y_{j-1}; \sigma_{j-1}, \delta_{j-1}][y_{j+1}; \sigma_{j+1}^{(j+1)*}] \dots [y_N; \sigma_N^{(j+1)*}][y_j; \sigma'_j]. \quad (8)$$

Finally we apply [Hay08, Proposition 3.6] to conclude that S_j is a multiplicatively closed set of regular elements in both $A^{(j+1)}$ and $A^{(j)}$, satisfying the two-sided Ore condition, and that

$$\begin{aligned} \mathrm{Im}(f)S_j^{-1} &= \widehat{A^{(j+1)}}[x_j; \sigma'_j, \delta'_j]S_j^{-1}, \\ A^{(j)}S_j^{-1} &= A^{(j+1)}S_j^{-1}. \end{aligned}$$

Thus assertion (III) is proved.

The property $y_l y_j = \lambda_{l,j} y_j y_l$, along with the fact that $\lambda_{l,j} = \lambda_{j,l}^{-1}$, allows us to rearrange (8) to obtain

$$A^{(j)} = \mathbb{K}[y_1] \dots [y_{j-1}; \sigma_{j-1}, \delta_{j-1}][y_j; \sigma_j^{(j)}][y_{j+1}; \sigma_{j+1}^{(j)}] \dots [y_N; \sigma_N^{(j)}]. \quad (9)$$

Defining

$$A_{j-1}^{(j)} := \mathbb{K}[y_1] \dots [y_{j-1}; \sigma_{j-1}, \delta_{j-1}]$$

we see that $A_{j-1}^{(j)} \cong A_{j-1}$ and there is an isomorphism

$$\begin{aligned} A^{(j)} &= \mathbb{K}[y_1] \dots [y_{j-1}; \sigma_{j-1}, \delta_{j-1}][y_j; \sigma_j^{(j)}][y_{j+1}; \sigma_{j+1}^{(j)}] \dots [y_N; \sigma_N^{(j)}] \\ &\cong \mathbb{K}[X_1] \dots [X_{j-1}; \sigma_{j-1}, \delta_{j-1}][X_j; \sigma_j^{(j)}][X_{j+1}; \sigma_{j+1}^{(j)}] \dots [X_N; \sigma_N^{(j)}] \end{aligned} \quad (10)$$

sending y_l to X_l for all $l \in \llbracket 1, N \rrbracket$, where the maps (as defined on suitable subalgebras of $A^{(j)}$) are as follows:

$$(a') \quad \sigma_j^{(j)} = \sigma'_j|_{A_{j-1}^{(j)}} = \sigma_j;$$

$$(b') \quad \sigma_i^{(j)}|_{A_{j-1}^{(j)}} = \sigma_i^{(j+1)*}|_{A_{j-1}^{(j)}} = \sigma_i^{(j+1)}|_{A_{j-1}^{(j)}} \text{ for all } i \in \llbracket j+1, N \rrbracket;$$

$$(c') \quad \sigma_i^{(j)}(y_l) = \lambda_{i,l} y_l \text{ for all } i \in \llbracket j+1, N \rrbracket \text{ and } l \in \llbracket 1, i-1 \rrbracket.$$

Using the isomorphism in (10) along with the observations (a')–(c') above we can prove assertion (II) for all $i \in \llbracket j, N \rrbracket$: Observation (a') proves both parts of assertion (II) when $i = j$, since σ_j satisfies assertion (II) by definition (see H1.2.1 and H1.2.4). When $i \in \llbracket j+1, N \rrbracket$, observation (b') proves (II)(ii), since $\sigma_i^{(j+1)}$ satisfies H2.2.2, and observation (c') proves (II)(i). \square

Remark 2.4. If A is an algebra satisfying Hypothesis 1.2 then Hypothesis 2.2 is satisfied for $j = N + 1$. Theorem 2.3 then implies that Hypothesis 2.2 is also satisfied for all $j \in \llbracket 2, N + 1 \rrbracket$.

Corollary 2.5. *The algebra $A' := A^{(2)}$ is a quantum affine space. More precisely, by setting $T_i := X_i^{(2)}$ for all $i \in \llbracket 1, N \rrbracket$ and $\Lambda := (\lambda_{i,j}) \in M_N(\mathbb{K})$ to be the multiplicatively antisymmetric matrix, we obtain:*

$$A' = \mathbb{K}_\Lambda[T_1, \dots, T_N].$$

Remark 2.6. For all $j \in \llbracket 1, N \rrbracket$, we say that $A^{(j+1)}$ is the algebra obtained from A after $N - j$ steps of the deleting derivations algorithm.

2.2 Ring of fractions

In order to be able to track the completely prime ideals along the deleting derivations algorithm we need the following two results regarding the division ring of fractions of the algebras $A^{(j)}$ at each step of the algorithm. These results were proved in [Cau03a, Subsection 3.3] in the generic setting and can be applied directly to our setting thanks to the above results.

Let Σ be the multiplicatively closed set in A' generated by the elements T_1, \dots, T_N . For $j \in \llbracket 2, N \rrbracket$, define sets Σ_j :

$$\begin{aligned} \Sigma_2 &:= \Sigma, \\ \Sigma_{j+1} &= A^{(j+1)} \cap \Sigma_j \quad \text{for } j \in \llbracket 2, N \rrbracket. \end{aligned}$$

Proposition 2.7. *For all $j \in \llbracket 2, N + 1 \rrbracket$ the following are true:*

- (i) Σ_j is a multiplicatively closed set of regular elements in $A^{(j)}$ containing $X_{j-1}^{(j)}, \dots, X_N^{(j)}$;
- (ii) Σ_j satisfies the two-sided Ore condition in $A^{(j)}$;
- (iii) The algebras $A^{(j)}\Sigma_j^{-1} \subset \text{Frac}(A)$ are all equal.

Proof. We proceed by induction on j . When $j = 2$, statements (i) and (ii) are trivially true from the definition of Σ and the fact that the generators (and monomials in these generators) of a quantum affine space are regular and normal. Let $j \in \llbracket 2, N \rrbracket$ and suppose statements (i) and (ii) hold for j . We will show they also hold for $j + 1$ and that $A^{(j)}\Sigma_j^{-1} = A^{(j+1)}\Sigma_{j+1}^{-1}$.

Recall the notation from the previous section: $\{x_i\}_i = \{X_i^{(j+1)}\}_i$ and $\{y_i\}_i = \{X_i^{(j)}\}_i$ where $x_i = y_i$ for all $i \geq j$. By the induction hypothesis, Σ_j is a multiplicatively closed set of regular elements in $A^{(j)}$ containing y_{j-1}, \dots, y_N . Therefore $\Sigma_{j+1} = A^{(j)} \cap \Sigma_j$ is a multiplicatively closed set of regular elements of $A^{(j+1)}$ containing $y_j = x_j, \dots, y_N = x_N$, thus proving statement (i).

Recall the set $S_j = \{x_j^n \mid n \in \mathbb{N}\} = \{y_j^n \mid n \in \mathbb{N}\} \subset \Sigma_j \cap \Sigma_{j+1}$ and use Theorem 2.3(III) to obtain the inclusions

$$A^{(j+1)} \subset A^{(j+1)}S_j^{-1} = A^{(j)}S_j^{-1} \subset A^{(j)}\Sigma_j^{-1}. \quad (11)$$

Since $\Sigma_{j+1} \subset \Sigma_j$ then Σ_{j+1} must be invertible in $A^{(j)}\Sigma_j^{-1}$. We use this to show that an element $a \in A^{(j)}\Sigma_j^{-1}$ can be rewritten as an element in $A^{(j+1)}\Sigma_{j+1}^{-1}$. Write $a = yu^{-1}$, with $y \in A^{(j)}$ and $u \in \Sigma_j$. Since

$$\Sigma_j \subset A^{(j)} \subset A^{(j)}S_j^{-1} = A^{(j+1)}S_j^{-1},$$

we can write $u = vs_1^{-1}$ and $y = xs_2^{-1}$, with $v, x \in A^{(j+1)}$ and $s_1, s_2 \in S_j$. Then a becomes

$$a = xs_2^{-1}(vs_1^{-1})^{-1} = xs_2^{-1}s_1v^{-1} = xs_1s_2^{-1}v^{-1} = xs_1(vs_2)^{-1}.$$

Observe that $vs_2 = (vs_1^{-1})s_1s_2 = us_1s_2 \in \Sigma_j$ since $u \in \Sigma_j$ and $s_1, s_2 \in S_j \subset \Sigma_j$. Also, $v \in A^{(j+1)}$ and $s_2 \in S_j \subset A^{(j+1)}$, therefore $vs_2 \in A^{(j+1)} \cap \Sigma_j = \Sigma_{j+1}$. Similarly, $xs_1 \in A^{(j+1)}$ so we can write

$$a = bc^{-1} \in A^{(j+1)}\Sigma_{j+1}^{-1},$$

where $b = xs_1 \in A^{(j+1)}$ and $c = vs_2 \in \Sigma_{j+1}$. From the inductive hypothesis we know that $A^{(j)}\Sigma_j^{-1} = \Sigma_j^{-1}A^{(j)}$, so if $a \in A^{(j)}\Sigma_j^{-1}$ then it must also be true that $a \in \Sigma_j^{-1}A^{(j)}$. We also know from Theorem 2.3(III) that S_j is an Ore set in $A^{(j)}$ and $A^{(j+1)}$, thus $A^{(j)}S_j^{-1} = S_j^{-1}A^{(j)}$ and $A^{(j+1)}S_j^{-1} = S_j^{-1}A^{(j+1)}$. Using these results we can follow a similar method to before to rewrite $a \in A^{(j)}\Sigma_j^{-1} = \Sigma_j^{-1}A^{(j)}$ as

$$a = c'^{-1}b' \in \Sigma_{j+1}^{-1}A^{(j+1)},$$

with $c' \in \Sigma_{j+1}$ and $b' \in A^{(j+1)}$.

If we can prove that Σ_{j+1} is a two-sided Ore set in $A^{(j+1)}$ then the working above implies that $A^{(j)}\Sigma_j^{-1} \subseteq A^{(j+1)}\Sigma_{j+1}^{-1}$. Furthermore, $A^{(j+1)} \subset A^{(j)}\Sigma_j^{-1}$ and $\Sigma_{j+1} \subset \Sigma_j$ so we also have $A^{(j+1)}\Sigma_{j+1}^{-1} \subseteq A^{(j)}\Sigma_j^{-1}$. Hence statement (iii) is true if we can prove that statement (ii) holds.

