On the Epstein zeta function and the zeros of a class of Dirichlet series

Pedro Ribeiro, Semyon Yakubovich

Abstract

By generalizing the classical Selberg-Chowla formula, we establish the analytic continuation and functional equation for a large class of Epstein zeta functions. This continuation is studied in order to provide new classes of theorems regarding the distribution of zeros of Dirichlet series in their critical lines. Due to the symmetries provided by the representation via the Selberg-Chowla formula, some generalizations of well-known formulas in analytic number theory are given as examples.

Keywords : Dirichlet series; Epstein zeta function; Modified Bessel functions; Selberg-Chowla Formula; Hardy's Theorem

2020 Mathematics Subject Classification : 11E45, 11M41, 44A15, 33C10

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*Department of Mathematics, Faculty of Sciences of University of Porto, Rua do Campo Alegre, 687; 4169-007 Porto (Portugal). E-mail: up201403460@edu.fc.up.pt

1 Introduction

Let $Q(x,y) = ax^2 + bxy + cy^2$ be a real and positive definite quadratic form. The classical Epstein zeta function is defined as the Dirichlet series

$$Z_2(s,\, Q) = \sum_{m,n\neq 0} \frac{1}{Q(m,n)^s}, \quad \mathrm{Re}(s) > 1, \tag{1.1}$$

where the notation given in the subscript, $m,\, n \neq 0$, here means that only the term $m = n = 0$ is omitted from the infinite series.

In 1949, S. Chowla and A. Selberg [80] announced the following formula for (1.1), valid in the entire complex plane,

$$\begin{aligned} a^s\Gamma(s)\, Z_2(s,\, Q) &= 2\Gamma(s)\zeta(2s) + 2k^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1) \\ &+ 8k^{\frac{1}{2}-s}\pi^s \sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\, \cos\left(n\pi b/a\right) K_{s-\frac{1}{2}}\left(2\pi k n\right), \end{aligned} \tag{1.2}$$

where $d := b^2 - 4ac$ is the discriminant of the quadratic form, $k^2 := |d|/4a^2$ and $\sigma_\nu(n) = \sum_{d|n} d^\nu$ is the generalized divisor function of index ν. Also, $\zeta(s)$ denotes the classical Riemann zeta function and $K_\nu(z)$ is the modified Bessel function.

According with Berndt's revision [13], after its first announcement, (1.2) was later proved by Rankin [77], Bateman and Grosswald [5], Chowla and Selberg [81] (in a paper which appeared 18 years after the statement of (1.2)) and Motohashi [68], although the latter author had the main goal of proving directly the first Kronecker limit formula. As it should be expected, all these proofs used the theory of Fourier series and invoke directly the Poisson summation formula.

Berndt himself [13] gave a very interesting proof of (1.2), which was based on a previous theorem due to him regarding the Bessel expansion of the so called generalized Dirichlet series [7]. In fact, Berndt's proof of (1.2) was considered for a more general Epstein zeta function of the form

$$Z_2(s,\, Q;\, g,\, h) = \sum_{m,n\neq 0} \frac{e^{2\pi i(h_1 m + h_2 n)}}{Q(m+g_1,\, n+g_2)^s}, \quad \mathrm{Re}(s) > 1,$$

where g_1, g_2, h_1 and h_2 are real. Although only written for the particular case (1.2), a simplified version of Berndt's proof was later given by Kuzumaki [65]. It is remarkable that Kuzumaki's proof, as well as Berndt's, depends explicitly on the functional equation for the Hurwitz zeta function $\zeta(s,\, a)$. Although this functional equation can be derived immediately from the $\theta-$reflection formula formula [41]

$$\frac{1}{2} + \sum_{n=1}^{\infty} e^{-\alpha n^2}\cos(\beta n) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \sum_{n=-\infty}^{\infty} e^{-\frac{1}{\alpha}\left(\pi n + \frac{\beta}{2}\right)^2}, \tag{1.3}$$

(which is obviously connected with the Poisson summation formula, thus with all original proofs of (1.2)) it can nevertheless be established in strikingly different ways [1].

The generalization of (1.2) for Epstein zeta functions attached to quadratic forms with n variables was given by Terras [90], whose proof employed a multidimensional version of the Poisson summation formula. Suzuki [86] provided analogues of (1.2) for other Dirichlet series which apriori have nothing to do with (1.1). These Dirichlet series include the Riemann $\zeta-$function, the Dirichlet $L-$function and $L-$functions attached to holomorphic cusp forms.

The idea behind a Bessel expansion of a Dirichlet series of the form (1.1) seems to have been motivated by a formula discovered by Watson [[95], p. 99, eq. (4)], which was later generalized by H. Kober [63] (see Example 5.1), and perhaps the first explicit form of (1.2) appearing in the literature is due to Taylor [89]. Although Taylor himself attributed (1.2) to Kober, it is in Taylor's paper where the continuation of the Epstein zeta function via (1.2) is made explicit for the first time. Taylor's proof employed a Mellin-Barnes representation of the left-hand side of (1.2) involving the Gauss hypergeometric function. Although Taylor did not prove the functional equation for $Z_2(s, Q)$ directly from (1.2), as Selberg and Chowla did, his derivation of both results used the same ideas and this seems to be the first indication of the connection between (1.2) and the symmetric properties of the continuation of $Z_2(s, Q)$.

The Selberg-Chowla formula has found many applications in Number Theory. By using it, Selberg and Chowla [81] proved the existence of a real zero for $Z_2(s, Q)$ in the open interval $(\frac{1}{2}, 1)$, when the quadratic form is $Q(x, y) = x^2 + cy^2$ and c is large enough. This result was later improved by Bateman and Grosswald [[5], p. 367, Thm 3.] by showing that, for a general positive definite and real quadratic form Q, $Z_2(s, Q)$ has a real zero between $\frac{1}{2}$ and 1 if $k := \sqrt{|d|}/2a > 7.0556$.

With the aid of (1.2), Stark [88] and Fujii [42] gave very detailed descriptions about the distribution of the complex zeros of $Z_2(s, Q)$ in bounded regions. H. Ki [60] also proved a remarkable result regarding the zeros of finite approximations of the right-hand side of (1.2). This idea of studying the zeros of the Epstein zeta function via equivalent representations to (1.2) seems to have originated in Deuring's work [32].

In a very interesting paper by the same author [31], in which most of the approaches used later to prove (1.2) were already given[2], Deuring connected the Fourier expansion for the simple Epstein $\zeta-$function $Z_2(s, \mathbb{I}_2)$ (where $b = 0$, $a = c = 1$) with the existence of infinitely many zeros of $\zeta(s)$ on the critical line $\text{Re}(s) = \frac{1}{2}$. From this relation, he was able to deduce Hardy's Theorem for $\zeta(s)$. Although Deuring's proof of Hardy's Theorem requires more motivation than the standard ones given by Hardy [48], Landau [67] or

[1] see, for instance, a very recent paper by A. Dixit and R. Kumar [34], which established the functional equation for $\zeta(s, \alpha)$ via its Hermite integral representation. See also the discussion in Remark 3.2.

[2] For example, a closer look at Deuring's argument shows also that the fundamental ideas employed by Motohashi in his proof of the Kronecker limit formula (see [[68], p. 615] and the evaluation of (5) by using the Cahen-Mellin representation (1.23)) were already been used by Deuring [[31], p. 587, eq. (5), (6)].

Fekete [40], it is nevertheless an interesting proof, since the way it seeks the contradiction is implied from a subconvex estimate for the Epstein zeta function on the critical line, due to E. C. Titchmarsh [91],

$$Z_2\left(\frac{1}{2}+it,\mathbb{I}_2\right)=O\left(|t|^{\frac{1}{3}+\varepsilon}\right), \tag{1.4}$$

which in its turn is analogous to the classical Van der Corput estimate for $\zeta\left(\frac{1}{2}+it\right)$ [55].

In the most part of the classical proofs of Hardy's Theorem, one arrives at a contradiction by using the reflection formula for the Jacobi $\theta-$function, which ensures a suitable decay for $\theta\left(e^{i\left(\frac{\pi}{4}-\varepsilon\right)}\right)$, as $\varepsilon\rightarrow 0^{+}$ [48,67].

Although much more difficult to ensure than the asymptotic properties of Jacobi's $\theta-$function, a condition like (1.4) is deep since it connects two fundamental properties of two drastically different Dirichlet series.

Our goal in this paper is to extend the scope of the Selberg-Chowla formula (1.2) and use it to generalize Deuring's argument concerning the infinitude of zeros on the critical line of a large class of Dirichlet series.

We start by making a distinction between two cases:

1. **The diagonal Epstein zeta function**: for $\text{Re}(s)$ sufficiently large, one of the goals of this paper is to study an infinite series of the form

$$\sum_{m,n\neq 0}^{\infty}\frac{a_1(m)\,a_2(n)}{(\lambda_m+\lambda_n')^s},$$

where $a_1(m)$, $a_2(n)$, λ_m and λ_n' will be specified later. This double series was also studied in Berndt's paper [13] and a formula of Selberg-Chowla type for it has been indicated there. Under additional assumptions, in section 2 we give two formulas analogous to (1.2) and use them to arrive at a functional equation for this double infinite series.

2. **The non-diagonal Epstein zeta function**: analogously to the previous item, for $\text{Re}(s)$ sufficiently large, we study a double infinite series of the form

$$\sum_{m,n\neq 0}\frac{a_1(m)\,a_2(n)}{Q(\lambda_m,\lambda_n')^s},$$

where $Q(x,y)$ denotes, just as in the classical case (1.1), a real and positive definite Quadratic Form. We study some conditions ensuring the analytic continuation and functional equation of such double series.

The study of the double series outlined above is made in Sections 2 and 3 respectively. By proving analogues of the Selberg-Chowla formula (1.2), we establish functional equations for these series. In section 4 we use the Selberg-Chowla formulas proved in the previous sections in order to generalize Deuring's argument [31] to a Class of Dirichlet series. In section 5 we furnish particular examples of diagonal and non-diagonal Epstein zeta functions and a description of their analytic continuation is given. We also prove certain generalizations of well-known identities in Number Theory, such as Watson's and Ramanujan-Guinand's formulas.

Most of our principal results are given in section 4 and complement the Theorems presented in [17, 18, 52], where analogues of Hardy's Theorem are proved for large classes of Dirichlet series. The corollaries derived from our main Theorem 4.1 contain also the key observations used in most of the well-known proofs of Hardy's Theorem. In fact, by simplifying Deuring's argument for $\zeta(s)$, we shall see how his proof of Hardy's Theorem, as well as our general approach, connects with the other proofs appearing in the literature, which usually involve the asymptotic behavior of the Jacobi $\theta-$function or classical estimates of some exponential integrals [40, 48, 67, 94]. As a rather simple example of our method, we will see that the final conclusion of Hardy's Theorem for $\zeta(s)$ will be provided by Jacobi's 4-square Theorem, which constitutes a curious connection between a purely analytic theorem involving deep properties of $\zeta(s)$ and a beautiful arithmetical property of $r_4(n)$ (see Example 5.4).

1.1 Notation and Definitions

To state the two generalizations of the Selberg-Chowla formula given in this paper, we need the following definition:

Definition 1.1. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ be two sequences of positive numbers strictly increasing to ∞ and $(a(n))_{n\in\mathbb{N}}$ and $(b(n))_{n\in\mathbb{N}}$ two sequences of complex numbers not identically zero. Consider the functions $\phi(s)$ and $\psi(s)$ representable as Dirichlet series

$$\phi(s)=\sum_{n=1}^{\infty}\frac{a(n)}{\lambda_n^s} \quad \text{and} \quad \psi(s)=\sum_{n=1}^{\infty}\frac{b(n)}{\mu_n^s} \tag{1.5}$$

with finite abcissas of absolute convergence σ_a and σ_b respectively. Let $\Delta(s)$ denote one of the following three gamma factors: $\Gamma(s)$, $\Gamma\left(\frac{s}{2}\right)$ and $\Gamma\left(\frac{s+1}{2}\right)$ and r be an arbitrary positive real number in the first case and 1 in the other two. We say that ϕ and ψ satisfy the functional equation

$$\Delta(s)\,\phi(s)=\Delta(r-s)\,\psi(r-s), \tag{1.6}$$

if there exists a meromorphic function $\chi(s)$ with the following properties:

1. $\chi(s)=\Delta(s)\,\phi(s)$ for $\mathrm{Re}(s)>\sigma_a$ and $\chi(s)=\Delta(r-s)\,\psi(r-s)$ for $\mathrm{Re}(s)<r-\sigma_b$;

2. $\lim_{|\mathrm{Im}(s)|\to\infty}\chi(s)=0$ uniformly in every interval $-\infty<\sigma_1\leq\mathrm{Re}(s)\leq\sigma_2<\infty$.

3. The singularities $\chi(s)$ are at most poles and are confined to some compact set.

Moreover, if $\Delta(s) = \Gamma\left(\frac{s+\delta}{2}\right)$, $\delta \in \{0,1\}$, we take a convention which extends $a(n)$ to the negative integers as follows $a(-n) = (-1)^{\delta} a(n)$.

We say that the pair of functions $(\phi,\, \psi)$ representable as Dirichlet series (1.5) satisfy **Hecke's functional equation** if they satisfy the conditions of the previous definition for $\Delta(s) = \Gamma(s)$. The particular case of (1.6) reads

$$\Gamma(s)\,\phi(s) = \Gamma(r-s)\,\psi(r-s), \quad r > 0. \tag{1.7}$$

Similarly, we say that the pair of functions $(\phi,\, \psi)$ representable as (1.5) with finite abcissas of absolute convergence σ_a and σ_b satisfy **Bochner's functional equation** if they satisfy the functional equation

$$\Gamma\left(\frac{s+\delta}{2}\right)\phi(s) = \Gamma\left(\frac{1+\delta-s}{2}\right)\psi(1-s), \tag{1.8}$$

in the sense of Definition 1.1. Moreover, if the pair $(\phi,\, \psi)$ satisfies (1.8) with $\delta = 0$, then we will say that $\phi(s)$ and $\psi(s)$ are even Dirichlet series belonging to the Bochner class. Otherwise, if (1.8) is satisfied for $\delta = 1$, we will say that $\phi(s)$ and $\psi(s)$ are odd Dirichlet series belonging to the Bochner class.

Although under slight variations, the class of Dirichlet series covered by Definition 1.1 was considered by a large number of authors. The Dirichlet series satisfying the functional equation (1.8) were studied by S. Bochner and K. Chandrasekharan in [26], where considerations of general analogues of Hamburger's theorem were given. K. Chandrasekharan and R. Narasimhan [27] proved the equivalence between several identities in Number theory and Hecke's functional equation (1.7). Included as a subclass in the context of generalized Dirichlet series are the classical Dirichlet series with signature $(\lambda,\, r,\, \gamma)$, considered in Hecke's works [7].

Before proceeding further, we shall denote $\sum_{m,n\neq 0}$ as the infinite sum over all integers m, n not simultaneously zero. If we assume that m and n are positive, we shall write $\sum_{m,n\neq 0}^{\infty}$ instead and we will always use the convention $\lambda_0 = \mu_0 = 0$ and $\lambda_{-n} = -\lambda_n$, $\mu_{-n} = -\mu_n$. Also, we will always write $\sigma_a := \max\{\sigma_{a_1},\, \sigma_{a_2}\}$ and σ_b analogously.

We will now define a double Dirichlet series which exhibits a similar behavior as the classical Epstein zeta function (1.1). We will often call it by the name "diagonal Epstein zeta function".

Definition 1.2. Let $(\lambda_n,\, \lambda_n^{\prime})_{n\in\mathbb{N}}$ be a pair of sequences of positive numbers strictly increasing to ∞ and $(a_1(n),\, a_2(n))_{n\in\mathbb{N}}$ a pair of sequences of complex numbers not identically zero. Also, let $\phi_1(s)$ and $\phi_2(s)$ be two functions representable as Dirichlet series

$$\phi_1(s) = \sum_{n=1}^{\infty} \frac{a_1(n)}{\lambda_n^{s}}, \quad \phi_2(s) = \sum_{n=1}^{\infty} \frac{a_2(n)}{\lambda_n^{\prime s}}, \tag{1.9}$$

with finite abcissas of absolute convergence σ_{a_1} and σ_{a_2} respectively. For the pair (ϕ_1, ϕ_2), we define the generalized diagonal Epstein zeta function as the double Dirichlet series

$$\mathscr{Z}_2\left(s, a_1, a_2, \lambda, \lambda'\right) = \sum_{m,n\neq 0}^{\infty} \frac{a_1(m)\, a_2(n)}{(\lambda_m + \lambda_n')^s}, \quad \text{Re}(s) > 2\,\max\left\{\sigma_{a_1}, \sigma_{a_2}\right\}. \tag{1.10}$$

It is easily seen that the double series (1.10) is absolutely convergent if $\text{Re}(s) > 2\,\max\left\{\sigma_{a_1}, \sigma_{a_2}\right\}$. Examples of (1.10) include the classical Dirichlet associated with the sum of two squares

$$\frac{1}{4}\,\zeta_2(s) = \frac{1}{4}\sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = \sum_{m,n\neq 0}^{\infty} \frac{1}{(m^2+n^2)^s}, \quad \text{Re}(s) > 1 \tag{1.11}$$

which can be constructed if we take $\phi_1(s) = \phi_2(s) = \zeta(2s)$.

Just as (1.11), we may represent a general diagonal Epstein $\mathscr{Z}_2\left(s; a_1, a_2; \lambda, \lambda'\right)$ via the single Dirichlet series

$$\mathscr{Z}_2\left(s, a_1, a_2, \lambda, \lambda'\right) = \sum_{m,n\neq 0}^{\infty} \frac{a_1(m)\, a_2(n)}{(\lambda_m + \lambda_n')^s} = \sum_{n=1}^{\infty} \frac{\mathfrak{U}_2(n)}{\Lambda_n^s}, \quad \text{Re}(s) > 2\,\sigma_a \tag{1.12}$$

where $\mathfrak{U}_2(n)$ generalizes the arithmetical function $r_2(n)$ in the following way:

$$\mathfrak{U}_2(n) = \sum_{j_1, j_2 : \lambda_{j_1} + \lambda'_{j_2} = \Lambda_n} a_1(j_1)\, a_2(j_2), \tag{1.13}$$

and the sequence Λ_n is taken as $\Lambda_n := \left(\lambda_{j_1} + \lambda'_{j_2}\right)_{j_1, j_2 \in \mathbb{N}_0}$ and then rearranged in increasing order.

Since (1.11) constitutes a particular case of the Epstein zeta function $Z_2(s, Q)$ when Q is a diagonal quadratic form, we shall call to the double series (1.10) by the name "diagonal Epstein zeta function". In section 2 we prove that, if ϕ_1 and ϕ_2 satisfy definition 1.1 with $\Delta(s) = \Gamma(s)$, then it is possible to write two distinct Selberg-Chowla formulas for (1.10). We will also introduce a subclass of Dirichlet, named class $\mathscr{A}$, for which (1.10) will also obey to Hecke's functional equation (see definition 2.2).

In section 3 we will study the analytic continuation of the following double Dirichlet series:

Definition 1.3. Let Q be a real, binary and positive definite quadratic form given by $Q(x, y) = ax^2 + bxy + cy^2$ and (ϕ_1, ϕ_2) a pair of Dirichlet series similar to (1.9) with abcissas of absolute convergence σ_{a_1} and σ_{a_2}. Assume also that their arithmetical functions $a_i(n)$ can be extended to the negative integers as $a_i(n) = (-1)^{\delta} a_i(n)$, $\delta \in \{0, 1\}$, and take the convention $\lambda_0 = \lambda'_0 = 0$ and $\lambda_{-n} = -\lambda_n$, $\lambda'_{-n} = -\lambda'_n$.

We define the non-diagonal Epstein zeta function associated with ϕ_1 and ϕ_2 as the double Dirichlet series

$$\mathscr{Z}_2\left(s; Q; a_1, a_2; \lambda, \lambda'\right) = \sum_{m,n\neq 0} \frac{a_1(m)\, a_2(n)}{(a\lambda_m^2 + b\,\lambda_m\lambda_n' + c\lambda_n'^2)^s}, \quad \text{Re}(s) > \sigma_a := \max\{\sigma_{a_1}, \sigma_{a_2}\} \tag{1.14}$$

where the infinite series is taken over the pair of integers which are not both zero.

It is easily seen that the double Dirichlet series on the right-hand side converges absolutely if $\mathrm{Re}(s) > \max\{\sigma_{a_1}, \sigma_{a_2}\}$. We will always adopt the convention that $a_1(n)$ and $a_2(n)$ have the same parity since, otherwise, the previous double series would be identically zero.

In section 3 we prove analogues of the Selberg-Chowla formula for (1.14) when ϕ_1 and ϕ_2 satisfy definition 1.1 with $\Delta(s) = \Gamma\left(\frac{s+\delta}{2}\right)$.

