Brauer and Geršgorin locations of zeros of perturbed Chebyshev polynomials of second kind by dilation

Zélia da Rocha
Departamento de Matemática - CMUP
Faculdade de Ciências da Universidade do Porto
Rua do Campo Alegre n.687, 4169 - 007 Porto, Portugal
Phone: 00351 220402215; Fax: 00351 220402108; Email: mrdioh@fc.up.pt

Dedicated to the memory of Professor Manuel R. J. da Silva

Abstract:
We consider some perturbed of the Chebyshev polynomials of second kind obtained by modifying by dilation one of its recurrence coefficients at an arbitrary order. By applying Brauer and Geršgorin theorems to Jacobi matrices associated to such perturbed sequences, we obtain some locations of their zeros.

Key words: Perturbed Chebyshev polynomials; zeros; Jacobi matrices; Geršgorin circle theorems; *Mathematica*®.

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1 Introduction

It is well known that a polynomial sequence $\{P_n(x)\}_{n\geq 0}$ is *orthogonal* if and only if it satisfies a recurrence relation of order two (see (3)-(4)) defined by two sequences of coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$, the so-called *recurrence coefficients*. If we modify, at the order r, the β-coefficient by translation, $\beta_r \to \beta_r + \mu_r$, or the γ-coefficient by dilation, $\gamma_r \to \lambda_r \gamma_r$, by means of some parameters μ_r or λ_r, we obtain a *perturbed orthogonal sequence*, the so-called in literature generalized co-recursive or co-dilated polynomials, respectively [17]. In the last years, several authors have worked about perturbed orthogonal polynomials often taking as study case the fundamental example of the Chebyshev sequence of second kind (see (5)). With no attempt of completion, we cite [4, 5, 17, 18, 19, 21, 22]. Zeros of orthogonal polynomials, and in particular zeros of Chebyshev families, constitutes an important topic, because they are used in several methods in numerical analysis and have applications in applied sciences [12, 15, 20, 22]. With respect to properties of zeros of perturbed orthogonal polynomials see in particular [2, 3, 16, 21]. Furthermore, these families have some applications, in which their zeros play a role [11, 23]. I have also studied these perturbed sequences in the papers [6, 7, 8, 9], where the reader can find further references about this subject. In the present article, I study the location of zeros of perturbed Chebyshev sequence of second kind by dilation; the translation case will be treated in another work [10].

Is is well known that $[-1, 1]$ is the smallest interval that contains the set of zeros of the four Chebyshev sequences [20]. It is an interesting question to know how the above perturbations change the location of zeros. In this work, we furnish an answer to this question by applying *Geršgorin circle theorems* [13, 14, 24] and Brauer results [1, 24] to *Jacobi matrices* associated to perturbed Chebyshev polynomials of second kind by dilation [5, 20]. Then, we obtain some locations for their zeros. Those results by *Geršgorin* [13, 14, 24] and Brauer [1, 24] provide a simple way to determine certain intervals, circles or ovals whose union contains the eigenvalues of any $n \times n$ complex matrix.

Jacobi matrices J_n (29) are defined from the recurrence coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ of any orthogonal polynomial sequence in such a way that $P_n(x)$ is the characteristic polynomial of J_n (30). In other words, the eigenvalues of J_n are the zeros of $P_n(x)$ [5].

This article is organised as follows. After this introduction, in Section 2, we define the perturbed Chebyshev polynomials of second kind by dilation, and we give some of their properties. We keep the same type of notation adopted in previous works [8, 9]. In Section 3, we deduce a location of the real zeros of a perturbed Chebyshev polynomial of second kind by dilation in terms of the zeros of the Chebyshev polynomial of second kind with the same degree. In next section, we consider the associated Jacobi matrices, and we compute their Geršgorin disks and sets. We begin by fixing the order r of perturbation, then for each r, we should consider separately some initial values of the degree n, thereafter we obtain results for all the other values of n. In Section 5, we apply the Geršgorin and Brauer theorems and we obtain, in each case, a location for the set of zeros depending on the value of the parameter of perturbation λ_r. In some cases, we conclude about the advantage of the Brauer location with respect to the Geršgorin one. Also, we present a definition for the sharpness of the locations we obtained. In next section, in order to illustrate this work, we give some symbolic and numerical results, and we show some graphical representations. We finish this article with some conclusions.

2 Perturbed Chebyshev polynomials of second kind by dilation

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual space. We denote by $\langle u, p\rangle$ the effect of the form $u \in \mathcal{P}'$ on the polynomial $p \in \mathcal{P}$. In particular, $\langle u, x^n\rangle := (u)_n \,, n \geq 0$, represent the *moments* of u. A form u is said *regular* [18] if and only if there exists a polynomial sequence $\{P_n\}_{n\geq 0}$, such that

$$\langle u, P_n P_m\rangle \;=\; 0 \;,\; n \neq m \;,\; n, m \geq 0 \;, \tag{1}$$

$$\left\langle u, P_n^2\right\rangle = k_n \;\neq\; 0 \;,\; n \geq 0 \;. \tag{2}$$

Consequently $\deg P_n = n$, $n \geq 0$, and any P_n can be taken monic, then $\{P_n\}_{n\geq 0}$ is called a *monic orthogonal polynomial sequence* (MOPS) with respect to u. The sequence

$\{P_n\}_{n\geq 0}$ is regularly orthogonal with respect to u [18] if and only if there exists two sequences of coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$, with $\gamma_{n+1} \neq 0$, $n \geq 0$, such that $\{P_n\}_{n\geq 0}$ satisfies the following initial conditions and linear recurrence relation of order 2

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0, \tag{3}$$

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 2. \tag{4}$$

Furthermore, the recurrence coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ satisfy

$$\beta_n = \frac{\langle u, xP_n^2(x)\rangle}{k_n} \quad , \quad \gamma_{n+1} = \frac{k_{n+1}}{k_n}, \quad n \geq 0.$$

As usual, we suppose that, $\beta_n = 0$, $\gamma_{n+1} = 0$, and $P_n(x) = 0$, for all $n < 0$.

A form $u \in \mathcal{P}'$ is *positive definite* if and only if $< u, p >> 0$, $\forall p \in \mathcal{P}$: $p(x) \geq 0$, $p \not\equiv 0$, $x \in \mathbb{R}$. A form $u \in \mathcal{P}'$ is *symmetric* if and only if $(u)_{2n+1} = 0$, $n \geq 0$. A polynomial sequence $\{P_n\}_{n\geq 0}$ is symmetric if and only if $P_n(-x) = (-1)^n P_n(x)$, $n \geq 0$; then $P_{2n+1}(0) = 0$, and zeros of $P_n(x)$ are symmetric with respect to the origin. If $\{P_n\}_{n\geq 0}$ is a MOPS with respect to u, the symmetry of $\{P_n\}_{n\geq 0}$ is equivalent to $\beta_n = 0$, $n \geq 0$, and the positive definiteness of u is equivalent to β_n, $\gamma_{n+1} \in \mathbb{R}$, $\gamma_{n+1} > 0$, $n \geq 0$ [5]. If $\{P_n\}_{n\geq 0}$ is a symmetric MOPS, then $P_{2n}(0) \neq 0$, $n \geq 0$.

In this work, our starting point is the well known monic Chebyshev polynomial sequence of second kind $\{P_n(x)\}_{n\geq 0}$ [20] defined by the recurrence coefficients

$$\beta_n = 0 \quad ; \quad \gamma_{n+1} = \frac{1}{4} \ , \ n \geq 0 \ . \tag{5}$$

This is a symmetric and positive definite example. We are going to perturbe this sequence as follows. For some fixed order $r \geq 1$, we modify the recurrence coefficient γ_r by multiplying it by a parameter $\lambda_r \in \mathbb{R}$, $\lambda_r \neq 0$, $\lambda_r \neq 1$, that is, we consider the following recurrence coefficients

$$\beta_n^d = 0 \ ; \ \gamma_r^d = \frac{\lambda_r}{4} \ , \ \gamma_{n+1}^d = \frac{1}{4}, \ n \neq r-1 \ , \ n \geq 0 \ , \ r \geq 1 \ , \tag{6}$$

that determine the so-called *r-th perturbed Chebyshev sequence of second kind by dilation*, noted by $\{P_n^d(\lambda_r; r)(x)\}_{n\geq 0}$. This perturbation modifies the Chebyshev polynomials from the degree $r + 1$, as stated by the following relations

$$P_k^d(\lambda_r; r)(x) = P_k(x) \ , \ k = 0(1)r \ , \tag{7}$$

$$P_{r+1}^d(\lambda_r; r)(x) = xP_r(x) - \frac{\lambda_r}{4}P_{r-1}(x) \ , \tag{8}$$

$$P_{n+r+2}^d(\lambda_r; r)(x) = xP_{n+r+1}^d(\lambda_r; r)(x) - \frac{1}{4}P_{n+r}^d(\lambda_r; r)(x) \ , \ n \geq 0 \ . \tag{9}$$

From (6), we see that $\{P_n^d(\lambda_r; r)(x)\}_{n\geq 0}$ is symmetric; in addition, if $\lambda_r > 0$, then it is also positive definite.

The monic Chebyshev polynomial sequence of first kind $\{T_n(x)\}_{n\geq 0}$ [20] defined by the recurrence coefficients

$$\beta_n = 0 \ , \ n \geq 0 \quad ; \quad \gamma_1 = \frac{1}{2} \ , \ \gamma_{n+1} = \frac{1}{4} \ , \ n \geq 1 \ , \tag{10}$$

can be naturally considered as perturbed of $\{P_n(x)\}_{n\geq 0}$ as follows [7]

$$r = 1 \ , \ \lambda_r = 2 \ , \ T_n(x) = P_n^d(\lambda_r; r)(x) \ . \tag{11}$$

We recall the trigonometric definition of the monic Chebyshev sequence of second kind

$$P_n(x) = \frac{1}{2^n} \frac{\sin(n+1)t}{\sin t} \ , \ n \geq 0 \ , \tag{12}$$

with $x = \cos t$, $t \in [0, \pi]$; and the explicit formula of its zeros in increasing size [20]

$$\xi_k^{(n)} = \cos\left(\frac{n-k+1}{n+1}\pi\right), \ k = 1(1)n \ . \tag{13}$$

3 A location of real zeros of perturbed Chebyshev polynomial sequence of second kind by dilation

We begin by recalling some well known properties of zeros of orthogonal polynomial sequences $\{P_n(x)\}_{n\geq 0}$. In the positive definite case previously defined, all zeros are distinct real numbers [5]. Furthermore, there exists an *interlacing property* between zeros of polynomials of consecutive degrees, $P_n(x)$ and $P_{n+1}(x)$, that is,

$$\xi_k^{(n+1)} < \xi_k^{(n)} < \xi_{k+1}^{(n+1)} \ , \ k = 1(1)n \ , \ n \geq 1 \ .$$

Moreover, there are some *monotonicity properties*, namely, for $k \geq 1$, $\{\xi_k^{(n)}\}_{n=k}^{\infty}$ is a decreasing sequence, and $\{\xi_{n-k+1}^{(n)}\}_{n=k}^{\infty}$ is an increasing sequence. Also,

$$\lim_{n\to+\infty} \xi_i^{(n)} = \rho_i \quad , \quad \lim_{n\to+\infty} \xi_{n-j+1}^{(n)} = \eta_j \ ,$$

being $[\rho_1, \eta_1]$ the smallest closed interval that contains all zeros of $\{P_n(x)\}_{n\geq 0}$.

