INTEGRABILITY OF VECTOR FIELDS AND MEROMORPHIC SOLUTIONS

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Abstract. Let $\mathcal{F}$ be a foliation defined on a complex projective manifold M of dimension n and admitting a holomorphic vector field X tangent to it along some non-empty Zariski-open set. In this paper we prove that if X has sufficiently many integral curves that are given by meromorphic functions defined on $\mathbb{C}$ then the restriction of $\mathcal{F}$ to any invariant complex 2-dimensional analytic set admits a first integral of Liouvillean type. In particular, on $\mathbb{C}^3$, every rational vector fields whose solutions are meromorphic functions defined on $\mathbb{C}$ admits a non-empty invariant analytic set of dimension 2 where the restriction of the vector field yields a Liouvillean integrable foliation.

1. Introduction

This paper is a contribution to a classical subject going back to the works of Painlevé, Chazy, and Garnier that is also of considerable interest in distribution value theory, namely: the study of differential equations whose solutions are meromorphic functions defined on $\mathbb{C}$. These equations are of interest in Physics, as they clearly satisfy Painlevé property, but also in Complex analysis due to their remarkable properties and connections with Nevanlinna theory, see for example the survey [8]. In this paper, however, we will consider not necessarily a single differential equation but rather a *system of differential equations* or, in modern language, general vector fields. The elementary theory of differential equations, allows us to turn any ordinary (system of) differential equation into a vector field defined on a larger phase space, see for example [18]. For example, the classical Painlevé equations give rise to a rational vector field on $\mathbb{C}^3$. Nonetheless vector fields, say on $\mathbb{C}^3$, obtained out of second order differential equations have a very special form and the analogous problem for "systems of differential equations" aka general vector fields is much harder and less developed (see for example [9]). The main results in this paper hold in the general case of vector fields. The most general result obtained in this work is Theorem A below:

We refer the reader to Section 2 for accurate definitions of meromorphic solution of a vector field and of singular foliations associated with a meromorphic vector field. The definition of Liouvillean integrability can be found in Section 4 or, alternatively, in any standard reference such as [31]. With this terminology in place, we have:

Theorem A. *Let M be a compact projective manifold and denote by X a meromorphic vector field on M. Assume that the set of leaves of $\mathcal{F}$ that are meromorphically parametrized by $\mathbb{C}$ has strictly positive logarithmic capacity. Then the restriction of $\mathcal{F}$ to every invariant irreducible 2-dimensional analytic set (not contained in the singular set of $\mathcal{F}$) admits a non-constant first integral of Liouvillean type.*

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Theorem A becomes particularly effective in dimension 3 and to explain this, it suffices to consider the case of rational vector fields on $\mathbb{C}^3$ all of whose solutions are meromorphic on $\mathbb{C}$ (this will actually follow, for example, if it can ensured that the set of leaves meromorphically parametrized by $\mathbb{C}$ has non-empty interior). Then we have the following easy application of Theorem A:

Theorem B. *Let X be a rational vector field on $\mathbb{C}^3$ and recall that its underlying foliation $\mathcal{F}$ admits a holomorphic extension to all of $\mathbb{CP}^3$ which has singular set of dimension at most 1. Assume that the integral curves of X are meromorphic functions defined on $\mathbb{C}$. Then all of the following hold:*

(1) *The foliation $\mathcal{F}$ leaves invariant a certain 2-dimensional analytic set $\mathcal{S} \subset \mathbb{CP}^3$ (note that $\mathcal{S}$ is never contained in the singular set of $\mathcal{F}$ since the latter has dimension at least 1)*
(2) *The restriction of $\mathcal{F}$ to each irreducible component of $\mathcal{S}$ admits a Liouvillian first integral.*

Theorem B is actually a kind of missing piece in a more comprehensive approach to the classification of the vector fields in question. A somehow similar though technically easier problem would be to first classify the singular points of these vector fields and there, again, Theorem B plays an important role. These issues are further detailed in Section 5 (see also [29]) and, at several different levels, they follow ideas put forward in [17], [21], [12], and [27]

Both Theorem A and Theorem B can be made slightly more accurate as the nature of the Liouvillean first integral can be made explicit as it will be indicated in Section 4.

Let us close this introduction with a short outline of our argument. The main analytic tools used in this work are somehow due to M. Brunella: we will make fundamental use of his theorem on the plurisubharmonic variation of the foliated Poincaré metric [3], [5]. Another deep theorem that will be required for our discussion is McQuillan's theorem [22] on entire maps tangent to foliations on surfaces. For the latter, however, we will follow the presentation given in Brunella's notes [4]. Roughly speaking, up to a suitably adapted definition of "leaf", the plurisubharmonic variation of the leafwise Poincaré metric basically allows us to conclude that every "leaf" of a foliation must be a quotient of $\mathbb{C}$ or a rational curve provided that a "large number" of them are so. In particular, leaves contained in two-dimensional invariant analytic sets must themselves be either rational curves or quotients of $\mathbb{C}$ which, in turn, ensures that the classification results of the corresponding foliations provided in [22], [4] can be brought to bear. Carrying out this strategy successfully, however, requires a few classical geometric ideas from foliation theory as well as some more standard tools from differential equations. For example, in order to use Brunella's theorem in [3], [5] we need to control the set of leaves carrying non-trivial holonomy and for this Haefliger's notion of *compactly/countably generated pseudogroup* [13] comes in handy. Also the conclusion about the existence of Liouvillean first integral requires some minor facts about Kleinian groups and about growth rate of leaves. For the basics of foliation theory used in this work, we refer to any of the standard books in the area, for example [10] is well adapted to our needs.

Finally, in Section 5, we outline how Theorem B can be combined with resolution of singularities in dimension 3 [23], [25], see also [28], and with techniques from [12], [11] to adapt a powerful and well-known two-dimensional technique to the problem of describing vector fields with meromorphic solutions in three-dimensional manifolds.

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2. Vector fields, foliations, and enlarged leaves

The notions of holomorphic/meromorphic vector fields on an open set $U \subset \mathbb{C}^n$ are easy to define by taking advantage of the standard coordinates $(x_1, \ldots, x_n)$ of $\mathbb{C}^n$. Denoting by $\partial/\partial x_i$ the constant complex vector field arising from the i^{th} element in the canonical basis of $\mathbb{C}^n$, we have:

Definition 2.1. A complex vector field

$$X = f_1 \partial/\partial x_1 + \cdots + f_n \partial/\partial x_n$$

is said to be holomorphic (resp. meromorphic) on $U \subset \mathbb{C}^n$ if each of the functions f_i, $i = 1, \ldots, n$ is holomorphic (resp. meromorphic) on U.