In analogy with the arithmetical function (1.12), we can write the non-diagonal Epstein zeta function (1.14) as the following (single) Dirichlet series

$$\mathscr{Z}_2(s; Q; a_1, a_2; \lambda, \lambda') = \sum_{n=1}^{\infty} \frac{\mathfrak{U}_Q(n)}{\Lambda_n^s}$$

where

$$\mathfrak{U}_Q(n) = \sum_{Q(\lambda_{j_1}, \lambda'_{j_2}) = \Lambda_n} a_1(j_1)\, a_2(j_2). \tag{1.15}$$

Before giving the preliminary results, we remark that throughout this paper we shall use the following notation and conventions:

- Let Q denote the matrix representation of the Quadratic form $Q(x, y)$ given in definition 1.3. If d is the discriminant of Q, we will define the inverse quadratic form as $Q^{-1}(\mathbf{x}) = -\frac{d}{4}\,\mathbf{x}^T Q^{-1}\,\mathbf{x}$. Note that $Q^{-1}(x, y) = cx^2 - bxy + ay^2$.

- In order to make our general formulas resemble the original formulation (1.2), we will use the symbol $\sigma_z(\nu_j; b_1, a_2; \mu, \lambda')$ to denote a generalized version of the divisor function (see Definition 2.1). At some points (see Example 5.2), we will simplify the notation and reduce it to $\sigma_z(\nu_j; b_1, a_2)$.

- Whenever we use the term "critical line" for a given Dirichlet series $\phi(s)$ satisfying Definition 1.1 we will be referring to $\mathrm{Re}(s) = \frac{r}{2}$ if $\phi(s)$ satisfies Hecke's functional equation (1.7) and to $\mathrm{Re}(s) = \frac{1}{2}$ if $\phi(s)$ satisfies Bochner's functional equation (1.8). The property "$\phi(s)$ has infinitely many zeros at its critical line" will sometimes be reformulated as "$\phi(s)$ satisfies Hardy's Theorem".

- We will write $H_{r_1}(s; b_1, a_2; \mu, \lambda')$ and $H_{r_2}(s; b_2, a_1; \mu', \lambda)$ to denote the entire complex functions (2.1) and (2.4) appearing in the Selberg-Chowla representations for the diagonal Epstein zeta function. For the entire complex functions appearing in the Selberg-Chowla representations for the non-diagonal Epstein zeta function, (3.11) and (3.12), we shall use alternatively $H_1(s; Q; b_1, a_2; \mu, \lambda')$ and $H_2(s; Q; b_2, a_1, \mu', \lambda)$. In analogy with the generalized divisor function, we will sometimes simplify the notation by writing these entire functions as $H_1(s; Q; b_1, a_2)$ or $H_2(s; Q; b_2, a_1)$.

- If, in Definitions 1.2 and 1.3, the Dirichlet series are such that $\phi_1 = \phi_2 = \phi$, we will write $\mathscr{Z}_2(s; a; \lambda)$ and $\mathscr{Z}_2(s; Q; a; \lambda)$ to respectively denote (1.10) and (1.14).

1.2 Preliminary results

In several occasions throughout this paper, we shall need to estimate the asymptotic order of certain integrals involving a combination of Gamma functions and Dirichlet series, $\phi(s)$, satisfying Definition 1.1. To justify most of the steps, we will often invoke the following version of Stirling's formula

$$\Gamma(\sigma+it)=(2\pi)^{\frac{1}{2}}\,t^{\sigma+it-\frac{1}{2}}\,e^{-\frac{\pi t}{2}-it+\frac{i\pi}{2}(\sigma-\frac{1}{2})}\left(1+1/12(\sigma+it)+O\left(1/t^{2}\right)\right), \tag{1.16}$$

as $t\rightarrow\infty$, uniformly for $-\infty<\sigma_1\leq\sigma\leq\sigma_2<\infty$. A similar formula can be written for $t<0$ as t tends to $-\infty$ by using the fact that $\Gamma(\overline{s})=\overline{\Gamma(s)}$.

To estimate the order of $\phi(s)$ at the line $\text{Re}(s)=\sigma$ we shall need a version of the classical Phragmén-Lindelöf theorem given in [[93], p. 180, 5.65]. Since, for $\sigma>\sigma_b$, $\psi(\sigma+it)=O(1)$ as $|t|\rightarrow\infty$, it follows from the functional equation (1.6) and Stirling's formula (1.16) that, for $\sigma<r-\sigma_b$, $\phi(s)$ satisfies

$$\begin{aligned}\phi(\sigma+it)&=O\left(\frac{\Delta(r-s)}{\Delta(s)}\,\psi(r-s)\right)=O\left(\frac{\Delta(r-s)}{\Delta(s)}\right)\\ &=\begin{cases}O\left(|t|^{r-2\sigma}\right) & \text{if } \Delta(s)=\Gamma(s)\\ O\left(|t|^{\frac{1-2\sigma}{2}}\right) & \text{if } \Delta(s)=\Gamma\left(\frac{s+\delta}{2}\right)\end{cases},\end{aligned} \tag{1.17}$$

as $|t|$ tends to ∞. Since $\phi(s)=O(1)$ for $\sigma=\text{Re}(s)>\sigma_a$, it follows from property 2. in Definition 1.1 that:

1. If $\phi(s)$ satisfies Hecke's functional equation (1.7), then for any $\delta>0$ and $r-\sigma_a-\delta\leq\sigma:=\text{Re}(s)\leq\sigma_a+\delta$,

$$\phi(\sigma+it)=O\left(|t|^{\sigma_a+\delta-\sigma}\right),\quad |t|\rightarrow\infty. \tag{1.18}$$

2. If $\phi(s)$ satisfies Bochner's functional equation (1.8), then for any $\varepsilon>0$ and $1-\sigma_a-\varepsilon\leq\sigma:=\text{Re}(s)\leq\sigma_a+\varepsilon$,

$$\phi(s)=O\left(|t|^{\frac{\sigma_a+\varepsilon-\sigma}{2}}\right),\quad |t|\rightarrow\infty. \tag{1.19}$$

Since Hardy's short note [48] and the subsequent quantitative proofs given by Landau, Fekete, Hardy and Littlewood [40,49,66], as well as later adaptations to other Dirichlet series by Kober [62] and Hecke [52], the study of the zeros of $\zeta(s)$ and other general Dirichlet series is often dependent on the asymptotic behavior of the associated Jacobi $\theta-$function as a function in the upper half-plane. From now on, if $\phi(s)$ satisfies Hecke's functional equation (1.7) in the sense of Definition 1.1 and it is representable by the first Dirichlet series in (1.5), we shall use $\Theta(z;a;\lambda)$ to denote the generalized $\theta-$function

$$\Theta(z;a;\lambda):=\sum_{n=1}^{\infty}a(n)\,e^{-\lambda_n z},\quad \text{Re}(z)>0. \tag{1.20}$$

Following Bochner [25], for $\text{Re}(z)>0$, let $P(z)$ denote the residual function [10]

$$P(z)=\frac{1}{2\pi i}\int_C\chi(s)\,z^{-s}ds, \tag{1.21}$$

where C denotes a curve, or curves, encircling the singularities of $\chi(s)$ given in Definition 1.1. It was proved by Bochner [25] that Hecke's functional equation for $\phi(s)$ and $\psi(s)$ (1.7) and the modular relation

$$\sum_{n=1}^{\infty} a(n)\, e^{-\lambda_n z} = z^{-r} \sum_{n=1}^{\infty} b(n)\, e^{-\mu_n/z} + P(z). \tag{1.22}$$

are equivalent. It is interesting that the observation of this equivalence had its genesis in B. Riemann's revolutionary memoir, where one of the implications was proved for the first time. There, a second proof of the functional equation for $\pi^{-s}\zeta(2s)$ (which satisfies Hecke's functional equation with $r=\frac{1}{2}$) was given by observing the reflection formula for the classical Jacobi $\theta-$function, which is a particular case of (1.22).

The converse is obtained upon the use of the Cahen-Mellin integral [[49], p. 120, eq. (1.II)]

$$e^{-z} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\, z^{-s} ds, \tag{1.23}$$

valid for $c>0$, $\text{Re}(z)>0$ and z^{-s} having its principal value.

We will also need a slightly more general version of (1.22) when $\phi(s)$ satisfies one of Bochner's functional equations (1.8). This will be done in Lemma 3.1 of the paper. In order to establish it, we will require the representation of the confluent hypergeometric function ${}_1F_1$ as the following Mellin transform [[47], p. 503, eq. 3.952.8], valid for $\text{Re}(\alpha)>0$, $\text{Re}(s)>0$ and $\beta\in\mathbb{C}$,

$$\int_0^{\infty} x^{s-1}\, e^{-\alpha x^2}\cos(\beta x)\, dx = \frac{\alpha^{-\frac{s}{2}} e^{-\frac{\beta^2}{4\alpha}}}{2}\,\Gamma\left(\frac{s}{2}\right)\, {}_1F_1\left(\frac{1-s}{2};\frac{1}{2};\frac{\beta^2}{4\alpha}\right). \tag{1.24}$$

Since the real function $e^{-\alpha x^2}\cos(\beta x)$ is smooth, we are allowed to integrate the left-hand side of (1.24) by parts an arbitrary number of times. Doing so, we see that its asymptotic estimate must be of the form $O(|s|^{-N})$ as $|s|\to\infty$ for any $N\in\mathbb{N}$. A simple argument by continuation shows that an estimate like this holds for every $s\in\mathbb{C}$. This gives a suitable decaying behavior for the right-hand side of (1.24) which will be useful in the proof of Lemma 3.1. Of course, stronger estimates for the right-hand side of (1.24) could be given by invoking asymptotic expansions for the Whittaker function [47], but since we will not need these expansions we shall not write them.

Since we will apply Mellin's inversion formula to the right-hand side of (1.24) (see eq. (3.3) below), the confluent hypergeometric function will have to satisfy a reflection formula compatible with Bochner's functional equation. As we shall see, the right symmetries are expected due to Kummer's transformation formula for formula for ${}_1F_1$ [[1], p. 191, eq. (4.1.11)]

$${}_1F_1\left(a;c;x\right) = e^{x}\, {}_1F_1\left(c-a;c;-x\right). \tag{1.25}$$

In the sequel, we will employ several times well-known integral representations for the modified Bessel function $K_{\nu}(z)$. For a matter of clarity in our exposition, we will invoke them solely when they are needed.

2 The Selberg-Chowla formula and the analytic continuation of the Diagonal Epstein zeta function

Before stating the properties of the special class $\mathscr{A}$, for which functional equations will be given, we will consider the following general Theorem, which establishes the analytic continuation of the Diagonal Epstein zeta function (1.10). As usual, the following result provides a formula valid in a region contained in the plane of absolute convergence of the Double series (1.10). However, as one should expect, they may be taken to other regions of $\mathbb{C}$ by analytic continuation.

Theorem 2.1 (A general Selberg-Chowla formula [[13], p. 166])**.** *Let ϕ_1 and ϕ_2 be the Dirichlet series given in the above definition 1.1 and satisfying Hecke's functional equation (1.7). Also, let s be a complex number such that $Re(s)>\mu>\max\left\{0,2\sigma_a,2\sigma_b,\frac{r'}{2}\right\}$, where $\sigma_a=\max\{\sigma_{a_1},\sigma_{a_2}\}$, $\sigma_b=\max\{\sigma_{b_1},\sigma_{b_2}\}$ and $r'=\max\{r_1,r_2\}$. Let $H_{r_1}(s;b_1,a_2;\mu,\lambda')$ denote the double series*

$$H_{r_1}(s;b_1,a_2;\mu,\lambda')=\sum_{m,n=1}^{\infty}b_1(m)\,a_2(n)\left(\frac{\mu_m}{\lambda'_n}\right)^{\frac{s-r_1}{2}}K_{r_1-s}\left(2\sqrt{\mu_m\lambda'_n}\right). \tag{2.1}$$

Then, for $Re(s)>\mu$, the following identity for the generalized Epstein zeta function holds

$$\Gamma(s)\,\mathscr{Z}_2\left(s;a_1,a_2;\lambda,\lambda'\right)=a_1(0)\,\Gamma(s)\,\phi_2(s)+a_2(0)\,\Gamma(s)\,\phi_1(s)+\sum_{n=1}^{\infty}\frac{a_2(n)}{\lambda_n'^{s}}R_1\left(s,\frac{1}{\lambda'_n}\right)$$
$$+2H_{r_1}\left(s;b_1,a_2;\mu,\lambda'\right), \tag{2.2}$$

where $b_1(m)$ is the arithmetical function associated to the Dirichlet series $\psi_1(s)$ and $R_1(s,x)$ denotes the sum of the residues of the meromorphic function

$$R_1(z)=\Gamma(z)\,\phi_1(z)\,\Gamma(s-z)\,x^{-z} \tag{2.3}$$

at the poles of $\Gamma(z)\,\phi_1(z)$. Equivalently, if $H_{r_2}(s;b_2,a_1;\mu',\lambda)$ denotes the double infinite series,

$$H_{r_2}(s;b_2,a_1;\mu',\lambda)=\sum_{m,n=1}^{\infty}b_2(m)\,a_1(n)\left(\frac{\mu'_m}{\lambda_n}\right)^{\frac{s-r_2}{2}}K_{r_2-s}\left(2\sqrt{\mu'_m\lambda_n}\right), \tag{2.4}$$

then $\mathscr{Z}_2(s;a_1,a_2;\lambda,\lambda')$ can be also described by the formula

$$\Gamma(s)\,\mathscr{Z}_2\left(s;a_1,a_2;\lambda,\lambda'\right)=a_1(0)\,\Gamma(s)\,\phi_2(s)+a_2(0)\,\Gamma(s)\,\phi_1(s)+\sum_{m=1}^{\infty}\frac{a_1(m)}{\lambda_m^{s}}R_2\left(s,\frac{1}{\lambda_m}\right)$$
$$+2H_{r_2}\left(s;b_2,a_1;\mu',\lambda\right), \tag{2.5}$$

whenever $Re(s)>\mu$. Here, in analogy with (2.3), $R_2(s,x)$ denotes the sum of the residues of the meromorphic function

$$R_2(z)=\Gamma(z)\,\phi_2(z)\,\Gamma(s-z)\,x^{-z} \tag{2.6}$$

at the poles of $\Gamma(z)\,\phi_2(z)$.

Proof. The idea of the proof follows [[7], p. 311] and its sketched version in [13]. Let $\mu > \max\left\{0, 2\sigma_a, 2\sigma_b, \frac{r'}{2}\right\}$ as in the statement and consider a fixed $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > \mu$. Under this condition, the Beta transform integral [[38], p. 349, 7.3.15]

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,x^{-z}dz = \frac{\Gamma(s)}{(1+x)^s}, \quad x>0 \tag{2.7}$$

holds. Since our Epstein series (1.10) is defined as a double infinite series which converges absolutely for $\mathrm{Re}(s) > \mu \geq 2\,\sigma_a$, we may write it as

$$\mathscr{Z}_2\left(s; a_1, a_2; \lambda, \lambda'\right) = a_1(0)\phi_2(s) + a_2(0)\phi_1(s) + \sum_{n=1}^{\infty} a_2(n)\,\ell_n\left(s, a_1\right), \tag{2.8}$$

where $\ell_n(s, a_1)$ denotes the generalized Dirichlet series [7] given by

$$\ell_n\left(s, a_1\right) = \sum_{m=1}^{\infty}\frac{a_1(m)}{\left(\lambda_m+\lambda_n'\right)^s}, \quad \mathrm{Re}(s) > \sigma_{a_1}.$$

By the Mellin representation (2.7), we see that we can write $\ell_n\left(s, a_1\right)$ as the contour integral

$$\lambda_n'^s\,\Gamma(s)\,\ell_n\left(s, a_1\right) = \sum_{m=1}^{\infty}\frac{a_1(m)\,\Gamma(s)}{\left(1+\frac{\lambda_m}{\lambda_n'}\right)^s} = \sum_{m=1}^{\infty}\frac{a_1(m)}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\left(\frac{\lambda_m}{\lambda_n'}\right)^{-z}dz, \tag{2.9}$$

valid whenever $\mathrm{Re}(s) > \sigma_{a_1}$, which is the case as $\mathrm{Re}(s) > \mu > \max\{0, 2\sigma_a\} \geq \max\{0, 2\sigma_{a_1}\}$. From the definition of μ, we know that $\sum a_1(m)\,\lambda_m^{-\mu}$ converges absolutely. Moreover, from a corollary of Stirling's formula (1.16),

$$|\Gamma(\mu+it)\,\Gamma(s-\mu-it)| = O\left(|t|^{\mathrm{Re}(s)-1}e^{-\pi|t|}\right), \quad |t| \to \infty, \tag{2.10}$$

we are allowed to interchange the orders of integration and summation in (2.9). This gives the integral representation for the generalized Dirichlet series $\ell_n(s, a_1)$,

$$\lambda_n'^s\Gamma(s)\,\ell_n(s, a_1) = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\phi_1(z)\,\Gamma(z)\,\Gamma(s-z)\left(\frac{1}{\lambda_n'}\right)^{-z}dz, \;\; \mu > \max\left\{0, \sigma_a, \sigma_b, \frac{r_1}{2}\right\}.$$

Let us now move the line of integration to $z = r_1 - \mu + it$ (which is on the left of $\mathrm{Re}(z) = \mu$ because $\mu > \frac{r'}{2} \geq \frac{r_1}{2}$ by hypothesis) and integrate along a positively oriented rectangular contour $\mathscr{R}_1$ containing the vertices $\mu \pm iT$ and $r_1 - \mu \pm iT$, where $T > \sup_{\rho\in\mathscr{S}}|\mathrm{Im}(\rho)|$, with $\mathscr{S}$ denoting the set of singularities[3] of $\chi_1(z)\,\Gamma(s-z)$. By the Residue Theorem, we have the equality

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mu-iT}^{\mu+iT}\phi_1(z)\,\Gamma(z)\,\Gamma(s-z)\,\lambda_n'^z\,dz &= \frac{1}{2\pi i}\int_{r_1-\mu-iT}^{r_1-\mu+iT}\phi_1(z)\,\Gamma(z)\,\Gamma(s-z)\,\lambda_n'^z\,dz \\ +\frac{1}{2\pi i}\int_{\mu\pm iT}^{r_1-\mu\pm iT}\phi_1(z)\,\Gamma(z)\,\Gamma(s-z)\,\lambda_n'^z\,dz &+ \sum_{\rho\in\mathscr{R}_1}\mathrm{Res}_{z=\rho}\left(\phi_1(z)\,\Gamma(z)\,\Gamma(s-z)\,\left(1/\lambda_n'\right)^{-z}\right),\end{aligned} \tag{2.11}$$

[3]Note that this supremum taken over $\mathscr{S}$ is finite from the assumption that the singularities of $\chi_1(z)$ are contained in a compact set and the observation that all the singularities of $\Gamma(s-z)$ have their imaginary part equal to $\mathrm{Im}(s)$.

where the second integral represents the integrals over the horizontal lines. We now see that

$$\int_{\mu\pm iT}^{r_1-\mu\pm iT}\phi_1(z)\,\Gamma(z)\,\Gamma(s-z)\,\lambda_n'^{z}\,dz\to 0\ \text{ as }\ T\to\infty, \tag{2.12}$$

by condition 2 on Definition 1.1. Despite this imposition, we should note that (2.12) could easily come from an immediate application of Stirling's formula (1.16) together with the Phragmén-Lindelöf estimate (1.18) and a weaker condition of the form $\phi(s)=O\left(\exp|s|^{K}\right)$, for some $K>0$, as $|s|\to\infty$ [7, 17, 18]. However, by the adoption of Definition 1.1, this justification becomes even easier. Letting $T\to\infty$ in (2.11), we obtain

$$\begin{aligned}\lambda_n'^{s}\Gamma(s)\,\ell_n(s,\,a_1) &= \frac{1}{2\pi i}\int_{r_1-\mu-i\infty}^{r_1-\mu+i\infty}\phi_1(z)\,\Gamma(z)\,\Gamma(s-z)\,\left(\frac{1}{\lambda_n'}\right)^{-z}dz\\ &\quad+\sum_{\rho\in\mathscr{R}_1}\text{Res}_{z=\rho}\left(\phi_1(z)\,\Gamma(z)\,\Gamma(s-z)\,\left(1/\lambda_n'\right)^{-z}\right).\end{aligned} \tag{2.13}$$

We now claim that the last term of the previous equality is precisely $R_1(s,\,1/\lambda_n')$, whose definition is given at the statement of this result. It is easy to check by the functional equation for $\phi_1(z)$ (1.7) and the definition of μ that all poles of $\chi_1(z)=\Gamma(z)\,\phi_1(z)$ are on the interior of $\mathscr{R}_1$. Moreover, since $\text{Re}(s)>\mu$ by hypothesis, we clearly have that $\text{Re}(s-z)>0$ for any $z\in\mathscr{R}_1$, so that $\Gamma(s-z)$ has no poles inside $\mathscr{R}_1$ and the only poles taken into account in the sum in (2.13) are precisely the ones coming from the function $\Gamma(z)\,\phi_1(z)$. Thus, the previous equality reduces to

$$\lambda_n'^{s}\Gamma(s)\,\ell_n(s,\,a_1)=\frac{1}{2\pi i}\int_{r_1-\mu-i\infty}^{r_1-\mu+i\infty}\phi_1(z)\,\Gamma(z)\,\Gamma(s-z)\,\left(\frac{1}{\lambda_n'}\right)^{-z}dz+R_1\left(s,\,\frac{1}{\lambda_n'}\right).$$

Let us now invoke the functional equation for the Dirichlet series $\phi_1(z)$ (1.7) and use it in the contour integral given above. Doing this and taking the change of variables $z\leftrightarrow r_1-z$, we arrive at

$$\begin{aligned}\lambda_n'^{s}\Gamma(s)\,\ell_n(s,\,a_1) &= \lambda_n'^{r_1}\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\psi_1(z)\,\Gamma(z)\,\Gamma(s-r_1+z)\,\lambda_n'^{-z}\,dz+R_1\left(s,\,\frac{1}{\lambda_n'}\right)\\ &= \lambda_n'^{r_1}\sum_{m=1}^{\infty}\frac{b_1(m)}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma(s-r_1+z)\,\left(\mu_m\lambda_n'\right)^{-z}dz+R_1\left(s,\,\frac{1}{\lambda_n'}\right),\end{aligned} \tag{2.14}$$

where in the last step we have also used the absolute convergence of the Dirichlet series $\psi_1(z)$ for $\text{Re}(z)=\mu>\sigma_b\geq\sigma_{b_1}$ and Stirling's formula for the product of $\Gamma-$functions.