Zeros of Chebyshev polynomials satisfy all those properties in the interval $[-1, 1]$ [20]. Perturbed Chebyshev polynomials of second kind by dilation, $\{P_n^d(\lambda_r; r)(x)\}_{n\geq 0}$, with $\lambda_r \in \mathbb{R}$, $\lambda_r > 0$, satisfy those properties as well in some interval. If $\lambda_r < 0$, we are not in the positive definite case, then zeros have a different behaviour, namely there exists some conjugate pairs of complex zeros. In both cases, perturbed Chebyshev polynomials are *semi-classical sequences* [19], consequently their zeros are always simple [18, p.123].

The location of zeros we are looking for will be accomplish above taking as starting point the next result obtained in a previously article.

Proposition 3.1 *[8, p.107] If $\{P_n(x)\}_{n\geq 0}$ is the Chebyshev polynomial sequence of second kind and $\{P^d_n(\lambda_r;r)(x)\}_{n\geq 0}$ is its r-th perturbed sequence by dilation, with $\lambda_r \neq 0$, $\lambda_r \neq 1$, $r \geq 1$, then the following connection relations hold*

$$P^d_k(\lambda_r;r)(x) = P_k(x) \ , \ k = 0(1)r \ ,$$

$$P^d_k(\lambda_r;r)(x) = P_k(x) + \frac{1-\lambda_r}{4}\sum_{i=1}^{k-r}\frac{1}{4^{i-1}}P_{k-2i}(x) \ , \ k = r+1(1)2r-1 \ , \tag{14}$$

$$P^d_{n+2r}(\lambda_r;r)(x) = P_{n+2r}(x) + \frac{1-\lambda_r}{4}\sum_{i=1}^{r}\frac{1}{4^{i-1}}P_{n+2(r-i)}(x) \ , \ n \geq 0 \ . \tag{15}$$

Proposition 3.2 *If $\{P_n(x)\}_{n\geq 0}$ is the Chebyshev polynomial sequence of second kind, being $\xi^{(k)}_i$, $i = 1(1)k$, the zeros of $P_k(x)$ by increasing size, and $\{P^d_n(\lambda_r;r)\}_{n\geq 0}$ is its r-th perturbed sequence by dilation with $\lambda_r \in \mathbb{R}$, $\lambda_r \neq 0$, $\lambda_r \neq 1$, being $\xi^{(k)}_i(\lambda_r;r)$, $i = 1(1)k$, the zeros of $P^d_k(\lambda_r;r)(x)$ by increasing size, then for $k \geq r+1$ and $r \geq 1$, it holds*

1. $sgn\big[P^d_k(\lambda_r;r)(\xi^{(k)}_k)\big] = sgn(1-\lambda_r)$.
2. $sgn\big[P^d_k(\lambda_r;r)(\xi^{(k)}_1)\big] = (-1)^k sgn(1-\lambda_r)$.
3. *If $\lambda_r < 1$, then*
   (a) $\forall y \geq \xi^{(k)}_k$, $P^d_k(\lambda_r;r)(y) > 0$. *Thus there are no real zeros of $P^d_k(\lambda_r;r)(x)$ greater than or equal to $\xi^{(k)}_k$.*
   (b) $\forall z \leq \xi^{(k)}_1$, $(-1)^k P^d_k(\lambda_r;r)(z) > 0$. *Thus there are no real zeros of $P^d_k(\lambda_r;r)(x)$ less than or equal to $\xi^{(k)}_1$.*
   (c) *All real zeros of $P^d_k(\lambda_r;r)(x)$ belong to the interval $\left]\xi^{(k)}_1, \xi^{(k)}_k\right[$.*
4. *If $\lambda_r > 1$, then the number of real zeros of $P^d_k(\lambda_r;r)(x)$ less than $\xi^{(k)}_1$ or greater than $\xi^{(k)}_k$ is odd. In particular,*

$$\xi^{(k)}_1(\lambda_r;r) < \xi^{(k)}_1 \quad , \quad \xi^{(k)}_k(\lambda_r;r) > \xi^{(k)}_k. \tag{16}$$

Proof. We shall use the properties of zeros of Chebyshev polynomials mentioned in the beginning of this section and the trivial fact that for any monic polynomials it holds

$$\lim_{x\to+\infty} P_n(x) = +\infty \ , \ \lim_{x\to-\infty} P_{2n}(x) = +\infty \ , \ \lim_{x\to-\infty} P_{2n+1}(x) = -\infty. \tag{17}$$

As perturbed polynomials $P^d_k(\lambda_r;r)(x)$ are symmetric, their zeros are symmetric with respect to the origin. This fact is important in items 3 and 4. We are going to deal with the recurrence relation (14). The same could be done with (15). We remark that in (14) the polynomials under sum have degrees smaller than the degree k of $P_k(x)$ and have the same parity of it; moreover all their zeros belong to $]\xi_1^{(k)}, \xi_k^{(k)}[$. Next, we will evaluate (14) at the first and the last zeros of $P_k(x)$, and at certain y and z, and we note the signs of polynomials at those points according with (17).

$$P_k^d(\lambda_r; r)(\xi_k^{(k)}) = \underbrace{P_k(\xi_k^{(k)})}_{=\,0} + (1-\lambda_r) \sum_{i=1}^{k-r} \frac{1}{4^i} \underbrace{P_{k-2i}(\xi_k^{(k)})}_{>\,0} , \tag{18}$$

$$P_k^d(\lambda_r; r)(y) = \underbrace{P_k(y)}_{>\,0} + (1-\lambda_r) \sum_{i=1}^{k-r} \frac{1}{4^i} \underbrace{P_{k-2i}(y)}_{>\,0} , \ \forall y > \xi_k^{(k)} , \tag{19}$$

$$P_{2k}^d(\lambda_r; r)(\xi_1^{(2k)}) = \underbrace{P_{2k}(\xi_1^{(2k)})}_{=\,0} + (1-\lambda_r) \sum_{i=1}^{2k-r} \frac{1}{4^i} \underbrace{P_{2k-2i}(\xi_1^{(2k)})}_{>\,0} , \tag{20}$$

$$P_{2k+1}^d(\lambda_r; r)(\xi_1^{(2k+1)}) = \underbrace{P_{2k+1}(\xi_1^{(2k+1)})}_{=\,0} + (1-\lambda_r) \sum_{i=1}^{2k-r+1} \frac{1}{4^i} \underbrace{P_{2k-2i+1}(\xi_1^{(2k+1)})}_{<\,0} \tag{21}$$

$$P_{2k}^d(\lambda_r; r)(z) = \underbrace{P_{2k}(z)}_{>\,0} + (1-\lambda_r) \sum_{i=1}^{2k-r} \frac{1}{4^i} \underbrace{P_{2k-2i}(z)}_{>\,0} , \ \forall z < \xi_1^{(2k)} , \tag{22}$$

$$P_{2k+1}^d(\lambda_r; r)(z) = \underbrace{P_{2k+1}(z)}_{<\,0} + (1-\lambda_r) \sum_{i=1}^{2k-r+1} \frac{1}{4^i} \underbrace{P_{2k-2i+1}(z)}_{<\,0}, \forall z < \xi_1^{(2k+1)} . \tag{23}$$

From the above considerations, we can easily deduce the following conclusions. Item 1 follows from (18); and item 2 follows from (20)-(21). If $\lambda_r < 1$, then (18) $\Longrightarrow P_k^d(\lambda_r; r)(\xi_k^{(k)}) > 0$ and (20) $\Longrightarrow P_k^d(\lambda_r; r)(y) > 0$, then we get item 3(a); (20)-(23) $\Longrightarrow$ 3(b); items 3(a) and 3(b) $\Longrightarrow$ item 3(c). If $\lambda_r > 1$, then (18) $\Longrightarrow$ $P_k^d(\lambda_r; r)(\xi_k^{(k)}) < 0$, and $P_k^d(\lambda_r; r)(\xi_k^{(k)}) < 0 \Longrightarrow$ item 4 by (17) and the symmetry. ■

4 Brauer and Geršgorin sets of Jacobi matrices

4.1 Brauer and Geršgorin sets of any matrix

We begin by recalling some results by Geršgorin [13, 14, 24] and Brauer [1, 24] to locate the eigenvalues of any matrix $A_n = (a_{ij})_{1 \le i,j \le n} \in \mathbb{C}^{n \times n}$.

Geršgorin disks (G-disks) are defined by

$$\mathcal{D}_i^{(n)} = \left\{ z \in \mathbb{C} : |z - a_{ii}| \le r_i^{(n)} \right\} , \ r_i^{(n)} = \sum_{j=1, j \ne i}^{n} |a_{ij}| , \quad 1 \le i \le n . \tag{24}$$

Geršgorin set (G-set) is the union of all G-disks, that is,

$$\mathcal{D}^{(n)} = \bigcup_{i=1}^{n} \mathcal{D}_i^{(n)} \ , \ n \geq 1 \ . \tag{25}$$

Geršgorin circle theorem [13] states that the set of eigenvalues of A_n, the so-called *spectra* of A_n, $\sigma(A_n)$, satisfies $\sigma(A_n) \subseteq \mathcal{D}^{(n)}$. Furthermore, if there exist $m < n$ G-disks disjoint from the remaining ones, then their union contains exactly m eigenvalues of A_n.

Brauer-Cassini ovals of A_n [1] (B-ovals) are defined by

$$\mathcal{B}_{ij}^{(n)} = \left\{ z \in \mathbb{C} : |z - a_{ii}|\,|z - a_{jj}| \leq r_{ij}^{(n)} \right\}, \ r_{ij}^{(n)} = r_i^{(n)} r_j^{(n)} \ , \ 1 \leq i < j \leq n \ , n \geq 2 \ , \tag{26}$$

and the Brauer set (B-set) is

$$\mathcal{B}^{(n)} = \bigcup_{\substack{i,j=1 \\ i<j}}^{n} \mathcal{B}_{ij}^{(n)} \ , \ n \geq 2 \ . \tag{27}$$

We remark that there exist $\binom{n}{2} = n(n-1)/2$ B-ovals. The Brauer theorem [1] states that

$$\sigma(A_n) \subseteq \mathcal{B}^{(n)} \subseteq \mathcal{D}^{(n)} \ . \tag{28}$$

more precisely

$$\mathcal{B}_{ij}^{(n)} \subseteq \left(\mathcal{D}_i^{(n)} \cup \mathcal{D}_j^{(n)} \right) .$$

Furthermore, there exists an analogous of Geršgorin result on disjoint subsets for Brauer-Cassini ovals. In the sequel, we will refer B-ovals as B-subsets.