Let M be a complex manifold along with a covering $\{(U_k, \varphi_k)\}$ of M by coordinate charts. A holomorphic (resp. meromorphic) vector field on M consists of a collection of holomorphic (resp. meromorphic) vector fields $\{X_k\}$ defined on $\varphi_k(U_k) \subset \mathbb{C}^n$ such that for every k_1, k_2 we have

$$Z_{k_1} = (\varphi_{k_2} \circ \varphi_{k_1}^{-1})^* Z_{k_2} ,$$

provided that $U_{k_1} \cap U_{k_2} \neq \emptyset$. In particular, a holomorphic vector field defines a (possibly empty) divisor of zeros on M while a meromorphic vector fields may define a divisor of zeros and a divisor of poles.

The definition of singular holomorphic foliation on M being slightly different, it is convenient to make it explicit so as to understand its usefulness when compared with vector fields.

Definition 2.2. Let M and $\{(U_k, \varphi_k)\}$ be as above. A singular holomorphic foliation $\mathcal{F}$ on M consists of a collection of holomorphic vector fields Y_k satisfying the following conditions:

- For every k, Y_k is a holomorphic vector field defined on $\varphi_k(U_k) \subset \mathbb{C}^n$ whose singular set has codimension at least 2.
- Whenever $U_{k_1} \cap U_{k_2} \neq \emptyset$, we have $\varphi_{k_1}^* Z_{k_1} = g_{k_1 k_2} \varphi_{k_2}^* Z_{k_2}$ for some nowhere vanishing holomorphic function $g_{k_1 k_2}$ defined on $U_{k_1} \cap U_{k_2}$.

The *singular set* $\text{Sing}\,(\mathcal{F})$ of a foliation $\mathcal{F}$ is then defined as the union over k of the sets $\varphi_k^{-1}(\text{Sing}\,(Z_k)) \subset M$, where $\text{Sing}\,(Z_k)$ stands for the singular set of Z_k. Therefore the singular set of any holomorphic foliation has codimension at least two. In particular, a foliation *has no divisor of either poles or zeros.*

Remark 2.3. Note that the foliations introduced in Definition 2.2 are all of complex dimension 1 and, for this reason, they are often referred to in the literature as 1-dimensional holomorphic foliations or yet as foliations by Riemann surfaces. This issue about dimension is not emphasized in Definition 2.2 since this is the only type of foliation that will be considered in this paper.

To large extent, the interest of Definition 2.2 stems from the following observation.

Lemma 2.4. *Let X be a meromorphic vector field defined on a complex manifold M. Then the local orbits of X induce a holomorphic foliation $\mathcal{F}$ defined on all of M.*

Proof. Consider a point $p \in M$. By definition of meromorphic vector field, in a suitable coordinate $\varphi_k : U \subset M \to \mathbb{C}^n$ defined around p, the vector field X is represented by a meromorphic vector field X_k defined on $\varphi_k(U) \subset \mathbb{C}^n$. Owing to the standard Hilbert nullstellensatz in codimension 1, the divisor of zeros and poles of X_k can be factored out so as to have

$$X_k = \frac{f}{g} Y_k$$

where f and g are holomorphic functions and where Y_k is a vector field with empty divisor of zeros and poles. In other words, Y_k is a holomorphic vector field whose zero-set has codimension at least 2. It is now immediate to check that the collection of vector fields $\{Y_k\}$ satisfies the condition in Definition 2.2 and hence defines a holomorphic foliation $\mathcal{F}$ on all of M. □

The foliation $\mathcal{F}$ arising from a vector field X as in Lemma 2.4 will be called the *foliation associated with* X, or yet, the *underlying foliation of* X.

The leaves of a holomorphic foliation $\mathcal{F}$ on a manifold M can easily be defined as follows. First note that the standard flow box theorem for holomorphic vector fields ensures that $\mathcal{F}$ coincides with a *regular foliation* by 2-dimensional real surfaces on the manifold $M \setminus \mathrm{Sing}\,(\mathcal{F})$. We then define the *leaves of the singular foliation* $\mathcal{F}$ *on* M to be the leaves of the regular foliation on $M \setminus \mathrm{Sing}\,(\mathcal{F})$ obtained as restriction of $\mathcal{F}$.

The preceding raises an important issue for us. Consider a meromorphic vector field X defined on M and assume that the divisor of poles (or zeros) of X is not empty. This divisor of poles cannot be contained in the singular set of the holomorphic foliation $\mathcal{F}$ associated with X since the latter set has codimension at least 2. Hence the "generic" point of the divisor of poles of X is regular for $\mathcal{F}$ and the divisor may or may not be invariant by the foliation $\mathcal{F}$. The last question, however, is directly related with the existence of meromorphic solutions, cf. Lemma 2.5 below.

At this point, it is convenient to make accurate the notion of meromorphic solution for a vector field. Consider a meromorphic vector field X on a complex manifold M and denote by $\mathcal{F}$ its associated foliation. Now consider a point $p \in M$ lying away from the divisor of poles of X and such that $X(p) \neq 0$ and denote by ϕ a local solution of X satisfying $\phi(0) = p$. We say that the solution of X through p is a meromorphic function on $\mathbb{C}$ if ϕ the following holds:

- There is a discrete set $\mathcal{D} = \{t_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ such that ϕ possesses a holomorphic extension Φ to $\mathbb{C} \setminus \mathcal{D}$.
- The points $t_i \in \mathcal{D}$ are poles for Φ.

Clearly if the solution of X through p is a meromorphic function on $\mathbb{C}$ then the same applies to every point in the orbit of p. We will then say that *the leaf* L *of* $\mathcal{F}$ *through* p *is (meromorphically) parametrized by* $\mathbb{C}$ *via* X. Note that the latter definition makes sense for every point p regular for $\mathcal{F}$ even though p belongs to the divisor of poles (or zeros) of X.

Next, we have:

Lemma 2.5. *Let X be a meromorphic vector field defined on a complex manifold M and denote by S an irreducible (non-empty) component of the divisor of poles of X. If $p \in S$ is a regular point of $\mathcal{F}$ at which $\mathcal{F}$ and S are transverse, then the leaf L of $\mathcal{F}$ through p is not parametrized by $\mathbb{C}$ via X.*

Proof. Consider the restriction $X_{|L}$ of X to L viewed as a meromorphic vector field on a Riemann surface (L). Let z be a local coordinate around $p \in L$, $p \simeq 0 \in \mathbb{C}$. Since X has a pole at p, there follows that X takes on the form

$$X = z^{-k} f(z) \partial/\partial z$$

in the coordinate z. Here $k \geq 1$ and f is a holomorphic function satisfying $f(0) \neq 0$. In fact, the elementary theory of normal forms for 1-dimensional complex vector fields ensures that the coordinate z can therefore be chosen so as to have $f = 1$ (constant). In other words, without loss of generality we can suppose that $X = z^{-k}\partial/\partial z$. The integral curve of X with initial point $z_0 \neq 0$ is therefore $\phi(t) = \sqrt[k+1]{(1+k)t + z_0^{k+1}}$. This function is multivalued since $k+1 \geq 2$ and therefore does not yield a meromorphic function defined on $\mathbb{C}$. This establishes the lemma. □

Remark 2.6. The statement of Lemma 2.5 actually holds for semicomplete vector fields, see for example [12]. In turn, it is not hard to check that vector fields all of whose solutions are meromorphic will necessarily be semicomplete.