From the inclusion $A^{(j+1)}\Sigma_{j+1}^{-1} \subseteq A^{(j)}\Sigma_j^{-1}$ we can write any $a = bc^{-1} \in A^{(j+1)}\Sigma_{j+1}^{-1}$ as $a \in A^{(j)}\Sigma_j^{-1}$ and, applying the above working, we see that there exist $c' \in \Sigma_{j+1}$ and $b' \in A^{(j+1)}$ such that $a = c'^{-1}b' \in \Sigma_{j+1}^{-1}A^{(j+1)}$. This verifies the two-sided Ore condition on $\Sigma_{j+1} \subset A^{(j+1)}$ necessary for proving statement (ii) and, by the comment earlier, statement (iii). \square

From the above proposition it is clear that:

Theorem 2.8. (i) *There exists a multiplicatively closed set of regular elements $S \subseteq A$ such that $AS^{-1} = A'\Sigma^{-1} = \mathbb{K}_\Lambda[T_1^{\pm 1}, \dots, T_N^{\pm 1}]$.*

(ii) $\text{Frac}(A^{(j)}) = \text{Frac}(A)$ for all $j \in \llbracket 2, N+1 \rrbracket$ and, in particular, $\text{Frac}(A) = \text{Frac}(A')$.

(iii) *A is a PI algebra if and only if the $\lambda_{i,j}$ are roots of unity for all $i, j \in \llbracket 1, N \rrbracket$ and, in this case, $\text{PI-deg}(A) = \text{PI-deg}(A')$.*

Proof. Let $S = \Sigma_{N+1}$. Proposition 2.7(iii) shows that $AS^{-1} = A'\Sigma^{-1}$ and that all the algebras $A^{(j)}$ have a common localisation, thus proving (i) and (ii). [CP93, Proposition 7.1] states that A' is PI if and only if the $\lambda_{i,j}$ are roots of unity for all $1 \leq i, j \leq N$ and [Hay08, Corollary 4.7] states the same result for A . Therefore they have the same PI degree since they have equal total rings of fractions. This proves (iii). \square

3 Deleting Derivations Algorithm on completely prime quotients

In this section we set up a canonical embedding $\psi : \text{C.Spec}(A) \rightarrow \text{C.Spec}(A')$, from the completely prime spectrum of A into the completely prime spectrum of A' , and use this to extend the algorithm of the previous section to quotient algebras. We also define criteria for some ideal $Q \in \text{C.Spec}(A')$ to lie in the image of ψ .

Many of the results of this section are analogues to those found in the generic setting [Cau03a, Sections 4 and 5] and their proofs follow in almost the same way, thanks to the results of the previous section. The proofs concerning membership criteria for the canonical embedding in Subsection 3.2 closely mirror those in the Poisson setting, which can be found in [LL17, Sections 2.3 and 2.4].

3.1 Canonical Embedding and partition of completely prime spectra

3.1.1 Embedding

Let A satisfy Hypothesis 1.2 and define the following sets, which we endow with the induced Zariski topology:

$$\begin{aligned}\mathcal{P}_j^0(A^{(j)}) &:= \{P \in \text{C.Spec}(A^{(j)}) \mid y_j \notin P\}, \\ \mathcal{P}_j^1(A^{(j)}) &:= \{P \in \text{C.Spec}(A^{(j)}) \mid y_j \in P\}, \\ \mathcal{P}_j^0(A^{(j+1)}) &:= \{P \in \text{C.Spec}(A^{(j+1)}) \mid x_j \notin P\}, \\ \mathcal{P}_j^1(A^{(j+1)}) &:= \{P \in \text{C.Spec}(A^{(j+1)}) \mid x_j \in P\}.\end{aligned}$$

In the following results, by the term *increasing* we mean that the homeomorphism is order-preserving with respect to inclusion of ideals. By *bi-increasing* we mean that both the homeomorphism and its inverse are increasing homeomorphisms.

Lemma 3.1. *There is an increasing homeomorphism $\psi_j^0 : \mathcal{P}_j^0(A^{(j+1)}) \rightarrow \mathcal{P}_j^0(A^{(j)})$ given by $\psi_j^0(P) := PS_j^{-1} \cap A^{(j)}$. Its inverse is defined by $(\psi_j^0)^{-1}(Q) := QS_j^{-1} \cap A^{(j+1)}$ and is also increasing. Hence ψ_j^0 is bi-increasing.*

Proof. Extension and contraction maps provide bi-increasing inverse homeomorphisms between $\text{C.Spec}(A^{(j)}S_j^{-1})$ and $\mathcal{P}_j^0(A^{(j)})$. Similarly, we obtain a bi-increasing homeomorphism between $\mathcal{P}_j^{(0)}(A^{(j+1)})$ and $\text{C.Spec}(A^{(j+1)}S_j^{-1})$. Since $A^{(j)}S_j^{-1} = A^{(j+1)}S_j^{-1}$ then their completely prime spectra, as topological spaces, are equal. Therefore the two bi-increasing homeomorphisms defined here give rise to the bi-increasing homeomorphism in the statement of this lemma. \square

Next we turn our attention to the sets $\mathcal{P}_j^1(A^{(j)})$ and $\mathcal{P}_j^1(A^{(j+1)})$.

Lemma 3.2. *There is a surjective algebra homomorphism $g_j : A^{(j)} \rightarrow A^{(j+1)}/\langle x_j \rangle$ which takes $y_i \mapsto x_i + \langle x_j \rangle$, for all $1 \leq i \leq N$, where $x_i + \langle x_j \rangle$ is the canonical image of x_i in $A^{(j+1)}/\langle x_j \rangle$.*

Proof. By Theorem 2.3 we have

$$\begin{aligned}A^{(j+1)} &:= \mathbb{K}\langle x_1, \dots, x_N \rangle \cong \mathbb{K}[X_1] \cdots [X_j; \sigma_j, \delta_j][X_{j+1}; \sigma_{j+1}^{(j+1)}] \cdots [X_N; \sigma_N^{(j+1)}], \\ A^{(j)} &:= \mathbb{K}\langle y_1, \dots, y_N \rangle \cong \mathbb{K}[X_1] \cdots [X_{j-1}; \sigma_{j-1}, \delta_{j-1}][X_j; \sigma_j^{(j)}] \cdots [X_N; \sigma_N^{(j)}].\end{aligned}$$

Restricting these algebras to $R := \mathbb{K}\langle x_1, \dots, x_{j-1} \rangle$ and $S := \mathbb{K}\langle y_1, \dots, y_{j-1} \rangle$ we see that there is an isomorphism $S \rightarrow R$ sending $y_i \mapsto x_i$ for all $i \in \llbracket 1, j-1 \rrbracket$. Since $R \subseteq A^{(j+1)}$ we can compose this isomorphism with the natural surjection $A^{(j+1)} \rightarrow A^{(j+1)}/\langle x_j \rangle$ to obtain the algebra homomorphism $f : S \rightarrow A^{(j+1)}/\langle x_j \rangle$, sending $y_i \mapsto x_i + \langle x_j \rangle$ for all $i \in \llbracket 1, j-1 \rrbracket$. Using the commutation rules for $A^{(j+1)}$ and the fact that $x_j + \langle x_j \rangle = 0$, we see that $(x_j + \langle x_j \rangle)(x_i + \langle x_j \rangle) = \lambda_{j,i}(x_i + \langle x_j \rangle)(x_j + \langle x_j \rangle) = 0$ for all $i \in \llbracket 1, j-1 \rrbracket$. The relations on $x_i + \langle x_j \rangle \in A^{(j+1)}/\langle x_j \rangle$ therefore agree with those on $y_i \in S[y_j; \sigma_j^{(j)}] \cdots [y_N; \sigma_N^{(j)}] = A^{(j)}$, for all $i \in \llbracket 1, N \rrbracket$. Applying the universal property of Ore extensions allows us to conclude the construction of the surjective homomorphism g_j . \square

Lemma 3.3. *There is an increasing injective map $\psi_j^1 : \mathcal{P}_j^1(A^{(j+1)}) \rightarrow \mathcal{P}_j^1(A^{(j)})$ taking $P \mapsto \psi_j^1(P) := g_j^{-1}(P/\langle x_j \rangle)$, where $P/\langle x_j \rangle$ denotes the canonical image of P in $A^{(j+1)}/\langle x_j \rangle$, which induces a bi-increasing homeomorphism between $\mathcal{P}_j^1(A^{(j+1)})$ and the image $\psi_j^1(\mathcal{P}_j^1(A^{(j+1)}))$.*

Proof. Using the First Isomorphism Theorem for algebras, we restrict the map g_j from Lemma 3.2 to yield an isomorphism $g'_j : A^{(j)}/\ker(g_j) \xrightarrow{\sim} A^{(j+1)}/\langle x_j \rangle$. This induces the following bi-increasing homeomorphisms between sets endowed with the Zariski topology:

$$\begin{aligned} f_1 : \mathcal{P}_j^1(A^{(j+1)}) &\longrightarrow \text{C.Spec}(A^{(j+1)}/\langle x_j \rangle), \\ f_2 : \text{C.Spec}(A^{(j+1)}/\langle x_j \rangle) &\longrightarrow \text{C.Spec}(A^{(j)}/\ker(g_j)), \\ f_3 : \text{C.Spec}(A^{(j)}/\ker(g_j)) &\longrightarrow \{Q \in \text{C.Spec}(A^{(j)}) \mid \ker(g_j) \subseteq Q\}. \end{aligned}$$

The composition of these maps gives a bi-increasing homeomorphism

$$\begin{aligned} f_3 \circ f_2 \circ f_1 : \mathcal{P}_j^1(A^{(j+1)}) &\longrightarrow \{Q \in \text{C.Spec}(A^{(j)}) \mid \ker(g_j) \subseteq Q\} \\ P &\longmapsto g_j^{-1}(P/\langle x_j \rangle). \end{aligned}$$

Note that $g_j(y_j) = x_j + \langle x_j \rangle = 0$ so $\langle y_j \rangle \subseteq \ker(g_j)$, which leads to the inclusion

$$\{Q \in \text{C.Spec}(A^{(j)}) \mid \ker(g_j) \subseteq Q\} \subseteq \mathcal{P}_j^1(A^{(j)}).$$

Therefore, from $f_3 \circ f_2 \circ f_1$, we can define an increasing injective map

$$\begin{aligned} \psi_j^1 : \mathcal{P}_j^1(A^{(j+1)}) &\longrightarrow \mathcal{P}_j^1(A^{(j)}) \\ P &\longmapsto g_j^{-1}(P/\langle x_j \rangle), \end{aligned}$$

which induces a bi-increasing homeomorphism on its image, $\{Q \in \text{C.Spec}(A^{(j)}) \mid \ker(g_j) \subseteq Q\}$. \square

Using the two previous results we define the map $\psi_j : \text{C.Spec}(A^{(j+1)}) \rightarrow \text{C.Spec}(A^{(j)})$ where, for $P \in \text{C.Spec}(A^{(j+1)})$, we set

$$\psi_j(P) := \begin{cases} \psi_j^0(P) = PS_j^{-1} \cap A^{(j)} & \text{if } P \in \mathcal{P}_j^0(A^{(j+1)}); \\ \psi_j^1(P) = g_j^{-1}(P/\langle x_j \rangle) & \text{if } P \in \mathcal{P}_j^1(A^{(j+1)}). \end{cases}$$

The next result follows immediately.