4.2 Brauer and Geršgorin sets of Jacobi matrices associated to monic orthogonal polynomials

Let us consider the sequence of *Jacobi matrices* $\{J_n\}_{n\geq 1}$ associated with any monic orthogonal polynomial sequence $\{P_n(x)\}_{n\geq 0}$ given by (3)-(4), taking α_n such that, $\alpha_n^2 = \gamma_n \neq 0, \ n \geq 1$,

$$J_n = \begin{pmatrix} \beta_0 & \alpha_1 & & & & \\ \alpha_1 & \beta_1 & \alpha_2 & & & \\ & \alpha_2 & \beta_2 & \alpha_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{n-2} & \beta_{n-2} & \alpha_{n-1} \\ & & & & \alpha_{n-1} & \beta_{n-1} \end{pmatrix} \ , \ n \geq 1 \ . \tag{29}$$

It can be proved that $P_n(x)$ is the characteristic polynomial of J_n, that is,

$$P_n(x) = \det(xI_n - J_n) \ , \ n \geq 1 \ , \tag{30}$$

where I_n is the identity matrix of order n. Thus, zeros of $P_n(x)$ are eigenvalues of J_n. In this work, we are going to apply the Geršgorin [13] and Brauer [1] theorems to the Jacobi matrix J_n, in order to obtain some locations of zeros of $P_n(x)$.

G-disks of J_n are

$$\mathcal{D}_1^{(1)} = \{\beta_0\} \ ; \ \mathcal{D}_i^{(n)} = \{z \in \mathbb{C} : |z - \beta_{i-1}| \leq r_i^{(n)}\}, \ 1 \leq i \leq n, \ n \geq 2 \ ,$$

with radius given by

$$r_1^{(1)} = 0 \ ;$$
$$r_1^{(n)} = |\alpha_1| \ , \ r_i^{(n)} = |\alpha_{i-1}| + |\alpha_i| \ , \ 2 \leq i \leq n-1, \ r_n^{(n)} = |\alpha_{n-1}| \ , \ n \geq 2 \ . \quad (31)$$

We recall that $r_1^{(1)} \neq r_1^{(2+m)}$ and $r_n^{(n)} \neq r_n^{(n+1+m)}$, $m \geq 0$, thus the upper index in the notation is necessary.

B-subsets of J_n are

$$\mathcal{B}_{ij}^{(n)} = \{z \in \mathbb{C} : |z - \beta_{i-1}||z - \beta_{j-1}| \leq r_{ij}^{(n)}\}, \ r_{ij}^{(n)} = r_i^{(n)} r_j^{(n)}, \ 1 \leq i < j \leq n, \ n \geq 2.$$

Noting the zeros of $P_n(x)$ by $\xi_k^{(n)}$, $k = 1(1)n$, Geršgorin and Brauer theorems allow us to conclude that

$$Z^{(n)} = \left\{\xi_k^{(n)}\right\}_{k=0}^{n} \subseteq \mathcal{B}^{(n)} \subseteq \mathcal{D}^{(n)} \ . \quad (32)$$

In the symmetric case ($\beta_n = 0$, $n \geq 0$), G-sets and B-sets are the following disks centred at the origin

$$\mathcal{D}^{(n)} = \left\{z \in \mathbb{C} : |z| \leq g^{(n)}\right\} \ , \ g^{(n)} = \max\left\{r_i^{(n)}, \ 1 \leq i \leq n\right\} \ , \ n \geq 1 \ ; \quad (33)$$

$$\mathcal{B}^{(n)} = \left\{z \in \mathbb{C} : |z| \leq b^{(n)}\right\} \ , \ b^{(n)} = \sqrt{\max\left\{r_{ij}^{(n)}, \ 1 \leq i < j \leq n\right\}} \ , \ n \geq 2 \ . \quad (34)$$

It is worthy to note that, from (32), we have

$$b^{(n)} \leq g^{(n)} \ , \ n \geq 2. \quad (35)$$

We remark that J_n is symmetric ($J_n = J_n^T$), but if $\exists i$: $1 \leq i \leq n-1$: $\gamma_i < 0$, then α_i is a pure imaginary number and J_n is not an hermitian matrix ($J_n \neq (\overline{J_n})^T$). In the symmetric positive definite case ($\beta_n = 0$, $\gamma_{n+1} > 0$, $n \geq 0$), those disks (33)-(34) are reduced to intervals since all zeros (eigenvalues) are real numbers.

4.3 Brauer and Geršgorin sets of Jacobi matrices associated to Chebyshev polynomials of second kind

From (5) and (29), we obtain the sequence of Jacobi matrices $\{\mathcal{J}_n\}_{n\geq 1}$ associated to the monic Chebyshev sequence of second kind

$$\mathcal{J}_n = \begin{pmatrix} 0 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/2 & 0 & 1/2 \\ & & & 1/2 & 0 \end{pmatrix} \ , \ n \geq 1 \ . \quad (36)$$

Radius of G-disks of $\mathcal{J}_n$ are given by

$$r_1^{(1)} = 0 \ ; \ r_1^{(2)} = r_2^{(2)} = \frac{1}{2} \ ; \ r_1^{(n)} = r_n^{(n)} = \frac{1}{2} \ , \ r_i^{(n)} = 1 \ , \ 2 \leq i \leq n-1 \ , \ n \geq 3 \ . \quad (37)$$

Radius of G-sets and B-sets of $\mathcal{J}_n$ are given by

$$g^{(1)} = 0 \ ; \ g^{(2)} = \frac{1}{2} \ ; \ g^{(n)} = 1 \ , \ n \geq 3 \quad ; \quad b^{(2)} = \frac{1}{2} \ ; \ b^{(3)} = \frac{\sqrt{2}}{2} \ ; \ b^{(n)} = 1 \ , \ n \geq 4 \ .$$

With the notation

$$\mathcal{I}_{\frac{1}{2}} = \left[-\frac{1}{2}, \frac{1}{2}\right] \quad , \quad \mathcal{I}_1 = [-1, 1] \ , \quad (38)$$

G-sets and B-sets are the following intervals

$$\mathcal{D}^{(1)} = \{0\} \ , \ \mathcal{D}^{(2)} = \mathcal{I}_{\frac{1}{2}} \ , \ \mathcal{D}^{(n)} = \mathcal{I}_1 \ , \ n \geq 3 \ ;$$
$$\mathcal{B}^{(2)} = \mathcal{I}_{\frac{1}{2}} \ , \ \mathcal{B}^{(3)} = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \ , \ \mathcal{B}^{(n)} = \mathcal{I}_1 \ , \ n \geq 4 \ .$$

We consider that $\mathcal{J}_n$ is constituted by the first n rows and n columns of an infinite dimensional Jacobi matrix $\mathcal{J}$ corresponding to all the Chebyshev sequence $\{P_n\}_{n\geq 0}$.

4.4 Geršgorin sets of Jacobi matrices associated to perturbed Chebyshev polynomials of second kind by dilation

From (6) and (29), we obtain the following infinite dimensional Jacobi matrix associated to perturbed monic Chebyshev sequences of second kind by dilation

$$J^d(\lambda_r; r) = \begin{pmatrix} 0 & \frac{1}{2} & & & & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & \frac{1}{2} & 0 & \frac{\sqrt{\lambda_r}}{2} & & & & \\ & & & \frac{\sqrt{\lambda_r}}{2} & 0 & \frac{1}{2} & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & \frac{1}{2} & 0 & \frac{1}{2} & \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix} , \begin{matrix} \leftarrow \text{row } r \\ \leftarrow \text{row } r+1 \end{matrix} \quad (39)$$

such that, for $n \geq 1$, $J_n^d(\lambda_r; r)$ notes the matrix constituted by the first n rows and n columns of it. For $1 \leq n \leq r$, $J_n^d(\lambda_r; r) \equiv \mathcal{J}_n$ given by (36).

Let us introduce some notation related to $J_n^d(\lambda_r; r)$. We note its G-disks by $\mathcal{D}_i^{d(n)}(\lambda_r; r)$, with radius $r_i^{d(n)}(\lambda_r; r)$; the G-set by $\mathcal{D}^{d(n)}(\lambda_r; r)$, with radius $g^{(n)}(\lambda_r; r)$; and the B-set by $\mathcal{B}^{d(n)}(\lambda_r; r)$, with radius $b^{(n)}(\lambda_r; r)$, respectively.

If $\lambda_r > 0$, $\lambda_r \neq 1$, then

$$\begin{aligned} \mathcal{D}^{(n)}(\lambda_r;r) &= [-g^{(n)}(\lambda_r;r), g^{(n)}(\lambda_r;r)] \ , \ n \geq 1 \ , &(40)\\ \mathcal{B}^{(n)}(\lambda_r;r) &= [-b^{(n)}(\lambda_r;r), b^{(n)}(\lambda_r;r)] \ , \ n \geq 2 \ . &(41) \end{aligned}$$

If $\lambda_r < 0$, then

$$\begin{aligned} \mathcal{D}^{(n)}(\lambda_r;r) &= \left\{ z \in \mathbb{C} : |z| \leq g^{(n)}(\lambda_r;r) \right\} \ , \ n \geq 1 \ , &(42)\\ \mathcal{B}^{(n)}(\lambda_r;r) &= \left\{ z \in \mathbb{C} : |z| \leq b^{(n)}(\lambda_r;r) \right\} \ , \ n \geq 2 \ . &(43) \end{aligned}$$

We note the set of zeros of $P_n^d(\lambda_r;r)(x)$ by

$$Z^{(n)}(\lambda_r;r) = \left\{ \xi_k^{(n)}(\lambda_r;r) \right\}_{k=1}^{n} . \tag{44}$$

Thus, Brauer and Geršgorin theorems applied to $J_n^d(\lambda_r;r)$ state that

$$Z^{(n)}(\lambda_r;r) \subseteq \mathcal{B}^{(n)}(\lambda_r;r) \subseteq \mathcal{D}^{(n)}(\mu_r;r) \ . \tag{45}$$

In the sequel, our goal is to make explicit G-sets and B-sets of $J_n^d(\lambda_r;r)$.

Perturbation by dilation modify only the rows of orders r and $r+1$ of $\mathcal{J}$, the Jacobi matrix associated to the Chebyshev sequence of second kind. Consequently, it modifies only the G-disks of orders $i = r$ and $i = r+1$, whose radius depend if $n = r+1$ or if $n \geq r+2$, and also if $r = 1$ or if $r \geq 2$. With the notation

$$A_r := A(\lambda_r) = \frac{\sqrt{|\lambda_r|}}{2} \quad , \quad C_r := C(\lambda_r) = \frac{1 + \sqrt{|\lambda_r|}}{2} \ , \tag{46}$$

radius of G-disks are given as follows

$$\begin{aligned} n = r+1 \Longrightarrow \quad & \Big\{ r = 1 \Rightarrow \left\{ r_1^{d(2)}(\lambda_1;1) = r_2^{d(2)}(\lambda_1;1) = A_1 \right\}, &(47)\\ & \quad r \geq 2 \Rightarrow \left\{ r_r^{d(r+1)}(\lambda_r;r) = C_r \ , \ r_{r+1}^{d(r+1)}(\lambda_r;r) = A_r \right\} \Big\} \ ; \\ n \geq r+2 \Longrightarrow \quad & \Big\{ r = 1 \Rightarrow \left\{ r_1^{d(n)}(\lambda_1;1) = A_1 \ ; \ r_2^{d(n)}(\lambda_1;1) = C_1 \right\} , \\ & \quad r \geq 2 \Rightarrow \left\{ r_r^{d(n)}(\lambda_r;r) = r_{r+1}^{d(n)}(\lambda_r;r) = C_r \right\} \Big\} . &(48) \end{aligned}$$

The i-th G-disk of $J_n^d(\lambda_r;r)$, for $i \neq r$ and $i \neq r+1$, coincides with the one of $\mathcal{J}_n$, thus

$$r_i^{d(n)}(\lambda_r;r) = r_i^{(n)} \ , \ i \neq r \ , \ i \neq r+1 \ , \ 1 \leq i \leq n \ , \ n \geq 1 \ , r \geq 1 \ ,$$

where $r_i^{(n)}$ is given by (37). For $i = r$ or $i = r+1$, the radius $r_r^{d(n)}(\lambda_r;r)$ and $r_{r+1}^{d(n)}(\lambda_r;r)$, of the corresponding G-disks of $J_n^d(\lambda_r;r)$ are the followings.