Recalling the Mellin representation for the Modified Bessel function [[38], p. 349, 7.3 (17)]

$$K_{\nu}(x)=\left(\frac{x}{2}\right)^{\nu}\frac{1}{4\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\Gamma(z-\nu)\,\Gamma(z)\,\left(\frac{x^2}{4}\right)^{-z}dz,\ \ \lambda>\max\{0,\,\text{Re}(\nu)\}, \tag{2.15}$$

we see that, since $\mu>\max\{0,\,r_1-\text{Re}(s)\}$, we may represent the contour integral on the right-hand side of (2.14) as

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma(s-r_1+z)\,\left(\mu_m\lambda_n'\right)^{-z}dz=2\,\left(\mu_m\lambda_n'\right)^{-\frac{r_1-s}{2}}K_{r_1-s}\left(2\sqrt{\mu_m\lambda_n'}\right). \tag{2.16}$$

Combining (2.14) and (2.16), we arrive at the following representation of the generalized Dirichlet series $\ell_n(s, a_1)$ (c.f. [7])

$$\lambda_n'^s \Gamma(s)\,\ell_n(s, a_1) = 2\,\lambda_n'^{r_1} \sum_{m=1}^{\infty} b_1(m)\,\left(\mu_m\lambda_n'\right)^{-\frac{r_1-s}{2}} K_{r_1-s}\left(2\sqrt{\mu_m\lambda_n'}\right) + R_1\left(s, \frac{1}{\lambda_n'}\right).$$

We can easily see that the double series on the right side of (2.8) converges absolutely if $\mathrm{Re}(s) > \sigma_{a_1} + \sigma_{a_2}$ and so it must converge absolutely for $\mathrm{Re}(s) > \mu$. Since $\phi_1(s)$ is analytic in the region $\mathrm{Re}(s) > \sigma_{a_1}$, it is clear from the expression for $R_1(s, x)$ that the series $\sum a_2(n)\,\lambda_n'^{-s} R_1(s, \lambda_n'^{-1})$ converges absolutely whenever $\mathrm{Re}(s) > \sigma_{a_2} + \max\{0, \sigma_{a_1}\}$. Hence, assuming that $\mathrm{Re}(s) > \mu \geq \sigma_{a_2} + \max\{0, \sigma_a\}$ and summing $a_2(n)\,\ell_n(s, a_1)$ with respect to n as in (2.8), we arrive at the representation (2.2). To finish, we only need to argue that, under the assumption $\mathrm{Re}(s) > \mu$, the double series which we have obtained by summing with respect to n, (2.1), converges absolutely. This is immediate due to the asymptotic estimate for the modified Bessel function

$$K_\nu(x) = O\left(x^{-\frac{1}{2}}\, e^{-x}\right), \quad x \to \infty \tag{2.17}$$

which implies that, for some positive constant C,

$$\left|H_{r_1}(s; b_1, a_2; \mu, \lambda')\right| \leq C \sum_{m,n=1}^{\infty} b_1(m)\, a_2(n) \left(\frac{\mu_m}{\lambda_n'}\right)^{\frac{s-r_1}{2}} \left(\mu_m\lambda_n'\right)^{-\frac{1}{4}} e^{-2\sqrt{\mu_m\lambda_n'}}$$

For any $\alpha > 0$, we know that $e^{-2\sqrt{\mu_m\lambda_n'}} \leq (\mu_m\lambda_n')^{-\alpha}$ for $m, n > N$, where N is sufficiently large[4]. Choose $\alpha > \max\left\{\sigma_a - \frac{1}{4} + \frac{r_1-s}{2}, \sigma_b - \frac{1}{4} + \frac{s-r_1}{2}\right\}$: from this choice and the absolute convergence of ϕ_2 and ψ_1 for $\mathrm{Re}(s) > \max\{\sigma_a, \sigma_b\}$, we immediately see that the series defining H_{r_1} converges absolutely for $\mathrm{Re}(s) > \mu$.

In a completely analogous way, the second formula (2.5) may be obtained, with the same choice of μ. In order to ensure that the identities (2.2) and (2.5) represent the same diagonal Epstein zeta function, we need to impose the possibility of reversing the order of summation in the double series (1.10). This is the case when we take $\mathrm{Re}(s) > \max\{0, 2\sigma_a\}$, so our choice of μ allows to reverse the order. This concludes the proof. □

Since the number of poles of ϕ_1 and ϕ_2 is finite (as they are contained in a compact set, by condition 3. in definition 1.1), we can easily study the continuation of the residual series

$$\varphi_1(s) := \sum_{m=1}^{\infty} \frac{a_1(m)}{\lambda_m^s} R_2\left(s, \frac{1}{\lambda_m}\right), \quad \varphi_2(s) := \sum_{n=1}^{\infty} \frac{a_2(n)}{\lambda_n'^s} R_1\left(s, \frac{1}{\lambda_n'}\right). \tag{2.18}$$

Thus, if we warrant that any one of the Dirichlet series $(\phi_1, \phi_2, \psi_1, \psi_2)$ has analytic continuation to the entire complex plane, the only remaining step towards the extension of the Selberg-Chowla formula to any

[4] note that $(\nu_j)_{j\in\mathbb{N}} := (\mu_m\lambda_n')_{m,n\in\mathbb{N}}$ may be arranged into an increasing sequence and we can impose the inequality $e^{-2\sqrt{\nu_j}} \leq \nu_j^{-\alpha}$ for $j \geq N_0$.

$s \in \mathbb{C}$ is the analytic continuation of the functions $H_{r_j}(s, a)$. As in the case of the classical Selberg-Chowla formula [5], we will show that H_{r_1} and H_{r_2} represent entire functions of s. Although the proof by Bateman and Grosswald [5] works mutatis mutandis in our case, we add, for completeness, an alternative proof of this fact, which employs the Mellin representation (2.15) of the modified Bessel function. A similar proof, adapted in a different context, is given in [[13], p. 181].

Proposition 2.1 (The analytic continuation of H)**.** *Let H_{r_1} and H_{r_2} be the double series given in (2.1) and (2.4). Then they represent entire functions in the variable $s \in \mathbb{C}$ and obey to the reflection formula*

$$H_{r_1}\left(s; a_1, b_2; \lambda, \mu'\right) = H_{r_2}\left(r_1 + r_2 - s; b_2, a_1; \mu', \lambda\right). \tag{2.19}$$

Proof. As in the previous Theorem, let $\mu > \max\left\{0, 2\sigma_a, 2\sigma_b, \frac{r'}{2}\right\}$ and take $\eta > \mu$. For any $\varepsilon > 0$, consider a rectangle $\mathscr{R}_{\varepsilon,\eta}(T)$, $T > 0$, whose vertices are $r' - \mu + \sigma_a + \varepsilon \pm iT$ and $\eta \pm iT$. Assuming that $s \in \mathscr{R}_{\varepsilon,\eta}(T)$ and returning to the representation (2.14), we can write the function H_{r_1} as

$$H_{r_1}\left(s; a_1; b_2; \lambda, \mu'\right) := \sum_{m,n=1}^{\infty} b_1(m)\, a_2(n)\, \lambda_n'^{r_1 - s} \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \Gamma(z)\Gamma(s - r_1 + z) \left(\mu_m \lambda_n'\right)^{-z} dz. \tag{2.20}$$

It is readily seen from (2.20) that we can argue the uniform convergence of the previous double series in $\mathscr{R}_{\varepsilon,\eta}(T)$ by estimating the contour integral appearing on the right-hand side of (2.20). To do this, we will set $s := \sigma + it \in \mathscr{R}_{\varepsilon,\eta}(T)$ and split the integral $(\mu - i\infty, \mu + i\infty)$ into three parts: $(\mu - i\infty, \mu - i(T+1))$, $(\mu - i(T+1), \mu + i(T+1))$ and $(\mu + i(T+1), \mu + i\infty)$. The bound for the first of these integrals is easy to obtain and is simply given by

$$\left| \int_{\mu - i(T+1)}^{\mu + i(T+1)} \Gamma(z)\Gamma(s - r_1 + z) \left(\mu_m \lambda_n'\right)^{-z} dz \right| \leq C \left(\mu_m \lambda_n'\right)^{-\mu} \Gamma(\sigma - r_1 + \mu)\Gamma(\mu) \leq C' \left(\mu_m \lambda_n'\right)^{-\mu} \tag{2.21}$$

where C' only depends on the maximum value that $\Gamma(x)$ attains in the interval $\sigma_a + r' - r_1 + \varepsilon < x < \eta + \mu - r_1$ and thus (2.21) does not depend of $s \in \mathscr{R}_{\varepsilon,\eta}(T)$. From Stirling's formula, it is also immediate to arrive at the second bound

$$\begin{aligned} \left| \int_{\mu + i(T+1)}^{\mu + i\infty} \Gamma(z)\Gamma(s - r_1 + z) \left(\mu_m \lambda_n'\right)^{-z} dz \right| &\leq C \left(\mu_m \lambda_n'\right)^{-\mu} \int_{T+1}^{\infty} y^{\mu - \frac{1}{2}} e^{-\pi y} (y + t)^{\sigma - r_1 + \mu - \frac{1}{2}} dy \\ &\leq C \left(\mu_m \lambda_n'\right)^{-\mu} \int_{T+1}^{\infty} y^{\mu - \frac{1}{2}} e^{-\pi y} (y + t)^{\eta - r_1 + \mu - \frac{1}{2}} dy \leq C' (\mu_m \lambda_n')^{-\mu}, \end{aligned} \tag{2.22}$$

where, once more, C' does not depend also on $s \in \mathscr{R}_{\varepsilon,\eta}(T)$. Due to the symmetry $\Gamma(\overline{z}) = \overline{\Gamma(z)}$, it is simple to see that a similar estimate to (2.22) holds when we integrate from $\mu - i\infty$ to $\mu - i(T+1)$. By (2.20), we see that

$$\left| H_{r_1}\left(s; a_1; b_2; \lambda, \mu'\right) \right| \leq C \sum_{m=1}^{\infty} \frac{|b_1(m)|}{\mu_m^{\mu}} \cdot \sum_{n=1}^{\infty} \frac{|a_2(n)|}{\lambda_n'^{\mu + \sigma - r_1}},$$

where $\sigma > r' - \mu + \sigma_a + \varepsilon$ (since $s \in \mathscr{R}_{\varepsilon,\eta}(T)$) and C does not depend on s. Since the right-hand side of the previous inequality is the product of two absolutely convergent series, an application of the Weierstrass M-test leads to the conclusion that the infinite series appearing in (2.20) converges absolutely and uniformly on $\mathscr{R}_{\varepsilon,\eta}(T)$. Since μ can be taken sufficiently large, we know that every bounded subset of $\mathbb{C}$ is contained in some rectangle of the form $\mathscr{R}_{\varepsilon,\eta}(T)$ and this shows that the double series (2.46) converges absolutely and uniformly in every bounded subset of the complex plane. From the fact that the double sequence of functions $f_{m,n}(s) = 2\,(\mu_m/\lambda_n')^{\frac{s-r_1}{2}}\,K_{r_1-s}\left(2\sqrt{\mu_m\lambda_n'}\right)$ represents, for each pair of integers $(m,n)\in\mathbb{N}^2$, an analytic function of s, we thus have that $H_{r_1}(s;b_1,a_2;\mu,\lambda')$ is an entire function.

Note that, in accordance with the previous proof, the fact that H_{r_1} is analytic in $\mathscr{R}_{\varepsilon,\eta}(T)$ shows the validity of (2.2) for $s\in\{z\in\mathbb{C}:\mathrm{Re}(z)>\mu\}\cup\mathscr{R}_{\varepsilon,\eta}(T)$. Since we can take ε arbitrarily small and T arbitrarily large, this argument shows that (2.2) holds in every half-plane of the form $\mathrm{Re}(s) > r' - \mu + \sigma_a$. Taking μ as large as desired, the analytic continuation of the Epstein Dirichlet series (1.10) can be taken to all complex numbers.

Analogously, one may see that H_{r_2} converges uniformly on any rectangle $\mathscr{R}_{\varepsilon,\eta}(T)$. The same can be concluded after an application of the reflection formula (2.19). By analytic continuation, this shows that both representations (2.2) and (2.5) hold for the Epstein zeta function (1.10). Finally, it is clear that (2.19) comes immediately from the property for the Modified Bessel function $K_\nu(z) = K_{-\nu}(z)$. □

Remark 2.1. It should be also pointed out that, even if one of the Dirichlet series ϕ_1 or ϕ_2 does not possess a functional equation of Hecke type (1.7), one of the formulas (2.2) or (2.5) is still valid, since the only tool required to prove these is the Functional equation of only one of the Dirichlet series. Although the appropriate Selberg-Chowla formula provides the analytic continuation of $\mathscr{Z}_2$ in this case, as it will be clear, we cannot obtain a suitable functional equation for it. For example in [[13], p. 165, eq. (6.1)] it is stated the analytic continuation of a Epstein zeta function of the type $\sum_{m,n} c(n)/Q(m,n)^s$, with $c(n)$ not necessarily being attached to a Dirichlet series with a given functional equation. See also [[78], p. 146, Remark 4.2.], as well as a character analogue of the first Kronecker limit formula on [[78], p. 165, eq. (4.107)] arising from this observation. Similar comments can be made in later sections of this paper, such as in Theorem 3.1.

Now we remark that the Selberg-Chowla formulas (2.2) and (2.5) can be written in terms of a generalized version of the divisor function, which we now define.

Definition 2.1 (A Generalized Divisor function)**.** Let $\{\nu_j\}_{j\in\mathbb{N}}$ represent the "product sequence" $\nu_j = \{\mu_m\cdot\lambda_n'\}_{m,n\in\mathbb{N}}$ and then arrange this product in increasing order. We define the generalized divisor function as

$$\sigma_z\left(\nu_j;b_1,a_2;\mu,\lambda'\right) = \sum_{(m,n):\,\mu_m\cdot\lambda_n'=\nu_j} b_1(m)\,a_2(n)\,\mu_m^z. \tag{2.23}$$

With the notation introduced in Definition 2.1, we have that H_{r_1} can be described by the series

$$H_{r_1}(s;b_1,a_2;\mu,\lambda') = \sum_{j=1}^{\infty} \sigma_{s-r_1}\left(\nu_j;b_1,a_2;\mu,\lambda'\right) \nu_j^{\frac{r_1-s}{2}} K_{r_1-s}(2\sqrt{\nu_j}), \tag{2.24}$$

which strongly resembles the entire part appearing in the classical Selberg-Chowla formula (1.2). Analogously, if we write $\{\nu_j'\}_{j\in\mathbb{N}} = \{\mu_m'\lambda_n\}_{m,n\in\mathbb{N}}$, it is also evident that we can write H_{r_2} as

$$H_{r_2}(s;b_2,a_1;\mu',\lambda) = \sum_{j=1}^{\infty} \sigma_{s-r_2}\left(\nu_j';b_2,a_1;\mu',\lambda\right) \nu_j'^{\frac{r_2-s}{2}} K_{r_2-s}\left(2\sqrt{\nu_j'}\right).$$

Taking $z=0$ in (2.23), we get a generalization of the standard divisor function $d(n)=\sum_{d|n}1$,

$$d\left(\nu_j;b_1,a_2;\mu,\lambda'\right) = \sum_{(m,n):\mu_m\lambda_n'=\nu_j} b_1(m)\,a_2(n). \tag{2.25}$$

Having proved that H_{r_i}, for each $i=1,2$, is entire, we are ready to establish the analytic continuation of the Epstein series $\mathscr{Z}_2(s;a_1,a_2;\lambda,\lambda')$. To do this, we need to understand the continuation of the residual functions given in the proof of Theorem 2.1, $\varphi_1(s)$ and $\varphi_2(s)$, appearing in (2.18). These will have to be fairly symmetric and need to obey to a functional equation similar to the reflection formula (2.19). In order to ensure this, we introduce the following class of Dirichlet series:

Definition 2.2. Let $\phi(s)$ be a Dirichlet series satisfying Definition 1.1 with $\Delta(s)=\Gamma(s)$. We say that ϕ belongs to the class $\mathscr{A}$ if additionally:

1. ϕ and ψ have analytic continuations to the entire complex plane and are analytic on $\mathbb{C}$ except for possible simple poles located at $s=r$ with residues ρ and $\rho^{\star}$ respectively.

2. $a(n)$ can be defined at 0 and has the value $-\phi(0)$. $b(n)$ can be analogously extended to $n=0$ as $-\psi(0)$.

Remark 2.2. Note that, by the functional equation (1.7) and condition 1 in the previous definition, $\phi_i(0) = -\rho_i^{\star}\Gamma(r_i)$, while $\psi_i(0) = -\rho_i\Gamma(r_i)$. Under the symmetry imposed by the functional equation and the first condition, it is clear that $\phi\in\mathscr{A}$ if and only if $\psi\in\mathscr{A}$. The purpose of introducing this class is to mimic as much as possible the class of Dirichlet series with a given signature, which is a clear subclass of $\mathscr{A}$. Besides, this class reproduces in a general form what happens in the classical cases of diagonal Epstein zeta functions. As a corollary of the Selberg-Chowla formulas given in Theorem 2.1, in the next result we prove that, if $\phi_1,\phi_2\in\mathscr{A}$, then $\mathscr{Z}_2\in\mathscr{A}$. Hence, this class is closed under composition of Dirichlet series attached to diagonal Epstein zeta functions. A similar class was also considered in [[6], p. 221].

From the functional equation for ϕ and under the hypotheses given for the class $\mathscr{A}$, it is simple to see that the residual function $R_1(s,x)$, appearing in (2.2) and described in (2.3), is given by

$$R_1(s,x) = \phi_1(0)\,\Gamma(s) + \rho_1\Gamma(r_1)\,\Gamma(s-r_1)\,x^{-r_1}, \tag{2.26}$$

so that, for $\mathrm{Re}(s) > \mu > \max\left\{0, 2\sigma_a, 2\sigma_b, \frac{r'}{2}\right\}$, we can easily see that

$$\varphi_2(s) = \sum_{n=1}^{\infty} \frac{a_2(n)}{\lambda_n'^s} R_1\left(s, 1/\lambda_n'\right) = \phi_1(0)\,\Gamma(s)\,\phi_2(s) + \rho_1\Gamma(r_1)\,\Gamma(s-r_1)\,\phi_2\left(s-r_1\right). \tag{2.27}$$

Likewise, one may find $R_2(s,x)$ and $\varphi_1(s)$. Invoking the fact that $a_i(0) = -\phi_i(0)$, we see that formulas (2.2) and (2.5), when considered over the class $\mathscr{A}$, are

$$\begin{aligned}\Gamma(s)\,\mathscr{Z}_2\left(s; a_1, a_2; \lambda, \lambda'\right) = -\phi_2(0)\,\Gamma(s)\,\phi_1(s) + \rho_1\Gamma(r_1)\,\Gamma(s-r_1)\,\phi_2\left(s-r_1\right)\\ +2H_{r_1}\left(s; b_1, a_2; \mu, \lambda'\right)\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\Gamma(s)\,\mathscr{Z}_2\left(s; a_1, a_2; \lambda, \lambda'\right) = -\phi_1(0)\,\Gamma(s)\,\phi_2(s) + \rho_2\Gamma(r_2)\,\Gamma(s-r_2)\,\phi_1\left(s-r_2\right)\\ +\,2H_{r_2}(s; b_2, a_1; \mu', \lambda).\end{aligned} \tag{2.29}$$

Using both representations (2.28) and (2.29) as well as the type of continuation which they yield, we now establish the following Corollary.