Proposition 3.4. *For $j \in \llbracket 2, N \rrbracket$ the map $\psi_j : \text{C.Spec}(A^{(j+1)}) \rightarrow \text{C.Spec}(A^{(j)})$ is injective. For $\epsilon \in \{0, 1\}$, ψ_j induces (by restriction) a bi-increasing homeomorphism $\mathcal{P}_j^\epsilon(A^{(j+1)}) \rightarrow \psi_j(\mathcal{P}_j^\epsilon(A^{(j+1)}))$ which is a closed subset of $\mathcal{P}_j^\epsilon(A^{(j)})$.*

Using these maps, we define the canonical embedding:

Definition 3.5. Set $\psi := \psi_2 \circ \dots \circ \psi_N$ to be the injective map $\psi : \text{C.Spec}(A) \rightarrow \text{C.Spec}(A')$. We call ψ the *canonical embedding* of $\text{C.Spec}(A)$ into $\text{C.Spec}(A')$.

3.1.2 Partition of $\text{C.Spec}(A)$

Let $\mathcal{W} := \mathbb{P}(\llbracket 1, N \rrbracket)$ denote the power set of $\llbracket 1, N \rrbracket$ and, for all $w \in \mathcal{W}$, set

$$\text{C.Spec}_w(A') := \{Q \in \text{C.Spec}(A') \mid Q \cap \{T_1, \dots, T_N\} = \{T_i \mid i \in w\}\},$$

where $\{T_i\}_{i=1}^N$ are the generators of the quantum affine space A' . From [GL98, Section 2.1] we get:

Lemma 3.6. *The sets $\{\text{C.Spec}_w(A')\}_{w \in \mathcal{W}}$ provide a partition of $\text{C.Spec}(A')$.*

We use ψ to pull this partition back to one on $\text{C.Spec}(A)$. For each $w \in \mathcal{W}$ we define

$$\text{C.Spec}_w(A) := \psi^{-1}(\text{C.Spec}_w(A'))$$

and let $\mathcal{W}' \subseteq \mathcal{W}$ denote the set of all $w \in \mathcal{W}$ such that $\text{C.Spec}_w(A) \neq \emptyset$. (Note that \mathcal{W}' depends on the expression of the algebra A as an iterated Ore extension.) We immediately obtain a partition of $\text{C.Spec}(A)$.

Theorem 3.7. *The set $\text{C.Spec}(A)$ has a partition indexed by the family \mathcal{W}' so that,*

$$\text{C.Spec}(A) = \bigsqcup_{w \in \mathcal{W}'} \text{C.Spec}_w(A), \quad \text{where } |\mathcal{W}'| \leq |\mathcal{W}| = 2^N.$$

Definition 3.8. We call the partition $\{\text{C.Spec}_w(A)\}_{w \in \mathcal{W}'}$ the *canonical partition* of $\text{C.Spec}(A)$, and we call each $w \in \mathcal{W}'$ a *Cauchon diagram* of A .

3.2 Properties of the Canonical Embedding

In order to use the deleting derivations algorithm for the purpose of calculating the PI degree of quotient algebras, we need to be able to test whether a completely prime ideal of A' lies in the image of the canonical embedding.

Lemma 3.9. *Fix some $j \in \llbracket 2, N \rrbracket$ and let $Q \in \text{C.Spec}(A^{(j)})$. Then,*

$$Q \in \text{Im}(\psi_j) \iff [\text{Either } x_j = y_j \notin Q \text{ or } \ker(g_j) \subseteq Q].$$

Proof. Apply Lemmas 3.1 and 3.3 to the cases $y_j \notin Q$ and $y_j \in Q$ respectively. \square

We define injective maps $f_j : \text{C.Spec}(A^{(j+1)}) \rightarrow \text{C.Spec}(A')$, for all $j \in \llbracket 1, N \rrbracket$, with $f_1 := \text{id}_{\text{C.Spec}(A')}$, the identity on $\text{C.Spec}(A')$ and, for all $j \in \llbracket 2, N \rrbracket$, $f_j := \psi_2 \circ \cdots \circ \psi_j$, so that $f_N = \psi$.

Proposition 3.10. *Let $Q \in \text{C.Spec}(A')$. The following are equivalent:*

- $Q \in \text{Im}(\psi)$.
- For all $j \in \llbracket 2, N \rrbracket$ we have $Q \in \text{Im}(f_{j-1})$ and either $X_j^{(j)} = X_j^{(j+1)} \notin f_{j-1}^{-1}(Q)$ or $\ker(g_j) \subseteq f_{j-1}^{-1}(Q)$.

Proof. Let $Q \in \text{C.Spec}(A')$. Suppose $Q \in \text{Im}(\psi)$. Then $Q = \psi(P)$ for some $P \in \text{C.Spec}(A)$. Since $\psi = f_{j-1} \circ \psi_j \circ \cdots \circ \psi_N$ then $Q = f_{j-1}(P_j)$ for all $j \in \llbracket 2, N \rrbracket$, where $P_j = \psi_j \circ \cdots \circ \psi_N(P)$. Hence $Q \in \text{Im}(f_{j-1})$ for all $j \in \llbracket 2, N \rrbracket$. From this we see that $f_{j-1}^{-1}(Q) \in \text{Im}(\psi_j)$, for all $j \in \llbracket 2, N \rrbracket$, and we apply Lemma 3.9 to $f_{j-1}^{-1}(Q)$ to conclude that either $X_j^{(j)} = X_j^{(j+1)} \notin f_{j-1}^{-1}(Q)$ or $\ker(g_j) \subseteq f_{j-1}^{-1}(Q)$.

Now suppose the second statement holds. By Lemma 3.9, $f_{j-1}^{-1}(Q) \in \text{Im}(\psi_j)$ for all $j \in \llbracket 2, N \rrbracket$. Let $P' := f_{N-1}^{-1}(Q)$ so that $P' \in \text{Im}(\psi_N)$. Then $P' = \psi_N(P)$ for some $P \in \text{C.Spec}(A)$ and

$$Q = f_{N-1}(P') = f_{N-1}(\psi_N(P)) = \psi(P) \in \text{Im}(\psi).$$

\square

We finish this subsection by giving a sufficient condition for a completely prime ideal in A' to be in the image of the canonical embedding. These next two results are not used in this paper, however they are stated here for the interested reader (proofs can be found in [Rog19, Section 4]).

Theorem 3.11. [Rog19, Theorem 4.17] *Let $w \in \mathcal{W}'$. Then $\psi(\text{C.Spec}_w(A))$ is a (non-empty) closed subset of $\text{C.Spec}_w(A')$ and the map ψ induces (by restriction) a bi-increasing homeomorphism from $\text{C.Spec}_w(A)$ to $\psi(\text{C.Spec}_w(A))$.*

Proposition 3.12. [Rog19, Proposition 4.18] *Let $w \in \mathcal{W}'$, $P \in \text{C.Spec}_w(A)$, and $Q \in \text{C.Spec}_w(A')$. If $\psi(P) \subseteq Q$ then $Q \in \text{Im}(\psi)$.*

3.3 Completely Prime Quotients

In this subsection we extend the algorithm in Section 2 to apply to completely prime quotient algebras of quantum nilpotent algebras. The results of the previous sections allow us to construct proofs in a similar way to those found in [Cau03a, Section 5.3]. This extended algorithm will allow us to pull certain irreducible representations on A'/Q (clarified in Section 5) back to irreducible representations on A/P .

3.3.1 $A^{(j)}/P$ and $A^{(j+1)}/\psi_j(P)$

We start by extending one step of the algorithm to completely prime quotients. For some $j \in \llbracket 2, N \rrbracket$, let $P \in \text{C.Spec}(A^{(j+1)})$ and $Q = \psi_j(P) \in \text{C.Spec}(A^{(j)})$ be its image under the canonical embedding. Set

$$B^{(j+1)} := A^{(j+1)}/P, \quad B^{(j)} := A^{(j)}/Q.$$

We denote by $\bar{x}_1, \dots, \bar{x}_N \in B^{(j+1)}$ and $\bar{y}_1, \dots, \bar{y}_N \in B^{(j)}$ the canonical images of the generators $x_1, \dots, x_N \in A^{(j+1)}$ in $B^{(j+1)}$ and $y_1, \dots, y_N \in A^{(j)}$ in $B^{(j)}$, respectively.