- For $r \geq 1$ and $n = 1$, $r_1^{d(1)}(\lambda_r;r) = 0$. • For $r = 1$ and $n \geq 2$, $r_1^{d(n)}(\lambda_1;1) = A_1$.

• For $r \geq 2$ and $n \geq r+1$, $r_r^{d(n)}(\lambda_r;r) = C_r$. • For $r \geq 1$ and $n = r+1$, $r_{r+1}^{d(r+1)}(\lambda_r;r) = A_r$.

• For $r \geq 1$ and $n \geq r+2$, $r_{r+1}^{d(n)}(\lambda_r;r) = C_r$.

Finally, we obtain the radius of G-sets of $J_n^d(\lambda_r;r)$ given next.

• For $r = 1$,

$$\begin{aligned} & g^{(1)}(\lambda_1;1) = 0,\ n = 1\ ;\ g^{(2)}(\lambda_1;1) = A_1,\ n = 2\ ; \\ & g^{(3)}(\lambda_1;1) = C_1,\ n = 3\ ;\ g^{(n)}(\lambda_1;1) = \max\{1, C_1\},\ n \geq 4\ . \end{aligned} \tag{49}$$

• For $r = 2$,

$$\begin{aligned} & g^{(1)}(\lambda_2;2) = 0,\ n = 1\ ;\ g^{(2)}(\lambda_2;2) = \frac{1}{2},\ n = 2\ ; \\ & g^{(3)}(\lambda_2;2) = g^{(4)}(\lambda_2;2) = C_2,\ n = 3\ ,\ n = 4\ ; \\ & g^{(n)}(\lambda_2;2) = \max\{1, C_2\}\ ,\ n \geq 5. \end{aligned}$$

• For $r \geq 3$,

$$\begin{aligned} & g^{(1)}(\lambda_r;r) = 0,\ n = 1\ ;\ g^{(2)}(\lambda_r;r) = \frac{1}{2}\ ,\ n = 2\ ; \\ & g^{(k)}(\lambda_r;r) = 1\ ,\ n = k\ ,\ k = 3(1)r\ ; \\ & g^{(n)}(\lambda_r;r) = \max\{1, C_r\}\ ,\ n \geq r+1. \end{aligned} \tag{50}$$

5 Brauer and Geršgorin location of zeros of perturbed Chebyshev polynomials of second kind by dilation

5.1 Geršgorin location of zeros of perturbed Chebyshev polynomials of second kind by dilation

I shall deduce the desired location from the radius $g^{(n)}(\lambda_r;r)$ of the Geršgorin sets, given by (49)-(50), obtained in the previous section.

Proposition 5.1 *The radius $g^{(n)}(\lambda_r;r)$, given by (49)-(50), of the Geršhgorin sets $\mathcal{D}^{(n)}(\lambda_r;r)$ (40)-(42) of the Jacobi matrices $J_n^d(\lambda_r;r)$ (39), for $\lambda_r \in \mathbb{R}$, $\lambda_r \neq 0$, $\lambda_r \neq 1$, $r \geq 1$, and $n \geq r+1$, satisfy the following*

- *For $r = 1$, and $n = 2$ or $n = 3$,*
  - *If $-1 \leq \lambda_1 < 0$ or $0 < \lambda_1 < 1$, then $g^{(2)}(\lambda_1;1) < \frac{1}{2}$ and $g^{(3)}(\lambda_1;1) < 1$.*
  - *If $|\lambda_1| > 1$, then $g^{(2)}(\lambda_1;1) > \frac{1}{2}$ and $g^{(3)}(\lambda_1;1) > 1$.*
- *For $r = 2$, and $n = 3$ or $n = 4$,*
  - *If $-1 \leq \lambda_2 < 0$ or $0 < \lambda_2 < 1$, then $g^{(3)}(\lambda_2;2) = g^{(4)}(\lambda_2;2) < 1$.*
  - *If $|\lambda_2| > 1$, then $g^{(3)}(\lambda_2;2) = g^{(4)}(\lambda_2;2) > 1$.*

- *For $r = 1$ and $n \geq 4$, or for $r = 2$ and $n \geq 5$, or for and $r \geq 3$ and $n \geq r+1$, the following holds, with the notation* (46)

  - *If $-1 \leq \lambda_r < 0$ or $0 < \lambda_r < 1$, then $g^{(n)}(\lambda_r; r) = 1$.*

  - *If $|\lambda_r| > 1$, then $g^{(n)}(\lambda_r; r) = C_r > 1$.*

$\mathcal{D}^{(n)}(\lambda_r; r)$ constitutes a location of zeros of $P_n(\lambda_r; r)(x)$, the r-th perturbed Chebyshev polynomial of second kind by dilation of degree n, by (45).

Corollary 5.2 *The Geršhgorin sets of the Jacobi matrices associated to the monic Chebyshev sequence of first kind $\{T_n(x)\}_{n\geq 0}$* (11) *are the followings*

$$\mathcal{D}^{d(1)}(2;1) = \{0\},\ g^{(1)} = 0\ ,\ n = 1\ ;$$
$$\mathcal{D}^{d(2)}(2;1) = \left[-g^{(2)}, g^{(2)}\right],\ g^{(2)} = \frac{\sqrt{2}}{2}\ ,\ n = 2\ ;$$
$$\mathcal{D}^{d(n)}(2;1) = \left[-g^{(n)}, g^{(n)}\right] \supset \mathcal{I}_1\ ,\ g^{(n)} = \frac{1+\sqrt{2}}{2} > 1\ ,\ n \geq 3\ .$$

Proof. In the last proposition, just take $r = 1$ and $\lambda_r = 2$. ∎

Remark that the well known location of zeros of $\{T_n(x)\}_{n\geq 0}$ is the interval $\mathcal{I}_1$ [20], thus in this case the Geršgorin location is not sharp.

Proposition 5.3 *If $\lambda_r > 1$, then the two extremal zeros of $P_n(\lambda_r; r)(x)$, the perturbed Chebyshev polynomial of second kind by dilation of degree n, $\xi_1^{(n)}(\lambda_r; r)$ and $\xi_n^{(n)}(\lambda_r; r)$, have the following location in terms of the two extremal zeros of $P_n(x)$, the Chebyshev polynomial of second kind of degree n, $\xi_1^{(n)}$ and $\xi_n^{(n)}$ given by* (13)*, and the bounds of Geršhgorin sets obtained in Proposition 5.1. Using the notation* (46)*, it holds*

- *For $r = 1$ and $n = 2$,*

$$-A_r \leq \xi_1^{(n)}(\lambda_r; r) < \xi_1^{(n)} \quad , \quad \xi_n^{(n)} < \xi_n^{(n)}(\lambda_r; r) \leq A_r.$$

- *For $r = 1$ and $n \geq 3$, or for $r \geq 2$ and $n \geq r+1$,*

$$-C_r \leq \xi_1^{(n)}(\lambda_r; r) < \xi_1^{(n)} \quad , \quad \xi_n^{(n)} < \xi_n^{(n)}(\lambda_r; r) \leq C_r.$$

5.2 Brauer location of zeros of perturbed Chebyshev polynomials of second kind by dilation

I shall deduce the desired location from the radius $r^{(n)}(\lambda_r; r)$ of the Geršgorin disks obtained in the previous section.

Proposition 5.4 *The radius $b^{(n)}(\lambda_r; r)$ of the Brauer sets $\mathcal{B}^{(n)}(\lambda_r; r)$* (41)-(43) *of the Jacobi matrices $J_n^d(\lambda_r; r)$* (39)*, for $\lambda_r \in \mathbb{R}$, $\lambda_r \neq 0$, $\lambda_r \neq 1$, $r \geq 1$, and $n \geq r+1$, with the notation* (46) *and*

$$A'_r = \sqrt{|\lambda_r|}\ ,\ B_r = \sqrt{A_r} = \sqrt[4]{|\lambda_r|}\ ,\ C'_r = 1 + \sqrt{|\lambda_r|}\ ,\ D_r = \sqrt{C'_r} = \sqrt{1+\sqrt{|\lambda_r|}}\ ,\quad (51)$$

satisfy the following

- *For* $r=1$, $b^{(n)} := b^{(n)}(\lambda_1;1)$, $n \geq 2$,
  - $b^{(2)} = \frac{1}{2}A'_1 = A_1 \ , \ n=2 \ ;$
  - $b^{(3)} = \max\left\{ \frac{1}{2}D_1 \ , \ \frac{1}{2}B_1D_1 \right\} \ , \ n=3 \ ,$

  $-1 \leq \lambda_1 < 0 \text{ or } 0 < \lambda_1 < 1 \Rightarrow b^{(3)} = \frac{1}{2}D_1 \ ; \ |\lambda_1| > 1 \Rightarrow b^{(3)} = \frac{1}{2}B_1D_1 \ ;$
  - $b^{(4)} = \max\left\{ \frac{1}{\sqrt{2}}D_1 \ , \ \frac{1}{2}B_1D_1 \right\} \ , \ n=4 \ ,$

  $-4 \leq \lambda_1 < 0 \text{ or } 0 < \lambda_1 < 1 \text{ or } 1 < \lambda_1 \leq 4 \Rightarrow b^{(4)} = \frac{1}{\sqrt{2}}D_1 \ ,$

  $|\lambda_1| > 4 \Longrightarrow b^{(4)} = \frac{1}{2}B_1D_1 \ ;$
  - $b^{(n)} = \max\left\{ 1 \ , \ \frac{1}{\sqrt{2}}D_1 \ , \ \frac{1}{2}B_1D_1 \right\} \ , \ n \geq 5 \ ,$

  $-1 \leq \lambda_1 < 0 \text{ or } 0 < \lambda_1 < 1 \Rightarrow b^{(n)} = 1 \ ;$

  $-4 \leq \lambda_1 < -1 \text{ or } 1 < \lambda_1 \leq 4 \Rightarrow b^{(n)} = \frac{1}{\sqrt{2}}D_1 \ ; \ |\lambda_1| > 4 \Rightarrow b^{(n)} = \frac{1}{2}B_1D_1.$

- *For* $r=2$, $b^{(n)} := b^{(n)}(\lambda_2;2)$, $n \geq 3$,
  - $b^{(3)} = \max\left\{ \frac{1}{2}D_2 \ , \ \frac{1}{2}B_2D_2 \right\} \ , \ n=3 \ ,$