Corollary 2.1. *Let ϕ_1 and ϕ_2 satisfy the conditions of the class $\mathscr{A}$ and having residues ρ_1 and ρ_2 at $s = r_1$ and $s = r_2$ respectively. Then the Selberg-Chowla formula (2.2) provides the analytic continuation of the diagonal Epstein $\zeta-$function (1.10) as a meromorphic function having a simple pole at $s = r_1 + r_2$ with residue $\frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1+r_2)}\,\rho_1\rho_2$. Moreover, the continuation of $\mathscr{Z}_2$ obeys to the functional equation*

$$\Gamma(s)\,\mathscr{Z}_2\left(s; a_1, a_2; \lambda, \lambda'\right) = \Gamma(r_1+r_2-s)\,\mathscr{Z}_2\left(r_1+r_2-s; b_1, b_2; \mu, \mu'\right). \tag{2.30}$$

Proof. First, we shall use the first representation for $\mathscr{Z}_2$ (2.2), which, under restriction to the class $\mathscr{A}$, reduces to (2.28). We have seen in the previous proposition that H_{r_1} is an entire function, so the meromorphic part of the Epstein $\zeta-$function $\mathscr{Z}_2(s; a_1, a_2; \lambda, \lambda')$ comes from the first two terms in (2.28), i.e., from the meromorphic function

$$G_{r_1}\left(s; a_1, a_2; \lambda, \lambda'\right) = -\phi_2(0)\,\phi_1(s) + \rho_1\Gamma(r_1)\,\frac{\Gamma(s-r_1)}{\Gamma(s)}\,\phi_2\left(s-r_1\right). \tag{2.31}$$

Clearly, from a standard verification, G_{r_1} has removable singularities at $s = r_1 - k$, $k \in \mathbb{N}_0$. Since, by the definition of the class $\mathscr{A}$, $\phi_1(s)$ and $\Gamma(s-r_1)$ are analytic in a neighbourhood of $s = r_1 + r_2$, G_{r_1} has at most one pole at $s = r_1 + r_2$ coming from the factor $\phi_2(s-r_1)$. The residue is easily seen to be $\frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1+r_2)}\,\rho_1\rho_2$.

To prove the functional equation (2.30), let us use both representations (2.28, 2.29) for $\mathscr{Z}_2$ and replace s by r_1+r_2-s on (2.28). From the reflection formula for H_{r_1}, we see that

$$H_{r_1}\left(r_1+r_2-s;\, b_1,\, a_2;\, \mu,\, \lambda'\right)=H_{r_2}\left(s;\, a_2,\, b_1;\, \lambda',\, \mu\right).$$

Thus, to describe $\mathscr{Z}_2(r_1+r_2-s;\,\cdot)$, we need to see how G_{r_1} behaves under the reflection $s\leftrightarrow r_1+r_2-s$. Since $\phi_i(0)=-\rho_i^{\star}\Gamma(r_i)$ and $\psi_i(0)=-\rho_i\Gamma(r_i)$ (by condition 2. on the class $\mathscr{A}$ and the functional equation for $\phi_i(s)$), we have that

$$\begin{aligned}\Gamma(r_1+r_2-s)\, G_{r_1}\left(r_1+r_2-s;\, a_1,\, a_2;\, \lambda,\, \lambda'\right) &= -\phi_2(0)\,\Gamma(s-r_2)\,\psi_1(s-r_2)+\rho_1\Gamma(r_1)\,\Gamma(s)\,\psi_2(s)\\ &= \rho_2^{\star}\,\Gamma(r_2)\,\Gamma(s-r_2)\,\psi_1(s-r_2)-\psi_1(0)\,\Gamma(s)\,\psi_2(s), \qquad (2.32)\end{aligned}$$

where in the penultimate equality we have used the functional equation for ϕ_i directly. Now, by (2.32) and the reflection formula for H_{r_i}, together with (2.28), we see that

$$\begin{aligned}\Gamma(r_1+r_2-s)\, Z_2\left(r_1+r_2-s,\, a_1,\, a_2,\, \lambda,\, \lambda'\right) &= -\psi_1(0)\,\Gamma(s)\,\psi_2(s)+\rho_2^{\star}\,\Gamma(r_2)\,\Gamma(s-r_2)\,\psi_1(s-r_2)+ \qquad (2.33)\\ &\quad +2H_{r_2}\left(s;\, a_2,\, b_1;\, \lambda',\, \mu\right).\end{aligned}$$

Using the second representation of the Selberg-Chowla formula (2.29), and replacing there a_1 by b_1 and a_2 by b_2 or, by other words, using the fact that $\psi_1,\,\psi_2\in\mathscr{A}$ and applying the work done in Theorem 2.1 for the diagonal Epstein $\zeta-$function,

$$\mathscr{Z}_2\left(s,\, b_1,\, b_2,\, \mu,\, \mu'\right)=\sum_{m,n\neq 0}^{\infty}\frac{b_1(m)\,b_2(n)}{\left(\mu_m+\mu_n'\right)^{s}},\quad \mathrm{Re}(s)>2\,\sigma_b, \qquad (2.34)$$

we find immediately that the right-hand side of (2.33) is precisely the same as the one given by (2.29) under the above mentioned substitutions. This establishes the functional equation (2.30) and completes the proof. □

Remark 2.3. Alternative proofs of Theorem 2.1 and Corollary 2.1 can be given by using Bochner's modular relation (1.22) to one of the Dirichlet series $\phi_1(s)$ or $\phi_2(s)$. Note that the left-hand side of (2.2) can be described, for $\mathrm{Re}(s)>\mu>\max\left\{0,\, 2\sigma_a,\, 2\sigma_b,\, \frac{r'}{2}\right\}$, as the Mellin transform

$$\begin{aligned}\Gamma(s)\,\mathscr{Z}_2\left(s;\, a_1,\, a_2;\, \lambda,\, \lambda'\right) &= \int_0^{\infty} x^{s-1}\sum_{m,n\neq 0}^{\infty} a_1(m)\,a_2(n)\,e^{-(\lambda_m+\lambda_n')\,x}\,dx\\ = a_1(0)\int_0^{\infty} x^{s-1}\Theta\left(x;\, a_2;\, \lambda'\right)dx &+ a_2(0)\int_0^{\infty} x^{s-1}\Theta\left(x;\, a_1;\, \lambda\right)dx+\int_0^{\infty} x^{s-1}\Theta\left(x;\, a_1;\, \lambda\right)\Theta\left(x;\, a_2;\, \lambda'\right)dx.\end{aligned} \qquad (2.35)$$

It is then evident that, if we invoke the reflection formula for the Theta function $\Theta(x;\, a_1;\, \lambda)$ (1.22) on the third integral above and then use the definition of the residual function $P_1(x)$ (1.21), as well as arguing by absolute convergence, the first two summands and the residual terms coming on the third integral give the first three terms appearing on the right-hand side of (2.2). Lastly, the remaining integral can be evaluated as follows

$$\int_0^\infty x^{s-r_1-1}\Theta\left(\frac{1}{x};b_1;\mu\right)\Theta\left(x;a_2;\lambda'\right)dx=\sum_{m,n=1}^\infty b_1(m)\,a_2(n)\int_0^\infty x^{s-r_1-1}\exp\left(-\lambda_n'x-\mu_m/x\right)dx$$
$$=2\sum_{m,n=1}^\infty b_1(m)\,a_2(n)\left(\frac{\mu_m}{\lambda_n'}\right)^{\frac{s-r_1}{2}}K_{r_1-s}\left(2\sqrt{\mu_m\lambda_n'}\right):=2H_{r_1}(s;b_1,a_2;\mu,\lambda'),$$

where we have used the representation of the Modified Bessel function $K_\nu(x)$ as the Mellin convolution of exponentials [[47], eq. 3.471.9, p. 368]

$$\int_0^\infty x^{s-1}e^{-\beta x}e^{-\gamma/x}dx=2\left(\frac{\gamma}{\beta}\right)^{\frac{s}{2}}K_s\left(2\sqrt{\beta\gamma}\right),\ \text{Re}(\beta),\text{Re}(\gamma)>0. \tag{2.36}$$

The proof of the functional equation (2.30) for $\phi_1,\phi_2\in\mathscr{A}$ can be given by invoking the reflection formula for both theta functions $\Theta(x;a_1;\lambda)$ and $\Theta(x;a_2;\lambda')$. If we denote the theta function of the Dirichlet series $\mathscr{Z}_2(s;\cdot)$ by $\Theta_2(x;a_1,a_2;\lambda,\lambda')$ we can derive a particular case of the modular relation (1.22),

$$\Theta_2(x;a_1,a_2;\lambda,\lambda')=x^{-r_1-r_2}\Theta_2\left(x^{-1};b_1,b_2;\mu,\mu'\right)+\Gamma(r_1)\Gamma(r_2)\rho_1\rho_2x^{-r_1-r_2}$$
$$-\Gamma(r_1)\Gamma(r_2)\rho_1^\star\rho_2^\star,\quad x>0, \tag{2.37}$$

which is equivalent to (2.30) (see also [[10], p. 342, Thm. 8.1] for an analogous argument with a single series). In Theorem 3.1 we shall prove the Selberg-Chowla for the non-diagonal Epstein zeta function (1.14) using this observation. Note also that Suzuki's proof [86] of the analogues of the Selberg-Chowla formula for $\zeta(s)$ and other Dirichlet series such as $L(s,\chi)$ and $L(s,a)$ (see example 5.6) follows the lines given in this remark.

We now introduce a class of multidimensional Epstein zeta functions and prove functional equations for them by mimicking the argument given in Corollary 2.1. If $\{\phi_i\}_{i=1,...,k}$ denotes a k-tuple of Dirichlet series satisfying Hecke's functional equation with abcissas of absolute convergence σ_{a_i} and Hecke's parameter r_i, representable by the series

$$\phi_i(s)=\sum_{n=1}^\infty\frac{a_i(n)}{\lambda_{n,i}^s},\quad\text{Re}(s)>\sigma_{a_i},$$

then we can define the multidimensional Epstein zeta function as

$$\mathscr{Z}_k(s;a_1,...,a_k;\lambda_1,...,\lambda_k):=\sum_{n_1,...,n_k\neq0}^\infty\frac{a_1(n_1)\cdot...\cdot a_k(n_k)}{(\lambda_{n_1,1}+...+\lambda_{n_k,k})^s},\ \ \text{Re}(s)>k\sigma_a, \tag{2.38}$$

where $\sigma_a=\max_{1\leq i\leq k}\sigma_{a_i}$. As in representation (1.12), we can write $\mathscr{Z}_k$ as the Dirichlet series

$$\mathscr{Z}_k(s;a_1,...,a_k;\lambda_1,...,\lambda_k)=\sum_{n=1}^\infty\frac{\mathfrak{U}_k(n)}{\Lambda_n^s},\ \ \text{Re}(s)>k\sigma_a \tag{2.39}$$

where, in analogy with $\mathfrak{U}_2(n)$ given in (1.13),

$$\mathfrak{U}_k(n) = \sum_{\lambda_{j_1,1}+\ldots+\lambda_{j_k,k}=\Lambda_n} a_1(j_1)\cdot\ldots\cdot a_k(j_k).$$

Suppose also that $\{\psi_i\}_{i=1,\ldots,k}$ denotes the k-tuple of Dirichlet series conjugate to each ϕ_i via Hecke's functional equation (1.7) and being representable by the series

$$\psi_i(s) = \sum_{n=1}^{\infty} \frac{b_i(n)}{\mu_{n,i}^s}, \quad \text{Re}(s) > \sigma_{b_i},$$

then we can write $\mathscr{Z}_k(s; b_1,\ldots, b_k; \mu_1,\ldots, \mu_k)$ also as a single Dirichlet series of the form

$$\mathscr{Z}_k(s; b_1,\ldots, b_k; \mu_1,\ldots, \mu_k) = \sum_{n=1}^{\infty} \frac{\mathfrak{V}_k(n)}{\Omega_{n,k}^s}, \;\; \text{Re}(s) > k\sigma_b, \tag{2.40}$$

where $\mathfrak{V}_k(n)$ plays the same role as $\mathfrak{U}_k(n)$ and $\sigma_b = \max_{1\leq i\leq k} \sigma_{b_i}$. In analogy with (2.30) one should expect a functional equation connecting (2.39) and (2.40). In fact, this is the case and the next corollary establishes it.

Corollary 2.2 (The Analytic Continuation of the Multidimensional diagonal Epstein zeta function)**.** *Assume that $\{\phi_i\}_{i=1,\ldots,k}$ are Dirichlet series satisfying Hecke's functional equation with parameter $r_i > 0$ and belonging to the class $\mathscr{A}$. Then (2.38) has an analytic continuation as a Dirichlet series belonging to the class $\mathscr{A}$. This continuation has at most a simple pole located at $s = \sum_{j=1}^k r_j$ and with residue*

$$Res_{s=\sum_{j=1}^k r_j} \mathscr{Z}_k(s; a_1,\ldots, a_k; \lambda_1,\ldots, \lambda_k) = \frac{\prod_{j=1}^k \Gamma(r_j)\,\rho_j}{\Gamma\left(\sum_{j=1}^k r_j\right)}. \tag{2.41}$$

Moreover, $\mathscr{Z}_k$ obeys to the functional equation

$$\Gamma(s)\,\mathscr{Z}_k(s; a_1,\ldots, a_k; \lambda_1,\ldots, \lambda_k) = \Gamma(r_1+\ldots+r_k-s)\,\mathscr{Z}_k(r_1+\ldots+r_k-s; b_1,\ldots, b_k; \mu_1,\ldots,\mu_k). \tag{2.42}$$

Proof. We have seen in Corollary 2.1 that (2.41) and (2.42) hold for $k = 2$. By induction, assume now that $\mathscr{Z}_{k-1}(s; a_1,\ldots, a_{k-1}; \lambda_1,\ldots, \lambda_{k-1})$ belongs to the class $\mathscr{A}$ and satisfies Hecke's functional equation with parameter $r = \sum_{j=1}^{k-1} r_j$ and has a residue at $s = \sum_{j=1}^{k-1} r_j$ given by

$$\text{Res}_{s=\sum_{j=1}^{k-1} r_j} \mathscr{Z}_{k-1} = \frac{\prod_{j=1}^{k-1} \Gamma(r_j)\,\rho_j}{\Gamma\left(\sum_{j=1}^{k-1} r_j\right)}.$$

By the first Selberg-Chowla formula (2.28) applied to $\mathscr{Z}_{k-1}$ and $\phi_k(s)$ (both satisying Hecke's functional equation (1.7) by the induction hypothesis) and using the representation (2.40), we see that $\mathscr{Z}_k$ can be written as

$$-\phi_k(0)\,\mathscr{Z}_{k-1}(s; a_1,\ldots, a_{k-1}; \lambda_1,\ldots, \lambda_{k-1}) + \prod_{j=1}^{k-1}\Gamma(r_j)\,\rho_j\,\frac{\Gamma(s-r_1-\ldots-r_{k-1})}{\Gamma(s)}\,\phi_k(s-r_1-\ldots-r_{k-1})$$
$$+\frac{2}{\Gamma(s)}\sum_{m,n=1}^{\infty} \mathfrak{V}_{k-1}(m)\,a_k(n)\left(\frac{\Omega_{m,k-1}}{\lambda_n}\right)^{\frac{s-r_1-\ldots-r_{k-1}}{2}} K_{r_1+\ldots+r_{k-1}-s}\left(2\sqrt{\Omega_{m,k-1}\lambda_n}\right). \tag{2.43}$$

It is clear from a simple adaptation of Proposition 2.1 that the last term in (2.43) represents an entire function. Hence, the possible poles of (2.38) come from the sum of the first two terms. It is clear once more that $\mathscr{Z}_k$ has removable singularities at $s=r_1+...+r_{k-1}-n$, $n\in\mathbb{N}_0$. This comes from the functional equation for ϕ_k and the induction hypothesis on $\mathscr{Z}_{k-1}$. Thus, the only possible pole of $\mathscr{Z}_k$ comes from the factor $\phi_k\left(s-r_1-...-r_{k-1}\right)$ on the second term of the previous expression and it is located at $s=r_1+...+r_k$. The residue immediately gives (2.41).

For proving the functional equation, it suffices to compare the representation (2.43) with the second representation coming from (2.29)

$$-\mathscr{Z}_{k-1}(0;a_1,...,a_{k-1};\lambda_1,...,\lambda_{k-1})\,\phi_k(s)+\rho_k\Gamma(r_k)\,\frac{\Gamma(s-r_k)}{\Gamma(s)}\times\mathscr{Z}_{k-1}\left(s-r_k;a_1,...,a_{k-1};\lambda_1,...,\lambda_{k-1}\right)$$
$$+\frac{2}{\Gamma(s)}\sum_{m,n=1}^{\infty}b_k(m)\,\mathfrak{U}_{k-1}(n)\left(\frac{\mu_m}{\Lambda_{n,k-1}}\right)^{\frac{s-r_k}{2}}K_{r_k-s}\left(2\sqrt{\mu_m\Lambda_{n,k-1}}\right)$$

and use the induction hypothesis. By adapting the argument given in the proof of Corollary 2.1 and replacing the roles of ϕ_1 and ϕ_2 there by ϕ_k and $\mathscr{Z}_{k-1}$, we arrive immediately at (2.42). □

Remark 2.4 (The dyadic Epstein zeta function)**.** For the case where $\phi_1=\phi_2=...=\phi_{2^k}:=\phi\in\mathscr{A}$, with $\phi(s)$ satisfying Hecke's functional equation with parameter r, one of the $2^{k+1}-1$ ways of constructing the Epstein zeta function $\mathscr{Z}_{2^{k+1}}$ is by taking the symmetric construction

$$\mathscr{Z}_{2^k}\left(s;a;\lambda\right)=\sum_{m=1}^{\infty}\frac{\mathfrak{U}_{2^k}(m)}{\Lambda_m^s},\quad\mathscr{Z}_{2^{k+1}}\left(s;a;\lambda\right)=\sum_{m,n\neq0}^{\infty}\frac{\mathfrak{U}_{2^k}(m)\,\mathfrak{U}_{2^k}(n)}{\left(\Lambda_m+\Lambda_n\right)^s}. \tag{2.44}$$

Thus, each $\mathscr{Z}_{2^k}(s;a;\lambda)$ plays the role of ϕ_1 and ϕ_2 in Theorem 2.1 and Corollary 2.1. By an application of Corollary 2.2, it is clear that we can write a Selberg-Chowla formula in the form

$$\Gamma(s)\,\mathscr{Z}_{2^{k+1}}\left(s;a;\lambda\right)=-\mathscr{Z}_{2^k}\left(0;a;\lambda\right)\Gamma(s)\,\mathscr{Z}_{2^k}\left(s;a;\lambda\right)+\Gamma^{2^k}(r)\,\rho^{2^k}\,\Gamma(s-2^kr)\,\mathscr{Z}_{2^k}\left(s-2^kr;a;\lambda\right)$$
$$+2H_{2^kr}(s;a;\lambda), \tag{2.45}$$

where $H_{2^kr}(s;a;\lambda)$ denotes the entire function

$$H_{2^kr}(s;a;\lambda)=\sum_{m,n=1}^{\infty}\mathfrak{V}_{2^k}(m)\,\mathfrak{U}_{2^k}(n)\left(\frac{\Omega_m}{\Lambda_n}\right)^{\frac{s}{2}-2^{k-1}r}K_{2^kr-s}\left(2\sqrt{\Omega_m\Lambda_n}\right), \tag{2.46}$$

with the arithmetical functions $\mathfrak{U}_{2^k}(n)$ and $\mathfrak{V}_{2^k}(n)$ appearing respectively in the Dirichlet series representations (2.39) and (2.40). Formulas of the type (2.45) will be used in the penultimate section of the paper, where we shall prove a class of Hardy Theorems for Dirichlet series with narrow critical strips (see Lemma 4.1 and Theorem 4.1). The multidimensional Epstein zeta function $\mathscr{Z}_{2^k}\left(s;a;\lambda\right)$ satisfies Hecke's functional equation (2.42) with parameter 2^kr.

Remark 2.5. By the equivalence between Bochner's modular relation (1.22) and Hecke's functional equation (1.7), we have that, under the restrictions of the class $\mathscr{A}$, (2.42) is equivalent to the modular relation

$$\Theta_k(z; a_1,...,a_k; \lambda_1,...,\lambda_k) = z^{-r_1-...-r_k}\,\Theta_k(z^{-1}; b_1,...,b_k; \mu_1,...,\mu_k) + \prod_{j=1}^{k}\Gamma(r_j)\,\rho_j\,z^{-r_1-...-r_k} - \prod_{j=1}^{k}\Gamma(r_j)\,\rho_j^{\star},\ \ \mathrm{Re}(z)>0, \tag{2.47}$$

where Θ_k denotes the multi-dimensional theta function

$$\Theta_k(z; a_1,...,a_k; \lambda_1,...,\lambda_k) := \sum_{m_1,...,m_k\neq 0}^{\infty} a_1(m_1)\cdot ...\cdot a_k(m_k)\,\exp\left\{-\sum_{i=1}^{k}\lambda_{m_i,i}\,z\right\}.$$

3 The analytic continuation of a Non-Diagonal Epstein Zeta function

Although the previous section furnished an analogue of the Epstein $\zeta-$function, the double Dirichlet series there presented is only useful when our study is reduced to diagonal quadratic forms.