Lemma 3.13. *Suppose $\bar{x}_j = 0$. Then there exists an algebra isomorphism $B^{(j)} \rightarrow B^{(j+1)}$ sending $\bar{y}_i \mapsto \bar{x}_i$ for all $i \in \llbracket 1, N \rrbracket$.*

Proof. Since $x_j \in P$ then $Q = \psi_j^1(P) = g_j^{-1}(P/\langle x_j \rangle)$. Concatenating g_j with the natural surjection $\pi : A^{(j+1)}/\langle x_j \rangle \rightarrow A^{(j+1)}/P$ gives the following surjective algebra homomorphism:

$$\begin{array}{ccccc} A^{(j)} & \xrightarrow{g_j} & A^{(j+1)}/\langle x_j \rangle & \xrightarrow{\pi} & A^{(j+1)}/P \\ y_i & \mapsto & x_i + \langle x_j \rangle & \mapsto & \bar{x}_i, \end{array}$$

The desired isomorphism may then be constructed upon noting that $\ker(\pi \circ g_j) = g_j^{-1}(P/\langle x_j \rangle) = Q$. \square

Lemma 3.14. *Suppose $\bar{x}_j \neq 0$ and let $Z_j := \{\bar{x}_j^n \mid n \in \mathbb{N}\}$. Then the following hold:*

- (i) Z_j is a multiplicative set of regular elements of $B^{(j+1)}$, which satisfies the two-sided Ore condition.
- (ii) There exists an injective algebra homomorphism $\gamma : B^{(j)} \rightarrow B^{(j+1)}Z_j^{-1}$ defined on the generators of $B^{(j)}$ in the following way:

$$\gamma(\bar{y}_i) = \begin{cases} \bar{x}_i & \text{if } i \geq j; \\ \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} \lambda_{j,i}^{-n} \overline{d_{j,n}(x_i)} \bar{x}_j^{-n} & \text{if } i < j, \end{cases}$$

where $\overline{d_{j,n}(x_i)}$ denotes the canonical image of $d_{j,n}(x_i)$ in $B^{(j+1)}$.

- (iii) If we identify $B^{(j)}$ with its image $\gamma(B^{(j)}) \subseteq B^{(j+1)}Z_j^{-1}$ then Z_j is a multiplicative set of regular elements in $B^{(j)}$ satisfying the two-sided Ore condition. Furthermore, $B^{(j)}Z_j^{-1} = B^{(j+1)}Z_j^{-1}$.

Proof. Since $x_j \notin P$ then $Q = \psi_j^0(P) = PS_j^{-1} \cap A^{(j)}$, where $S_j = \{x_j^n \mid n \in \mathbb{N}\}$ is the multiplicatively closed set of regular elements in $A^{(j+1)}$ and $A^{(j)}$ satisfying the two-sided Ore condition (Theorem 2.3(III)). Denote the subalgebra $A^{(j+1)}S_j^{-1} = A^{(j)}S_j^{-1} \subseteq F$ by Ω and the completely prime ideal $PS_j^{-1} = QS_j^{-1}$ by $\Theta \triangleleft \Omega$. Note that $\Theta \cap A^{(j+1)} = P$ and $\Theta \cap A^{(j)} = Q$.

We define an injective algebra homomorphism $\gamma' : B^{(j+1)} \rightarrow \Omega/\Theta$; $a + P \mapsto a1^{-1} + \Theta$ and identify $B^{(j+1)}$ with its image. Since $Z_j = \{\bar{x}_j^n \mid n \in \mathbb{N}\} = \{x_j^n + P \mid n \in \mathbb{N}\} = S_j + P$, its image under γ' becomes

$$\gamma'(Z_j) = Z_j 1^{-1} + \Theta = (S_j + P)1^{-1} + \Theta = S_j 1^{-1} + \Theta \subseteq B^{(j+1)} \subseteq \Omega/\Theta.$$

Identifying Z_j with its image $\gamma'(Z_j)$, we observe that all elements of the set Z_j are invertible in $B^{(j+1)}$. We can therefore write any element $b \in \Omega/\Theta$ as

$$b = a_1 s_1^{-1} + \Theta = s_2^{-1} a_2 + \Theta, \quad (12)$$

where $a_1, a_2 \in A^{(j+1)}$ and $s_1, s_2 \in S_j$. Let $b_1, b_2 \in B^{(j+1)}$ and $z_1, z_2 \in Z_j$ such that

$$b_1 = a_1 1^{-1} + \Theta, \quad b_2 = a_2 1^{-1} + \Theta, \quad z_1 = s_1 1^{-1} + \Theta, \quad z_2 = s_2 1^{-1} + \Theta.$$

Using (12), we see that, for all $b \in \Omega/\Theta$,

$$b = b_1 z_1^{-1} = z_2^{-1} b_2.$$

This shows that the set $Z_j \subset B^{(j+1)}$ satisfies the two-sided Ore condition, thus proving property (i) of the lemma. We have also proved the equality $B^{(j+1)} Z_j^{-1} = \Omega/\Theta$, i.e. $(A^{(j+1)}/P) Z_j^{-1} = A^{(j+1)} S_j^{-1} / P S_j^{-1}$.

For part (ii) we use the fact that $\Theta \cap A^{(j)} = Q$ and $B^{(j+1)} Z_j^{-1} = \Omega/\Theta$ to define an injective homomorphism

$$\begin{aligned} \gamma : B^{(j)} &\longrightarrow \Omega/\Theta \\ a + Q &\longmapsto a 1^{-1} + \Theta. \end{aligned}$$

Recall that the generators $y_i \in A^{(j)}$ are defined as:

$$y_i := \begin{cases} x_i & \text{if } i \geq j; \\ \sum_{n=0}^{\infty} q_j \frac{n(n+1)}{2} (q_j - 1)^{-n} \lambda_{j,i}^{-n} d_{j,n}(x_i) x_j^{-n} & \text{if } i < j. \end{cases}$$

It is straightforward to check that $\gamma(\bar{y}_i) = y_i 1^{-1} + \Theta$ gives the desired results for $i < j$ and $i \geq j$.

Part (iii) is proved in the same way as (i) by identifying $B^{(j)}$ with its image $\gamma(B^{(j)})$. \square

Lemmas 3.13 and 3.14 imply:

Lemma 3.15. $\text{Frac}(B^{(j+1)}) = \text{Frac}(B^{(j)})$.

3.3.2 A/P and $A'/\psi(P)$

We continue to extend the algorithm to apply to completely prime quotients A/P . Let $P \in \text{C.Spec}(A)$ and $Q = \psi(P) \in \text{C.Spec}(A')$ and set the following notation:

- $B := A/P$ and set $\bar{X}_1, \dots, \bar{X}_N \in B$ to be the canonical images of $X_1, \dots, X_N \in A$.
- $B' := A'/Q$ and set $t_1, \dots, t_N \in B'$ to be the canonical images of $T_1, \dots, T_N \in A'$.
- For $j \in \llbracket 2, N+1 \rrbracket$, denote by $P_j := \psi_j \circ \dots \circ \psi_N(P) \in \text{C.Spec}(A^{(j)})$ the image of P after $N - j + 1$ steps of the deleting derivations algorithm.

- For each $j \in \llbracket 2, N+1 \rrbracket$, define the algebra $B^{(j)} := A^{(j)}/P_j$ and denote by $\bar{X}_1^{(j)}, \dots, \bar{X}_N^{(j)}$ the canonical images of $X_1^{(j)}, \dots, X_N^{(j)}$ in $B^{(j)}$. Note: $B^{(N+1)} = B$ with $(\bar{X}_1^{(N+1)}, \dots, \bar{X}_N^{(N+1)}) = (\bar{X}_1, \dots, \bar{X}_N)$, and $B^{(2)} = B'$ with $(\bar{X}_1^{(2)}, \dots, \bar{X}_N^{(2)}) = (t_1, \dots, t_N)$.

Proposition 3.16. *For each $j \in \llbracket 2, N+1 \rrbracket$, $B^{(j)}$ is a subalgebra of $\text{Frac}(B)$ generated by $\bar{X}_1^{(j)}, \dots, \bar{X}_N^{(j)}$ and there is an algebra homomorphism,*

$$\begin{aligned} f_j : A^{(j)} &\longrightarrow \text{Frac}(B) \\ X_i^{(j)} &\longmapsto \bar{X}_i^{(j)} \end{aligned}$$

with image $B^{(j)}$ and kernel P_j .

Proof. Since $B^{(j)}$ is (trivially) a subalgebra of $\text{Frac}(B^{(j)})$ generated by $\bar{X}_1^{(j)}, \dots, \bar{X}_N^{(j)}$, Lemma 3.15 implies that $B^{(j)}$ is a subalgebra of $\text{Frac}(B)$ generated by these same elements. The homomorphism f_j is the concatenation of the natural embedding $B^{(j)} \hookrightarrow \text{Frac}(B)$ with the canonical surjection, $\pi_j : A^{(j)} \rightarrow A^{(j)}/P_j = B^{(j)}$. The stated image and kernel are easily verified. \square

Proposition 3.17. *Let $j \in \llbracket 2, N+1 \rrbracket$.*

- (i) *If $\bar{X}_j^{(j+1)} = 0$ then $\bar{X}_i^{(j)} = \bar{X}_i^{(j+1)}$ for all $i \in \llbracket 1, N \rrbracket$.*
- (ii) *Suppose $\bar{X}_j^{(j+1)} \neq 0$ and set $(x_1, \dots, x_N) := (X_1^{(j+1)}, \dots, X_N^{(j+1)})$. Then the generators of $B^{(j)}$ are can be obtained as follows:*

$$\bar{X}_i^{(j)} = \begin{cases} \bar{X}_i^{(j+1)} & \text{if } i \geq j; \\ \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} \lambda_{j,i}^{-n} f_{j+1} \circ d_{j,n}(x_i) (\bar{X}_j^{(j+1)})^{-n} & \text{if } i < j, \end{cases}$$

where f_{j+1} is the map defined in Proposition 3.16.