  $-1 \leq \lambda_2 < 0 \text{ or } 0 < \lambda_2 < 1 \Rightarrow b^{(3)} = \frac{1}{2}D_2 \ ; \ |\lambda_2| > 1 \Rightarrow b^{(3)} = \frac{1}{2}B_2D_2 \ ;$
  - $b^{(4)} = b^{(5)} = \frac{1}{2}C'_2 = C_2 \ ; \ n=4 \ , \ n=5 \ ;$
  - $b^{(n)} = \max\left\{ 1 \ , \ \frac{1}{2}C'_2 \ , \ \frac{1}{\sqrt{2}}D_2 \right\} \ , \ n \geq 6 \ ,$

  $-1 \leq \lambda_2 < 0 \text{ or } 0 < \lambda_2 < 1 \Rightarrow b^{(n)} = 1 \ ; \ |\lambda_2| > 1 \Rightarrow b^{(n)} = \frac{1}{2}C'_2 = C_2.$

- *For* $r=3$, $b^{(n)} := b^{(n)}(\lambda_3;3)$, $n \geq 4$,
  - $b^{(4)} = \max\left\{ \frac{1}{\sqrt{2}}B_3 \ , \ \frac{1}{\sqrt{2}}D_3 \ , \ \frac{1}{2}B_3D_3 \right\} \ , \ n=4 \ ,$

  $-4 \leq \lambda_3 < 0 \text{ or } 0 < \lambda_3 < 1 \text{ or } 1 < \lambda_3 \leq 4 \Rightarrow b^{(4)} = \frac{1}{\sqrt{2}}D_3 \ ,$

  $|\lambda_3| > 4 \Rightarrow b^{(4)} = \frac{1}{2}B_3D_3 \ ;$
  - $b^{(5)} = \max\left\{ \frac{1}{\sqrt{2}} \ , \ \frac{1}{2}C'_3 \ , \ \frac{1}{\sqrt{2}}D_3 \right\}, \ n=5 \ ,$

  $-1 \leq \lambda_3 < 0 \text{ or } 0 < \lambda_3 < 1 \Rightarrow b^{(5)} = \frac{1}{\sqrt{2}}D_3 \ ; \ |\lambda_3| > 1 \Rightarrow b^{(5)} = \frac{1}{2}C'_3 = C_3 \ ;$

- $b^{(n)} = \max\left\{1\ ,\ \frac{1}{2}C_3'\ ,\ \frac{1}{\sqrt{2}}D_3\right\},\ n \geq 6\ ,$

  $-1 \leq \lambda_3 < 0$ or $0 < \lambda_3 < 1 \Rightarrow b^{(n)} = 1\ ;\ |\lambda_3| > 1 \Rightarrow b^{(n)} = \frac{1}{2}C_3' = C_3.$

- *For* $r \geq 4$, $b^{(n)} := b^{(n)}(\lambda_r; r)$, $n \geq r+1$,

  - $$b^{(r+1)} = \max\left\{1\ ,\ \frac{1}{\sqrt{2}}B_r\ ,\ \frac{1}{\sqrt{2}}D_r\ ,\ \frac{1}{2}B_rD_r\right\}\ ,\ n = r+1\ , \tag{52}$$

    $-1 \leq \lambda_r < 0$ or $0 < \lambda_r < 1 \Rightarrow b^{(r+1)} = 1\ ;$

    $-4 \leq \lambda_r < -1$ or $1 < \lambda_r \leq 4 \Rightarrow b^{(r+1)} = \frac{1}{\sqrt{2}}D_r\ ;\ |\lambda_r| > 4 \Rightarrow b^{(r+1)} = \frac{1}{2}B_rD_r\ ;$

  - $b^{(n)} = \max\left\{1\ ,\ \frac{1}{2}C_r'\ ,\ \frac{1}{\sqrt{2}}D_r\right\},\ n \geq r+2\ ,$

    $-1 \leq \lambda_r < 0$ or $0 < \lambda_r < 1 \Rightarrow b^{(n)} = 1\ ;\ |\lambda_r| > 1 \Rightarrow b^{(n)} = \frac{1}{2}C_r' = C_r.$

$\mathcal{B}^{(n)}(\lambda_r; r)$ *constitutes a location of zeros of* $P_n(\lambda_r; r)(x)$*, the* r*-th perturbed Chebyshev polynomial of second kind by dilation of degree* n*, By* (45)*.*

Proof. By (34), we have

$$b^{(n)}(\lambda_r; r) = \sqrt{\max\left\{r_{ij}^{(n)}(\lambda_r; r),\ 1 \leq i < j \leq n\right\}}, r_{ij}^{(n)}(\lambda_r; r) = r_i^{(n)}(\lambda_r; r) r_j^{(n)}(\lambda_r; r)\ ,$$

with $r_i^{(n)}(\lambda_r; r)$ given by (37) and (47)-(48). In order to determine the maximum, observe that $B_r < D_r$, $C_r > 1$, and $D_r < C_r$. Next, we demonstrate de last case for $n = r+1$. Other cases are similar. The equality (52) is obtained by direct computations. If $|\lambda_r| \leq 1$, then the maximum is 1, because

$$\left\{\left(B_r \leq 1 \Rightarrow \frac{B_r}{\sqrt{2}} < 1\right) \wedge \left(D_r \leq \sqrt{2} \Rightarrow \frac{D_r}{\sqrt{2}} \leq 1\right)\right\} \Longrightarrow \frac{B_r}{\sqrt{2}}\frac{D_r}{\sqrt{2}} = \frac{1}{2}B_rD_r < 1\ .$$

If $1 < |\lambda_r| < 4$, then the maximum is $\frac{D_r}{\sqrt{2}}$, because

$$\left(\frac{B_r}{\sqrt{2}} < 1 \Rightarrow \frac{B_r}{\sqrt{2}}\frac{D_r}{\sqrt{2}} = \frac{1}{2}B_rD_r < \frac{D_r}{\sqrt{2}}\right)\ \wedge\ \frac{B_r}{\sqrt{2}} < \frac{D_r}{\sqrt{2}}\ .$$

If $|\lambda_r| \geq 4$, then the maximum is $\frac{1}{2}B_rD_r$, because

$$\frac{B_r}{\sqrt{2}} \geq 1 \Rightarrow \frac{B_r}{\sqrt{2}}\frac{D_r}{\sqrt{2}} = \frac{1}{2}B_rD_r \geq \frac{D_r}{\sqrt{2}} > \frac{B_r}{\sqrt{2}}\ .$$

■

Corollary 5.5 *Radius* $b^{(n)} := b^{(n)}(2;1)$ *of Brauer sets of Jacobi matrices associated to the Chebyshev sequence of first kind* $\{T_n(x)\}_{n\geq 0}$ (11) *are the followings, being* $g^{(n)} := g^{(n)}(2;1)$ *the radius of Geršgorin sets given by Corollary 5.2,*

$$b^{(2)} = \frac{\sqrt{2}}{2} = g^{(2)} \ ; \ b^{(3)} = \frac{1}{2}\sqrt[4]{2}\sqrt{1+\sqrt{2}} = \frac{1}{\sqrt[4]{2}}\sqrt{g^{(3)}} \ ; \tag{53}$$

$$b^{(n)} = \frac{1}{\sqrt{2}}\sqrt{1+\sqrt{2}} = \sqrt{g^{(n)}} \ , \ n \geq 4 \ . \tag{54}$$

Then,

$$b^{(n)} < g^{(n)}, \ n \geq 3 \ ; \ b^{(n)} > 1, \ n \geq 4 \ . \tag{55}$$

Proof. In Proposition 5.4, taking $r = 1$ and $\lambda_r = 2$, we obtain the first equalities of (53)-(54); the second ones are due to the expressions of $g^{(n)}$ given by Corollary 5.2. That result states, in particular, that $g^{(n)} > 1$, $n \geq 3$, then from (53)-(54), we get (55). Also, direct computations give

$$g^{(n)} \approx 1.207, \ n \geq 3 \ ; \ b^{(3)} \approx 0.934 \ , \ b^{(n)} \approx 1.099, \ n \geq 4 \ .$$

Thus, we conclude that the Brauer location of $\{T_n(x)\}_{n\geq 0}$ is better than the Geršgorin one given by Corollary 5.2, but still it is not sharp, because zeros of these polynomials belong to $\mathcal{I}_1$ [20]. ∎

Recall that, $b^{(n)}(\lambda_r;r) \leq g^{(n)}(\lambda_r;r)$, by (55). In the sequel, our goal will be to evaluate how much better is the Brauer location with respect to the Geršgorin one. For that, in all cases of Propositions 5.1 and 5.4, we are going to compare the radius of Geršgorin set $g^{(n)}(\lambda_r;r)$, given also by (49)-(50), with the radius of Brauer set $b^{(n)}(\lambda_r;r)$.

Proposition 5.6 *Radius of Brauer set* $b^{(n)} := b^{(n)}(\lambda_r;r)$ *given by Proposition 5.1, and radius of Geršgorin set* $g^{(n)} := g^{(n)}(\lambda_r;r)$ *given by Proposition 5.4, of Jacobi matrices* $J_n^d(\lambda_r;r)$ (39)*, for* $\lambda_r \in \mathbb{R}$*,* $\lambda_r \neq 0$*,* $\lambda_r \neq 1$*,* $r \geq 1$*, and* $n \geq r+1$*, satisfy the following, with the notations* (46) *and* (51)*.*

- *For* $r = 1$*,* $g^{(n)} := g^{(n)}(\lambda_1;1)$*,* $b^{(n)} := b^{(n)}(\lambda_1;1)$*,* $n \geq 2$*,*
  - *For* $n = 2:$ $b^{(2)} = g^{(2)} = A_1$.
  - *For* $n = 3:$ *if* $-1 \leq \lambda_1 < 0$ *or* $0 < \lambda_1 < 1$*, then* $b^{(3)} = \dfrac{1}{\sqrt{2}}\sqrt{g^{(3)}} < g^{(3)}$;

    if $|\lambda_1| > 1$*, then* $b^{(3)} = \dfrac{B_1}{\sqrt{2}}\sqrt{g^{(3)}} < g^{(3)}$;
  - *For* $n = 4:$ *if* $\lambda_1 = -1$*, then* $b^{(4)} = g^{(4)} = 1$;

    if $-1 < \lambda_1 < 0$ *or* $0 < \lambda_1 < 1$*, then* $b^{(4)} = \sqrt{C_1} < g^{(4)} = 1$;

    if $1 < |\lambda_1| \leq 4$*, then* $b^{(4)} = \sqrt{g^{(4)}} < g^{(4)}$;

    if $|\lambda_1| > 4$*, then* $b^{(4)} = \dfrac{B_1}{\sqrt{2}}\sqrt{g^{(4)}} < g^{(4)}$.

- *For* $n \geq 5:$ *if* $-1 \leq \lambda_1 < 0$ or $0 < \lambda_1 < 1,$ *then* $b^{(n)} = g^{(n)} = 1$;
  if $1 < |\lambda_1| \leq 4,$ *then* $b^{(n)} = \sqrt{g^{(n)}} < g^{(n)}$;
  if $|\lambda_1| > 4,$ *then* $b^{(n)} = \dfrac{B_1}{\sqrt{2}}\sqrt{g^{(n)}} < g^{(n)}$.