In order to have a proper environment to develop Selberg-Chowla formulas for Epstein zeta functions attached to non-diagonal quadratic forms, we need to replicate in a certain way the conditions obeyed by the Riemann $\zeta-$function, which is behind the analytic continuation of the classical Epstein zeta function $Z_2(s, Q)$. Doing so requires to introduce the following subclass $\mathscr{B}$ of Bochner Dirichlet series, which has a similar role as the class $\mathscr{A}$ had in the previous section.

Definition 3.1 (A Subclass of Bochner Dirichlet series)**.** Let $\phi(s)$ be a Dirichlet series satisfying Definition 1.1 with $\Delta(s)=\Gamma\left(\frac{s+\delta}{2}\right)$, $\delta\in\{0,1\}$, $r=1$ (Bochner class). We say that ϕ belongs to the class $\mathscr{B}$ if

- For $\delta=0$, ϕ and ψ have analytic continuations to the entire complex plane and are analytic everywhere in $\mathbb{C}$ except for possible simple poles located at $s=1$ with residues ρ and $\rho^{\star}$ respectively. For $\delta=1$, ϕ and ψ can be analytically continued as entire Dirichlet series.

- $a(n)$ can be defined at 0 as $a(0):=-2\phi(-\delta)$. Moreover, $a(n)$ can be defined for negative values of n as $a(-n)=(-1)^{\delta}\,a(n)$ and the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ can be extended to $n\in\mathbb{Z}$ with the values $\lambda_0=\mu_0=0$ and $\lambda_{-n}=-\lambda_n$, $\mu_{-n}=-\mu_n$. A similar extension holds for $b(n)$ with $b(0):=-2\psi(-\delta)$ and $b(-n)$ analogously taken.

Remark 3.1. Note that definition 3.1 mimics in some way the properties that the Dirichlet $L-$functions and the Riemann $\zeta-$function have. By the functional equation for Bochner Dirichlet series (1.8), it is effortless to see that $\phi(0)=-\frac{\rho^{\star}}{2}\sqrt{\pi}$, while $\psi(0)=-\frac{\rho}{2}\sqrt{\pi}$. Moreover, if $\phi(s)$ is entire, then it comes from the functional equation and the second item that $a(0)=0$. We remark that, in analogy with Theorem 2.1, it is possible to prove a general Selberg-Chowla for $\mathscr{Z}_2(s; Q; a_1, a_2; \lambda, \lambda')$ with the general setting of Definition

1.1. However, in order to simplify the foregoing argument, we shall prove a Selberg-Chowla formula for (1.14) assuming apriori the conditions of the previous definition.

Since most proofs of the classical Selberg-Chowla formula when $\lambda_n = \sqrt{\pi}n$ employ Fourier Analysis (in particular, the classical Poisson summation formula), we should emphasize that the standard technique works so nicely in the classical case because of the equivalence between Poisson's summation formula and the functional equation for $\zeta(s)$. A generalization of some of the classical arguments to prove (1.2) would require a generalized version of Poisson's formula, with the standard sequence $\lambda_n = \sqrt{\pi}n$ being replaced by an arbitrary one λ_n. Although this is possible in the Bochner class, we shall omit such approach, preferring to give a more direct one. As remarked on the introduction to this paper, the Selberg-Chowla formula had its genesis in Watson's formula [95] so one should expect that its generalized version should also have a similar motivation (see Example 5.1).

To prove a Selberg-Chowla formula for the non-diagonal Epstein zeta function $\mathscr{Z}_2(s;Q;a_1,a_2;\lambda,\lambda')$, we use a generalization of Bochner's modular relation (1.22) for the class $\mathscr{B}$. In the well-known case $\phi_1(s) = \phi_2(s) = \pi^{-\frac{s}{2}}\zeta(s)$ the formula given below reduces to Jacobi's reflection formula. We remark that, when the non-diagonal Epstein zeta function reduces to (1.1), our method of proof of the generalized Selberg-Chowla formula for $\mathscr{Z}_2(s;Q;a_1,a_2;\lambda,\lambda')$ reduces to the one presented on the second section of Selberg and Chowla's paper [81].

Lemma 3.1 (A generalization of the Theta reflection formula for the Class $\mathscr{B}$). *Let $\phi(s)$ be a Dirichlet series belonging to the class $\mathscr{B}$. If $a(n)$ is an even arithmetical function (i.e., $\delta = 0$), then, for any $x > 0$, $Re(\alpha) > 0$ and $\beta \in \mathbb{C}$, the following identity holds*

$$\sum_{n=1}^{\infty} a(n)\,e^{-\alpha\lambda_n^2x^2}\cos(\beta\lambda_n x) = \phi(0) + \sqrt{\frac{\pi}{\alpha}}\frac{\rho}{2x}\,e^{-\frac{\beta^2}{4\alpha}} + \frac{1}{2\sqrt{\alpha}\,x}\sum_{n\neq 0} b(n)\,e^{-\frac{1}{\alpha}\left(\frac{\mu_n}{x}+\frac{\beta}{2}\right)^2}, \tag{3.1}$$

which generalizes Bochner's functional equation (1.22) attached to the case where $r = \frac{1}{2}$. If, by other hand, $a(n)$ is an odd arithmetical function, we have the identity

$$\sum_{n=1}^{\infty} a(n)\,e^{-\alpha\lambda_n^2x^2}\sin(\beta\lambda_n x) = \frac{1}{2\sqrt{\alpha}x}\sum_{n\neq 0} b(n)\,e^{-\frac{1}{\alpha}\left(\frac{\mu_n}{x}+\frac{\beta}{2}\right)^2}. \tag{3.2}$$

Proof. As in the proof of Theorem 2.1, we start by choosing a suitable parameter μ which allows absolute convergence in every side of the equality to be proven. In our case, assume $\mu > \max\{\sigma_a, \sigma_b, 1\}$ and use the observation that, for $\text{Re}(\alpha) > 0$, $\sigma = \text{Re}(z) > 0$ and $\beta \in \mathbb{C}$, the following Mellin inverse representation holds [[47], p. 503, eq. 3.952.8]

$$e^{-\alpha x^2}\cos(\beta x) = \frac{e^{-\frac{\beta^2}{4\alpha}}}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\alpha^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{\beta^2}{4\alpha}\right)x^{-z}dz, \tag{3.3}$$

which can be proved by invoking (1.24) together with Kummer's transformation (1.25). Although the key of this comparison is Mellin's inversion formula, (3.3) could be alternatively proved by invoking the absolutely convergent power series for ${}_1F_1$ and then interchanging the orders of summation and integration.

From the fact that the integrand in (3.3) is the Mellin transform of a smooth function and decays faster than any polynomial as $|s| \to \infty$, we can write the left-hand side of (3.1) as the contour integral

$$\sum_{n=1}^{\infty} a(n)\, e^{-\alpha\lambda_n^2 x^2} \cos\left(\beta\lambda_n x\right) = \frac{e^{-\frac{\beta^2}{4\alpha}}}{4\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Gamma\left(\frac{z}{2}\right)\phi(z)\, {}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{\beta^2}{4\alpha}\right)(\sqrt{\alpha}x)^{-z}\, dz. \tag{3.4}$$

Proceeding once more as in Theorem 2.1, let us now move the line of integration on (3.4) to $\mathrm{Re}(z) = 1-\mu < 0$ and integrate along a positively oriented rectangular contour $\mathscr{R}$ containing the vertices $\mu \pm iT$ and $1-\mu \pm iT$, $T > 0$. Once again, we are allowed to invoke the Phragmén-Lindelöf principle for $\phi(z)$ (1.19), together with the decay properties of the integrand on the right-hand side of (3.3) in order to conclude that the integrals along the horizontal segments $[1-\mu-iT, \mu-iT]$ and $[\mu+iT, 1-\mu+iT]$ must vanish when $T \to \infty$.

By the residue Theorem, we obtain the equality

$$\sum_{n=1}^{\infty} a(n)\, e^{-\alpha\lambda_n^2 x^2} \cos(\beta\lambda_n x) = \frac{e^{-\frac{\beta^2}{4\alpha}}}{4\pi i} \int_{1-\mu-i\infty}^{1-\mu+i\infty} \Gamma\left(\frac{z}{2}\right)\phi(z)\, {}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{\beta^2}{4\alpha}\right)\left(\sqrt{\alpha}x\right)^{-z} dz$$
$$+\frac{e^{-\frac{\beta^2}{4\alpha}}}{2}\sum_{\rho\in\mathscr{R}} \mathrm{Res}_{z=\rho}\left\{\Gamma\left(\frac{z}{2}\right)\phi(z)\, {}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{\beta^2}{4\alpha}\right)(\sqrt{\alpha}x)^{-z}\, dz\right\}. \tag{3.5}$$

By the conditions of the class $\mathscr{B}$, it is evident that $\phi(z)$ has simple zeros located at the negative even integers if $a(n) = a(-n)$. Also, from the fact that ${}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{\beta^2}{4\alpha}\right)$ is an entire function of z, we see that all the poles that we need to take into consideration on the shift of the line of integration are located at $z = 0$ and $z = 1$. It is immediate to see that these residues are explicitly given by

$$\mathrm{Res}_{z=0} = 2\phi(0)\, e^{\frac{\beta^2}{4\alpha}}, \quad \mathrm{Res}_{z=1} = \sqrt{\frac{\pi}{\alpha}}\frac{\rho}{x}.$$

Now, invoking the functional equation for $\phi(z)$ (1.8) with $\delta = 0$ and performing the change of variables $z \leftrightarrow 1-z$, we see that the first term on the right-hand side of (3.5) is equal to

$$\frac{e^{-\frac{\beta^2}{4\alpha}}}{2\sqrt{\alpha}x}\sum_{n=1}^{\infty}\frac{b(n)}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(\frac{z}{2}\right){}_1F_1\left(\frac{z}{2};\frac{1}{2};\frac{\beta^2}{4\alpha}\right)\left(\frac{\mu_n}{\sqrt{\alpha}x}\right)^{-z} dz+\phi(0)+\sqrt{\frac{\pi}{\alpha}}\frac{\rho}{2x}e^{-\frac{\beta^2}{4\alpha}}. \tag{3.6}$$

The contour integral inside the infinite series can be evaluated via (3.3) upon a substitution of β by $i\beta$ and the use of Kummer's reflection formula (1.25). Nevertheless, for completeness, we may present a closed evaluation. Note that the power series defining the confluent Hypergeometric function is absolutely convergent for arbitrary $\beta \in \mathbb{C}$ and α such that $\mathrm{Re}(\alpha) > 0$. Moreover, a simple application of Stirling's formula

(1.16) allows to reverse the orders of summation and integration once we employ this series representation and we get

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(\frac{z}{2}\right)\,{}_1F_1\left(\frac{z}{2};\frac{1}{2};\frac{\beta^2}{4\alpha}\right)y^{-z}dz=\sum_{k=0}^{\infty}\frac{\sqrt{\pi}}{k!\,\left(\frac{1}{2}\right)_k}\left(\frac{\beta^2}{4\alpha}\right)^k\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(\frac{z}{2}+k\right)y^{-z}dz$$
$$=\sum_{k=0}^{\infty}\frac{\sqrt{\pi}}{k!\,\left(\frac{1}{2}\right)_k}\left(\frac{\beta^2y^2}{4\alpha}\right)^k\int_{\mu+2k-i\infty}^{\mu+2k+i\infty}\Gamma\left(\frac{z}{2}\right)y^{-z}dz=2\,e^{-y^2}\sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\frac{\beta y}{\sqrt{\alpha}}\right)^{2k}=2\,e^{-y^2}\cosh\left(\frac{\beta y}{\sqrt{\alpha}}\right). \quad (3.7)$$

Combining (3.7), (3.6) and (3.5) we obtain the generalization of the reflection formula for Jacobi's $\theta-$function,

$$\sum_{n=1}^{\infty}a(n)\,e^{-\alpha\lambda_n^2x^2}\cos\left(\beta\lambda_nx\right)=\phi(0)+\sqrt{\frac{\pi}{\alpha}}\frac{\rho}{2x}\,e^{-\frac{\beta^2}{4\alpha}}+\frac{e^{-\frac{\beta^2}{4\alpha}}}{\sqrt{\alpha}x}\sum_{n=1}^{\infty}b(n)\,e^{-\frac{\mu_n^2}{\alpha x^2}}\cosh\left(\frac{\beta\mu_n}{\alpha x}\right). \quad (3.8)$$

Since $b(n)$ is even by hypothesis and we have assumed that $\mu_{-n}=-\mu_n$, we can rewrite the last series in a symmetric form by summing over the non-zero integers. This gives the desired reflection formula (3.1).

For the odd case the proof is even simpler: proceed as above and use an integral representation similar to (3.3), i.e., [[47], p. 503, eq. 3.952.7]

$$e^{-\alpha x^2}\sin(\beta x)=\frac{\beta e^{-\frac{\beta^2}{4\alpha}}}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\alpha^{-\frac{s+1}{2}}\Gamma\left(\frac{1+s}{2}\right)\,{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{\beta^2}{4\alpha}\right)x^{-s}ds, \quad (3.9)$$

valid for $\text{Re}(\alpha)>0$ and $\sigma>-1$. Once more, (3.9) can be easily proved by appealing to Kummer's reflection formula and to the absolute convergence of the power series for ${}_1F_1$. By the hypothesis over the class $\mathscr{B}$ for odd arithmetical functions, the resulting integrand will be an entire function and so an application of the previous argument will not take into account residual terms. □

Remark 3.2. The previous lemma is connected with N. J. Fine's proof of the functional equation for the classical Hurwitz zeta function $\zeta(s,\alpha)$ [[41], p. 361, eq. (3)]. As done by Berndt for the case of Dirichlet series satisfying Hecke's functional equation [7], we could define the generalized Hurwitz zeta function[5] in the following way

$$\phi(s,\alpha)=\sum_{n=1}^{\infty}\frac{a(n)}{(\lambda_n+\alpha)^s},\quad \text{Re}(s)>\sigma_a,\ \alpha>0 \quad (3.10)$$

and derive the analytic continuation and functional equation for it by using the previous lemma. This observation connects our forthcoming proof of the Selberg-Chowla formula with the proofs by Berndt and Kuzumaki, which naturally employ the functional equation for $\zeta(s,\alpha)$ (see [[13], p. 160] and [[65], eq. (16), (24) and (25)]. There are other ways of studying the analytic continuation of (3.10) and one may even adapt the Bessel expansions proved by Berndt in [7] and [[11], p. 180, eq. (3.1)]. This would require to

[5]The Dirichlet series appearing in (3.10) is not a generalized/perturbed Dirichlet series in the sense of [[7], p.309] since it is not constructed from a sequence $(\lambda_n)_{n\in\mathbb{N}}$ attached to a Dirichlet series satisfying Hecke's functional equation.

evaluate a contour integral which can be expressed in terms of the generalized hypergeometric function ${}_1F_2$ and trigonometric functions. The resulting computations will not differ too much from those given in [34], which are derived from Hermite's integral representation for $\zeta(s,\alpha)$. Since Hermite's representation for $\zeta(s,\alpha)$ is usually obtained via the Abel-Plana summation formula [[97], p. 145, example 7], it would be also interesting to establish general analogues of the latter formula.

Theorem 3.1 (Selberg-Chowla for positive definite binary quadratic forms)**.** *Let ϕ_1 and ϕ_2 be two Dirichlet series belonging to the class $\mathscr{B}$ and s be a complex number such that $Re(s)>\mu>\max\{\sigma_a,\sigma_b\}$. If a_1, a_2 are even arithmetical functions, then we have the following Selberg-Chowla formula*

$$a^s\Gamma(s)\,\mathscr{Z}_2\left(s;Q;a_1,a_2;\lambda,\lambda'\right)=-4\phi_2(0)\Gamma(s)\phi_1(2s)+2\sqrt{\pi}\rho_1\Gamma\left(s-\frac{1}{2}\right)k^{1-2s}\phi_2(2s-1)$$
$$+4k^{\frac{1}{2}-s}\sum_{j=1}^{\infty}\sigma_{2s-1}\left(\nu_j;b_1,a_2;\mu,\lambda'\right)\cos\left(\frac{b\nu_j}{a}\right)\nu_j^{\frac{1}{2}-s}K_{s-\frac{1}{2}}\left(2k\nu_j\right), \tag{3.11}$$

where $(\nu_j)_{j\in\mathbb{N}}=\{\mu_m\lambda_n'\}_{m,n=1}^{\infty}$ is the product sequence arranged in increasing order and σ_ν denotes the generalized weighted divisor function of power ν (see Definition 2.1). Moreover, $d:=b^2-4ac$ is the discriminant of the quadratic form and $k^2:=|d|/4a^2$.

Analogously, if a_1 and a_2 are odd, we have the Selberg-Chowla formula

$$a^s\Gamma(s)\,\mathscr{Z}_2\left(s;Q;a_1,a_2;\lambda,\lambda'\right)=-4k^{\frac{1}{2}-s}\sum_{j=1}^{\infty}\sigma_{2s-1}\left(\nu_j;b_1,a_2;\mu,\lambda'\right)\sin\left(\frac{b\nu_j}{a}\right)\nu_j^{\frac{1}{2}-s}K_{s-\frac{1}{2}}\left(2k\nu_j\right). \tag{3.12}$$

Under the same assumptions, $\mathscr{Z}_2(s;Q;a_1,a_2;\lambda,\lambda')$ can be described by the second Selberg-Chowla formulas,

$$c^s\Gamma(s)\,\mathscr{Z}_2\left(s;Q;a_1,a_2;\lambda,\lambda'\right)=-4\phi_1(0)\Gamma(s)\phi_2(2s)+2\sqrt{\pi}\rho_2\Gamma\left(s-\frac{1}{2}\right)k'^{1-2s}\phi_1(2s-1)$$
$$+4k'^{\frac{1}{2}-s}\sum_{j=1}^{\infty}\sigma_{2s-1}\left(\nu_j';b_2,a_1;\mu',\lambda\right)\cos\left(\frac{b\nu_j'}{c}\right)\nu_j'^{\frac{1}{2}-s}K_{s-\frac{1}{2}}\left(2k'\nu_j'\right), \tag{3.13}$$

for a_1, a_2 even and

$$c^s\Gamma(s)\,\mathscr{Z}_2\left(s;Q;a_1,a_2;\lambda,\lambda'\right)=-4k'^{\frac{1}{2}-s}\sum_{j=1}^{\infty}\sigma_{2s-1}\left(\nu_j';b_2,a_1;\mu',\lambda\right)\sin\left(\frac{b\nu_j'}{c}\right)\nu_j'^{\frac{1}{2}-s}K_{s-\frac{1}{2}}\left(2k'\nu_j'\right), \tag{3.14}$$

for a_1, a_2 odd. Here $k'^2:=|d|/4c^2$ and the sequence ν_j' is analogously defined by $\nu_j'=\mu_m'\lambda_n$.

