Examples of surfaces with canonical map of degree 4

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Abstract

We give two examples of surfaces with canonical map of degree 4 onto a canonical surface.
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1 Introduction

Let S be a smooth minimal surface of general type with geometric genus $p_g \geq 3$. Denote by $\phi : S \dashrightarrow \mathbb{P}^{p_g-1}$ the canonical map and let $d := \deg(\phi)$. The following Beauville's result is well-known.

Theorem 1 ([Bea79]). *If the canonical image $\Sigma := \phi(S)$ is a surface, then either:*

(A) $p_g(\Sigma) = 0$, *or*

(B) Σ *is a canonical surface (in particular $p_g(\Sigma) = p_g(S)$).*

Moreover, in case (A) $d \leq 36$ and in case (B) $d \leq 9$.

The question of which pairs (d, p_g) can actually occur has been object of study for some authors. Several examples were given for case (A), but case (B) is still mysterious. It is known that if $d > 3$, then $p_g \leq 9$, but so far only the case $(d, p_g) = (5, 4)$ has been shown to exist (independently by Tan [Tan92] and by Pardini [Par91b]). We refer the recent preprint by Mendes Lopes and Pardini [MLP21] for a more detailed account on the subject. They leave some open problems, this note is motivated by their last question:

For what pairs (d, p_g), with $d > 3$, are there examples of surfaces in case (B) of Theorem 1?

Here we give examples for the cases $(d, p_g) = (4, 5)$ and $(4, 7)$, with canonical images a 40-nodal complete intersection surface in $\mathbb{P}^4$ and a 48-nodal complete intersection surface in $\mathbb{P}^6$, respectively (Beauville also paid some attention to such nodal surfaces, see [Bea17]).

We work explicitely with the equations of a 40-nodal surface from [RRS19], all computations are implemented with Magma [BCP97].

Notation

As usual the holomorphic Euler characteristic of a surface S is denoted by $\chi(S)$, the geometric genus by $p_g(S)$, the irregularity by $q(S)$, and a canonical divisor by K_S. A $(-m)$-curve is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-m$. We say that a set of nodes of S is 2-divisible if the sum $\sum A_i$ of the corresponding (-2)-curves in the smooth minimal model of S is 2-divisible in the Picard group.

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2 Construction

Let X_{40} be the surface in $\mathbb{P}^4$ given by the equations

$$\begin{aligned} 5\left(x^2+y^2+z^2+w^2+t^2\right)-7\left(x+y+z+w+t\right)^2=0 \\ 4\left(x^4+y^4+z^4+w^4+t^4+h^4\right)-\left(x^2+y^2+z^2+w^2+t^2+h^2\right)^2=0 \end{aligned} \tag{1}$$

where

$$h:=-(x+y+z+w+t).$$

It is a canonical surface with invariants $p_g=5$, $q=0$ and $K^2=8$. Its singular set is the union of 40 nodes $N_1,\dots,N_{40}$ (see [RRS19]).

Let $\widetilde{X}_{40}$ be the smooth minimal model of X_{40} and denote by A_i the (-2)-curves in $\widetilde{X}_{40}$ corresponding to the nodes N_i, $i=1,\dots,40$. We show in Section 3.1 that one can write

$$A_1+\cdots+A_{40}=D_a+D_b+D_c+D_{abc}+D_{bc}+D_{ac}+D_{ab}$$

where each of D_a,D_b,D_c,D_{abc} is a sum of 4 (-2)-curves, each of D_{bc},D_{ac},D_{ab} is a sum of 8 (-2)-curves, and such that there exist divisors L_1,L_2,L_3 satisfying:

$$\begin{aligned} D_a+D_{abc}+D_{ac}+D_{ab}\equiv 2L_1 \\ D_b+D_{abc}+D_{bc}+D_{ab}\equiv 2L_2 \\ D_c+D_{abc}+D_{bc}+D_{ac}\equiv 2L_3 \end{aligned} \tag{2}$$

This implies the existence of divisors $L_4,\dots,L_7$ such that:

$$\begin{aligned} D_a+D_b+D_c+D_{abc}\equiv 2L_4 \\ D_a+D_b+D_{bc}+D_{ac}\equiv 2L_5 \\ D_a+D_c+D_{bc}+D_{ab}\equiv 2L_6 \\ D_b+D_c+D_{ac}+D_{ab}\equiv 2L_7 \end{aligned} \tag{3}$$

Now identifying a,b,c with the generators of the group $(\mathbb{Z}/2)^3$, it follows from [Cat08, Proposition 7.6] or [Par91a] that these data define a $(\mathbb{Z}/2)^3$-covering $\pi:\widetilde{Y}\to\widetilde{X}_{40}$ branched on the (-2)-curves A_i, equivalently a $(\mathbb{Z}/2)^3$-covering $\psi:Y\to X_{40}$ branched on the nodes of X_{40}, where Y is the minimal model of $\widetilde{Y}$.