As mentioned, a fundamental ingredient in this paper is Brunella's theorem on the regularity of the leafwise Poincaré metric. To call on his result, however, a slightly more specific notion of "leaf" is required. To avoid misunderstandings, throughout this paper, the phrase "leaf of a singular (holomorphic) foliation" will be taken in the above introduced sense. In turn, the variant of it needed for Brunella's theorem to hold will be referred to as *enlarged leaf* and is defined as follows (cf. [5]).

In general, an *enlarged leaf* $\widehat{L}$ of $\mathcal{F}$ is obtained out of a leaf L of $\mathcal{F}$ by adding some singular points of $\mathcal{F}$ to L according to the rule described below. To begin with, consider a singular point of $\mathcal{F}$ identified with the origin of $\mathbb{C}^n$ by means of a local coordinate.

In the local setting, consider the n-dimensional polydisc $\mathbb{D}^n$ about the origin. This polydisc comes equipped with the trivial fibration $\mathbb{D}^n = \mathbb{D}^{n-1} \times \mathbb{D} \to \mathbb{D}^{n-1}$. A meromorphic map $f : \mathbb{D}^n \to M$ is said to be a *foliated meromorphic immersion* if the indeterminacy set $I(f)$ of f intersects each vertical fiber of $\mathbb{D}^n$ in a discrete set and if f satisfies the following additional conditions:

- f is an immersion on the complement of $I(f)$.
- In the complement of $I(f)$, f takes vertical fibers to leaves of $\mathcal{F}$.

If L is a leaf of $\mathcal{F}$, we say that a closed subset $K \subset L$ is a *vanishing end* of L if the following conditions are satisfied:

(1) K is isomorphic to the punctured disc and the holonomy of the restriction of $\mathcal{F}$ to $M \setminus \mathrm{Sing}\,(\mathcal{F})$ corresponding to the cycle ∂K has finite order k.
(2) There is a foliated meromorphic immersion $f : \mathbb{D}^n \to M$ such that
  (2a) $I(f) \cap (\{0\} \times \mathbb{D})$ is reduced to the origin of $\mathbb{C}^n$ (and where "$\{0\}$" stands for the origin of $\mathbb{D}^{n-1} \subset \mathbb{C}^{n-1}$).
  (2b) The image of f restricted to $(\{0\} \times \mathbb{D})$ coincides with the interior of K. Furthermore $f : (\{0\} \times \mathbb{D}) \to \mathrm{Int}\,(K)$ is a regular covering of degree k, where $\mathrm{Int}\,(K)$ stands for the interior of K.

Let now p be a point in $M \setminus \mathrm{Sing}\,(\mathcal{F})$ and denote by L_p the leaf of $\mathcal{F}$ through p. To define the *enlarged leaf* $\widehat{L}_p$ of $\mathcal{F}$ through p we proceed as follows. If L_p possesses no vanishing ends, then we set $\widehat{L}_p = L_p$. Otherwise, the enlarged leaf $\widehat{L}_p$ will consist of L_p with the ends of the vanishing ends added to it where the operation of adding an end to L_p should be understood in the sense of orbifolds; the multiplicity of the added point will be the order k of the holonomy relative to ∂K. These orbifolds can then be turned into Riemann surfaces by standard normalization.

For reference, it is convenient to point out the following immediate consequence of the construction above.

Lemma 2.7. *Given $p \in M \setminus \mathrm{Sing}\,(\mathcal{F})$, let L_p and $\widehat{L}_p$ denote respectively the leaf and the enlarged leaf of $\mathcal{F}$ through p. Then the set $\widehat{L}_p \setminus L_p$ is a discrete set of $\widehat{L}_p$.* □

3. Meromorphic solutions, parabolic leaves, and invariant surfaces

The purpose of this section is to prove the theorem below:

Theorem 3.1. *Let M be a complex projective manifold of dimension n equipped with a meromorphic vector field X whose associated foliation is denoted by $\mathcal{F}$. Denote by $(X)_\infty$ the divisor of poles of X and assume that the following condition is satisfied: there exists a foliated set $\mathcal{I} \subset M \setminus (X)_\infty$ having positive logarithmic capacity and such that for every point $p \in \mathcal{I}$, the corresponding solution of X yields a meromorphic function defined on all of $\mathbb{C}$.*

Assume now that $N \subset M$ is a, (singular, irreducible) surface invariant by $\mathcal{F}$ which is not contained in the singular set of $\mathcal{F}$. Denote by $\mathcal{F}_N$ the foliation naturally induced on N by restriction of $\mathcal{F}$. Then, up to birational equivalence, the foliation $\mathcal{F}_S$ is of one of the following types:

(1) *a fibration.*
(2) *a Riccati foliation.*
(3) *a turbulent foliation.*

As it will be seen, Theorem 3.1 is by far the main ingredient in the proof of Theorem A. In fact, all the deep material from complex analysis and algebraic geometry involved in the proofs of the mentioned theorems of Brunella and of McQuillan (cf. [3], [5], [22], [4]) are encoded in the proof of this theorem.

Some terminology, however, is needed to clarify the statement of the above theorem. Let then N denote a (possibly singular) compact complex surface equipped with a singular holomorphic foliation $\mathcal{F}$. The pair $(N, \mathcal{F})$ will be regarded up to birational equivalence as it is standard in complex geometry.

Recall that a *fibration* on a complex surface N is nothing but a non-constant holomorphic map with connected fibers $\mathcal{P} : N \to S$ where S stands for some compact Riemann surface. Clearly the set of critical values of $\mathcal{P}$ is a finite subset $\{p_1, \ldots, p_l\}$ of S and the fibers of $\mathcal{P}$ over the critical values are said to be the *singular fibers* of the fibration (or of π). Note also that the map $\mathcal{P}$ induces a $\mathcal{C}^\infty$-bundle structure from $M \setminus \bigcup_{i=1}^{l} \mathcal{P}^{-1}(p_i)$ to $S \setminus \{p_1, \ldots, p_l\}$. The genus of the corresponding fiber is called the genus of $\mathcal{P}$. Thus a fibration is of genus 0 if and only if its generic fiber is a rational curve. Similarly it is of genus 1 if and only if its generic fiber is an elliptic curve.