Proof. Throughout this proof we shall assume that $a_i(n)$, $i=1,2$, are even arithmetical functions, as analogous computations can be given for the odd case. As in the proof of Theorem 2.1, we start the argument by writing the double series (1.14) as

$$\mathscr{Z}_2\left(s;Q;a_1,a_2;\lambda,\lambda'\right)=2a_2(0)a^{-s}\phi_1(2s)+2a_1(0)c^{-s}\phi_2(2s)+\sum_{m\neq0,n\neq0}\frac{a_1(m)\,a_2(n)}{(a\lambda_m^2+b\lambda_m\lambda_n'+c\lambda_n'^2)^s}.$$

Also, we choose a parameter μ satisfying $\mu > \max\{\sigma_a, \sigma_b\}$ and a fixed $s\in\mathbb{C}$ such that $\text{Re}(s)>\mu$. Of course, in order to compute $\mathscr{Z}_2$, we need to evaluate the double series

$$\sum_{n\neq 0} a_2(n)\sum_{m\neq 0}\frac{a_1(m)}{(a\lambda_m^2+b\lambda_m\lambda_n'+c\lambda_n'^2)^s}.$$

In analogy to the generalized Dirichlet series appearing in the proof of Theorem 2.1, for a fixed n let us denote by $\ell_n(s,Q,a_1)$ the inner infinite series with respect to m appearing above. For $\text{Re}(s)>\mu$, $\ell_n(s,Q,a_1)$ is explicitly given by

$$\begin{aligned}\ell_n(s,Q,a_1) &= \sum_{m\neq 0}\frac{a_1(m)}{(a\lambda_m^2+b\lambda_m\lambda_n'+c\lambda_n'^2)^s}=\frac{1}{\Gamma(s)}\sum_{m\neq 0}a_1(m)\int_0^\infty y^{s-1}e^{-(a\lambda_m^2+b\lambda_m\lambda_n'+c\lambda_n'^2)y}dy\\ &=\frac{1}{\Gamma(s)}\int_0^\infty y^{s-1}e^{-\frac{|d|}{4a}\lambda_n'^2y}\sum_{m\neq 0}a_1(m)\,e^{-ay\left(\lambda_m+\frac{b}{2a}\lambda_n'\right)^2}dy\end{aligned}$$

where the last step can be justified by the fact that $\text{Re}(s)>\mu>\sigma_a$ and by absolute convergence. Invoking the reflection formula (3.1) with the roles of ϕ and ψ reversed and taking the substitutions $\alpha=\frac{1}{ay}>0$, $\beta=\frac{b}{a}\lambda_n'$ and $x=1$, we obtain that

$$\sum_{m\neq 0}a_1(m)\,e^{-ay\left(\lambda_m+\frac{b}{2a}\lambda_n'\right)^2}=\frac{2}{\sqrt{ay}}\left\{\sum_{m=1}^\infty b_1(m)\,e^{-\frac{\mu_m^2}{ay}}\cos\left(\frac{b}{a}\mu_m\lambda_n'\right)-\psi_1(0)-\frac{\rho_1^\star\sqrt{\pi ay}}{2}\,e^{-\frac{b^2}{4a}\lambda_n'^2y}\right\}$$

and so $\ell_n(s,Q,a_1)$ can be given by

$$\begin{aligned}&\frac{2a^{-\frac12}}{\Gamma(s)}\int_0^\infty y^{s-\frac32}e^{-\frac{|d|}{4a}\lambda_n'^2y}\left\{\sum_{m=1}^\infty b_1(m)\,e^{-\frac{\mu_m^2}{ay}}\cos\left(\frac{b}{a}\mu_m\lambda_n'\right)-\psi_1(0)-\frac{\rho_1^\star\sqrt{\pi ay}}{2}\,e^{-\frac{b^2}{4a}\lambda_n'^2y}\right\}dy\\ &=\frac{2\phi_1(0)}{\lambda_n'^{2s}}c^{-s}-\frac{2a^{-\frac12}\,\Gamma\left(s-\frac12\right)\psi_1(0)}{\Gamma(s)\,\lambda_n'^{2s-1}}\left(\frac{|d|}{4a}\right)^{\frac12-s}+\frac{2\,a^{-\frac12}}{\Gamma(s)}\sum_{m=1}^\infty b_1(m)\cos\left(\frac{b}{a}\mu_m\lambda_n'\right)\int_0^\infty y^{s-\frac32}e^{-\frac{\mu_m^2}{ay}}e^{-\frac{|d|}{4a}\lambda_n'^2y}dy\\ &=\frac{2\phi_1(0)}{\lambda_n'^{2s}}c^{-s}+\rho_1\sqrt{\pi}\,\frac{a^{-\frac12}\,\Gamma\left(s-\frac12\right)}{\Gamma(s)\,\lambda_n'^{2s-1}}\left(\frac{|d|}{4a}\right)^{\frac12-s}+\frac{4\,(ka)^{\frac12-s}}{\Gamma(s)\sqrt{a}}\sum_{m=1}^\infty b_1(m)\cos\left(\frac{b}{a}\mu_m\lambda_n'\right)\left(\frac{\mu_m}{\lambda_n'}\right)^{s-\frac12}K_{s-\frac12}\left(2k\mu_m\lambda_n'\right)\end{aligned}$$

with the second equality being justified again by absolute convergence. We have also used the fact that $\psi_1(0)=-\frac{\rho_1}{2}\sqrt{\pi}$ and the integral representation of the modified Bessel function given in (2.36).

It suffices now to multiply the previous equality by $a_2(n)$ and to sum over the index $n\in\mathbb{Z}\setminus\{0\}$. Since we are assuming that $\text{Re}(s)>\mu$, all the Dirichlet series involved in this process converge absolutely. Similarly to what we have done in the final part of the proof of Theorem 2.1, we can argue that the resulting double series of the form

$$\sum_{m,n=1}^\infty b_1(m)\,a_2(n)\cos\left(\frac{b}{a}\mu_m\lambda_n'\right)\left(\frac{\mu_m}{\lambda_n'}\right)^{s-\frac12}K_{s-\frac12}\left(2k\,\mu_m\lambda_n'\right)$$

will converge absolutely due to the classical estimate for the Modified Bessel function (2.17). From the condition $\text{Re}(s)>\mu>\max\{\sigma_a,\sigma_b\}$, we see that the series defining the non-diagonal Epstein zeta function is absolutely convergent as a double series and the order of the summation of both series can be interchanged. This gives the second Selberg-Chowla formula for the even case (3.13), which is obtained after replacing the role of ϕ_1 by the one of ϕ_2 and then replace a by c and k by k'. Invoking the definition of the generalized divisor function (2.23), we obtain all the Selberg-Chowla formulas in the form above stated. □

Remark 3.3. It is also possible to generalize to this scope the proof given by Taylor [89], mentioned at the Introduction. We remark that Taylor's proof is incomplete, since it only provides the Selberg-Chowla formula for the subset of positive definite binary quadratic forms satisfying $b^2 < 2ac$ (see page 182 on Taylor's paper and the assumption on the variable Y). However, as remarked in the historical introduction, the purpose of Taylor's analysis was only to obtain Kober's result [63], which was itself a particular case of the now called Selberg-Chowla formula. Nevertheless, a simple argument of continuation provided by the contiguous relations for the hypergeometric function ${}_2F_1$ [[47], p. 1009, eq. (9.132.2)] allows to extend Taylor's proof to all positive definite binary quadratic forms.

By analytic continuation, it is possible to establish the validity of the formulas (3.11, 3.12, 3.13, 3.14) for all complex values of s. Mimicking the steps leading to the proof of Proposition 2.1, it is simple to see that the infinite series involving the generalized divisor function are entire functions of s and obey to a reflection formula which is analogous to (2.19).

Similarly to the entire functions H_{r_1} and H_{r_2}, we may see that the series on the first Selberg-Chowla formulas (3.11) and (3.12) can be written in a unified way as

$$H_1^{\delta}(s;Q;b_1,a_2)=\sum_{m,n=1}^{\infty}b_1(m)\,a_2(n)\left\{(1-\delta)\cos\left(\frac{b}{a}\mu_m\lambda_n'\right)+\delta\sin\left(\frac{b}{a}\mu_m\lambda_n'\right)\right\}\left(\frac{\mu_m}{\lambda_n'}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2k\,\mu_m\lambda_n'\right),$$

whose dependence on δ is explicit. Analogously, the divisor series on the second Selberg-Chowla formulas (3.13) and (3.14) can be also written as

$$H_2^{\delta}(s;Q;b_2,a_1)=\sum_{m,n=1}^{\infty}b_2(m)\,a_1(n)\left\{(1-\delta)\cos\left(\frac{b}{c}\mu_m'\lambda_n\right)+\delta\sin\left(\frac{b}{c}\mu_m'\lambda_n\right)\right\}\left(\frac{\mu_m'}{\lambda_n}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2k'\mu_m'\lambda_n\right).$$

To prove a reflection formula connecting both entire functions, substitute on the argument of $H_1^{\delta}(s,Q;\cdot)$ s by $1-s$ and Q by Q^{-1}. Since $Q^{-1}(x,y)=cx^2-bxy+ay^2$ and $K_\nu(z)=K_{-\nu}(z)$, one obtains without effort

$$H_1^{\delta}\left(1-s;Q^{-1};a_1,b_2;\lambda,\mu'\right)=(-1)^{\delta}H_2^{\delta}\left(s;Q;b_2,a_1,\mu',\lambda\right), \tag{3.15}$$

which is employed to prove our next result.

Corollary 3.1 (The Analytic Continuation of the non-diagonal Epstein zeta function)**.** *Under the hypothesis of Theorem 3.1, the non-diagonal Epstein zeta function (1.14) can be continued to the complex plane as:*

1. A meromorphic function with a simple pole located at $s=1$ with residue $\frac{2\pi\rho_1\rho_2}{\sqrt{|d|}}$ if a_1 and a_2 are even arithmetical functions and ρ_1, ρ_2 respectively denote the residues of ϕ_1, ϕ_2 at $s=1$. In particular, if at least one of the Dirichlet series ϕ_i is entire, then $\mathscr{Z}_2$ is entire.

2. An entire function if a_1, a_2 are odd arithmetical functions.

Thus, $\mathscr{Z}_2$ is a Dirichlet series belonging to the class $\mathscr{A}$ and satisfying Hecke's functional equation

$$\left(\frac{2}{\sqrt{|d|}}\right)^{-s}\Gamma(s)\,\mathscr{Z}_2\left(s;Q;a_1,a_2;\lambda,\lambda'\right)=(-1)^{\delta}\left(\frac{2}{\sqrt{|d|}}\right)^{s-1}\Gamma(1-s)\,\mathscr{Z}_2\left(1-s;Q^{-1};b_1,b_2;\mu,\mu'\right). \tag{3.16}$$

Proof. The proof of the continuation is similar to the proof of Corollary 2.1: since H_1^{δ} and H_2^{δ} are entire functions of the complex variable s, then it follows from the Selberg-Chowla formula (3.12) that $\mathscr{Z}_2$ is entire if a_1 and a_2 are odd arithmetical functions. Furthermore, the functional equation (3.16) with $\delta=1$ is an immediate consequence of (3.15).

It remains to show the corollary for the case where a_1 and a_2 are even arithmetical functions: of course, the first terms on (3.11) are the only ones contributing to the singularities of $\mathscr{Z}_2$ so that we just need to study the meromorphic part

$$G(s,Q)=-4a^{-s}\phi_2(0)\,\phi_1(2s)+2\sqrt{\pi}\,\rho_1\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}a^{-s}k^{1-2s}\phi_2(2s-1).$$

It is clear that $G(s,Q)$ has removable singularities at the points $s=\frac{1}{2}-k$, $k\in\mathbb{N}$. This easily comes from the functional equation and the fact that $\phi_1\in\mathscr{B}$. Now, since $\phi_1(2s)$ and $\Gamma\left(s-\frac{1}{2}\right)$ are analytic in a neighbourhood of $s=1$, we see that $\mathscr{Z}_2$ must have a pole at $s=1$ coming from the contribution of $\phi_2(2s-1)$. The residue is very easy to compute and it is given by $\frac{2\pi\rho_1\rho_2}{\sqrt{|d|}}$.

As in Corollary 2.1, the proof of the functional equation is made by comparing both representations of $\mathscr{Z}_2$, (3.11) and (3.13): start by multiplying (3.11) by a^{-s} and then substitute s by $1-s$, Q by Q^{-1} and a_1, a_2 by b_1, b_2. We obtain

$$\begin{aligned}\Gamma(1-s)\,\mathscr{Z}_2\left(1-s;Q^{-1};b_1,b_2;\mu,\mu'\right)&=-4c^{s-1}\psi_2(0)\,\Gamma(1-s)\,\psi_1(2-2s)+\\+2\sqrt{\pi}\,\rho_1\,\Gamma\left(\frac{1}{2}-s\right)c^{s-1}\,k'^{2s-1}\,\psi_2(1-2s)&+4\,\left(k'c\right)^{s-\frac{1}{2}}\sqrt{\frac{1}{c}}H_1^0(1-s;Q^{-1};a_1,b_2;\lambda,\mu'),\end{aligned} \tag{3.17}$$

which, when combined with the reflection formula (3.15) and the functional equation for ψ_1 and ϕ_2, yields

$$\begin{aligned}\Gamma(1-s)\,\mathscr{Z}_2\left(1-s;Q^{-1};b_1,b_2;\mu,\mu'\right)&=-4c^{s-1}\psi_2(0)\,\Gamma\left(s-\frac{1}{2}\right)\,\phi_1\left(2s-1\right)+\\+2\sqrt{\pi}\,\rho_1\,c^{s-1}\,k'^{2s-1}\,\Gamma(s)\,\phi_2(2s)&+4\,\left(k'c\right)^{s-\frac{1}{2}}\sqrt{\frac{1}{c}}H_2^0(s;Q;b_2,a_1,\mu',\lambda).\end{aligned} \tag{3.18}$$

Invoking now the simple properties of the class $\mathscr{B}$, $\phi_i(0) = -\frac{\rho_i^\star}{2}\sqrt{\pi}$ and $\psi_i(0) = -\frac{\rho_i}{2}\sqrt{\pi}$, and making these substitutions on the first two terms of (3.18), we arrive precisely at the right-hand side of the second Selberg-Chowla formula (3.13) multiplied by an extra factor of $(2/\sqrt{|d|})^{1-2s}\, c^{-s}$. This completes the proof.

□

Remark 3.4. Note that, if $a_1 = a_2$ and $\lambda = \lambda'$, $\mu = \mu'$, it is clear that the equality

$$\mathscr{Z}_2\left(s; Q^{-1}; b_1, b_2; \mu, \mu'\right) = \mathscr{Z}_2\left(s; Q; b_1, b_2; \mu, \mu'\right)$$

holds for any $s \in \mathbb{C}$ and so the functional equation (3.16) assumes a symmetric form

$$\left(\frac{2}{\sqrt{|d|}}\right)^{-s} \Gamma(s)\, \mathscr{Z}_2\left(s; Q; a_1, a_1; \lambda, \lambda\right) = \left(\frac{2}{\sqrt{|d|}}\right)^{s-1} \Gamma(1-s)\, \mathscr{Z}_2\left(1-s; Q; b_1, b_1; \mu, \mu\right). \tag{3.19}$$

When $\phi_1(s) = \phi_2(s) = \pi^{-\frac{s}{2}}\zeta(s)$, the functional equation for the classical Epstein zeta function $Z_2(s, Q)$ is presented in this form.

Remark 3.5. If $a(n)$ is an even arithmetical function and $\phi \in \mathscr{B}$, then $\phi(2s) \in \mathscr{A}$ with $r = 1$. Analogously, if $a(n)$ is an odd arithmetical function and $\phi \in \mathscr{B}$ then $\phi(2s-1) \in \mathscr{A}$ with $r = \frac{3}{2}$. In fact, by letting $Q(x, y) = x^2 + y^2$, we have that (3.11) is a reformulation of the first Selberg-Chowla formula restricted to the class $\mathscr{A}$, (2.28) for $r_1 = r_2 = \frac{1}{2}$. This is why we have named the Epstein zeta function (1.10) as "diagonal". Despite this, it is clear that the Selberg-Chowla formulas for odd arithmetical functions and some particular quadratic form $Q(x, y)$ cannot be seen as reformulations or as particular cases of (2.28) and (2.29). This happens because the arithmetical function associated to the Dirichlet series $\varphi(s) := \phi(2s-1)$ is $\mathfrak{c}(n) = a(n)\, \lambda_n$ and not $a(n)$. Note that, by item 2. in Definition 3.1, $\mathfrak{c}(n)$ is an even arithmetical function and it is possible to establish a Selberg-Chowla formula for its non-diagonal Epstein zeta function. Although we cannot apply Theorem 3.1 in a direct way (because $a(n)\, \lambda_n$ is not attached to a Bochner Dirichlet series), it is not difficult to derive an analogue of Lemma 3.1 valid for this case. Let $\mathscr{Z}_2(s; Q; \mathfrak{c}_1, \mathfrak{c}_2; \lambda, \lambda')$ denote the following version of the non-diagonal Epstein zeta function

$$\mathscr{Z}_2(s; Q; \mathfrak{c}_1, \mathfrak{c}_2; \lambda, \lambda') = \sum_{m,n\neq 0} \frac{a_1(m)\lambda_m\, a_2(n)\lambda_n'}{Q(\lambda_m, \lambda_n')^s}, \quad \mathrm{Re}(s) > \sigma_a + 1. \tag{3.20}$$

Then, for any $\mathrm{Re}(s) > \mu > \max\{\sigma_a + 1, \sigma_b + 1\}$, the following Selberg-Chowla formula holds

$$\begin{aligned} a^s\Gamma(s)\, \mathscr{Z}_2(s; Q; \mathfrak{c}_1, \mathfrak{c}_2; \lambda, \lambda') = 4k^{\frac{3}{2}-s} \sum_{j=1}^{\infty} \sigma_{2s-3}\left(\nu_j; b_1, a_2; \mu, \lambda'\right) \nu_j^{\frac{5}{2}-s} \\ \times \left\{ \cos\left(\frac{b}{a}\nu_j\right) K_{\frac{3}{2}-s}(2k\nu_j) + \frac{2b}{\sqrt{|d|}} \sin\left(\frac{b}{a}\nu_j\right) K_{\frac{1}{2}-s}(2k\nu_j) \right\}, \end{aligned} \tag{3.21}$$

which gives the analytic continuation of the Dirichlet series (3.20) as an entire complex function. In particular, if $b = 0$ and $a = c = 1$, (3.21) reduces to (2.28) with $r_1 = r_2 = \frac{3}{2}$.

Remark 3.6. As in Corollary 2.2, we can define multidimensional analogues of the non-diagonal Epstein zeta function and prove functional equations for them, generalizing the argument given in [[90], p. 481, Theorem 2]. Just like the proof presented in Terras' paper, which invokes a multidimensional version of Poisson's summation formula [[90], p. 480], our generalization would require higher dimensional analogues of (3.1) and (3.2). As expected, one would obtain that this multidimensional version of (1.14) would also satisfy Hecke's functional equation with parameter $r := n/2$, where n is the number of variables of Q.

Since we have not found any application of these higher dimensional formulas to the study of the zeros of Dirichlet series (besides the diagonal ones described by Corollary 2.2), we shall leave the study of these formulas outside the scope of this paper.

4 The zeros of a class of Dirichlet series

It is immediate to check from the Selberg-Chowla (or the functional equation) that $\mathscr{Z}_2(s;\,a_1,a_2;\,\lambda,\,\lambda')$ has trivial zeros located at the negative integers, this is, $s=-n$, $n\in\mathbb{N}$. A similar comment can be made for $\mathscr{Z}_2(s;\,Q;\,a_1,a_2;\,\lambda,\,\lambda')$. Moreover, in a completely analogous way to the results given in [[5], p. 367, Theorem 3], one can exhibit subclasses of the Epstein zeta functions studied in the previous sections which do not obey to a generalized Riemann hypothesis, i.e., which fail the condition of having all their nontrivial zeros located in the critical line $\text{Re}(s)=r$. The next corollary of the Selberg-Chowla formula gives this result. In order to state it, we introduce the following notation: for a given $\xi>0$, we write $\mathscr{Z}_2(s;\,a_1,a_2;\,\lambda,\,\xi\lambda')$ as the analogue of the diagonal Epstein zeta function given by the double Dirichlet series

$$\mathscr{Z}_2(s;\,a_1,a_2;\,\lambda,\,\xi\lambda')=\sum_{m,n\neq0}^{\infty}\frac{a_1(m)\,a_2(n)}{(\lambda_m+\lambda_n'\xi)^s},\quad \text{Re}(s)>2\,\sigma_a. \tag{4.1}$$

Corollary 4.1 (Failure of the Riemann Hypothesis for $\mathscr{Z}_2$)**.** *Suppose that ϕ_1 and ϕ_2 are real Dirichlet series satisfying Hecke's functional equation and belonging to the class $\mathscr{A}$. Assume also that $\phi_i(0)<0$ for any $i=1,2$. Then, for sufficiently large $\xi>0$, $\mathscr{Z}_2(s;\,a_1,a_2;\,\lambda,\,\xi\lambda')$ has a real zero on the interval (r_1,r_1+r_2) for $r_2<r_1$.*

Moreover, if ϕ_i, $i=1,2$, are real Bochner Dirichlet series belonging to the class $\mathscr{B}$ and such that $\phi_i(0)<0$ then, for a quadratic form Q having a sufficiently large $k:=\sqrt{|d|}/2a$, $\mathscr{Z}_2(s;\,Q;\,a_1,a_2;\,\lambda,\,\lambda')$ has a real zero on the interval $\left(\frac{1}{2},1\right)$.