Since ψ is ramified only on nodes, we have $K_Y\equiv\psi^*(K_{X_{40}})$ and then $K_Y^2=8K_{X_{40}}^2=64$. We show in Section 3.1 that

$$h^0\left(\widetilde{X}_{40},\mathcal{O}_{\widetilde{X}_{40}}\left(K_{\widetilde{X}_{40}}+L_4\right)\right)=2$$

and

$$h^0\left(\widetilde{X}_{40}, \mathcal{O}_{\widetilde{X}_{40}}\left(K_{\widetilde{X}_{40}} + L_i\right)\right) = 0 \quad \text{for} \quad i \neq 4,$$

thus

$$p_g(Y) = p_g(X_{40}) + 2 + 0 + \cdots + 0 = 7.$$

We have

$$\chi(Y) = 8\chi\left(\widetilde{X}_{40}\right) + \frac{1}{2}\sum_1^7 L_i\left(K_{\widetilde{X}_{40}} + L_i\right) = 48 - (8 + 6 \times 12)/2 = 8,$$

and this implies $q(Y) = 0$.

The covering ψ factors as

$$\begin{array}{ccccc} Y & \longrightarrow & Y_{32} & \longrightarrow & Y_{48} \\ \downarrow & & & & \downarrow \\ X_{16} & \longrightarrow & X_{32} & \longrightarrow & X_{40} \end{array}$$

where the subscript n means a surface with singular set the union of n nodes (for $X_{16} \to X_{40}$ take for instance the bidouble covering given by L_1, L_2). All these surfaces are regular because $q(Y) = 0$.

It is easy to compute that $\chi(X_{16}) = 6$, thus $p_g(X_{16}) = p_g(X_{40}) = 5$, and we conclude that

the $(\mathbb{Z}/2)^2$-covering $X_{16} \to X_{40}$ is the canonical map of X_{16}.

Analogously, $p_g(Y) = p_g(Y_{48}) = 7$ and we claim that

the $(\mathbb{Z}/2)^2$-covering $Y \to Y_{48}$ is the canonical map of Y.

For this it suffices to show that Y_{48} is a canonical surface.

We follow Beauville [Bea17] and show that Y_{48} can be embedded in $\mathbb{P}^6$ as a complete intersection of 4 quadrics in the following way. The linear system L of quadrics through the branch locus of the covering $Y_{48} \to X_{40}$ (16 nodes) is of dimension 2. Using computer algebra it is not difficult to show that L contains quadrics B, C, D such that the surface X_{40} is given by $Q = 0$, $B^2 - CD = 0$, where Q is the quadric from (1) (we write the quadrics as general elements of L, thus depending on some parameters; then we obtain a variety on these parameters by imposing that the hypersurfaces $Q = 0$ and $B^2 - CD = 0$ are tangent at the 24 nodes of X_{40} which are disjoint from the 16 nodes of $B^2 - CD = 0$; finally we compute points in this variety).

Then Y_{48} is given in $\mathbb{P}^6(x, y, z, w, t, u, v)$ by equations

$$u^2 - C = v^2 - D = uv - B = Q = 0.$$

We give these equations in Section 3.2 and verify that Y_{48} is as stated.

Let us explain how we find 2-divisible sets of nodes in X_{40}. The surface X_{40} contains 40 *tropes*, which are hyperplane sections $H_i = 2T_i$ with $T_i \subset X_{40}$ a reduced curve through 12 nodes of X_{40}, and smooth at these points. Thus in $\widetilde{X}_{40}$ the pullback of such a trope can be written as

$$\widetilde{H}_i = 2\widehat{T}_i + \sum_{j \in J} A_j, \quad \#J = 12.$$

Thus for each pair of tropes the sum of nodes contained in their union and not contained in their intersection is 2-divisible.