- *For* $r = 2,$ $g^{(n)} := g^{(n)}(\lambda_2; 2),$ $b^{(n)} := b^{(n)}(\lambda_2; 2),$ $n \geq 3,$
  - *For* $n = 3:$ *if* $-1 \leq \lambda_2 < 0$ *or* $0 < \lambda_2 < 1,$ *then* $b^{(3)} = \dfrac{1}{\sqrt{2}}\sqrt{g^{(3)}} < g^{(3)} = C_2;$
  if $|\lambda_2| > 1,$ *then* $b^{(3)} = \dfrac{B_2}{\sqrt{2}}\sqrt{g^{(3)}} < g^{(3)} = C_2$.
  - *For* $n = 4:$ $b^{(4)} = g^{(4)} = C_2$.
  - *For* $n = 5:$ *if* $\lambda_2 = -1,$ *then* $b^{(5)} = g^{(5)} = 1$;
  if $-1 < \lambda_2 < 0$ *or* $0 < \lambda_2 < 1,$ *then* $b^{(5)} = C_2 < g^{(5)} = 1$;
  if $|\lambda_2| > 1,$ then $b^{(5)} = g^{(5)} = C_2$.
  - *For* $n \geq 6:$ *if* $-1 \leq \lambda_2 < 0$ *or* $0 < \lambda_2 < 1,$ *then* $b^{(n)} = g^{(n)} = 1$;
  if $|\lambda_2| > 1,$ *then* $b^{(n)} = g^{(n)} = C_2$.

- *For* $r = 3,$ $g^{(n)} := g^{(n)}(\lambda_3; 3),$ $b^{(n)} := b^{(n)}(\lambda_3; 3),$ $n \geq 4,$
  - *For* $n = 4:$ *if* $\lambda_3 = -1,$ then $b^{(4)} = g^{(4)} = 1$;
  if $-1 < \lambda_3 < 0$ *or* $0 < \lambda_3 < 1,$ *then* $b^{(4)} = \dfrac{1}{\sqrt{2}} D_3 < g^{(4)} = 1$;
  if $1 < |\lambda_3| \leq 4,$ *then* $b^{(4)} = \sqrt{g^{(4)}} < g^{(4)}$;
  if $|\lambda_3| > 4,$ *then* $b^{(4)} = \dfrac{B_3}{\sqrt{2}}\sqrt{g^{(4)}} < g^{(4)}$.
  - *For* $n = 5:$ *if* $\lambda_3 = -1,$ then $b^{(5)} = g^{(5)} = 1$;
  if $-1 < \lambda_3 < 0$ *or* $0 < \lambda_3 < 1,$ *then* $b^{(5)} = \dfrac{1}{\sqrt{2}} D_3 < g^{(5)} = 1$;
  if $|\lambda_3| > 1,$ *then* $b^{(5)} = g^{(5)} = C_3$.
  - *For* $n \geq 6:$ *if* $-1 \leq \lambda_3 < 0$ *or* $0 < \lambda_3 < 1,$ *then* $b^{(n)} = g^{(n)} = 1$;
  if $|\lambda_3| > 1,$ *then* $b^{(n)} = g^{(n)} = C_3$.

- *For* $r \geq 4,$ $g^{(n)} := g^{(n)}(\lambda_r; r),$ $b^{(n)} := b^{(n)}(\lambda_r; r),$ $n \geq r + 1,$
  - *For* $n = r + 1:$ *if* $-1 \leq \lambda_r < 0$ *or* $0 < \lambda_r < 1,$ then $b^{(r+1)} = g^{(r+1)} = 1$;
  if $1 < |\lambda_r| \leq 4,$ *then* $b^{(r+1)} = \sqrt{g^{(r+1)}} < g^{(r+1)} = C_r$;
  if $|\lambda_r| > 4,$ *then* $b^{(r+1)} = \dfrac{B_r}{\sqrt{2}}\sqrt{g^{(r+1)}} < g^{(r+1)} = C_r$.
  - *For* $n \geq r + 2:$ *if* $-1 \leq \lambda_r < 0$ *or* $0 < \lambda_r < 1,$ then $b^{(r+1)} = g^{(r+1)} = 1$;
  if $|\lambda_r| > 1,$ *then* $b^{(n)} = g^{(n)} = C_r$.

Proof. Let us prove some items of the case $r = 1$. For $n = 3$, and if $-1 \le \lambda_1 < 0$ or $0 \le \lambda_1 < 1$, then $b^{(3)} = \dfrac{1}{2}\sqrt{1+\sqrt{|\lambda_1|}} =< \dfrac{1+\sqrt{|\lambda_1|}}{2} = g^{(3)}$, since $1+\sqrt{|\lambda_1|} > 1$. If $|\lambda_1| > 1$, then $b^{(3)} = \dfrac{1}{2}\sqrt[4]{|\lambda_1|}\sqrt{1+\sqrt{|\lambda_1|}} < \dfrac{1+\sqrt{|\lambda_1|}}{2} = g^{(3)} \Longleftrightarrow 0 < 1+\sqrt{|\lambda_1|}$, which is a true proposition. For $n = 4$, if $-1 \le \lambda_1 < 0$ or $0 < \lambda_1 < 1$, then $g^{(4)} = 1$, and $b^{(4)} = \frac{1}{\sqrt{2}}\sqrt{1+\sqrt{|\lambda_1|}}$. If $\lambda_1 = -1$, then $b^{(4)} = g^{(4)}$. If $\lambda_1 \ne -1$, then $b^{(4)} < 1$ and consequently $b^{(4)} < g^{(4)}$. If $-4 \le \lambda_1 < -1$ or $1 < \lambda_1 \le 4$, then $b^{(4)} = \dfrac{1}{\sqrt{2}}\sqrt{1+\sqrt{|\lambda_1|}} = \sqrt{g^{(4)}} < g^{(4)}$, since $g^{(4)} > 1$. Other items or cases are similar to one we just proved. ■

From this proposition, we conclude about the advantage of the Brauer location for $r = 1$, $|\lambda_1| > 1$ and $n \ge 5$; and also for other values of r and some initial values of n, as can be seen from the graphical representations of $g^{(n)} := g^{(n)}(\lambda_r; r)$ and $b^{(n)} := b^{(n)}(\lambda_r; r)$ presented in Figures 1 and 2.

Definition 5.7 *A measure of the sharpness of the Brauer or Geršgorin locations of the zeros* $\xi_k^{(n)}(\lambda_r; r)$, $k = 1(1)n$, *of* $P_n(\lambda_r; r)(x)$, *for* $\lambda_r \in \mathbb{R}$, $\lambda_r \ne 0$, $\lambda_r \ne 1$, $r \ge 1$, *and* $n \ge r+1$, *is defined by*

$$\mathcal{S}^{(n)}(\lambda_r; r) = \left(R^{(n)}(\lambda_r; r) - M\right)/M \ , \ M = \max\left\{\left|\xi_k^{(n)}(\lambda_r; r)\right| : k = 1(1)n\right\},$$

where $R^{(n)}(\lambda_r; r)$ *is equal to the radius* $g^{(n)}(\lambda_r; r)$ *or* $b^{(n)}(\lambda_r; r)$ *of the Brauer or Geršgorin sets* $\mathcal{B}^{(n)}(\lambda_r; r)$ *or* $\mathcal{D}^{(n)}(\lambda_r; r)$ (40)-(43) *of the Jacobi matrices* $J_n^d(\lambda_r; r)$ (39).

Observe that the sharpness measures the relative error between the radius of the smallest circle or interval that contains all zeros and the radius of the G-set or the B-set.

6 Symbolic, numerical, and graphical results

In this section we give some symbolic, numerical, and graphical results obtained with the software *Mathematica*® in order to exemplify and illustrate the main results of this work. These kind of experiments are useful in the investigation on this topic, because they can serve as a verification tool, as a way to get negative answers, to formulate conjectures or to make some discoveries, and therefore allow to direct some aspects of the theoretical study.

To get concrete results, we should fix the order r of perturbation and the degree n. Taking, as in [9], $r = 6$, $n = 18$, and a symbolic parameter of perturbation λ_6, starting from the initial conditions (3), using the recurrence relation (4) with the recurrence coefficients (6), we compute recursively the polynomials with increase degrees until obtaining $P_{18}^d(6; \lambda_6)(x)$ presented next. We have used the command *Expand* to get polynomials in the canonical basis.

$$\begin{aligned} P_{18}^d(\lambda_6;6)(x) &= x^{18}-\frac{1}{4}(\lambda_6+16)\,x^{16}+\frac{1}{8}(7\lambda_6+53)\,x^{14}-\frac{1}{64}(79\lambda_6+376)\,x^{12} \\ &+\frac{1}{256}(230\lambda_6+771)\,x^{10}-\frac{(367\lambda_6+920)}{1024}x^8+\frac{(157\lambda_6+305)}{2048}x^6 \\ &-\frac{3\,(43\lambda_6+67)}{16384}x^4+\frac{9\,(2\lambda_6+3)}{65536}x^2-\frac{1}{262144}\,. \end{aligned}$$

Fixing the parameter of perturbation, taking for example $\lambda_6=-2$, we obtain polynomials that can be plotted and study numerically.

$$\begin{aligned} P_{18}^d(-2;6)(x) &= x^{18}-\frac{7x^{16}}{2}+\frac{39x^{14}}{8}-\frac{109x^{12}}{32}+\frac{311x^{10}}{256}-\frac{93x^8}{512}-\frac{9x^6}{2048}+\frac{57x^4}{16384} \\ &-\frac{9x^2}{65536}-\frac{1}{262144}\,. \end{aligned}$$

We compute their zeros with the command *NSolve*. They are

$$\begin{aligned} \{\xi_k^{(18)}(-2;6)\}_{k=1}^{18} &= \{-0.967703,-0.877088-0.0130199I,-0.877088+0.0130199I, \\ &-0.719435,-0.535283-0.053851I,-0.535283+0.053851I, \\ &-0.27156,0.-0.13711I,0.+0.13711I,0.-0.338328I, \\ &0.+0.338328I,0.27156,0.535283-0.053851I, \\ &0.535283+0.053851I,0.719435,0.877088-0.0130199I, \\ &0.877088+0.0130199I,0.967703\}\,. \end{aligned}$$

In Figure 3, we present some representations of zeros and G-sets for $P_{18}^d(\lambda_6;6)(x)$, with values of the parameter λ_6 corresponding to last items of Proposition 5.1. Also, we give the measures of sharpness of G-sets, numerical values of extremal zeros, and some comments. We notice the existence of pairs of close zeros.

Figures 4 and 5 represent the numerical evolution of the sharpness of Geršgorin and Brauer locations with the degree n, for $r=1$ and $r=6$, and several values of the parameter of perturbation. We remark that the sharpnesses decrease even slightly, which means that the locations improve with the increase of the degree.

7 Conclusions

We consider that Geršgorin circles theorems allowed to obtain, in a simple way, quite good locations for the zeros of perturbed Chebyshev polynomials of second kind by dilation. The Brauer location is more difficult to obtain, but is better in some cases. The more the degree n is big, the more the locations are relevant, because computing zeros of polynomials of hight degree could be inaccurate and is time consuming.

The methodology applied in this article constitutes a starting point for treating more complicated perturbed orthogonal sequences and getting some locations for their zeros.

Figure 1: Illustration of Proposition 5.6. Graphs of radius of G-sets, $g^{(n)} := g^{(n)}(\lambda_r; r)$ (**blue trace**), and radius of B-sets, $b^{(n)} := b^{(n)}(\lambda_r; r)$ (**dashed red trace**), as functions of $\lambda_\mathbf{r}$, for $-\mathbf{10} \leq \lambda_\mathbf{r} \leq \mathbf{10}$.