Naturally a fibration is a special type of foliation. Next, assume N is equipped with a fibration $\mathcal{P} : N \to S$ and with a singular foliation $\mathcal{F}$. The foliation $\mathcal{F}$ is said to be *transverse to* $\mathcal{P}$ if the leaves of $\mathcal{F}$ are transverse to the fibers of π away from finitely many fibers of $\mathcal{P}$, which are necessarily invariant under $\mathcal{F}$. Finally, if $\mathcal{F}$ is transverse to some genus 0 fibration, then $\mathcal{F}$ is called a *Riccati foliation*. Similarly a foliation transverse to a genus 1 fibration is called a *turbulent foliation*.

The following general lemma will also be useful in what follows.

Lemma 3.2. *Let $\mathcal{F}$ be a singular holomorphic foliation on a complex Kähler manifold M. Then the set of leaves of $\mathcal{F}$ carrying non-trivial holonomy has volume equal to zero. In fact, this set has zero logarithmic capacity.*

Proof. The idea goes back to [7] and makes an important use of Haefliger's notion of "holonomy pseudogroup" along with its property of being "countably generated" on general open manifolds, see [13], [2]. Consider $\mathcal{F}$ as a regular holomorphic foliation on the open complex manifold $M \setminus \mathrm{Sing}\,(\mathcal{F})$ where $\mathrm{Sing}\,(\mathcal{F})$ stands for the singular set of $\mathcal{F}$. Let L be a leaf of $\mathcal{F}$ and denote by $\sigma : [0,1] \to L$ a loop in L. Unless the holonomy map $\rho_L(\sigma)$ obtained from σ is identically zero, L is contained in the set of fixed points of $\rho_L(\sigma)$ and the latter is a (local) proper complex analytic subset of $M \setminus \mathrm{Sing}\,(\mathcal{F})$. The countably generated nature of Haefliger's holonomy pseudogroup allows us now to conclude that the set of leaves of $\mathcal{F}$ carrying non-trivial holonomy is contained in a countable union of locally defined (proper) analytic subsets of $M \setminus \mathrm{Sing}\,(\mathcal{F})$. Since the latter set must have null measure with respect to Kähler form on M, and in fact zero logarithmic capacity, the statement follows. □

We are now able to begin our approach to Theorem 3.1. To some extent, the content of this theorem was previously indicated in [27] but in what follows we provide full detail and accurate formulations.

Let M, X, and $\mathcal{F}$ be as in Theorem 3.1. By using standard terminology, a Riemann surface is said to be *parabolic* if it is a quotient of $\mathbb{C}$. Similarly, a Riemann surface that is a quotient of the hyperbolic disc $\mathbb{D}$ is called *hyperbolic*. According to the uniformization theorem, every Riemann surface different from $\mathbb{CP}^1$ is either parabolic or hyperbolic. Now we have the following:

Lemma 3.3. *Every enlarged leaf of $\mathcal{F}$ is either $\mathbb{CP}^1$ or a parabolic Riemann surface.*

Proof. The point of considering enlarged leaves $\widehat{L}$, as opposed to standard leaves L, lies in the fact that the Poincaré metric on *enlarged leaves* varies in a pluri-subharmonic way, thanks to Brunella's theorem in [3], see also [5]. This means the following. Let every (enlarged) leaf of $\mathcal{F}$ be equipped with its Poincaré metric: the unique complete metric with curvature equals to -1) in the case of a hyperbolic Riemann surface and the totally vanishing "metric" otherwise. If Z is a (locally defined) holomorphic vector field tangent to $\mathcal{F}$, then we have the (locally defined) function P given by

$$P(q) = \log(\|Z(q)\|)$$

where $\| \,. Vert$ stands for the norm of X with respect to the above introduced foliated metric. Basically, Brunella's theorem states that the function f is *plurisubharmonic* unless it is constant equal to $-\infty$. Clearly the polar set of P coincides with non-hyperbolic enlarged leaves of $\mathcal{F}$. In particular, we have the following consequence of Brunella's theorem which will play in a key role in the proof of the lemma: if the set of *enlarged leaves* of $\mathcal{F}$ that *are not* hyperbolic Riemann surface has strictly positive logarithmic capacity then *every enlarged leaf of $\mathcal{F}$ is either parabolic or a rational curve.*

In view of the preceding, the proof of the lemma is reduced to checking that the set formed by enlarged leaves of $\mathcal{F}$ that are either rational curves or parabolic Riemann surfaces is sufficiently large in the sense that it must have strictly positive logarithmic capacity.

Consider now a regular point $p \in M$ for X and denote by L_p (resp. $\widehat{L}_p$) the leaf of $\mathcal{F}$ (resp. the enlarged leaf of $\mathcal{F}$) through p. Assume that L_p has trivial holonomy. We then denote by ϕ the integral curve of X satisfying $\phi(0) = p$. Owing to Lemma 3.2, to prove the lemma it suffices to show that $\widehat{L}_p$ is not a hyperbolic Riemann surface provided that ϕ is a meromorphic map defined on $\mathbb{C}$.

Taking advantage of the fact M is projective, i.e. can be embedded in some complex projective space, it is easy to give a meaning to the meromorphic nature of ϕ. Up to considering a hyperplane section on M, the corresponding Zariski-open set U is naturally isomorphic to an algebraic affine manifold of dimension n in $\mathbb{C}^N$ (for some sufficiently large $N \in \mathbb{N}$). In the standard coordinates of $\mathbb{C}^N$, we have $\phi(T) = (\phi_1(T), \ldots, \phi_N(T))$. Saying that ϕ is a meromorphic map defined on $\mathbb{C}$ then means that each of the coordinate maps ϕ_i, $1 \leq i \leq N$, is a meromorphic map from $\mathbb{C}$ to $\mathbb{C}$ as defined in Section 2. In particular, there is some discrete set $\mathcal{D} = \{t_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ such that ϕ defines a holomorphic map from $\mathbb{C} \setminus \mathcal{D} \to \mathbb{C}^N$. Moreover, ϕ has at worst poles at points of $\mathcal{D}$.

Let $\hat{t}$ be a point in $\mathcal{D}$ and $B(\epsilon) \subset \mathbb{C}$ a small disc around $\hat{t}$ containing no other points from $\mathcal{D}$. Clearly ϕ maps the punctured disc $B(\epsilon) \setminus \{\hat{t}\}$ to $L_p \subset U$. Our purpose is to show that ϕ extends to $\hat{t}$ as a holomorphic map taking values in the enlarged leaf $\widehat{L}_p$. Since M can be viewed as the projective compactification of $U \subset \mathbb{C}^N$ and since ϕ_i at worst a pole at $\hat{t}$, there follows that the limit $\lim_{t\to\hat{t}} \phi(t)$ exists in M. Set then $\lim_{t\to\hat{t}} \phi(t) = q \in M$.