Using these 2-divisibilities, the idea for finding configurations as in (2) is simple: we have used a computer algorithm to list and check possibilities.

3 Computations

The computations below are implemented with Magma V2.26-5.

3.1 The covering $Y \to X_{40}$

We start by defining the surface X_{40} and its singular set.

```
K:=Rationals();
R<r>:=PolynomialRing(K);
K<r>:=ext<K|r^2 + 15>;
P<x,y,z,w,t>:=ProjectiveSpace(K,4);
h:=-x-y-z-w-t;
Q:=5*(x^2+y^2+z^2+w^2+t^2)-7*(x+y+z+w+t)^2;
I:=4*(x^4+y^4+z^4+w^4+t^4+h^4)-(x^2+y^2+z^2+w^2+t^2+h^2)^2;
X40:=Surface(P,[Q,I]);
SX40:=SingularSubscheme(X40);
```

The partition of the 40 nodes:

```
Da:={P![3,3,-2,-2,3],P![4,-r+1,r-5,-r+1,4],
     P![-r+1,4,r-5,-r+1,4],P![r+1,r+1,-r-5,4,4]};
Db:={P![2,-3,-3,-3,2],P![4,r+1,r+1,-r-5,4],
     P![-r-5,r-5,r-5,-r-5,10],P![r-5,-r+1,-r+1,4,4]};
Dc:={P![-3,-3,2,-3,2],P![-r+1,-r+1,r-5,4,4],
     P![r-5,r-5,-r-5,-r-5,10],P![r+1,r+1,4,-r-5,4]};
Dabc:={P![-2,3,3,-2,3],P![-r-5,r+1,r+1,4,4],
       P![r-5,4,-r+1,-r+1,4],P![r-5,-r+1,4,-r+1,4]};
Dbc:={P![-2,-2,3,3,3],P![3,-2,-2,3,3],
      P![4,-r-5,r+1,r+1,4],P![4,-r+1,-r+1,r-5,4],
      P![4,r+1,-r-5,r+1,4],P![-r-5,r+1,4,r+1,4],
      P![-r+1,-r+1,4,r-5,4],P![r+1,-r-5,4,r+1,4]};
Dac:={P![-3,2,-3,-3,2],P![3,-2,3,-2,3],
      P![4,r-5,-r+1,-r+1,4],P![-r+1,r-5,4,-r+1,4],
      P![-r+1,r-5,-r+1,4,4],P![r-5,-r-5,r-5,-r-5,10],
      P![r+1,4,r+1,-r-5,4],P![r+1,-r-5,r+1,4,4]};
Dab:={P![-3,-3,-3,2,2],P![-2,3,-2,3,3],
      P![-r-5,4,r+1,r+1,4],P![-r+1,4,-r+1,r-5,4],
      P![-r-5,-r-5,r-5,r-5,10],P![-r-5,r-5,-r-5,r-5,10],
      P![r-5,-r-5,-r-5,r-5,10],P![r+1,4,-r-5,r+1,4]};
```

Verification that these are in fact the nodes:

```
&join[Da,Db,Dc,Dabc,Dbc,Dac,Dab] eq SingularPoints(X40);
HasSingularPointsOverExtension(X40) eq false;
```

Some of the tropes of X_{40} :

```
tropes:=[
    6*x + (-r - 9)*y + (r - 9)*z + (r - 9)*w + (-r - 9)*t,
    16*x + (-r - 9)*y + 16*z + (3*r + 11)*w + (3*r + 11)*t,
    16*x + (r - 9)*y + 16*z + (-3*r + 11)*w + (-3*r + 11)*t,
    6*x + (r - 9)*y + (-r - 9)*z + (r - 9)*w + (-r - 9)*t,
    16*x + (3*r + 11)*y + 16*z + (3*r + 11)*w + (-r - 9)*t,
    16*x + (-3*r + 11)*y + (-3*r + 11)*z + (r - 9)*w + 16*t,
    x + y + w,
    16*x + (r - 9)*y + (-3*r + 11)*z + (-3*r + 11)*w + 16*t,
    x + z + w
];
```

The reduced subscheme of these tropes:

```
red:=[ReducedSubscheme(Scheme(X40,q)):q in tropes];
&and[Degree(q) eq 4:q in red];
```