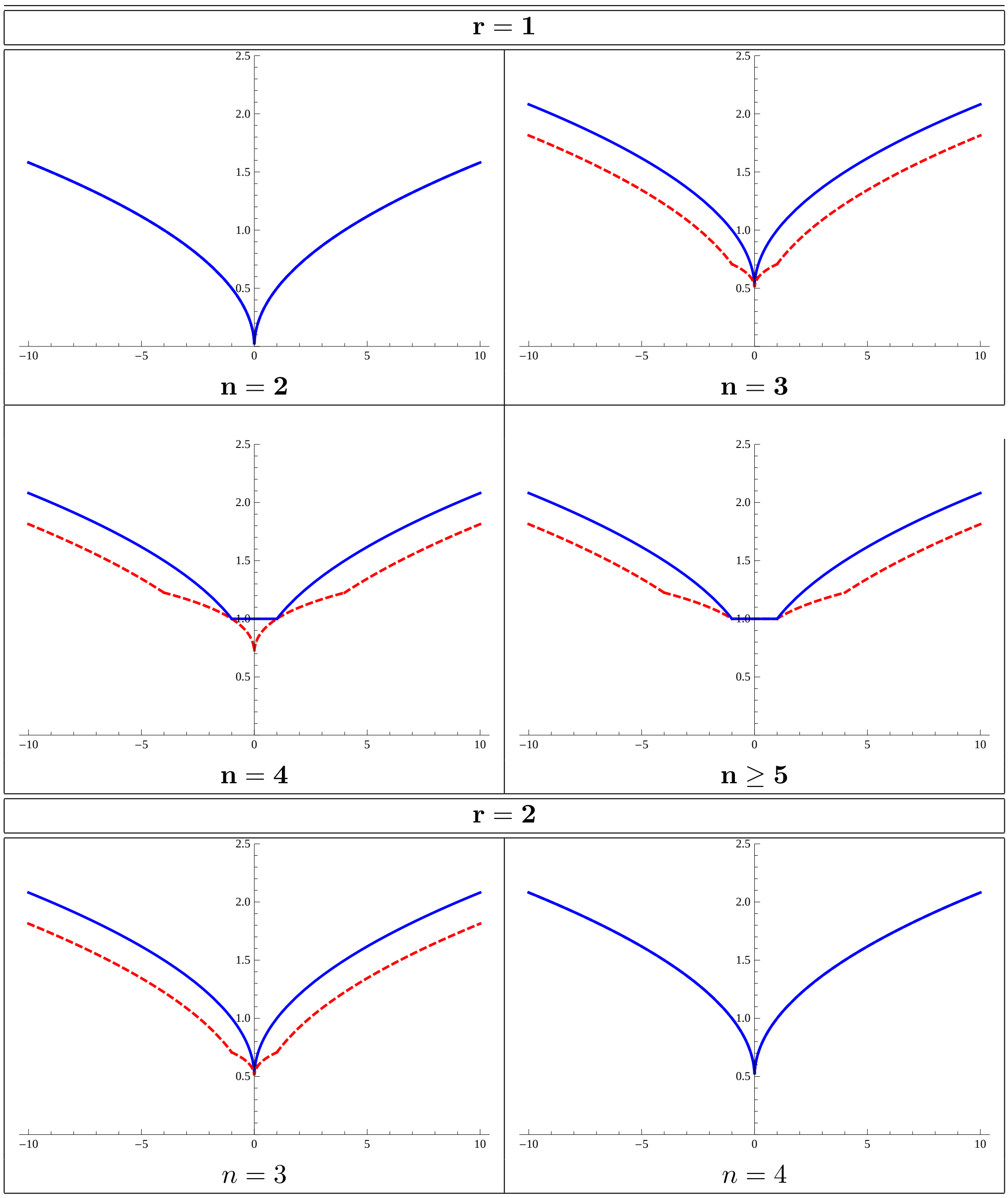


Figure 2: continuation of the previous figure.

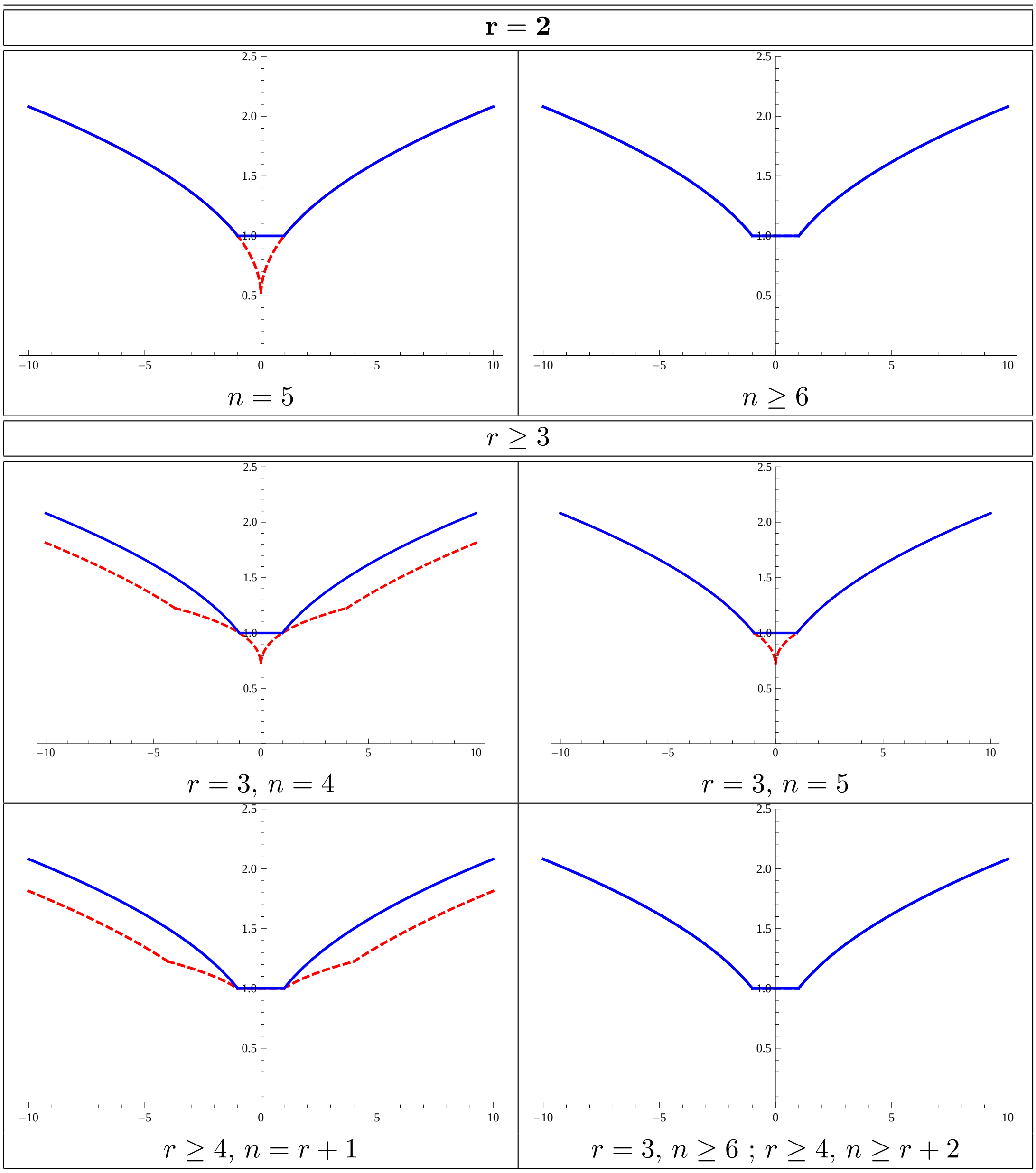
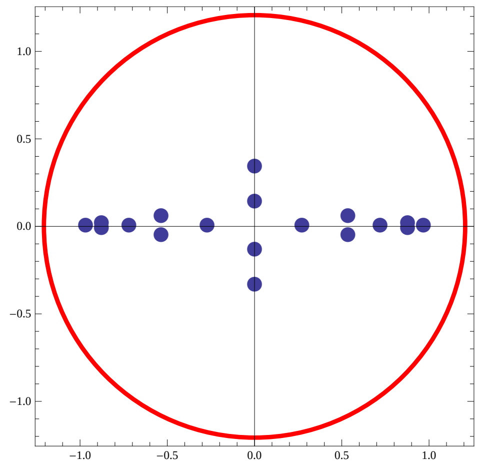


Figure 3: Illustration of Propositions 3.2, 5.1, and 5.3. **Zeros** (**blue dots**) and border of **G-sets** (traced in **red**) for some perturbed Chebyshev polynomials of second kind by dilation of order $\mathbf{r} = \mathbf{6}$ and degree $\mathbf{n} = \mathbf{18}$, $\mathbf{P_{18}}(\lambda_{\mathbf{6}};\mathbf{6})(\mathbf{x})$, for $\lambda_{\mathbf{6}} = \mathbf{-2}, \mathbf{-0.75}, \mathbf{0.75}, \mathbf{5}$. Zeros of $\mathbf{P_{18}}(\lambda_{\mathbf{6}};\mathbf{6})(\mathbf{x})$ are noted by $\xi_k^{d(18)}$, $k = 1(1)18$. Zeros of Chebyshev polynomial $P_{18}(x)$ are noted by $\xi_k^{(18)}$, $k = 1(1)18$. $\mathcal{S} := \mathcal{S}_G^{(18)}(\lambda_6; 6)$ is the sharpness of Geršgorin location. If $\mathbf{1} \neq \lambda_{\mathbf{6}} > \mathbf{0}$, all zeros are real. If $\lambda_{\mathbf{6}} < \mathbf{0}$, there exist some pairs of conjugate complex zeros. Real zeros are symmetric with respect to the origin.

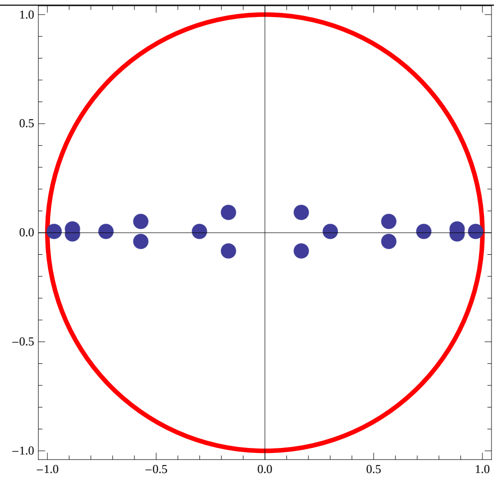


$\lambda_{\mathbf{6}} = \mathbf{-2} < \mathbf{-1}$, $\mathcal{S} \approx \mathbf{0.247}$

All real zeros are in $\left]\xi_1^{(18)}, \xi_{18}^{(18)}\right[\subset \mathcal{I}_{\mathbf{1}}$,

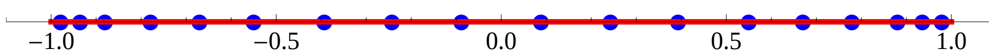
by Proposition 3.2.