- (iii) *Suppose $\bar{X}_j^{(j+1)} \neq 0$ and let $Z_j = \{(\bar{X}_j^{(j+1)})^n \mid n \in \mathbb{N}\} = \{(\bar{X}_j^{(j)})^n \mid n \in \mathbb{N}\}$ be a multiplicatively closed set of regular elements in $B^{(j)}$ and $B^{(j+1)}$. Then Z_j satisfies the two-sided Ore condition in both $B^{(j)}$ and $B^{(j+1)}$ and we have $B^{(j)}Z_j^{-1} = B^{(j+1)}Z_j^{-1}$.*

Proof. Lemmas 3.13 and 3.15 prove part (i). Part (ii) follows from Lemma 3.14 once one notes that $f_{j+1} \circ d_{j,n}(x_i) = \bar{d}_{j,n}(x_i) \cdot 1^{-1} \in \text{Frac}(B)$. Part (iii) also follows directly from Lemma 3.14. \square

Let $w \in \mathcal{W}'$ with $P \in \text{C.Spec}_w(A)$ and $Q = \psi(P)$. By the definition of $\text{C.Spec}_w(A')$, we have that $T_i \in Q$ if and only if $i \in w$ or, equivalently, $t_i = 0$ if and only if $i \in w$. Let $i \in \bar{w} := \{1, \dots, N\} \setminus w$ so that $t_i \neq 0$. Then, since T_i is normal in A' and Q is completely prime, t_i is normal and regular in B' , hence it is invertible in $\text{Frac}(B') = \text{Frac}(B)$. We denote by Σ the multiplicatively closed set of regular elements in B' generated by all t_i such that $i \in \bar{w}$. From this set we define, recursively, the sets $\Sigma_j \subset B^{(j)}$ for $j \in \llbracket 2, N+1 \rrbracket$ in the following way:

$$\Sigma_2 := \Sigma, \quad \Sigma_{j+1} := B^{(j+1)} \cap \Sigma_j.$$

The next result extends Proposition 2.7 to quotient algebras.

Proposition 3.18. *For each $j \in \llbracket 2, N+1 \rrbracket$ the following statements hold:*

- (i) Σ_j is a multiplicatively closed set of regular elements in $B^{(j)}$ which contains, as a subset, $\{\bar{X}_i^{(j)} \mid i \in \llbracket j-1, N \rrbracket \text{ and } \bar{X}_i^{(j)} \neq 0\}$;

(ii) Σ_j satisfies the two-sided Ore condition in $B^{(j)}$;

(iii) The algebras $B^{(j)}\Sigma_j^{-1}$, when considered as subalgebras of $\text{Frac}(B)$, are all equal.

Proof. We proceed by induction on j . When $j = 2$, statements (i) and (ii) are immediately satisfied by the discussion preceding this proposition. Fix $j \in \llbracket 2, N \rrbracket$ and assume statements (i) and (ii) are true. We show that these properties are also true when replacing j with $j + 1$ and that $B^{(j)}\Sigma_j^{-1} = B^{(j+1)}\Sigma_{j+1}^{-1}$. We consider two cases: $\bar{X}_j^{(j+1)} = 0$ and $\bar{X}_j^{(j+1)} \neq 0$.

When $\bar{X}_j^{(j+1)} = 0$ we apply Proposition 3.17(i) to obtain $\bar{X}_i^{(j)} = \bar{X}_i^{(j+1)}$ for all $i \in \llbracket 1, N \rrbracket$. Therefore $B^{(j)} = B^{(j+1)}$, and statements (i) and (ii) follow immediately by the inductive hypothesis and the fact that $\Sigma_{j+1} \subseteq \Sigma_j$.

Now suppose $\bar{X}_j^{(j+1)} \neq 0$. Applying Proposition 3.17(ii) gives $\bar{X}_i^{(j)} = \bar{X}_i^{(j+1)}$ for all $i \geq j$ and, by the induction hypothesis, we know that

$$\{\bar{X}_i^{(j)} \mid i \in \llbracket j-1, N \rrbracket, \bar{X}_i^{(j)} \neq 0\} \subseteq \Sigma_j.$$

Therefore,

$$\begin{aligned} \Sigma_{j+1} &:= B^{(j+1)} \cap \Sigma_j \supseteq B^{(j+1)} \cap \{\bar{X}_i^{(j)} \mid i \in \llbracket j-1, N \rrbracket, \bar{X}_i^{(j)} \neq 0\} \\ &= \{\bar{X}_i^{(j+1)} \mid i \in \llbracket j, N \rrbracket, \bar{X}_i^{(j+1)} \neq 0\}. \end{aligned}$$

The set Σ_j is multiplicatively closed, by the induction hypothesis, which means Σ_{j+1} is multiplicatively closed and it contains regular elements because P_{j+1} is a completely prime ideal. Hence $B^{(j+1)}$ is a domain in which all nonzero elements are regular. This proves part (i).

Note that $Z_j \subset \Sigma_{j+1}$ and $Z_j \subset \Sigma_j$. Applying Proposition 3.17(iii), we see that

$$B^{(j+1)} \subset B^{(j+1)}Z_j^{-1} = B^{(j)}Z_j^{-1} \subset B^{(j)}\Sigma_j^{-1}.$$

These inclusions mirror those found in (11), and we may apply the rest of the method used in the proof of Proposition 2.7 to conclude parts (ii) and (iii) of this proposition. \square

An immediate consequence of this proposition is that we can now show equivalence of the total rings of fractions of A/P and $A'/\psi(P)$ in the case where $P \in \text{C.Spec}_w(A)$, for some $w \in \mathcal{W}'$.

Theorem 3.19. *Let $w \in \mathcal{W}'$, $P \in \text{C.Spec}_w(A)$, and $Q = \psi(P) \in \text{C.Spec}_w(A')$. Let Σ be the multiplicatively closed set of elements in A'/Q which is generated by all the generators t_i such that $i \in \bar{w} = \{1, \dots, N\} \setminus w$, where t_i is the canonical image of T_i in A'/Q . Then,*

- (i) Σ is a multiplicatively closed set of regular, normal elements in A'/Q .
- (ii) There exists a multiplicatively closed set of regular elements, Γ , in A/P which satisfies the two-sided Ore condition in A/P and is such that $(A/P)\Gamma^{-1} = (A'/Q)\Sigma^{-1}$.
- (iii) $\text{Frac}(A/P) = \text{Frac}(A'/Q)$. Furthermore, A/P and A'/Q are PI algebras if all $\lambda_{i,j}$ with $i, j \in \bar{w}$ are roots of unity and, in this case, $\text{PI-deg}(A/P) = \text{PI-deg}(A'/Q)$.

The following ideals are defined for later use as they ensure that at the end of the deleting derivations algorithm on A/P we obtain a quantum affine space $A'/\psi(P)$.

Definition 3.20. Let $w \in \mathcal{W}'$. We call $P_w \in \text{C.Spec}_w(A)$ a *Cauchon ideal* if $\psi(P_w) = J_w := \langle T_i \in A' \mid i \in w \rangle \in \text{C.Spec}_w(A')$.

Remark 3.21. It can be shown that a rational torus action on A induces a rational torus action on A' and that ψ sends torus-invariant completely prime ideals in A to torus-invariant completely prime ideals in A' . This matches analogous results in the generic setting [Cau03a] and the Poisson setting [LL17]. Unlike in those settings, this observation does not immediately yield results allowing us to determine explicitly what W' is for specific algebras A . For this reason, we omit these results from this paper.

4 PI degree of completely prime quotients of quantum matrices at roots of unity

In this section, we consider iterated Ore extensions $A = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \dots [X_N; \sigma_N, \delta_N]$ satisfying Hypothesis 1.2 **at roots of unity**. That is, we assume that all the parameters $\lambda_{i,j}$ from Hypothesis 1.2 are roots of unity.

In that case, it follows from [Hay08, Theorem 1.2] that the algebra A is PI and that, with the notation of the previous section:

$$\text{PI-deg}(A) = \text{PI-deg}(A').$$

Our aim in this section is to provide techniques to compute the PI degree of completely prime quotients of A . In particular, we will focus on the case where A is a single parameter quantum matrix algebra (at roots of unity).

4.1 PI degree of completely prime quotients of A

Under our assumptions, A is a PI algebra and this implies that every completely prime quotient of A is a PI algebra too. Our next aim is to compute the PI degree and construct irreducible representations of maximal dimension of quotients by Cauchon ideals.

4.2 PI degree of quotients by a Cauchon ideal

In the case where P_w is the Cauchon ideal associated to a Cauchon diagram $w \in \mathcal{W}'$, we get that $\psi(P_w) = \langle T_i \mid i \in w \rangle$ and so A'/Q is a quantum affine space. Namely, $A'/Q = \mathcal{O}_{\Lambda_w}(\mathbb{K}^{N-|w|})$, where Λ_w is the matrix deduced from $\Lambda = (\lambda_{ij})$ by removing rows and columns indexed by elements of w .

In view of Theorem 3.19(iii) and the above discussion, we obtain the following result.

Theorem 4.1. *Let P_w be the Cauchon ideal associated to a Cauchon diagram $w \in \mathcal{W}'$. Then $\text{PI-deg}(A/P_w) = \text{PI-deg}(\mathcal{O}_{\Lambda_w}(\mathbb{K}^{N-|w|}))$.*

De Concini and Procesi developed techniques to compute the PI degree of a quantum affine space. More precisely, they prove the following result.

Theorem 4.2. [CP93, Proposition 7.1] *Let $S = (s_{ij})$ be a multiplicatively skew-symmetric $n \times n$ matrix with coefficients in \mathbb{K} . Then:*

1. *If all s_{ij} are roots of unity, then there exists a primitive root of unity $q \in \mathbb{K}^\times$ and integers a_{ij} such that $s_{ij} = q^{a_{ij}}$ for all i, j .*
2. *Assume $s_{ij} = q^{a_{ij}}$ for all i, j , where q is a primitive ℓ^{th} root of unity and $a_{ij} \in \mathbb{Z}$ for all i, j . Set $M := (a_{ij}) \in M_n(\mathbb{Z})$; this is a skew-symmetric matrix. Then the PI degree of the*

corresponding quantum affine space $\mathcal{O}_q(\mathbb{K}^n) := \mathcal{O}_S(\mathbb{K}^n)$ is \sqrt{h} , where h is the cardinality of the image of the homomorphism

$$\mathbb{Z}^n \xrightarrow{M} \mathbb{Z}^n \xrightarrow{\pi} (\mathbb{Z}/\ell\mathbb{Z})^n,$$

where π denotes the canonical epimorphism.

The above result together with Theorem 4.1 allows us to compute the PI degree of quotients by Cauchon ideals.

In the following section, we illustrate this in the case when A is a single parameter quantum matrix algebra.