They are smooth at the nodes of X_{40} :

```
&and[Dimension(SingularSubscheme(q) meet SX40) eq -1:q in red];
```

Two 2-divisible disjoint sets of 20 nodes, which confirm that the 40 nodes are 2-divisible:

```
s1:=Points(Scheme(SX40,tropes[1]*tropes[2])) diff
    Points(Scheme(SX40,[tropes[1],tropes[2]]));
s2:=Points(Scheme(SX40,tropes[6]*tropes[7])) diff
    Points(Scheme(SX40,[tropes[6],tropes[7]]));
&and[#s1 eq 20,#s2 eq 20,#(s1 join s2) eq 40];
```

We compute three 2-divisible sets of 24 nodes:

```
Sets:=[];
for q in [[2,5],[1,4],[3,8]] do
  pts:=Points(Scheme(SX40,tropes[q[1]]*tropes[q[2]])) diff
       Points(Scheme(SX40,[tropes[q[1]],tropes[q[2]]]));
  Append(~Sets,SingularPoints(X40) diff pts);
end for;
```

and use these sets to check the divisibilities in (2):

```
Da join Dabc join Dac join Dab eq Sets[1];
Db join Dabc join Dbc join Dab eq Sets[2];
Dc join Dabc join Dbc join Dac eq Sets[3];
```

Now we show that $h^0\left(\widetilde{X}_{40}, \mathcal{O}_{\widetilde{X}_{40}}\left(K_{\widetilde{X}_{40}} + L_4\right)\right) = 2$. Let $N_1, \ldots, N_{16}$ be the nodes in $D_a+D_b+D_c+D_{abc}$ and $A_1, \ldots, A_{16}$ be the corresponding (-2)-curves. Let H_1, H_2 be the tropes that give

$$\widetilde{H}_1 + \widetilde{H}_2 = 2\widehat{T}_1 + 2\widehat{T}_2 + \sum_{1}^{16} A_i + 2\sum_{17}^{20} A_i,$$

with $A_{17}, \dots, A_{20} \in \widetilde{H}_1 \cap \widetilde{H}_2$. Then

$$\sum_1^{16} A_i \equiv 2L_4, \quad \text{with} \quad K_{\widetilde{X}_{40}} + L_4 \equiv 2\widetilde{H} - \widehat{T}_1 - \widehat{T}_2 - \sum_{17}^{20} A_i.$$

We compute below that the system of quadrics through the curves $T_1, T_2 \subset \mathbb{P}^4$ is generated by 2 elements, modulo the quadric Q. Since the double covering $Y_{48} \to X_{40}$ is ramified exactly at $N_1, \dots, N_{16}$ and $\chi(Y_{48}) = 8$ implies $p_g(Y_{48}) \geq 7 = p_g(X_{40}) + 2$, then $h^0\left(\widetilde{X}_{40}, \mathcal{O}_{\widetilde{X}_{40}}\left(K_{\widetilde{X}_{40}} + L_4\right)\right) = 2$.

```
T1:=ReducedSubscheme(Scheme(X40,tropes[2]));
T2:=ReducedSubscheme(Scheme(X40,tropes[9]));
pts:=Points(SX40 meet (T1 join T2)) diff
     Points(SX40 meet T1 meet T2);
pts eq (Da join Db join Dc join Dabc);
L:=LinearSystem(LinearSystem(P,2),T1 join T2);
#Sections(LinearSystemTrace(L,X40)) eq 2;
```

Let us show that $h^0\left(\widetilde{X}_{40}, \mathcal{O}_{\widetilde{X}_{40}}\left(K_{\widetilde{X}_{40}} + L_i\right)\right) = 0$ for $i \neq 4$. Suppose the opposite. Let $A_1, \dots, A_{24}$ be the corresponding (-2)-curves. Then there is a curve $E \in \left|K_{\widetilde{X}_{40}} + L_i\right|$, and $EA_i = -1$ implies that the linear system $\left|K_{\widetilde{X}_{40}} + L_i - \sum_1^{24} A_j\right| = \left|K_{\widetilde{X}_{40}} - L_i\right|$ is nonempty. Therefore $\left|2K_{\widetilde{X}_{40}} - \sum_1^{24} A_j\right|$ is nonempty, which implies that there is at least one quadric in $\mathbb{P}^4$ through the corresponding nodes $N_1, \dots, N_{24}$ (modulo the quadric Q). We show below that this does not happen.

```
Sets:=[
Da join Dabc join Dac join Dab,
Db join Dabc join Dbc join Dab,
Dc join Dabc join Dbc join Dac,
Da join Db join Dbc join Dac,
Da join Dc join Dbc join Dab,
Db join Dc join Dac join Dab
];
for q in Sets do
  L:=LinearSystem(LinearSystem(P,2),[P!x:x in q]);
  #Sections(LinearSystemTrace(L,S)) eq 0;
end for;
```