We claim that the point q is a vanishing end of L_p so that, by definition, we must have $q \in \widehat{L}_p$. To show that q is a vanishing end, we need to construct an appropriate foliated meromorphic immersion f defined on $\mathbb{D}^{n-1} \times \mathbb{D}$. The existence of this meromorphic immersion is clear if q happens to be a regular point of $\mathcal{F}$. In the general case, we proceed as follows. Note that the restriction of ϕ to $B(\epsilon)$ (with $\phi(\hat{t}) = q$) yields a natural candidate for f restricted to $\{0\} \times D$. Moreover the holonomy of L_p is trivial by assumption and thus so is the holonomy associated with the cycle given by the boundary of $\phi(B(\epsilon))$. In particular, ϕ can be extended as a meromorphic immersion to the product of D^{n-1} with an annulus of type $B(\epsilon) \setminus B(\epsilon/2)$ (which is also defined on $\{0\} \times B(\epsilon)$). To show that the resulting map can further be extended to $\mathbb{D}^{n-1} \times \mathbb{D}$, we just need to use the parametrization of the leaves by the meromorphic integral curves of X. This proves that q is a vanishing end and hence that $q \in \widehat{L}_p$.

To complete the proof of the lemma we proceed as follows. Since $\lim_{t\to\hat{t}} \phi(t) = q \in \widehat{L}_p \subset M$, there follows from Riemann theorem that ϕ extends to $\hat{t}$ as a holomorphic map taking values in $\widehat{L}_p$. Since $\hat{t}$ is an arbitrary point of $\mathcal{D}$, we conclude that ϕ extends to a (non-constant) holomorphic map from all of $\mathbb{C}$ to $\widehat{L}_p$. The enlarged leaf $\widehat{L}_p$ is therefore either a rational curve or a parabolic Riemann surface. The lemma is proved. □

We are now able to derive Theorem 3.1 from McQuillan's theorem on foliations tangent to a Zariski-dense entire map, as presented by Brunella in [4].

Proof of Theorem 3.1. Owing to Lemma 3.3, we know that no enlarged leaf of the foliation $\mathcal{F}$ is a hyperbolic Riemann surface. Assume now that N is a 2-dimensional an irreducible algebraic set invariant by $\mathcal{F}$ and not contained in the singular set of $\mathcal{F}$. Denote by $\mathcal{F}_N$ the foliation induced on N by $\mathcal{F}$. Here an additional comment is needed to compare leaves of $\mathcal{F}_N$ with leaves of $\mathcal{F}$ *that are contained in* N. The issue giving rise to this difference lies in the fact that $\mathcal{F}_N$ may differ from the restriction of $\mathcal{F}$ to N. More precisely, note that $N \cap \mathrm{Sing}\,(\mathcal{F})$ may still contain algebraic curves as opposed to only isolated singular points. Away from these "singular curves", $\mathcal{F}_N$ coincides locally with the restriction of $\mathcal{F}$ to N. Around points lying in a singular curve, however, the foliation $\mathcal{F}_N$ is locally obtained after dividing the components of the foliation *restricted to* N by a non-trivial common factor (see Definition 2.2 where all singular foliations have singular set of codimension at least 2). In particular a point in $N \cap \mathrm{Sing}\,(\mathcal{F})$ may turn out to be *regular* for $\mathcal{F}_N$ and, as a consequence, the leaves (resp. enlarged leaves) of $\mathcal{F}$ contained in N may differ from the leaves (resp. enlarged leaves) of $\mathcal{F}_N$. Similarly, a point $q \in N$ may be a vanishing end for a leaf of $\mathcal{F}_N$ without being so for the corresponding leaf viewed as belonging to the foliation $\mathcal{F}$ (note, in particular, the condition on the holonomy associated with the cycle ∂K in the definition of vanishing end). Nonetheless, it is straightforward to check that taking a leaf (resp. enlarged leaf) of $\mathcal{F}$ contained in N and replacing it by the corresponding leaf (resp. enlarged leaf) of $\mathcal{F}_N$ can only add new points to the former leaf (resp. enlarged leaf). Since there are non-constant holomorphic maps from $\mathbb{C}$ to every enlarged leaf of $\mathcal{F}$ - and in particular to those enlarged leaves of $\mathcal{F}$ contained in N - the same maps also provide non-constant holomorphic maps from $\mathbb{C}$ to the enlarged leaves of $\mathcal{F}_N$. In other words, the enlarged leaves of $\mathcal{F}_N$ are never hyperbolic Riemann surfaces.

Now consider a enlarged leaf $\widehat{L}$ of the foliation $\mathcal{F}_N$. In view of the preceding, $\widehat{L}$ is not a hyperbolic Riemann surface. Since N is irreducible and of complex dimension 2, either $\widehat{L}$ is Zariski-dense in N or $\widehat{L}$ is contained in a compact curve (necessarily invariant under $\mathcal{F}_N$). In the latter case, note that the compact curve in question is either rational or elliptic since $\widehat{L}$ is not hyperbolic, though this is not important in what follows. In fact, a well-known theorem due to Jouanolou [19] asserts that $\mathcal{F}_N$ must admit a rational first integral provided that it leaves infinitely many (irreducible, compact) curves invariant. In turn, if $\mathcal{F}_N$ admits a rational first integral, then it is clearly equivalent to a fibration so that the statement of Theorem 3.1 holds.

Consider now the former case where $\widehat{L}$ is Zariski-dense. Being non-hyperbolic, there follows that $\widehat{L}$ is a quotient of $\mathbb{C}$. Hence $\mathcal{F}_N$ is a foliation that is tangent to an *Zariski-dense entire curve*. Owing to McQuillan's theorem (cf. Chapter 9 and Theorem 4 in page 131 of [4]), one of the following has to hold:

(1) Up to a finite ramified covering $\widetilde{N}$ of N, the foliation $\mathcal{F}_N$ is defined by a (global) holomorphic vector field.
(2) Up to birational equivalence $\mathcal{F}_N$ is a Riccati foliation or a turbulent foliation.
(3) A fibration.

As far as the statement of Theorem 3.1 is concerned, we can assume without loss of generality that (1) holds. Complex compact surfaces equipped with holomorphic vector fields have been studied since long and there are several ways to rely on the available information to derive Theorem 3.1. The argument provided below has the advantage of relying exclusively on very classical material in the realm of complex (algebraic) geometry.

Consider then the pair $\widetilde{N}$ equipped with a (non-trivial) holomorphic vector field Z whose underlying foliation is $\mathcal{F}_Z$. Here $\widetilde{N}$ is a possibly singular projective surface. We need to show that up to birational equivalence, the foliation $\mathcal{F}_Z$ either is a fibration or is as in item (2).