4.3 PI degree of quotients of single parameter quantum matrix algebras by Cauchon ideals

In this section, we assume that $\lambda_{ij} = q^{m_{ij}}$ for all i, j , where q is a primitive ℓ^{th} root of unity and $m_{ij} \in \mathbb{Z}$ for all i, j . In this case, we say that A is *uniparameter*. We set $M := (m_{ij}) \in M_N(\mathbb{K})$, so that M is a skew-symmetric integral matrix. It is a well-known result that every skew-symmetric integral matrix S is congruent (in the sense of [New72, Chapter IV]) to its skew-normal form, that is, to a block diagonal matrix of the form

$$S = \begin{pmatrix} 0 & h_1 & & & & & & \\ -h_1 & 0 & & & & & & \\ & & 0 & h_2 & & & & \\ & & -h_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & h_s & \\ & & & & & -h_s & 0 & \\ & & & & & & & \mathbf{0} \end{pmatrix},$$

where $\mathbf{0}$ is a square matrix of zeros of dimension $\dim(\ker(S))$ so that $2s = N - \dim(\ker(S))$, and $h_1, h_1, h_2, h_2, \dots, h_s, h_s \in \mathbb{Z} \setminus \{0\}$ are called the *invariant factors* of M . As they always come in pairs, from now on we will avoid repetition and list the invariant factors simply as h_1, h_2, \dots, h_s . These have the property that $h_i | h_{i+1}$ for all $i \in \llbracket 1, s-1 \rrbracket$.

Theorem 4.3. *Assume A is a uniparameter iterated Ore extension satisfying Hypothesis 1.2 with parameter q being a primitive ℓ^{th} root of unity. Let $w \in \mathcal{W}$ and $P_w := \psi^{-1}(J_w)$ be the corresponding Cauchon ideal. Then*

$$\text{PI-deg}(A/P_w) = \prod_{i=1}^{\frac{N - \dim(\ker(M(w)))}{2}} \frac{\ell}{\gcd(h_i, \ell)},$$

where the h_i are the invariant factors of the matrix $M(w)$ deduced from M by removing rows and columns indexed by w .

We finish this section by studying the case of single parameter quantum matrices. That is, we assume that $A = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$ with q being a primitive ℓ^{th} root of unity. Recall that the quantised coordinate ring of $m \times n$ matrices over \mathbb{K} denoted by $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ is the \mathbb{K} -algebra generated by $m \times n$ indeterminates $\{X_{i,j}\}_{i,j=1}^{m,n}$ subject to the following relations: for $(1, 1) \leq (i, j) < (s, t) \leq (m, n)$ (in lexicographic ordering), we have

$$X_{i,j}X_{s,t} = \begin{cases} X_{s,t}X_{i,j} & i < s, j > t; \\ qX_{s,t}X_{i,j} & (i = s, j < t) \text{ or } (i < s, j = t); \\ X_{s,t}X_{i,j} + (q - q^{-1})X_{i,t}X_{s,j} & i < s, j < t. \end{cases}$$

Under our assumption that q is a primitive ℓ^{th} root of unity, the algebra $A = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$ is a uniparameter iterated Ore extension (with the generators added in the lexicographic order) satisfying Hypothesis 1.2, see [Hay08, Section 5.3] for details.

Let w be a Cauchon diagram and P_w the corresponding Cauchon ideal in A . It follows from the above result that to compute the PI degree of A/P_w , we need to compute the dimension of $\ker(M(w))$ and the invariant factors of $M(w)$. This was done in [BCL12] and [BLR22] respectively. Before stating our main result, we recall necessary results from these two articles.

Given any diagram $w \in \mathcal{W}$, we associate an $m \times n$ diagram $D(w)$, where $D(w)$ is the $m \times n$ grid whose square in position (i, j) is coloured black if $(i, j) \in w$ and white if $(i, j) \notin w$. We may compute its *toric permutation* $\tau = \tau_w$, as defined in [BCL12, Section 4.1], by laying pipes over the squares such that we place a “cross” on each black square and a “hyperbola” on each white square. We label the sides of the diagram with the numbers $1, \dots, m+n$ such that each pair of opposite sides share the same labels in the same order. The permutation, τ_w , may then be read off this diagram by defining $\tau_w(i)$ to be the label (on the left or top side of $D(w)$) reached by following the pipe starting at label i (on the right or bottom side of $D(w)$). See Figure 1 for an example of a diagram with $\tau = (17)(26384)$.

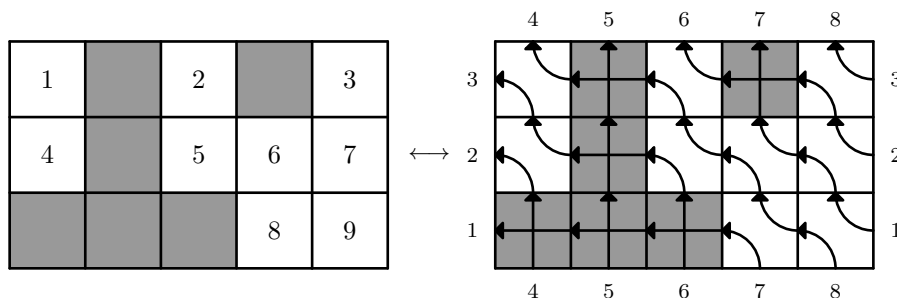


Figure 1: Labelled 3×5 diagram (left) with pipe dream construction (right) associated to the diagram $\{(1, 2), (1, 4), (2, 2), (3, 1), (3, 2), (3, 3)\}$ in $\mathcal{O}_q(M_{3,5}(\mathbb{K}))$.

It was proved in [BCL12] that $\dim(\ker(M(w)))$ is given by the number $r(w)$ of odd cycles in the disjoint cycle decomposition of its associated toric permutation τ_w . Moreover, [BLR22, Theorem 2.6] shows that all invariant factors of $M(w)$ are powers of 2.

Putting all these together, we obtain the following result.

Theorem 4.4. *Assume q is a primitive ℓ^{th} root of unity with ℓ odd. Let w be a Cauchon diagram for $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ and P_w be the corresponding Cauchon ideal. Then*

$$\text{PI-deg}(\mathcal{O}_q(M_{m,n}(\mathbb{K}))/P_w) = \sqrt{\ell^{mn-r(w)}},$$

where $r(w)$ is the number of odd cycles in the disjoint cycle decomposition of the toric permutation associated to w .

Note that we did not need an explicit description of the set \mathcal{W}' of Cauchon diagrams to obtain the above result. For completeness, we note that \mathcal{W}' coincides with the set of $m \times n$ Cauchon-Le diagrams (see [Rog19, Theorem 4.37] for details).

5 Irreducible Representations of Maximum Dimension

We take $1 \neq q \in \mathbb{K}^*$ to be arbitrary, unless stated otherwise. Let A be an algebra satisfying Hypothesis 1.2 such that $\lambda_{i,j} = q^{m_{i,j}}$ for some skew-symmetric matrix $M = (m_{i,j})_{i,j} \in M_N(\mathbb{Z})$. Fix $P \in \text{C.Spec}_w(A)$, for some Cauchon diagram $w \in \mathcal{W}'$. We make use of the deleting derivations algorithm to construct an irreducible representation of A/P given a “suitable” irreducible representation of $A'/\psi(P)$. When A is a PI algebra (which holds if q is a root of unity) and \mathbb{K} is algebraically closed, it turns out that any irreducible representation of $A'/\psi(P)$ is “suitable”.

Recall the notation set in Section 3.3: $B := A/P$ with generators $\bar{X}_i := X_i + P$ and $B' := A'/\psi(P)$ with generators $t_i := \bar{T}_i$. Set, again, $\Sigma := \Sigma_2 \subseteq B'$ to be the (two-sided) Ore set generated by the t_i , for $i \in \llbracket 1, N \rrbracket \setminus w$ and then define, recursively for all $j \in \llbracket 2, N \rrbracket$, $\Sigma_{j+1} := B^{j+1} \cap \Sigma_j$, with $\Gamma := \Sigma_{N+1}$ being a (two-sided) Ore set in B .

Proposition 5.1. *Let A be a \mathbb{K} -algebra satisfying Hypothesis 1.2, $P \in \text{C.Spec}_w(A)$ for some $w \in \mathcal{W}'$ and suppose that (ϕ', V) is an irreducible representation of B' where, for each $e \in \Sigma$, there exists some $\xi = \xi_e \in \mathbb{K}^*$ and $\ell = \ell_e \in \mathbb{N}_{>1}$ such that $\phi'(e)^\ell = \xi \text{Id}_V$. Then*

1. *For any $b \in B$, there exists some $b' \in B'$ and $e \in \Sigma$ such that*

$$b = b'e^{-1} \in B'\Sigma^{-1}$$

and ϕ' induces an irreducible representation of B on V (of the same dimension), defined by the algebra homomorphism

$$\begin{aligned} \phi : B &\longrightarrow \text{End}_{\mathbb{K}}(V) \\ b &\longmapsto \xi^{-1} \phi'(b') \phi'(e)^{\ell-1}. \end{aligned}$$

2. *For each $g \in \Gamma$, there exists some $\xi' \in \mathbb{K}^*$ and $\ell' \in \mathbb{N}_{>1}$ such that $\phi(g)^{\ell'} = \xi' \text{Id}_V$.*

Proof. Let (ϕ', V) be an irreducible representation of B' satisfying the conditions of the proposition. The condition on ϕ' implies that $\phi'(e)^{-1} = \xi_e^{-1} \phi'(e)^{\ell_e-1}$ and induces a representation of $B'\Sigma^{-1}$,

$$\begin{aligned} \hat{\phi} : B'\Sigma^{-1} &\longrightarrow \text{End}_{\mathbb{K}}(V) \\ b' &\longmapsto \phi'(b') \\ e^{-1} &\longmapsto \xi_e^{-1} \phi'(e)^{\ell_e-1}, \end{aligned}$$

from which we observe that $\hat{\phi}(e)^{-1} = \xi_e^{-1} \hat{\phi}(e)^{\ell_e-1}$, for all $e \in \Sigma$. The inclusion $B' \subseteq B'\Sigma^{-1}$ ensures that $(\hat{\phi}, V)$ is irreducible.

From Proposition 3.18 and Theorem 3.19 we see that $B'\Sigma^{-1} = B\Gamma^{-1}$, hence each element in $B\Gamma^{-1}$ can be written in terms of elements in $B'\Sigma^{-1}$, and vice versa. This allows us to view $\hat{\phi}$ as an algebra homomorphism $\hat{\phi} : B\Gamma^{-1} \rightarrow \text{End}_{\mathbb{K}}(V)$, hence $(\hat{\phi}, V)$ defines an irreducible representation of $B\Gamma^{-1}$.