Proof. We shall prove the result only for the diagonal Epstein zeta function $\mathscr{Z}_2(s;\,a_1,a_2;\,\lambda,\,\xi\lambda')$, since similar considerations hold for $\mathscr{Z}_2(s;\,Q;\,a_1,a_2;\,\lambda,\,\lambda')$. The idea of the proof is to write a Selberg-Chowla formula for (4.1). To do this, let us replace on (2.28) the sequence λ_n' by $\xi\lambda_n'$, where $\xi>0$. By virtue of

Hecke's functional equation, we also need to replace μ_n' by μ_n'/ξ, $b_2(n)$ by $\xi^{-r_2}b_2(n)$ and ρ_2 by $\xi^{-r_2}\rho_2$. From (2.28), the following Selberg-Chowla formula holds

$$\Gamma(s)\sum_{m,n\neq 0}^{\infty}\frac{a_1(m)\,a_2(n)}{(\lambda_m+\lambda_n'\xi)^s}=-\phi_2(0)\,\Gamma(s)\,\phi_1(s)+\rho_1\Gamma(r_1)\,\Gamma(s-r_1)\,\xi^{r_1-s}\,\phi_2\,(s-r_1)$$

$$+2\,\xi^{\frac{r_1-s}{2}}\sum_{m,n=1}^{\infty}\sigma_{s-r_1}\,(\nu_j;b_1,a_2)\,\nu_j^{\frac{r_1-s}{2}}K_{r_1-s}\left(2\sqrt{\mu_m\lambda_n'\,\xi}\right), \tag{4.2}$$

which provides the analytic continuation of (4.1). Assume now that $r_1>r_2$: from Corollary 2.1 we know that the only pole that $\mathscr{Z}_2$ possesses is located at $s=r_1+r_2$ with residue given by

$$\mathrm{Res}_{s=r_1+r_2}\,\mathscr{Z}_2(s;a_1,a_2;\lambda,\xi\lambda')=\frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1+r_2)}\,\xi^{-r_2}\,\rho_1\rho_2.$$

Thus, since $\rho_i>0$ (because $\phi_i(0)<0$ by hypothesis) and $\xi>0$, the latter is a positive real number, so that $\lim_{s\to r_1+r_2^-}\mathscr{Z}_2\,(s;a;\lambda;\xi)=-\infty$. We may now compute $\mathscr{Z}_2$ at $s=r_1$: to do this, let us write the Laurent series for $\phi_1(s)$ around $s=r_1$ as

$$\phi_1(s)=\frac{\rho_1}{s-r_1}+\rho_{0,1}+O\,(s-r_1)\,, \tag{4.3}$$

and use the meromorphic expansions for $\Gamma(s)$ and $\phi_1(s)$ around $s=0$,

$$\Gamma(s)=\frac{1}{s}-\gamma+O(s), \tag{4.4}$$

$$\phi_1(s)=\phi_1(0)+\phi_1'(0)\,s+O(s^2), \tag{4.5}$$

where γ is Euler's constant. Due to the uniform convergence of the infinite series in (4.2) (by Proposition 2.1), taking the limit $s\to r_1$ on the right-hand side of (4.2) gives

$$-\phi_2(0)\left\{\rho_{0,1}+\rho_1\log(\xi)+\rho_1\gamma+\rho_1\,\frac{\Gamma'(r_1)}{\Gamma(r_1)}\right\}+\rho_1\phi_2'(0)+\frac{2}{\Gamma(r_1)}\sum_{j=1}^{\infty}d\left(\nu_j;b_1,a_1;\mu,\lambda'\right)K_0\left(2\xi\sqrt{\nu_j}\right), \tag{4.6}$$

where the arithmetical function d is the generalization of the usual divisor function $d(n)$ provided by formula (2.25).

Now, we study the infinite series at the right-hand side of (4.6): from the inequality [[5], p. 368, Lemma 3]

$$K_0(x)\leq\left(\frac{\pi}{2x}\right)^{1/2}e^{-x}\left[1-\frac{1}{8x}+\frac{9}{128x^2}\right], \tag{4.7}$$

we can deduce that the infinite series in (4.6) is dominated by

$$\left(\frac{\pi}{4\xi}\right)^{1/2}\sum_{m,n=1}^{\infty}b(m)\,a(n)\,(\mu_m\lambda_n)^{-\frac{1}{4}}\,e^{-2\xi\sqrt{\mu_m\lambda_n}}\left[1-\frac{1}{16\xi\sqrt{\mu_m\lambda_n}}+\frac{9}{512\xi^2\mu_m\lambda_n}\right], \tag{4.8}$$

which clearly goes to zero when $\xi \to \infty$. Hence, it suffices to choose ξ for which the residual term in (4.6) is positive. But this condition is satisfied whenever

$$\xi > \exp\left\{\frac{\phi_2'(0)}{\phi_2(0)} - \frac{\rho_{0,1}}{\rho_1} - \gamma - \frac{\Gamma'(r_1)}{\Gamma(r_1)}\right\}. \tag{4.9}$$

Since there exists some ξ_0 such that, for any $\xi > \xi_0$, (4.8) is less than the right-hand side of (4.9), if we take $\xi > \xi_0$ we have that $\mathscr{Z}_2(r_1; a_1, a_2; \lambda, \xi\lambda') > 0$ and so there must be some real zero of the Epstein zeta function in the interval $(r_1, r_1 + r_2)$. □

Although in general most particular cases of the Epstein zeta functions defined in the previous sections may have infinitely many zeros off their critical lines, a large class of them will certainly admit infinitely many zeros on the critical line. For the classical Epstein zeta function $Z_2(s, Q)$, this fact was proved for the first time by Potter and Titchmarsh [73]. A simpler proof, similar to Hardy's proof of the same theorem for $\zeta(s)$, was given by Kober [62].

Since we have generalized the Selberg-Chowla formula and the continuation of Epstein zeta functions in the previous sections, in this section we use the advantage of representing the double Dirichlet series $\mathscr{Z}_2(s; a_1, a_2; \lambda, \lambda')$ in terms of the single series $\phi_1(s)$ and $\phi_2(s)$ in order to relate the distribution of zeros of $\phi_1(s)$ and $\phi_2(s)$ with the asymptotic order of $\mathscr{Z}_2$ in its critical line. Our method generalizes Deuring's [31] and it may be useful for a wide variety of examples, including those found by Suzuki [86]. We also prove that if $\mathscr{Z}_2$ satisfies Hardy's Theorem, then $\phi(s)$ must satify Hardy's Theorem as well.

Remark 4.1. In the next results, we shall consider the subsequence of multidimensional diagonal Epstein zeta functions, $\mathscr{Z}_{2^k}(s; a; \lambda)$, given in Remark 2.4. Recall that the abcissa of absolute convergence of $\mathscr{Z}_{2^k}(s; a; \lambda)$ is less than or equal to $2^k\sigma_a$ and so, from the Functional equation for $\mathscr{Z}_{2^k}$ (2.42) and the Phragmén-Lindelöf principle (1.18), we know that, for any $\delta > 0$ and $2^k r - 2^k\sigma_a - \delta \leq \sigma \leq 2^k\sigma_a + \delta$, one has the general estimate

$$\mathscr{Z}_{2^k}(\sigma + it; a; \lambda) = O\left(|t|^{2^k\sigma_a - \sigma + \delta}\right), \quad |t| \to \infty. \tag{4.10}$$

Moreover, since the dyadic Epstein zeta function satisfies Hecke's functional equation with parameter $2^k r$, the critical line for $\mathscr{Z}_{2^k}$ is the vertical line $\mathrm{Re}(s) = 2^{k-1}r$ and there we have the asymptotic order for $\mathscr{Z}_{2^k}$

$$\mathscr{Z}_{2^k}\left(2^{k-1}r + it; a; \lambda\right) = O\left(|t|^{2^k\sigma_a - 2^{k-1}r + \delta}\right), \quad |t| \to \infty. \tag{4.11}$$

Estimates (4.10) and (4.11) will be useful in the proof of Theorem 4.1. This theorem gives two main conditions imposed on the diagonal Epstein zeta functions $\mathscr{Z}_{2^k}$ assuring the existence of infinitely many zeros of $\phi(s)$ at the critical line $\mathrm{Re}(s) = \frac{r}{2}$. The bridge connecting the truth of Hardy's theorem for $\phi(s)$ and for $\mathscr{Z}_{2^k}$ lies precisely in Selberg-Chowla's formula (2.45). However, to derive in a precise form this connection, we still need to write it in a suitable form, i.e., involving an integral representation of ϕ on the critical line. In the next lemma we rewrite (2.45) in an integral form. There, we shall consider the subsequence of multidimensional diagonal Epstein zeta functions, $\mathscr{Z}_{2^k}(s;a;\lambda)$, defined in Remark 2.4. Note the convention that $\mathscr{Z}_{2^0}(s;a;\lambda):=\phi(s)$.

Lemma 4.1. *Let $\mathscr{Z}_{2^k}(s;a;\lambda)$ be the dyadic Epstein zeta function defined by (2.44). Then the following representation of $\mathscr{Z}_{2^{k+1}}(s;a;\lambda)$ at the critical line, $Re(s)=2^k r$, holds*

$$\begin{aligned}\Gamma(2^k r+2it)\,\mathscr{Z}_{2^{k+1}}\left(2^k r+2it;a;\lambda\right)&=-2\mathscr{Z}_{2^k}(0;a;\lambda)\,\Gamma(2^k r+2it)\,\mathscr{Z}_{2^k}\left(2^k r+2it;a;\lambda\right)+\\ +2\Gamma^{2^k}(r)\,\rho^{2^k}\,\Gamma(2it)\,\mathscr{Z}_{2^k}(2it;a;\lambda)&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(2^{k-1}r+i(y-t)\right)\mathscr{Z}_{2^k}\left(2^{k-1}r+i(y-t);b;\lambda\right)\times\\ &\times\Gamma\left(2^{k-1}r+i(y+t)\right)\mathscr{Z}_{2^k}\left(2^{k-1}r+i(y+t);a;\lambda\right)dy.\end{aligned}\tag{4.12}$$

Proof. Start with the Selberg-Chowla formula (2.45) and write $s=2^k r+2it$, $t\in\mathbb{R}$. We obtain the representation

$$\begin{aligned}\Gamma(2^k r+2it)\,\mathscr{Z}_{2^{k+1}}\left(2^k r+2it;a;\lambda\right)&=-\mathscr{Z}_{2^k}(0;a;\lambda)\,\Gamma(2^k r+2it)\,\mathscr{Z}_{2^k}\left(2^k r+2it;a;\lambda\right)+\\ +\Gamma^{2^k}(r)\,\rho^{2^k}\,\Gamma(2it)\,\mathscr{Z}_{2^k}(2it;a;\lambda)&+2\sum_{m,n=1}^{\infty}\mathfrak{V}_{2^k}(m)\,\mathfrak{U}_{2^k}(n)\left(\frac{\Omega_m}{\Lambda_n}\right)^{it}K_{2it}\left(2\sqrt{\Omega_m\Lambda_n}\right),\end{aligned}\tag{4.13}$$

where the definitions of the arithmetical functions $\mathfrak{U}_{2^k}(n)$ and $\mathfrak{V}_{2^k}(m)$ are given on the second section (see Remark 2.4). To proceed further, we need to give an integral representation for the double series appearing on the right-hand side of (4.13). Our idea is then to return to the integral representation for the modified Bessel function (2.15) and rewrite it in a more symmetric form

$$K_\nu(x)=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}2^{w-2}\Gamma\left(\frac{w+\nu}{2}\right)\Gamma\left(\frac{w-\nu}{2}\right)x^{-w}\,dw,\quad \mu>\mathrm{Re}(\nu).\tag{4.14}$$

Replacing ν by $2it$ in (4.14) and letting $\mu>\max\left\{0,\,2^k\sigma_a,\,2^k\sigma_b,\,2^{k+1}r\right\}$, we can write $2H_{2^k r}(2^k r+2it;a;\lambda)$ as

$$\begin{aligned}&\sum_{m,n=1}^{\infty}\mathfrak{V}_{2^k}(m)\,\mathfrak{U}_{2^k}(n)\left(\frac{\Omega_m}{\Lambda_n}\right)^{it}\frac{1}{\pi i}\int_{\mu-i\infty}^{\mu+i\infty}2^{w-2}\Gamma\left(\frac{w+2it}{2}\right)\Gamma\left(\frac{w-2it}{2}\right)\left(2\sqrt{\Omega_m\Lambda_n}\right)^{-w}dw\\ &=\frac{1}{4\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(\frac{w-2it}{2}\right)\mathscr{Z}_{2^k}\left(\frac{w-2it}{2};b;\lambda\right)\Gamma\left(\frac{w+2it}{2}\right)\mathscr{Z}_{2^k}\left(\frac{w+2it}{2};a;\lambda\right)dw,\end{aligned}\tag{4.15}$$

since this choice of μ allowed to interchange the double series and the Mellin integral (see the proof of Theorem 2.1). Let us now shift the line of integration in (4.15) to $\mathrm{Re}(w)=2^k r$ which, by the choice of μ, is located on the left of the line $\mathrm{Re}(w)=\mu$. Doing this requires once more to integrate along a positively oriented rectangular contour $\mathscr{R}$ with vertices $\mu\pm iT$ and $2^k r\pm iT$, $T>0$. The application of the Phragmén-Lindelöf principle for $\mathscr{Z}_{2^k}$ (4.10) and Stirling's formula allows to conclude that the integrals on the horizontal segments vanish when $T \to \infty$. By shifting the line of integration, we note that the integrand has two simple poles on the interior of $\mathscr{R}$ in the form $w = 2^{k+1} r \pm 2it$ and their residues are, respectively,

$$R_{2^{k+1} r+2it} = 2\Gamma^{2^k}(r)\,\rho^{\star 2^k}\Gamma(2^k r+2it)\,\mathscr{Z}_{2^k}\left(2^k r+2it;\, a;\, \lambda\right)$$
$$= -2\,\mathscr{Z}_{2^k}(0;\, a;\, \lambda)\,\Gamma(2^k r+2it)\,\mathscr{Z}_{2^k}\left(2^k r+2it;\, a;\, \lambda\right), \tag{4.16}$$

where we have used the fact that $\mathscr{Z}_{2^k}(s;\, a;\, \lambda) \in \mathscr{A}$ (see Corollary 2.2) and the simple property of the class $\mathscr{A}$ (see Remark 2.2), and

$$R_{2^{k+1} r-2it} = 2\Gamma^{2^k}(r)\,\rho^{2^k}\,\Gamma(2it)\,\mathscr{Z}_{2^k}(2it;\, a;\, \lambda)\,. \tag{4.17}$$

Using the residue residue theorem and adding the residues (4.16) and (4.17) to the residual terms in (4.13), we obtain the formula

$$\Gamma(2^k r+2it)\,\mathscr{Z}_{2^{k+1}}\left(2^k r+2it;\, a;\, \lambda\right) = -2\mathscr{Z}_{2^k}(0;\, a;\, \lambda)\,\Gamma(2^k r+2it)\,\mathscr{Z}_{2^k}\left(2^k r+2it;\, a;\, \lambda\right)+$$
$$+2\Gamma^{2^k}(r)\,\rho^{2^k}\,\Gamma(2it)\,\mathscr{Z}_{2^k}(2it;\, a;\, \lambda)+\frac{1}{4\pi i}\int_{r-i\infty}^{r+i\infty}\Gamma\left(\frac{w-2it}{2}\right)\mathscr{Z}_{2^k}\left(\frac{w-2it}{2};\, b;\, \lambda\right)\times$$
$$\times\Gamma\left(\frac{w+2it}{2}\right)\mathscr{Z}_{2^k}\left(\frac{w+2it}{2};\, a;\, \lambda\right)dw,$$

which yields (4.12) after we write $w = 2^k r+2iy$ on the contour integral. □

We now establish our Main Theorem:

Theorem 4.1. *Let $\phi(s)$ be a Dirichlet series satisfying Hecke's functional equation in the sense of Definition 1.1 and belonging to the class $\mathscr{A}$. Suppose that the parameters of ϕ and ψ satisfy $\overline{a}(n) = b(n)$ and $\lambda_n = \mu_n$. Also, let σ_a denote the abcissa of absolute convergence of ϕ. If one of the following conditions holds:*

1. *The abcissa of absolute convergence satisfies $\sigma_a < \min\{1,r\}+\frac{1}{2}\max\{1,r\}$ and, for some $k \geq 0$, the diagonal Epstein zeta function $\mathscr{Z}_{2^{k+1}}$, (2.44), associated with ϕ has infinitely many zeros in its critical line $Re(s) = 2^k r$.*

2. *The abcissa of absolute convergence satisfies $\sigma_a < \min\{1,r\}+\frac{1}{2}\max\{1,r\}$ and, for some $k \geq 0$, the diagonal Epstein zeta function $\mathscr{Z}_{2^{k+1}}$ obeys to a subconvex estimate on the critical line of the form*

$$\mathscr{Z}_{2^{k+1}}(2^k r+it;\, a;\, \lambda) = o\left(|t|^{2^k-\frac{1}{2}}\right),\quad |t| \to \infty. \tag{4.18}$$

Then $\phi(s)$ has infinitely many zeros on the critical line $Re(s) = \frac{r}{2}$.

Proof. Since $\sigma_a = \sigma_b$ (because $\overline{a}(n) = b(n)$ and $\lambda_n = \mu_n$) and $\phi_1 = \phi_2 = \phi \in \mathscr{A}$, the Selberg-Chowla representation for the diagonal Epstein zeta function $\mathscr{Z}_2(s; a_1, a_2; \lambda, \lambda')$ reduces to (4.13) with $k = 0$. For the case $k = 0$, (4.12) gives

$$\Gamma(r+2it)\,\mathscr{Z}_2\,(r+2it; a; \lambda) = -2\phi(0)\,\Gamma(r+2it)\,\phi\,(r+2it) + 2\,\Gamma(r)\,\rho\,\Gamma(2it)\,\phi\,(2it) \\ + \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+i(y-t)\right)\psi\left(\frac{r}{2}+i(y-t)\right)\Gamma\left(\frac{r}{2}+i(y+t)\right)\phi\left(\frac{r}{2}+i(y+t)\right)dy. \quad (4.19)$$

The idea of this proof is to estimate both sides of the previous equality when $|t| \to \infty$. We shall give an estimate for the right-hand side of (4.19) under the assumption that $\phi(s)$ does not possess infinitely many zeros at the line $\mathrm{Re}(s) = \frac{r}{2}$ which will contradict the assumptions of our Theorem.

By the hypothesis $b(n) = \overline{a}(n)$, we know that the integrand in the previous equation must be a real function of $y \in \mathbb{R}$. For convenience, we shall write $\Gamma(s)\,\phi(s)$ as $R_{\phi}(s)$ and $\Gamma(s)\,\psi(s)$ as $R_{\psi}(s)$. Supposing that $\phi\,(s)$ has finitely many zeros on the line $\mathrm{Re}(s) = \frac{r}{2}$, there exists some $T_0 > 0$ such that, if $|y-t| > T_0$ and $|y+t| > T_0$, the integrand has a constant sign. Hence, the following equality holds

$$\left|\int_{-t+T_0}^{t-T_0} R_{\psi}\left(\frac{r}{2}+i\,(y-t)\right)R_{\phi}\left(\frac{r}{2}+i\,(y+t)\right)dy\right| = \int_{-t+T_0}^{t-T_0}\left|R_{\psi}\left(\frac{r}{2}+i\,(y-t)\right)R_{\phi}\left(\frac{r}{2}+i\,(y+t)\right)\right|dy. \quad (4.20)$$

The idea now is to see that the right-hand side of (4.20) will provide a lower bound for the integral on the right-hand side of (4.19) which, in its turn, will contradict hypothesis 1. and 2. of our statement. If we take a partition of the integral in (4.19) as

$$\left\{\int_{t-T_0}^{\infty} + \int_{-\infty}^{-t+T_0} + \int_{-t+T_0}^{t-T_0}\right\} R_{\psi}\left(\frac{r}{2}+i\,(y-t)\right)R_{\phi}\left(\frac{r}{2}+i\,(y+t)\right)dy, \quad (4.21)$$

we see that, in the third of these, we can use the contradiction hypothesis (4.20).

Let us denote each integral in the previous partition by $\mathscr{A}_i(t)$. We first check that, for any $\delta > 0$, the first integrals satisfy the asymptotic order

$$\mathscr{A}_1(t),\, \mathscr{A}_2(t) = O\left(|t|^{\sigma_a - \frac{1}{2}+\delta}\,e^{-\pi|t|}\right). \quad (4.22)$$

The estimate (4.22) comes directly from Stirling's formula and a simple application of the Phragmén-Lindelöf principle (1.18) for ϕ and ψ on the critical line $\mathrm{Re}(s) = \frac{r}{2}$. It suffices to verify (4.22) only for $\mathscr{A}_1(t)$, which in its turn concludes the same verification for $\mathscr{A}_2(t)$ since, from the functional equation (1.7) for $\phi(s)$, $\mathscr{A}_1(t) = \mathscr{A}_2(t)$. From the fact that ϕ and ψ have the same abcissa of absolute convergence, σ_a, R_{ϕ} and R_{ψ} obey to the estimates

$$R_{\phi}\left(\frac{r}{2}+it\right),\, R_{\psi}\left(\frac{r}{2}+it\right) = O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}e^{-\frac{\pi}{2}|t|}\right), \ \ |t| \to \infty, \quad (4.23)$$

which can be used inside the integral for $\mathscr{A}_1(t)$, after we split $\mathscr{A}_1$ as follows

$$\int_{-T_0+t}^{\infty} R_{\psi}\left(\frac{r}{2}+i\,(y-t)\right)R_{\phi}\left(\frac{r}{2}+i\,(y+t)\right)dy = \left\{\int_{-T_0}^{T_0} + \int_{T_0}^{\infty}\right\} R_{\psi}\left(\frac{r}{2}+iu\right)R_{\phi}\left(\frac{r}{2}+i\,(u+2t)\right)du.$$

Thus, by a simple application of Stirling's formula one sees immediately that

$$\int_{-T_0}^{T_0} R_\psi\left(\frac{r}{2}+iu\right) R_\phi\left(\frac{r}{2}+i(u+2t)\right) du = O\left(\int_{-T_0}^{T_0}\left|R_\psi\left(\frac{r}{2}+iu\right)\right| |u+2t|^{\sigma_a-\frac{1}{2}+\delta} e^{-\frac{\pi}{2}|u+2t|}\, du\right)$$
$$= O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\, e^{-\pi|t|}\right) \tag{4.24}$$

while, analogously,

$$\int_{T_0}^{\infty} R_\psi\left(\frac{r}{2}+iu\right) R_\phi\left(\frac{r}{2}+i(u+2t)\right) du = O\left(|t|^{2\sigma_a+2\delta}\, e^{-\pi|t|}\int_{T_0/2t}^{\infty}(u^2+u)^{\sigma_a-\frac{1}{2}+\delta}\, e^{-2\pi ut}\, du\right)$$
$$= O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\, e^{-\pi|t|}\right) \tag{4.25}$$

providing (4.22).