3.2 The surface Y_{48}

Here we give the equations of Y_{48} as a complete intersection of 4 quadrics in $\mathbb{P}^6$. We start by defining $\mathbb{P}^6$ over a certain number field.

```
K:=Rationals(); R<x>:=PolynomialRing(K);
K<r,m>:=ext<K|x^2 + 15,x^2 - 95/42*x + 2855/2646>;
R<n>:=PolynomialRing(K);
K<n>:=ext<K|
n^2 + 443889677/206391214080000*r - 46942774543/619173642240000>;
P6<x,y,z,w,t,u,v>:=ProjectiveSpace(K,6);
```

The three quadrics B, C, D :

```
B:=(675/4802*r+334125/33614)*n*x*z+(-389475/67228*r+3266325/67228)*n*x*w+
 (34425/9604*r+451575/67228)*n*y*w+(-389475/67228*r+3266325/67228)*n*z*w+
 (-62100/16807*r+348300/16807)*n*w^2+(239625/33614*r+1541025/33614)*n*x*t
 +(-8100/2401*r+137700/16807)*n*y*t+(239625/33614*r+1541025/33614)*n*z*t
 +(6075/9604*r+3007125/67228)*n*w*t+(71550/16807*r+319950/16807)*n*t^2;
C:=x*y+1/154*(126*m-181)*y^2+1/42*(-42*m+95)*x*z+y*z+(1/1540*(14*m-25)*r
 +1/924*(-798*m+1997))*x*w+(1/420*(42*m-65)*r+1/308*(-294*m+767))*y*w
 +(1/1540*(14*m-25)*r+1/924*(-798*m+1997))*z*w+(1/385*(-119*m+185)*r
 +1/462*(-168*m+311))*w^2+(1/1540*(-14*m+25)*r+1/924*(-798*m+
 1997))*x*t+(1/420*(-42*m+65)*r+1/308*(-294*m+767))*y*t+(1/1540*(-14*m
 +25)*r+1/924*(-798*m+1997))*z*t+1/154*(126*m-71)*w*t+(1/385*(119*m-
 185)*r+1/462*(-168*m+311))*t^2;
D:=x*y+1/77*(-63*m+52)*y^2+m*x*z+y*z+(1/2310*(-21*m+10)*r+1/154*(133*m+
 32))*x*w+(1/70*(-7*m+5)*r+1/154*(147*m+51))*y*w+(1/2310*(-21*m+
 10)*r+1/154*(133*m+32))*z*w+(1/2310*(714*m-505)*r+1/154*(56*m-
 23))*w^2+(1/2310*(21*m-10)*r+1/154*(133*m+32))*x*t+(1/70*(7*m-5)*r
 +1/154*(147*m+51))*y*t+(1/2310*(21*m-10)*r+1/154*(133*m+32))*z*t+
 1/77*(-63*m+107)*w*t+(1/2310*(-714*m+505)*r+1/154*(56*m-23))*t^2;
```

We obtain alternative equations for X_{40} :

```
F:=B^2-C*D;
Q:=5*(x^2+y^2+z^2+w^2+t^2)-7*(x+y+z+w+t)^2;
X:=Scheme(P6,[F,Q,u,v]);
h:=-x-y-z-w-t;
I:=4*(x^4+y^4+z^4+w^4+t^4+h^4)-(x^2+y^2+z^2+w^2+t^2+h^2)^2;
X40:=Scheme(P6,[Q,I,u,v]);
X eq X40;
```

And finally the equations of Y_{48} in $\mathbb{P}^6$:

```
Y48:=Surface(P6,[u^2-C,v^2-D,u*v-B,Q]);
SY48:=SingularSubscheme(Y48);
Dimension(SY48) eq 0;
Degree(SY48) eq 48;
Degree(ReducedSubscheme(SY48)) eq 48;
```

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