Claim. Without loss of generality, $\widetilde{N}$ can be assumed to be smooth.

Proof of the Claim. Let $\mathrm{Sing}\,(\widetilde{N})$ denote the singular locus of $\widetilde{N}$. If not empty, the set $\mathrm{Sing}\,(\widetilde{N})$ is constituted by isolated points and irreducible curves. However, $\mathrm{Sing}\,(\widetilde{N})$ is clearly invariant under the flow of Z so that we actually have:

- Every isolated point of $\mathrm{Sing}\,(\widetilde{N})$ is invariant under Z (i.e. it is a singular point of Z).
- Every irreducible curve C contained in $\mathrm{Sing}\,(\widetilde{N})$ is globally invariant under Z. Note that if the curve C is singular, then its singular points are invariant by Z as well.

Now, we apply a resolution procedure to $\widetilde{N}$. This procedure consists of two types of operations, namely, normalizations and blow-ups (see for example [1]). A normalization must be centered at a smooth curve contained in $\mathrm{Sing}\,(\widetilde{N})$ and, since the curve is question is globally invariant by Z, the corresponding transform of Z still is a holomorphic vector field. In turn, blow-ups are centered at isolated points of $\mathrm{Sing}\,(\widetilde{N})$ or at singular points of a curve in $\mathrm{Sing}\,(\widetilde{N})$. In either case, they are invariant under Z so that again the corresponding transform of Z is a holomorphic vector field. Proceeding by induction, it is now clear that the singularities of $\widetilde{N}$ can be resolved in such way that the final transform of the vector field Z is still a holomorphic vector field. This establishes the claim. □

In the sequel we assume $\widetilde{N}$ to be smooth. We can also assume $\widetilde{N}$ not to contain (-1)-rational curves, i.e., $\widetilde{N}$ can be assumed to be minimal. In fact, the transform of a holomorphic vector field arising from collapsing a (-1)-rational curve is always holomorphic since it is holomorphic away from the point to which the curve collapsed. Since we are dealing with a surface, the vector field must then extend holomorphically to the point in question since it has codimension 2.

The list of compact complex surfaces carrying a *non-singular* holomorphic vector field was provided by Mizuhara [24] and it reads:

(1) A complex torus $\mathbb{C}^2/\Lambda$;
(2) A flat holomorphic fiber bundle over an elliptic curve;

(3) An elliptic surface without singular fibers or with singular fibers of type mI_0 only. In other words, the singular fibers are elliptic curves with finite multiplicity;
(4) A Hopf surface or a positive Inoue surface.

In turn, a compact complex surface carrying a holomorphic vector field exhibiting at least one singular point must either be rational or a surface of class VII as shown in [6]. Note that Hopf surfaces ans Inoue surfaces are themselves examples of class VII surfaces, see [1]. Since $\widetilde{N}$ is projective, and hence algebraic, surfaces of class VII can be ruled out of our discussion since they are never algebraic, see [1] page 244. Also, a minimal rational surface is either a Hirzebruch surface F_n, $n \neq 1$ or $\mathbb{CP}^2$.

It is immediate to check holomorphic vector fields on a complex torus are constant and hence satisfy the conditions of Theorem 3.1 provided that the torus in question is algebraic. Similarly, holomorphic vector fields on $\mathbb{CP}^2$ are "linear" and it is also immediate to check that they satisfy the conditions of the theorem in question.

In all the remaining cases, the surface $\widetilde{N}$ carries a (possibly singular) fibration of genus 0 or 1. Since Z is holomorphic, the classical Lemma of Blanchard applies to say that Z must preserve the fibration in question. If the fibers are individually preserved by Z, then Z is tangent to the mentioned fibration so that Theorem 3.1 holds. Finally, if instead, the flow of Z permutes the fibers of the mentioned fibration, then it is clear that the foliation $\mathcal{F}_Z$ will either be Riccati or turbulent depending only on the genus of the fibration. The proof of Theorem 3.1 is complete. □

4. Proofs for Theorems A and B

Let us begin by recalling the definition of *Liouvillian function*. Informally speaking, a Liouville function is a function that can be built up from rational functions by means of algebraic operations, exponentiation, and integrals. The following definition makes this notion accurate (see for example [31]).

Definition 4.1. Let $k \subset K$ be differential fields. The field K is said to be a Liouvillian extension of k if there is a tower $k = K_0 \subset \cdots \subset K_s = K$ of differential fields such that $K_i = K_{i-1}(t_i)$, for $i = 1, \ldots, s$, where at least one of the following holds:

(1) The derivative t_i' of t_i lies in K_{i-1} (i.e. t_i is the integral of an element in K_{i-1}).
(2) $t_i \neq 0$ and t_i'/t_i lies in K_{i-1} (i.e. t_i is the exponential of an element in K_{i-1}).
(3) t_i is algebraic over K_{i-1}

If N is a projective manifold, a *Liouvillian function* on N is therefore an element of a Liouvillian extension of the field of rational functions on N. If, in addition, N is equipped with a foliation $\mathcal{F}$, then a *Liouvillian first integral* for $\mathcal{F}$ is nothing but a Liouvillian function that is constant over the leaves of $\mathcal{F}$.

An immediate consequence of the above definition is that the existence of a Liouvillian first integral $\mathcal{F}$ is a birational invariant of the pair $(N, \mathcal{F})$. Similarly, a birational equivalence between two manifolds gives rise to a one-to-one correspondence between the corresponding algebras of Liouvillian functions over $\mathbb{C}$.

Remark 4.2. M. Singer has proved in [31] that a differential equation with Liouvillian first integrals must admit a transversely affine structure. This statement was later generalized to codimension 1 foliations and an interesting treatment of the topic can be found in [32]. These statements provide an alternative dynamical point of view in the notion of Liouvillian first integrals.

A converse is easy. If a codimension 1 foliation admits a transverse affine structure, then the developing map (see [10]) associated with the affine structure in question provides a Liouvillian first integral for the foliation.

In the sequel, we place ourselves in the setting of Theorem A. Thus M is a projective manifold endowed with a meromorphic vector field X whose underlying foliation is denoted by $\mathcal{F}$. We also assume that the set of leaves of $\mathcal{F}$ associated to integral curves of X giving rise to meromorphic maps defined on $\mathbb{C}$ has strictly positive logarithmic capacity.

As explained in Section 3, all enlarged leaves of $\mathcal{F}$ are either $\mathbb{CP}^1$ or a parabolic Riemann surface (cf. Lemma 3.3). Let now N be an irreducible (possibly singular) projective surface left invariant by $\mathcal{F}$ and denote by $\mathcal{F}_N$ the foliation induced by $\mathcal{F}$ on N. We can assume from now on that $\mathcal{F}_N$ has no meromorphic first integral, otherwise there is nothing to prove. Owing to Jouanolou's theorem [19] this ensures that only finitely many leaves of $\mathcal{F}_N$ fail to be Zariski-dense.