Every element $b \in B$ may be written as an element $b \cdot 1^{-1} \in B\Gamma^{-1}$, which we write simply as $b \in B\Gamma^{-1}$. By the equality of localisations, b may also be written as an element in $B'\Sigma^{-1}$, that is, $b = b'e^{-1} \in B'\Sigma^{-1}$. Restricting $\hat{\phi}$ to B gives the following algebra homomorphism:

$$\begin{aligned} \phi : B &\longrightarrow \text{End}_{\mathbb{K}}(V) \\ b &\longmapsto \hat{\phi}(b'e^{-1}) = \xi^{-1} \phi'(b') \phi'(e)^{\ell-1}. \end{aligned}$$

This defines a (not necessarily irreducible) representation of B on V . To show that this representation is irreducible we note that $\Gamma \subseteq \Sigma$. Therefore, for all $g \in \Gamma$, there exists some $\xi' \in \mathbb{K}^*$ and $\ell' \in \mathbb{N}$ such that $\hat{\phi}(g)^{\ell'} = \xi' \text{Id}_V$, hence

$$\phi(g)^{\ell'} = \hat{\phi}(g)^{\ell'} = \xi' \text{Id}_V.$$

Using the identity $B\Gamma^{-1} = B'\Sigma^{-1}$ we may write any element $b' \in B'$ as $b' = bg^{-1} \in B\Gamma^{-1}$. Thus

$$\phi'(b') = \hat{\phi}(b') = \hat{\phi}(bg^{-1}) = \xi'^{-1} \hat{\phi}(b) \hat{\phi}(g)^{\ell'-1} = \phi(\xi'^{-1} b g^{\ell'-1}),$$

which shows that $\phi'(B') \subseteq \phi(B)$. Therefore, since (ϕ', V) is irreducible, so too is (ϕ, V) . \square

Remark 5.2. If we take $P = \{0\}$ in Proposition 5.1 then we get $B' = A'$ and $B = A$ and the statement can be stated analogously given the results and notation of Proposition 2.7 and Theorem 2.8. This means we can construct an irreducible representation on A provided we have an irreducible representation on (ϕ', V) satisfying the condition $\phi'(e)^{\ell_e} = \xi_e \text{Id}_V$ for all elements e of the Ore set in A' generated by $\{T_i\}_{i=1}^N$ and for some $\ell \in \mathbb{N}_{>1}$.

Restricting the algebra A to the root of unity case and taking \mathbb{K} to be algebraically closed (needed for Schur's lemma used in the proof), we see that any irreducible representation on B' will satisfy the conditions of Proposition 5.1.

Theorem 5.3. *Take \mathbb{K} to be an algebraically closed field. Let A be a \mathbb{K} -algebra satisfying Hypothesis 1.2 and suppose that the $\lambda_{i,j}$ in H1.2.1 are of the form $\lambda_{i,j} = q^{m_{i,j}}$ for some skew-symmetric matrix $M = (m_{i,j})_{i,j} \in M_N(\mathbb{Z})$, where $q \in \mathbb{K}^*$ is a primitive ℓ^{th} **root of unity** with $\ell \in \mathbb{N}_{>1}$. Then:*

1. Any irreducible representation (ϕ', V) of B' satisfies the conditions of Proposition 5.1.
2. Any irreducible representation (ϕ', V) of B' induces an irreducible representation (ϕ, V) of B of the same dimension.

Proof.

1. A' is a prime affine PI algebra (Theorem 2.8(iii)), as is B' , hence any irreducible representation (ϕ', V) of B' is finite dimensional. Suppose $P \in \text{C.Spec}_w(A)$, for some $w \in \mathcal{W}'$. Then the quotient algebra, $B' = A'/\psi(P)$, is generated by all t_i such that $i \in \llbracket 1, N \rrbracket \setminus w$, where $t_i := T_i + \psi(P)$. Hence $t_i t_j = q^{m_{i,j}} t_j t_i$, for all $i, j \in \llbracket 1, N \rrbracket \setminus w$. We deduce that $t_i^\ell \in Z(B')$ for all $i \in \llbracket 1, N \rrbracket \setminus w$, thus, by Schur's Lemma, $\phi'(t_i)^\ell = \xi_i \text{Id}_V$ for some $\xi_i \in \mathbb{K}^*$. Since Σ is defined to be the multiplicatively closed set generated by $\{t_i \mid i \in \llbracket 1, N \rrbracket \setminus w\}$, then the conditions of Proposition 5.1 are satisfied.
2. Follows from Proposition 5.1.

\square

Remark 5.4. When P is a Cauchon ideal ($P = P_w$ for some $w \in \mathcal{W}'$) then, by definition, B' becomes a quantum affine space. If A then satisfies the conditions of Theorem 5.3 (i.e. we take a quantum nilpotent algebra at a root of unity, q) then

$$\text{PI-deg}(B') = \prod_{i=1}^{\frac{N - \dim(\ker(M(w)))}{2}} \frac{\ell}{\gcd(h_i, \ell)},$$

where the h_i are the invariant factors of the matrix $M(w)$ deduced from M by removing rows and columns indexed by w (see Theorem 4.3). We can construct an irreducible representation of

B' of maximal dimension using [BLR22, Proposition 3.4]. This will, by Theorem 5.3, induce an irreducible representation of B via the method of Proposition 5.1. These results therefore give a method to construct an explicit maximal dimensional irreducible representation of B for a given $w \in \mathcal{W}'$.

Proposition 5.1 and Theorem 5.3 have been stated for the two extreme algebras in the DDA. In practice, we would often deal with one step of the DDA at a time, that is we would deduce an irreducible representation of $A^{(j+1)}/P_{j+1}$ from an irreducible representation of $A^{(j)}/P_j$.

5.1 Example: A maximal irreducible representation of $U_q^+(\mathfrak{so}_5)/\langle z' \rangle$

To illustrate Remark 5.4 we will construct an irreducible representation of a quotient of $A := U_q^+(\mathfrak{so}_5)$ by the ideal $\langle z' \rangle$ (defined later in this section). This example will require applications of the deleting derivations algorithm on both A (Theorem 2.3) and $A/\langle z' \rangle$ (Theorem 3.19), computing the image of $\langle z' \rangle$ by the canonical embedding (Section 3.2), and the calculation of the PI degree of $A/\langle z' \rangle$ (Theorem 4.3). The full details of these computations are omitted as they are straightforward but lengthy, however they can be found in [Rog19, Sections 5.3.1 and 7.3.2].

For this example, we take q to be a primitive ℓ^{th} root of unity with $\ell \notin \{2, 4\}$.

5.1.1 Defining the algebra

$U_q^+(\mathfrak{so}_5)$ is the \mathbb{C} -algebra generated by two indeterminates E_1, E_2 subject to the the following relations:

$$\begin{aligned} E_1^3 E_2 - (q^2 + 1 + q^{-2}) E_1^2 E_2 E_1 + (q^2 + 1 + q^{-2}) E_1 E_2 E_1^2 - E_2 E_1^3 &= 0 \\ E_2^2 E_1 - (q^2 + q^{-2}) E_2 E_1 E_2 + E_1 E_2^2 &= 0. \end{aligned}$$

There is a PBW basis of $U_q^+(\mathfrak{so}_5)$ formed by monomials $E_1^{k_1} E_4^{k_4} E_3^{k_3} E_2^{k_2}$, where k_1, k_2, k_3, k_4 are nonnegative integers and E_3, E_4 are certain root vectors. This result can be found, for example, in [Lau06, Section 2.4]. The same paper also expresses this algebra as an iterated Ore extension over \mathbb{C} generated by these four indeterminates in the order E_1, E_4, E_3, E_2 . For easier application of the deleting derivations algorithm, we relabel the indeterminates:

$$X_1 := E_1, \quad X_2 := E_4, \quad X_3 := E_3, \quad X_4 := E_2.$$

The relations between X_1, X_2, X_3, X_4 in [Lau06] then become

$$\begin{aligned} X_2 X_1 &= q^{-2} X_1 X_2, \\ X_3 X_1 &= X_1 X_3 - (q + q^{-1}) X_2, & X_3 X_2 &= q^{-2} X_2 X_3, \\ X_4 X_1 &= q^2 X_1 X_4 - q^2 X_3, & X_4 X_2 &= X_2 X_4 - \frac{q^2 - 1}{q + q^{-1}} X_3^2, & X_4 X_3 &= q^{-2} X_3 X_4. \end{aligned}$$

This allows us to present $A := U_q^+(\mathfrak{so}_5)$ as the following iterated Ore extension:

$$U_q^+(\mathfrak{so}_5) = \mathbb{C}[X_1][X_2; \sigma_2][X_3; \sigma_3, \delta_3][X_4; \sigma_4, \delta_4]$$

where, using the notation $A_j := \mathbb{C}\langle X_1, \dots, X_j \rangle \subseteq A$, the automorphisms σ_i and skew-derivations δ_i are defined on the generators as:

$$\begin{array}{llll}
\sigma_2 : A_1 \longrightarrow A_1; & X_1 \longmapsto q^{-2}X_1, & & \\
\sigma_3 : A_2 \longrightarrow A_2; & X_1 \longmapsto X_1 & \delta_3 : A_2 \longrightarrow A_2; & X_1 \longmapsto -(q + q^{-1})X_2 \\
& X_2 \longmapsto q^{-2}X_2, & & X_2 \longmapsto 0, \\
\sigma_4 : A_3 \longrightarrow A_3; & X_1 \longmapsto q^2X_1 & \delta_4 : A_3 \longrightarrow A_3; & X_1 \longmapsto -q^2X_3 \\
& X_2 \longmapsto X_2 & & X_2 \longmapsto -\frac{q^2 - 1}{q + q^{-1}}X_3^2 \\
& X_3 \longmapsto q^{-2}X_3, & & X_3 \longmapsto 0.
\end{array} \tag{13}$$

5.1.2 Verifying that $U_q^+(\mathfrak{so}_5)$ satisfies Hypothesis 1.2

Routine computations on these maps using [Hay08, Theorem 2.8 and Lemma 5.3] show that A satisfies all properties of Hypothesis 1.2. In particular:

(H1.2.1) The $\lambda_{i,j}$'s are as in (13).