The G-set is $\left\{\mathbf{z} \in \mathbb{C} : |\mathbf{z}| \leq \frac{\mathbf{1}+\sqrt{|\mathbf{-2}|}}{\mathbf{2}} \approx \mathbf{1.21}\right\}$.



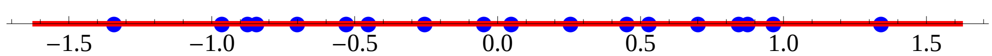
$\mathbf{-1} \leq \lambda_{\mathbf{6}} = \mathbf{-0.75} < \mathbf{0}$, $\mathcal{S} \approx \mathbf{0.0321}$

The G-set is the unit disk.



$\mathbf{0} < \lambda_{\mathbf{6}} = \mathbf{0.75} < \mathbf{1}$, $\mathcal{S} \approx \mathbf{0.0218}$. The G-set is $\mathcal{I}_{\mathbf{1}}$.

The extremal zeros are : $\xi_{\mathbf{1}}^{\mathbf{d(18)}} = \mathbf{-0.978712}$ and $\xi_{\mathbf{18}}^{\mathbf{d(18)}} = -\xi_{\mathbf{1}}^{\mathbf{d(18)}}$.



$\lambda_{\mathbf{6}} = \mathbf{5} > \mathbf{1}$, $\mathcal{S} \approx \mathbf{0.206}$. The G-set is $\left[-\frac{\mathbf{1}+\sqrt{\mathbf{5}}}{\mathbf{2}}, \frac{\mathbf{1}+\sqrt{\mathbf{5}}}{\mathbf{2}}\right] \approx [\mathbf{-1.62}, \mathbf{1.62}]$.

By Proposition 3.2, both extremal zeros, $\xi_{\mathbf{1}}^{\mathbf{d(18)}} = \mathbf{-1.34162}$ and $\xi_{\mathbf{18}}^{\mathbf{d(18)}} = -\xi_{\mathbf{1}}^{\mathbf{d(18)}}$,

are outside of $\left]\xi_1^{(18)}, \xi_{18}^{(18)}\right[$. Indeed, those zeros satisfy,

$-\frac{\mathbf{1}+\sqrt{\mathbf{5}}}{\mathbf{2}} \leq \xi_{\mathbf{1}}^{\mathbf{d(18)}} < \xi_{\mathbf{1}}^{\mathbf{(18)}} = \cos\left(\frac{\mathbf{18}}{\mathbf{19}}\pi\right) \approx \mathbf{-0.986}$, by Proposition 5.3 and (13).

All other zeros are in $\mathcal{I}_{\mathbf{1}}$.

Figure 4: Numerical evolution of the sharpness $\mathcal{S}_{\mathbf{G}}^{(\mathbf{n})}(\lambda_{\mathbf{r}};\mathbf{r})$ of G-sets (**blue dots**), and the sharpness $\mathcal{S}_{\mathbf{B}}^{(\mathbf{n})}(\lambda_{\mathbf{r}};\mathbf{r})$ of B-sets (**red squares**) with the degree $\mathbf{n}=\mathbf{2(1)25}$ for order $\mathbf{r}=\mathbf{1}$ and several values of the parameter of perturbation $\lambda_{\mathbf{1}}$. In each case, ranges of sharpnesses are indicated next to the value of $\lambda_{\mathbf{1}}$.

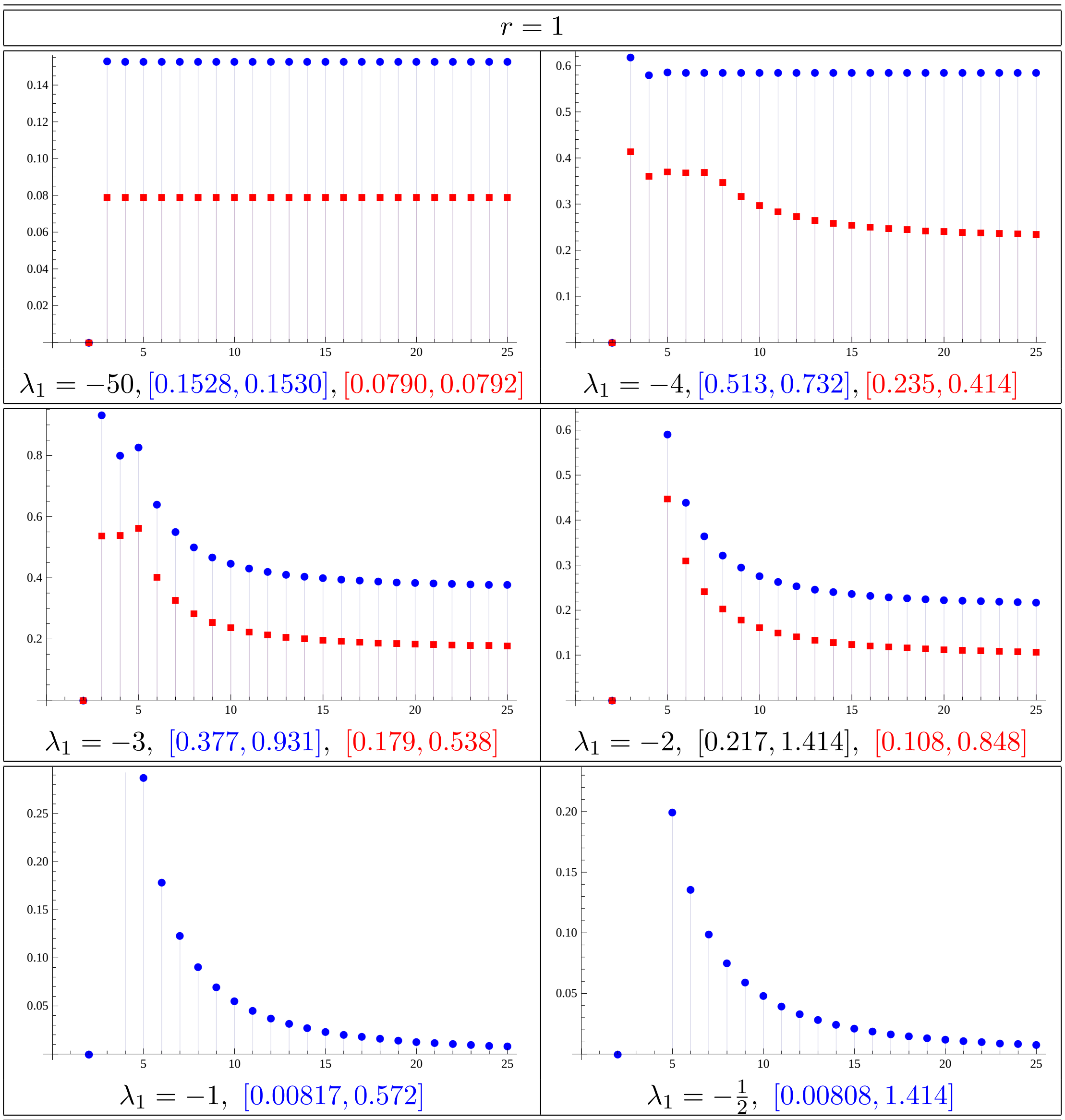
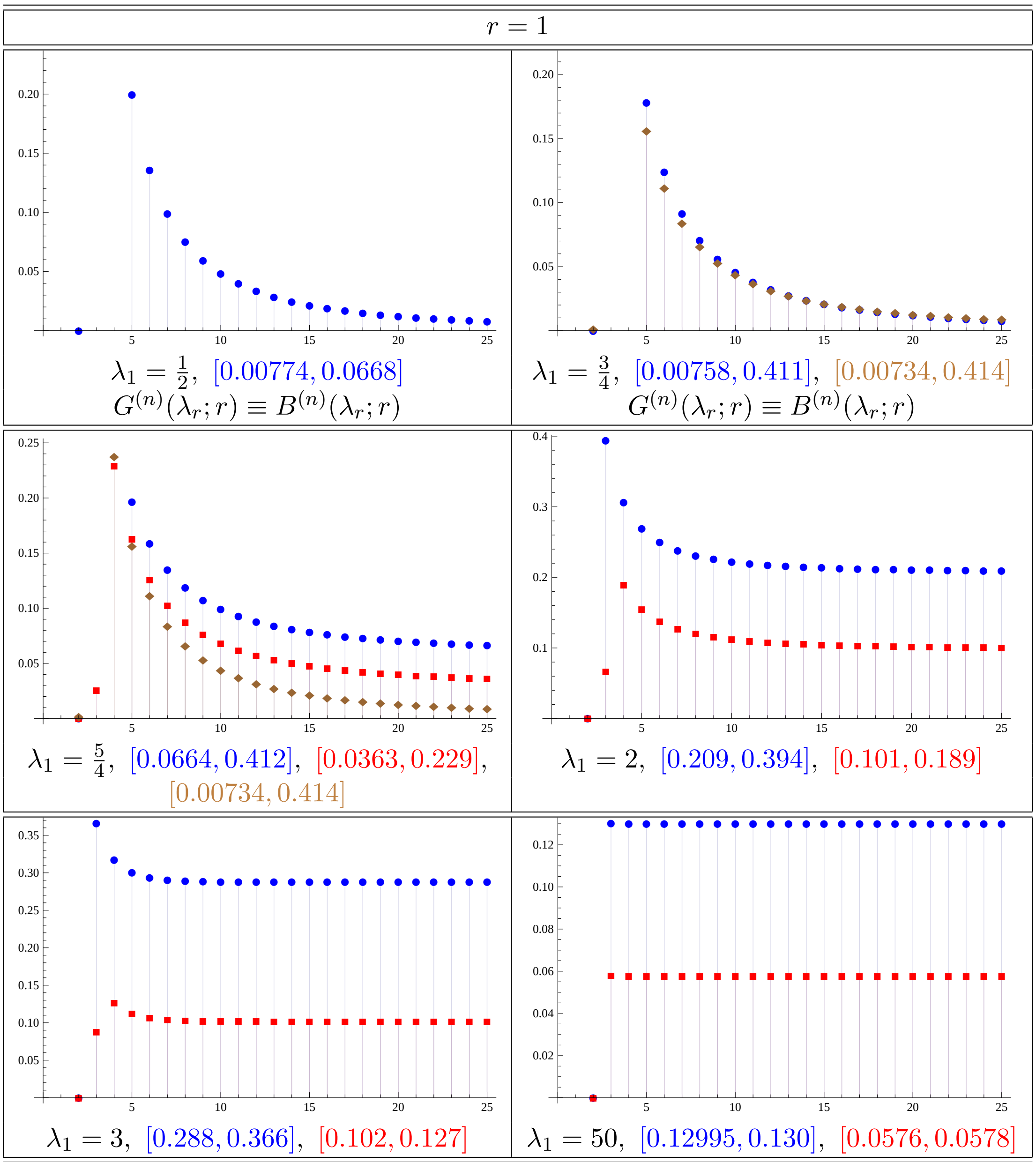


Figure 5: Continuation of the previous figure. For $\boldsymbol{\lambda_1 = 1 \pm \frac{1}{4}}$, that is, for $\boldsymbol{\lambda_1 = \frac{3}{4}}$ and $\boldsymbol{\lambda_1 = \frac{5}{4}}$, we represent also the sharpness of G-sets for Chebyshev polynomials of second kind (**brown diamonds**).



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