In view of Theorem 3.1, there follows that $\mathcal{F}_N$ is either a Riccati foliation or a turbulent one (up to birational equivalence). A slightly more accurate statement is, however, possible:

Lemma 4.3. *If $\mathcal{F}_N$ is a Riccati foliation, then its holonomy group is amenable. Similarly, if $\mathcal{F}_N$ is a turbulent foliation, then its holonomy group is solvable and virtually abelian (i.e. it contains an abelian subgroup of finite index).*

Proof. Let us first assume that $\mathcal{F}_N$ is a turbulent foliation and denote by $\mathcal{P} : N \to S$ the corresponding genus 1 fibration. Note that the regular fibers of $\mathcal{P}$ are pairwise equivalent as elliptic curves since local parallel transport along the leaves of $\mathcal{F}_N$ yields holomorphic diffeomorphisms between them. If E denote the "typical" fiber of $\mathcal{P}$, then the holonomy group of $\mathcal{F}$ is nothing but the image of the holonomy representation of the fundamental group of S in the group of automorphisms $\mathrm{Aut}\,(E)$ of the elliptic curve $E = \mathbb{C}/\Lambda$, where Λ stands for a lattice of $\mathbb{C}$. The possibilities for the automorphism group $\mathrm{Aut}\,(E)$ are well-known. First, $\mathrm{Aut}\,(E)$ is naturally realized as a subgroup of the affine group of $\mathbb{C}$. In particular, $\mathrm{Aut}\,(E)$ must be solvable. To see that $\mathrm{Aut}\,(E)$ is a finite extension of an abelian group, let $\mathrm{Aut}_0(E)$ stand for the subgroup of $\mathrm{Aut}\,(E)$ consisting of those automorphisms whose action on $H^1(E)$ is trivial coincides with the automorphisms induced by translations on $\mathbb{C}$ ($E = \mathbb{C}/\Lambda$). In particular, $\mathrm{Aut}_0(E)$ is abelian. In general, the index of $\mathrm{Aut}_0(E)$ in $\mathrm{Aut}\,(E)$ can take only on the following values: 2, 4, and 6 (cf. [14], page 321). It then follows that $\mathrm{Aut}\,(E)$ is a finite extension of an abelian group.

The case of Riccati foliations is slightly different in that the monodromy group is a subgroup of the automorphism group of the Riemann sphere, namely $\mathrm{PSL}\,(2, \mathbb{C})$. In general the holonomy group of a Riccati equation is not amenable. However, the enlarged leaves of $\mathcal{F}_N$ are parabolic Riemann surfaces and this imposes strong constraints on the holonomy group in question. In fact, a subgroup of $\mathrm{PSL}\,(2, \mathbb{C})$ that *is not* amenable contains a free subgroup on two generators (Tits alternative) and has therefore exponential growth: this forces the Zariski-dense leaves of $\mathcal{F}_E$ to have exponential growth as well (see [10] for the case of regular foliations, details in the specific case of Riccati foliations can be found in [27]). Clearly the exponential growth of the leaves means that they are hyperbolic Riemann surfaces. Since the passage of a leaf to the corresponding enlarged leaf only adds a discrete set of points to the former, the enlarged leaves must be hyperbolic themselves and this yields a contradiction with Lemma 3.3. The proof of the lemma is complete. □

We are now ready to prove Theorem A.

Proof of Theorem A. We keep the preceding notation. In particular, we can assume that the foliation $\mathcal{F}_N$ is transverse to a (singular) fibration $\mathcal{P} : N \to S$ of genus 0 or 1. Let us first consider the case in which this fibration has genus 1, i.e., the case where $\mathcal{F}_N$ is a turbulent foliation. Denote by Γ the corresponding holonomy group. Lemma 4.3 establishes that Γ is solvable and this yields the desired extension implied in the definition of Liouvillian functions. In more precise terms, Γ is a finite extension of the abelian group $\Gamma_0 = \Gamma \cap \mathrm{Aut}_0(E)$: the solutions are then expressed as a finite extension of elliptic functions. Alternatively, to make the connection with Singer's theorem in Remark 4.2, just observe that the natural "affine structure" inherited by $E = /\Lambda$ from $\mathbb{C}$ is naturally preserved by Γ since this group is naturally containe in the affine group of $\mathbb{C}$.

Let us now consider the case where $\mathcal{F}_N$ is a Riccati foliation. Again let Γ denote the corresponding holonomy group which is naturally a finitely generated subgroup of $\mathrm{PSL}\,(2,\mathbb{C})$. We can assume Γ is infinite, otherwise all leaves of $\mathcal{F}_N$ are compact and hence we have a rational first integral. In view of the discussion in Remark 4.2, it suffices to prove the existence of a transverse affine structure invariant by Γ. In particular, it is enough to check that Γ is conjugate to a subgroup of the affine group $\mathrm{Aff}\,(\mathbb{C}) \subset \mathrm{PSL}\,(2,\mathbb{C})$.

Assume first that Γ is discrete and hence a Kleinian group. The group Γ is, however, amenable (Lemma 4.3) and it is well known that (infinite) amenable Kleinian group are all *elementary* in the sense that their Limit sets consist of one or two points. The Limit set being naturally invariant under Γ, either Γ has a fixed point or Γ has a subgroup Γ_2 of index 2 having a fixed point. In the first case, the fixed point of Γ can be used to construct a conjugation between Γ and some subgroup of $\mathrm{Aff}\,(\mathbb{C})$. In the second case, Γ_2 has a Liouvillian first integral and so has Γ as a degree 2 extension of Γ_2.

Finally, assume now that Γ is not discrete so that its closure $\overline{\Gamma} \subset \mathrm{PSL}\,(2,\mathbb{C})$ is a Lie subgroup of $\mathrm{PSL}\,(2,\mathbb{C})$ with non-trivial Lie algebra. Clearly this Lie algebra must be strictly contained in the Lie algebra of $\mathrm{PSL}\,(2,\mathbb{C})$, otherwise Γ would be dense in $\mathrm{PSL}\,(2,\mathbb{C})$ and thus non-amenable. This Lie algebra is therefore either abelian or conjugate to an affine Lie algebra. From there it is again clear that the (amenable) group has a fixed point or, at least, contains a index 2 subgroup possessing a fixed point. The existence of the desired Liouvillian first integral then follows as above. The proof of Theorem A is complete. □

We can now prove Theorem B as well.