(H1.2.2) $\lambda_{i,j} = q^{m_{i,j}}$, with the $m_{i,j}$ defining the skew-symmetric matrix

$$M := \begin{pmatrix} 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \end{pmatrix} \in M_4(\mathbb{Z}).$$

(H1.2.3) (σ_3, δ_3) is q^2 -skew and (σ_4, δ_4) is q^4 -skew. So we set $q_3 = q^2$ and $q_4 = q^4$.

(H1.2.4) The higher q_j -skew σ_j -derivations are defined to be:

$$d_{j,n}(X_l) = \begin{cases} X_l & n = 0; \\ \delta_j(X_l) & n = 1; \\ 0 & n > 1, \end{cases} \tag{14}$$

for $j \in \{3, 4\}$ and $l \in \llbracket 1, j-1 \rrbracket$.

Since there are two nonzero derivations we apply Theorem 2.3 twice to obtain $A' = \mathbb{C}_{q^M}[T_1, T_2, T_3, T_4]$, where

$$T_4 := X_4, \quad T_3 := X_3, \quad T_2 := X_2 - \frac{q^4}{(q^2 + 1)(q + q^{-1})}X_3^2X_4^{-1}, \quad T_1 := X_1 - \frac{q^2(q + q^{-1})}{q^2 - 1}X_2X_3^{-1}. \tag{15}$$

5.1.3 Finding the PI degree of $U_q^+(\mathfrak{so}_5)/\langle z' \rangle$

Consider the ideal $\langle z' \rangle \triangleleft A$, generated by the central element (see [Lau06, Section 2.4])

$$z' := -(q^2 - q^{-2})(q + q^{-1})X_2X_4 + q^2(q^2 - 1)X_3^2.$$

Using localisation theory, Proposition 2.7(iii), and repeated application of Proposition 3.4 and Lemma 3.1, one can show that $\psi^{-1}(\langle T_2 \rangle) = \langle z' \rangle$. In particular this proves that $\langle z' \rangle \in \mathcal{C}\text{-Spec}_{\{2\}}(A)$ and therefore $\{2\} \in \mathcal{W}'$. In fact, since $A'/\langle T_2 \rangle = \mathbb{C}_{q^{M'}}[t_1, t_3, t_4]$, where $t_i := T_i + \langle T_2 \rangle$ for all $i \in \{1, 3, 4\}$, then $\langle z' \rangle$ is a Cauchon ideal.

By Theorem 3.19,

$$\text{PI-deg}(U_q^+(\mathfrak{so}_5)/\langle z' \rangle) = \text{PI-deg}(A'/\langle T_2 \rangle) = \text{PI-deg}(\mathbb{C}_{q^{M'}}[t_1, t_3, t_4])$$

where $M' = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$ is obtained from M by deleting the second row and second column. It is easily verified that the skew normal form of M' is $S = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, hence M' has a kernel of dimension 1 and one pair of invariant factors: $h_1 = 2$. Applying Theorem 4.3 with $P_{\{2\}} = \langle z' \rangle$ gives:

$$\text{PI-deg}(A/\langle z' \rangle) = \prod_{i=1}^{\frac{3-1}{2}} \frac{\ell}{\gcd(h_i, \ell)} = \frac{\ell}{\gcd(2, \ell)} = \begin{cases} \ell & \ell \text{ is odd;} \\ \ell/2 & \ell > 4 \text{ is even.} \end{cases}$$

5.1.4 Constructing an irreducible representation of $U_q^+(\mathfrak{so}_5)/\langle z' \rangle$ of maximum dimension

Recall that $B' := \mathbb{C}_{q^{M'}}[t_1, t_3, t_4]$, $B := U_q^+(\mathfrak{so}_5)/\langle z' \rangle$ and let $\Sigma \subseteq B'$ be the multiplicatively closed set generated by $\{t_1, t_3, t_4\}$. The quantum affine space associated to S is $D := \mathbb{C}_{q^S}[x_1, y_1, z_1]$. By [BLR22, Proposition 3.4(ii)] there is an ℓ -dimensional \mathbb{C} -vector space V , $\lambda, \xi \in \mathbb{C}^*$ and an algebra homomorphism $\varphi : D \rightarrow \text{End}_{\mathbb{C}}(V)$ whose image on the generators of D , upon fixing a basis $v_1, \dots, v_{\ell} \in V$, may be presented as matrices in the following way:

$$\varphi(x_1) = \begin{pmatrix} \lambda & & & & \\ & \lambda q^2 & & & \\ & & \lambda q^4 & & \\ & & & \ddots & \\ & & & & \lambda q^{(\ell-1)2} \end{pmatrix}, \quad \varphi(y_1) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}, \quad \varphi(z_1) = \xi \text{Id}_V.$$

The pair (φ, V) defines an irreducible representation of D .

To define an irreducible representation of $\mathbb{C}_{q^{M'}}[t_1, t_3, t_4]$ we apply [BLR22, Proposition 3.4(iii)] using $E^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, where $E \in M_3(\mathbb{Z})$ is the invertible matrix satisfying $EM'E^T = S$. The resulting algebra homomorphism $\phi' : \mathbb{C}_{q^{M'}}[t_1, t_3, t_4] \rightarrow \text{End}_{\mathbb{C}}(V)$, is defined on t_1, t_3, t_4 as

$$\begin{aligned} \phi'(t_1) &= \varphi(x_1^1 y_1^0 z_1^1) = \varphi(x_1) \varphi(z_1) = \xi \varphi(x_1), \\ \phi'(t_3) &= \varphi(x_1^{-1} y_1^0 z_1^0) = \varphi(x_1)^{-1} = \lambda^{-\ell} \varphi(x_1^{\ell-1}), \\ \phi'(t_4) &= \varphi(x_1^0 y_1^{-1} z_1^0) = \varphi(y_1)^{-1} = \varphi(y_1^{\ell-1}), \end{aligned} \tag{16}$$

and it defines an irreducible representation of $\mathbb{C}_{q^{M'}}[t_1, t_3, t_4]$ on V . Using the definition of φ , we see that

$$\begin{aligned} \phi'(t_1)^\ell &= \xi^\ell \varphi(x_1)^\ell = (\xi \lambda)^\ell \text{Id}_V, \\ \phi'(t_3)^\ell &= \lambda^{-\ell^2} \varphi(x_1)^{\ell^2 - \ell} = \lambda^{-\ell^2} \lambda^{\ell^2 - \ell} \text{Id}_V = \lambda^{-\ell} \text{Id}_V, \\ \phi'(t_4)^\ell &= \varphi(y_1)^{\ell^2 - \ell} = \text{Id}_V, \end{aligned}$$

hence the conditions of Proposition 5.1 are satisfied. Applying this proposition allows us to define an irreducible representation of B once we know how to write the generators $\bar{X}_1, \dots, \bar{X}_4 \in B$ in terms of the generators $t_1, t_3, t_4 \in B'$. Using (15) we see that

$$t_4 := \bar{X}_4, \quad t_3 = \bar{X}_3, \quad t_2 = \bar{X}_2 - \frac{q^4}{(q^2+1)(q+q^{-1})} \bar{X}_3^2 \bar{X}_4^{-1}, \quad t_1 = \bar{X}_1 - \frac{q^2(q+q^{-1})}{q^2-1} \bar{X}_2 \bar{X}_3^{-1}.$$

Rearranging these identities, and noting that $t_2 = 0$, allows us to write

$$\bar{X}_4 = t_4, \quad \bar{X}_3 = t_3, \quad \bar{X}_2 = \frac{q^4}{(q^2+1)(q+q^{-1})}t_3^2t_4^{-1}, \quad \bar{X}_1 = t_1 + \frac{q^4}{q^4-1}t_3t_4^{-1}.$$

Applying Proposition 5.1 and using (16), we deduce that there is an algebra homomorphism $\phi : B \rightarrow \text{End}_{\mathbb{C}}(V)$ defined on generators as

$$\begin{aligned} \phi(\bar{X}_4) &= \phi'(t_4) = \varphi(y_1)^{-1}, & \phi(\bar{X}_3) &= \phi'(t_3) = \varphi(x_1)^{-1}, \\ \phi(\bar{X}_2) &= \phi' \left(\frac{q^4}{(q^2+1)(q+q^{-1})}t_3^2t_4^{-1} \right) & \phi(\bar{X}_1) &= \phi' \left(t_1 + \frac{q^4}{q^4-1}t_3t_4^{-1} \right) \\ &= \frac{q^4}{(q^2+1)(q+q^{-1})}\varphi(x_1)^{-2}\varphi(y_1), & &= \xi\varphi(x_1) + \frac{q^4}{q^4-1}\varphi(x_1)^{-1}\varphi(y_1). \end{aligned}$$

which defines an irreducible representation (ϕ, V) of B . Substituting in the matrices for $\varphi(x_1)$ and $\varphi(y_1)$ and taking, for example, $\ell = 5$ gives the following explicit form of this irreducible representation:

$$\begin{aligned} \phi(\bar{X}_4) &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, & \phi(\bar{X}_3) &= \begin{pmatrix} \lambda^{-1} & 0 & 0 & 0 & 0 \\ 0 & \lambda^{-1}q^3 & 0 & 0 & 0 \\ 0 & 0 & \lambda^{-1}q & 0 & 0 \\ 0 & 0 & 0 & \lambda^{-1}q^4 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{-1}q^2 \end{pmatrix}, \\ \phi(\bar{X}_2) &= \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{(q^2+1)^2\lambda^2} \\ \frac{q}{(q^2+1)^2\lambda^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{q^2}{(q^2+1)^2\lambda^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{q^3}{(q^2+1)^2\lambda^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{q^4}{(q^2+1)^2\lambda^2} & 0 \end{pmatrix}, \\ \phi(\bar{X}_1) &= \begin{pmatrix} \xi\lambda & 0 & 0 & 0 & \frac{q^4}{(q^4-1)\lambda} \\ \frac{q^2}{(q^4-1)\lambda} & \xi\lambda q^2 & 0 & 0 & 0 \\ 0 & \frac{1}{(q^4-1)\lambda} & \xi\lambda q^4 & 0 & 0 \\ 0 & 0 & \frac{q^3}{(q^4-1)\lambda} & \xi\lambda q & 0 \\ 0 & 0 & 0 & \frac{q}{(q^4-1)\lambda} & \xi\lambda q^3 \end{pmatrix}. \end{aligned}$$

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