Proof of Theorem B. Theorem B follows from Theorem A once it is established the existence of a 2-dimensional analytic set $\mathcal{S} \subset \mathbb{CP}^3$ that happens to be invariant by $\mathcal{F}$. For this consider first X given as a rational vector field on $\mathbb{C}^3$. The divisor of poles $(X)_\infty$ of X on $\mathbb{C}^3$, if not empty, yields an algebraic set of dimension 2. Let then $\mathcal{S} \subset \mathbb{CP}^3$ be an irreducible component of the algebraic set in question. All we need to check is that $\mathcal{S}$ is invariant by $\mathcal{F}$. However, if this were not the case, then a leaf L of $\mathcal{F}$ transverse to $\mathcal{S}$ would be such that integral curves of X with initial conditions at (generic) points of L would be multivalued functions, rather than a a meromorphic function (cf. Lemma 2.5). The resulting contradiction proves the statement in this case.

It remains to consider the case in which $(X)_\infty$ is empty, i.e., the case in which X is actually a polynomial vector field. In this case, we consider the "plane at infinity" $\Delta_\infty = \mathbb{CP}^3 \setminus \mathbb{C}^3$. If X extends holomorphically to Δ_∞, then X is actually a holomorphic vector field on all of $\mathbb{CP}^3$. It is therefore a linear vector field, i.e. a vector field arising from the group $\mathrm{PSL}\,(4,\mathbb{C})$ of automorphisms of $\mathbb{CP}^3$. In this case, the existence of an invariant plane for $\mathcal{F}$ is immediate. Finally, if X has poles on Δ_∞, then the preceding argument shows that Δ_∞ must be invariant by $\mathcal{F}$ again. Theorem B is proved. □

5. Appendix - from dimension 2 to dimension 3

Both local and global theory of singular holomorphic foliations in dimension 2 are far more developed than its counterparts in higher dimensions. The issue stems from the fact that these *one-dimensional foliations* are also of *codimension* 1 when the ambient has dimension 2. This means that "invariant divisors" are merely a leaf with compact closure for the corresponding foliation $\mathcal{F}$. In contrast, once the dimension n satisfies $n \geq 3$, then invariant divisors carry a global foliation whose dynamics can be very complicate and this complication has a serious impact on the difficulty of the problems.

Whereas the survey [29] is referred to for a detailed discussion of the above mentioned issue, we will quickly revisit here a general and powerful method to address the dynamics of (certain) foliations on complex surfaces. In chronological terms, the first instance of this technique can be traced back to the work of Hudai-Verenov [15], later improved by Il'yashenko [17]. A definite step forward, however, was undertaken by Mattei and Moussu in [21] who, working with singularities, made systematic use of Seidenberg theorem [30] and settled the basics for the corresponding "semi-global analysis" of the singular points. While the contexts of [15] and [17] are in principle very different from the local problems treated in [21], they both fit in the same framework that is summarized below.

Consider a singular holomorphic foliation $\mathcal{F}$ on a complex surface M leaving invariant a curve $C \subset M$. The purpose is to investigate the structure of $\mathcal{F}$ on a neighborhood of C. To make the discussion less technical, let us assume that none of the singularities of $\mathcal{F}$ lying in C is *dicritical*. Here we recall that a singular point is said to be *dicritical* if there exists a finite sequence of blow-ups starting at the singular point in question and leading to a foliation transverse at generic points to at least one irreducible component of the resulting (total) exceptional divisor, see for example [21], [16]. To get insight into the structure of $\mathcal{F}$ around C, we then proceed as follows:

1. Apply Seidenberg resolution theorem [30] to the singularities of $\mathcal{F}$ lying in C. This allows us to replace $\mathcal{F}$ by a new foliation $\widetilde{\mathcal{F}}$ all of whose singular points are "simple", i.e., they have at least one eigenvalue different from zero. Moreover, $\widetilde{\mathcal{F}}$ is tangent to a (connected) tree-like divisor consisting of curves $C_1, \ldots, C_k$.
2. Having "understood" the singular points of $\widetilde{\mathcal{F}}$ (so that nothing is "hidden" in them), the (singular, reducible) curve $C_1 \cup \cdots \cup C_k$ is invariant by $\widetilde{\mathcal{F}}$ so that it gives rise to a holonomy representation in $\text{Diff}\,(\mathbb{C}, 0)$ (including "passage of corners" as detailed in [21]). As a consequence, essentially all of the dynamics of $\mathcal{F}$ (or $\widetilde{\mathcal{F}}$) becomes encoded in the dynamics of a group of local diffeomorphisms of $\mathbb{C}$ fixing $0 \in \mathbb{C}$.

The curve C becomes therefore a kind of *core for the dynamics* in the sense that a large part - if not all - of the dynamics of $\mathcal{F}$ arises from its local dynamics around C. From the global perspective, like in [15], [17], [20], if in addition that initial curve C is ample, then all regular leaves of $\mathcal{F}$ will intersect arbitrarily small neighborhoods of C. In particular, the local dynamics of $\mathcal{F}$ around C can be globally "propagated" on the ambient surface. An important success of this strategy closely related to problems discussed in this paper appears in [12] and [11].

Carrying out a similar approach in higher dimensions, or at least in dimension 3, is no longer out of reach. A suitable generalization of Seidenberg theorem has been obtained in [25], [23], see also [28]. Similarly much progress has been made about the dynamics of subgroups of $\text{Diff}\,(\mathbb{C}^n, 0)$, $n \geq 2$. Nonetheless the fact that invariant divisors (that exist in the case in question thanks to Lemma 2.5) are foliated and the dynamics of this global foliation is the source of huge difficulty. In fact, even if the structure of the foliation around (simple) singular points can be understood, the existence of non-trivial global dynamics makes it unclear if these singularities "interact among them". Along similar lines, the leaves of the foliation might have only cyclic holonomy groups and these might be essentially independent: this contrasts again with the 2-dimensional case where all of the information is encoded in what is essentially a single subgroup of Diff $(\mathbb{C}, 0)$. As a type of startling examples, the foliations constructed in [20] can be intrinsically attached to singular points of foliations, see [29].

Summarizing the preceding paragraph, the (potential) existence of dynamically complicated foliations on invariant divisors is the main obstacle to approach a large class of problems with the general strategy outlined above. Apparently, the only way around this difficulty consists of a priori describing the dynamics in question. In other words, given a specific problem (such as the classification of vector fields with meromorphic solutions), it is necessary to obtain an a priori result showing that the restriction of the foliation in question to an invariant divisor cannot have an excessive complicated dynamics. If this restricted foliation turns out to be simple enough, then the general strategy will look promising. In this respect, and concerning the classification of vector fields with meromorphic solutions in dimension 3, this is exactly the content of Theorem A and B: these foliations are integrable in the sense of Liouville and their possible dynamics are explicitly described.

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