In the interval of integration considered on the third integral $\mathscr{A}_3(t)$, we may invoke Stirling's formula once more, since we can take T_0 sufficiently large and, in this interval, we have $|y-t|>T_0$ and $|y+t|>T_0$. From (1.16), we have that

$$\left|\Gamma\left(\frac{r}{2}+i(y-t)\right)\Gamma\left(\frac{r}{2}+i(y+t)\right)\right| = 2\pi\left(t^2-y^2\right)^{\frac{r-1}{2}} e^{-\pi|t|}\left(1+O(|t|^{-1})\right). \tag{4.26}$$

Let c be the least positive integer such that $a(c)\neq 0$: invoking (4.26) on the right-hand side of (4.20), we obtain the inequality

$$\int_{-t+T_0}^{t-T_0}\left|R_\psi\left(\frac{r}{2}+i(y-t)\right) R_\phi\left(\frac{r}{2}+i(y+t)\right)\right| dy = \lambda_c^{-r}\int_{-t+T_0}^{t-T_0}\left|\lambda_c^{r+2iy} R_\psi\left(\frac{r}{2}+i(y-t)\right) R_\phi\left(\frac{r}{2}+i(y+t)\right)\right| dy$$
$$\geq 2\pi\lambda_c^{-r} e^{-\pi|t|}\left|\int_{\frac{r}{2}-i(t-T_0)}^{\frac{r}{2}+i(t-T_0)} \lambda_c^{2z}\left(t^2+\left(z-\frac{r}{2}\right)^2\right)^{\frac{r-1}{2}}\psi(z-it)\,\phi(z+it)dz\left\{1+O\left(|t|^{-1}\right)\right\}\right|. \tag{4.27}$$

By Cauchy's Theorem applied to the positively oriented rectangular contour with vertices $\frac{r}{2}\pm i(t-T_0)$ and $\sigma_a+1\pm i(t-T_0)$, we can write the contour integral on the right-hand side of (4.27) as

$$\left\{\int_{\frac{r}{2}-i(t-T_0)}^{\sigma_a+1-i(t-T_0)}+\int_{\sigma_a+1-i(t-T_0)}^{\sigma_a+1+i(t-T_0)}+\int_{\sigma_a+1+i(t-T_0)}^{\frac{r}{2}+i(t-T_0)}\right\}\lambda_c^{2z}\left(t^2+\left(z-\frac{r}{2}\right)^2\right)^{\frac{r-1}{2}}\psi(z-it)\,\phi(z+it)\,dz. \tag{4.28}$$

The first and third integrals in the above partition can be easily evaluated with the aid of Phragmén-Lindelöf estimates. Note that in the first of these, the factor $\phi(z+it)$ does not depend on t and since T_0 is a fixed number then we just need to apply (1.18) for $\psi(z-it)$. By symmetry, the same happens on the third integral with the roles of ϕ and ψ being reversed. Denoting each integral on (4.28) by $\mathscr{A}_{3j}$, we find that, for every $\delta>0$,

$$\mathscr{A}_{31}(t),\ \mathscr{A}_{33}(t) = O\left(|t|^{\sigma_a+\frac{r}{2}-1+\delta}\right). \tag{4.29}$$

In order to estimate the middle integral, $\mathscr{A}_{32}$, we can use the Dirichlet series representation of the product $\psi(z-it)\,\phi(z+it)$. We arrive at ($k:=\sigma_a+1-\frac{r}{2}$)

$$\int_{\sigma_a+1-i(t-T_0)}^{\sigma_a+1+i(t-T_0)}\lambda_c^{2z}\left(t^2+\left(z-\frac{r}{2}\right)^2\right)^{\frac{r-1}{2}}\psi(z-it)\,\phi(z+it)\,dz=|a(c)|^2\int_{k-i(t-T_0)}^{k+i(t-T_0)}\left(t^2+z^2\right)^{\frac{r-1}{2}}dz \qquad (4.30)$$
$$+\sum_{n=c+1}^{\infty}\overline{a}(m)\,a(n)\left(\frac{\lambda_m}{\lambda_n}\right)^{it}\left(\frac{\lambda_c^2}{\lambda_m\lambda_n}\right)^{\frac{r}{2}}\int_{k-i(t-T_0)}^{k+i(t-T_0)}\left(t^2+z^2\right)^{\frac{r-1}{2}}\left(\frac{\lambda_c^2}{\lambda_m\lambda_n}\right)^{z}dz=2i\,|a(c)|^2\,|t|^r+O\left(|t|^{r-1}\right)$$

upon an integration by parts for the integrals in the infinite series. Since $\sigma_a<\min\{1,r\}+\frac{1}{2}\max\{1,r\}$ by hypothesis, (4.27) gives

$$\int_{-t+T_0}^{t-T_0}\left|R_\psi\left(\frac{r}{2}+i\,(y-t)\right)R_\phi\left(\frac{r}{2}+i\,(y+t)\right)\right|dy\geq 2\pi\lambda_c^{-r}e^{-\pi|t|}\left|2i|a(c)|^2\,|t|^r+O(|t|^{r-1})+O\left(|t|^{\sigma_a+\frac{r}{2}-1+\delta}\right)\right|$$
$$=4\pi\,|a(c)|^2\lambda_c^{-r}\,|t|^r e^{-\pi|t|}+O\left(|t|^{r-1}\,e^{-\pi|t|}\right)+O\left(|t|^{\sigma_a+\frac{r}{2}-1}\,e^{-\pi|t|}\right). \qquad (4.31)$$

Now, let us note that the second O term in (4.31) always bounds the first one. As usual, define

$$\mu(\sigma)=\inf\left\{\xi\,:\,\phi(\sigma+it)=O(|t|^{\xi})\right\}.$$

From (1.17), the general theory of $\mu(\sigma)$ [[93], p. 299] and the hypothesis of Theorem 4.1, we know that $\mu(\sigma)\leq r-2\sigma$ for $\sigma\leq r-\sigma_a$. Since $\mu(\sigma)$ is decreasing, we must have that $0\leq\mu\left(r-\sigma_a\right)\leq 2\sigma_a-r$ and this implies $\sigma_a\geq\frac{r}{2}$. From the contradiction hypothesis (4.20), (4.31) gives the lower bound for $\mathscr{A}_3(t)$

$$|\mathscr{A}_3(t)|\geq 4\pi\,|a(c)|^2\lambda_c^{-r}\,|t|^r\,e^{-\pi|t|}+O\left(|t|^{\sigma_a+\frac{r}{2}-1}\,e^{-\pi|t|}\right),\quad |t|>T_0. \qquad (4.32)$$

Returning to the Selberg-Chowla formula (4.19), together with (4.32), the hypothesis (4.20) and the definition of the integrals $\mathscr{A}_i(t)$, we deduce the following inequality

$$2\,|a(c)|^2\lambda_c^{-r}\,|t|^r\,e^{-\pi|t|}+O\left(|t|^{\sigma_a+\frac{r}{2}-1}\,e^{-\pi|t|}\right)\leq\frac{1}{2\pi}\,|\mathscr{A}_3(t)|=|\Gamma\left(r+it\right)\,\mathscr{Z}_2(r+it;\,a;\,\lambda)-$$
$$-2\phi(0)\,\Gamma(r+2it)\,\phi\left(r+2it\right)+2\Gamma(r)\,\rho\,\Gamma(2it)\,\phi\left(2it\right)-\frac{1}{2\pi}\mathscr{A}_1(t)-\frac{1}{2\pi}\mathscr{A}_2(t)\Big|. \qquad (4.33)$$

To contradict hypothesis 1. and 2. for $k=1$, we just need to estimate the remaining terms appearing on the right-hand side of (4.33). We have seen already that $\mathscr{A}_1(t)$ and $\mathscr{A}_2(t)$ satisfy (4.22), so we just need to estimate the remaining two. From (1.18) and (1.16), it is immediate to see that, for every positive δ,

$$2\Gamma(r)\,\rho\,\Gamma(2it)\,\phi\left(2it\right)-2\phi(0)\,\Gamma(r+2it)\,\phi\left(r+2it\right)=O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\,e^{-\pi|t|}\right),\ \ |t|\to\infty. \qquad (4.34)$$

Using (4.33) and the fact that $\sigma_a < r+\frac{1}{2}$, we obtain the lower bound for the Epstein zeta function

$$|\Gamma(r+2it)\, \mathscr{Z}_2(r+2it;\, a;\, \lambda)| \geq 2\,|a(c)|^2 \lambda_c^{-r}\, |t|^r\, e^{-\pi|t|} + O\left(|t|^{\sigma_a+\frac{r}{2}-1}\, e^{-\pi|t|}\right) + O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\, e^{-\pi|t|}\right) \quad (4.35)$$

valid for any $|t| > T_0$. Using Stirling's formula (1.16) and enlarging T_0 if needed, (4.35) immediately implies that, for some positive constant B, the order of $\mathscr{Z}_2(s;\, a;\, \lambda)$ at the critical line satisfies the lower bound

$$|\mathscr{Z}_2(r+2it;\, a;\, \lambda)| > B\,|t|^{\frac{1}{2}}, \quad |t| > T_0, \qquad (4.36)$$

and so $\mathscr{Z}_2(s;\, a;\, \lambda)$ cannot have infinitely many zeros on the critical line $\text{Re}(s) = r$, since it is always greater than a positive function for every $|\text{Im}(s)| > 2T_0$. This also contradicts (4.18) for $k = 1$. To prove the Theorem for a general $k \in \mathbb{N}$, we shall use Lemma 4.1 and prove by induction that (4.36) implies (4.18).

The following claim contains the necessary assertion to finish the proof:

Claim 4.1. *Suppose that, for some $T_k > 0$, the 2^k-th Epstein zeta function has the lower bound on its critical line*

$$|\mathscr{Z}_{2^k}(2^{k-1}r+it;\, a;\, \lambda)| > B\,|t|^{\frac{2^k-1}{2}}, \quad |t| > T_k, \qquad (4.37)$$

for some $B > 0$. Then we have that $\mathscr{Z}_{2^{k+1}}$ has a similar bound of the form

$$|\mathscr{Z}_{2^{k+1}}(2^k r+it;\, a;\, \lambda)| > B'\,|t|^{\frac{2^{k+1}-1}{2}}, \quad |t| > 2T_k,\ B' > 0. \qquad (4.38)$$

Proof of Claim 4.1. The idea is to mimic the argument just given. Use (4.12) and estimate the residual terms. It is clear from (4.10) and Stirling's formula that these are bounded as $O\left(|t|^{2^k\sigma_a-\frac{1}{2}+\delta}\right)$. To estimate the integral, we take once more a partition similar to (4.21). As before, if we denote each one of these integrals by $\mathscr{A}_i^{(k)}(t)$, $i = 1, 2, 3$, we can mimic the proof of (4.22) to obtain

$$\mathscr{A}_1^{(k)}(t),\ \mathscr{A}_2^{(k)}(t) = O\left(|t|^{2^k\sigma_a-\frac{1}{2}+\delta}\, e^{-\pi|t|}\right), \quad |t| \to \infty.$$

On the third integral, $\mathscr{A}_3^{(k)}(t)$, we proceed as in the equality (4.20): by the inductive hypothesis (4.37), we know that $\mathscr{Z}_{2^k}(s;\, a;\, \lambda)$ has constant sign on the line $\text{Re}(s) = 2^{k-1}r$ for $|\text{Im}(s)| > T_k$. Employing directly (4.37) on the third integral $\mathscr{A}_3^{(k)}(t)$ we obtain, for any $|t| > T_k$,

$$\begin{aligned} \left|\mathscr{A}_3^{(k)}(t)\right| &= \int_{-t+T_k}^{t-T_k} \left|\Gamma\left(2^{k-1}r+i(y-t)\right)\, \mathscr{Z}_{2^k}\left(2^{k-1}r+i(y-t);\, b;\, \lambda\right) \times \right. \\ &\quad \left. \times \Gamma\left(2^{k-1}r+i(y+t)\right)\, \mathscr{Z}_{2^k}\left(2^{k-1}r+i(y+t);\, a;\, \lambda\right)\right| dy > \\ &> B\, e^{-\pi|t|} \left|\int_{-t+T_k}^{t-T_k} \left(t^2-y^2\right)^{2^{k-1}(r+1)-1} dy\right| = D\,|t|^{2^k(r+1)-1}\, e^{-\pi|t|} + O\left(|t|^{2^k(r+1)-2}\, e^{-\pi|t|}\right) \end{aligned}$$

for some $D > 0$. From the Selberg-Chowla formula (4.12) we have that, for $|t| > T_k$ and some positive constant D',

$$|\Gamma(2^k r+2it)\, \mathscr{Z}_{2^{k+1}}(2^k r+2it;\, a;\, \lambda)| > D'\,|t|^{2^k(r+1)-1}\, e^{-\pi|t|}, \qquad (4.39)$$

which implies (4.38). □

Returning to the proof of Theorem 4.1, the assumption that $\phi(s)$ does not have infinitely many zeros on the critical line implies (4.37) for $k=1$ and $T_1 := 2T_0$. But then the previous claim shows that this contradicts conditions 1. and 2. on the statement and the proof follows. □

Remark 4.2. (The Bochner case) Note that, if $\phi(s)$ is a Dirichlet series belonging to the class $\mathscr{B}$, then $\varphi(s) := \phi(2s-\delta) \in \mathscr{A}$ with abcissa of absolute convergence $\sigma_\varphi = \frac{\sigma_a+\delta}{2}$ and $r = \frac{1}{2}+\delta$. An immediate application of the previous theorem shows that, if the abcissa of absolute convergence of $\phi(s)$ satisfies $\sigma_a < 2+\frac{\delta}{2}$ and $\overline{a}(n) = b(n)$, $\lambda_n = \mu_n$, then if the Dirichlet series

$$\mathscr{Z}_2(s;\mathbb{I};a;\lambda) := \sum_{m,n\neq 0} \frac{a(m)\,a(n)\,(\lambda_m\lambda_n)^{\delta}}{(\lambda_m^2+\lambda_n^2)^s}$$

has infinitely many zeros on the critical line $\mathrm{Re}(s) = \frac{1}{2}+\delta$, $\phi(s)$ has infinitely many zeros on the line $\mathrm{Re}(s) = \frac{1}{2}$ as well.

Remark 4.3. Note that the condition $\sigma_a < 1+\frac{r}{2}$ was essential to move the line of integration on the third integral $\mathscr{A}_3(t)$ from the critical line $\mathrm{Re}(s) = \frac{r}{2}$ to $\mathrm{Re}(s) = \sigma_a + 1$. In fact, this condition is an essential tool to prove that the estimate (4.29) of the horizontal integrals is actually bounded by $|t|^r$, hence allowing to establish the lower bound (4.32). Although this condition and procedure are essential to establish the general result on Theorem 4.1, if we assume that the generalized Dirichlet series $\phi(s)$ is a Titchmarsh series [[74], Chapter 2, p. 39], then a Theorem of Ramachandra [[4], p. 570, Thm. 3] would still allow to conclude a lower bound of the form (4.36).

Examples of "Titchmarsh series" include the class of Hecke Dirichlet series with signature (λ, r, γ) and so the condition $\sigma_a < 1+\frac{r}{2}$ in the statement of the previous theorem can be suspended for this class.

Nevertheless, the imposition $\sigma_a < r+\frac{1}{2}$ needs to be kept since it assures that $|t|^r$ exceeds the estimates of the integrals $\mathscr{A}_1(t)$ and $\mathscr{A}_2(t)$ (4.22), as well as the estimates for the residual terms (4.34) appearing in the Selberg-Chowla formula. For an application of Ramachandra's result, see [69], where an adaptation of the Potter-Titchmarsh method [73], together with Ramachandra's Theorem, are essential to give a proof of Hardy's Theorem for a subclass of Dirichlet series belonging to the Selberg class with degree two.

Remark 4.4. Condition 2. (4.18) on the previous theorem was used by Deuring to prove the infinitude of zeros of $\zeta(s)$ at $\mathrm{Re}(s) = \frac{1}{2}$. This condition may be complicated to establish in general, since in all particular cases one needs to develop suitable exponential sums in order to prove it. Nevertheless, the first condition may add simplicity to the theorem, as the infinitude of zeros of $\mathscr{Z}_2(s;a;\lambda)$ can be established independently. This is certainly the case for the Dirichlet series $\phi(s) = \zeta_2(s) = \sum r_2(n)/n^s$, whose infinitude of zeros on the critical line $\mathrm{Re}(s) = \frac{1}{2}$ can be easily proved by invoking the fact that $\zeta_4(s)$ has infinitely many zeros on the line $\mathrm{Re}(s) = 1$. The infinitude of zeros of $\zeta_4(s)$ in the line $\mathrm{Re}(s) = 1$ can be established in an independent way, by invoking, for instance, Jacobi's 4-square Theorem (see Example 5.4).

Based upon the previous Theorem, we can derive several parallel results which are scattered in the literature, including proofs involving the behavior of the $\theta-$function. The following result seems unnoticed.

Corollary 4.2. *Assume that $\phi(s)$ is a Dirichlet series satisfying the conditions of Theorem 4.1 and $\Theta(z;\,a;\,\lambda)$ denote its generalized $\theta-$function (1.20). If*

$$\Theta\left(e^{i\left(\frac{\pi}{2}-\varepsilon\right)};\,a;\,\lambda\right)=o\left(\varepsilon^{-\frac{r+1}{2}}\right),\quad \varepsilon\to 0^{+}, \tag{4.40}$$

then $\phi(s)$ has infinitely many zeros on the critical line $Re(s)=\frac{r}{2}$.

Proof. Assume that $\phi(s)$ has finitely many zeros on the critical line $\mathrm{Re}(s)=\frac{r}{2}$. It follows from the proof of the previous Theorem that

$$|\Gamma(r+2it)\;\mathscr{Z}_2(r+2it;\,a;\,\lambda)\,|\geq 2\,|a(c)|^2\lambda_c^{-r}\,|t|^r\,e^{-\pi|t|}+O\left(|t|^{\sigma_a+\frac{r}{2}-1}\,e^{-\pi|t|}\right)+O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\,e^{-\pi|t|}\right),\quad |t|>T_0. \tag{4.41}$$

For $\mathrm{Re}(z)>0$ and $\mu>2\sigma_a$, we have from (1.23) that $\Theta_2(z;\,a;\,\lambda)$ admits the representation as the Mellin integral

$$\begin{aligned}\Theta_2(z;\,a;\,\lambda)&=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(s)\,\mathscr{Z}_2(s;\,a;\,\lambda)\,z^{-s}ds\\&=\frac{1}{2\pi i}\int_{C_1}\Gamma(s)\,\mathscr{Z}_2(s;\,a;\,\lambda)\,z^{-s}ds+\frac{1}{2\pi i}\int_{C_2}\Gamma(s)\,\mathscr{Z}_2(s;\,a;\,\lambda)\,z^{-s}ds,\end{aligned} \tag{4.42}$$

where C_1 and C_2 are the paths given by

$$C_1=(r-i\infty,\,r-2iT_0)\cup(r+2iT_0,\,r+i\infty)$$

and

$$C_2=(r-2iT_0,\,\mu-2iT_0)\cup(\mu-2iT_0,\,\mu+2iT_0)\cup(\mu+2iT_0,\,r+2iT_0)\,.$$

If we now take $z=\exp\left\{i\left(\frac{\pi}{2}-\varepsilon\right)\right\}$ in (4.42), we have that

$$\Theta_2\left(e^{i\left(\frac{\pi}{2}-\varepsilon\right)};\,a;\,\lambda\right)=\frac{1}{2\pi i}\int_{C_1}\Gamma(s)\,\mathscr{Z}_2(s;\,a;\,\lambda)\,e^{-i\left(\frac{\pi}{2}-\varepsilon\right)s}ds+O(1). \tag{4.43}$$

From the functional for $\mathscr{Z}_2(s;\,a;\,\lambda)$ (2.30) and the assumption that $b(n)=\overline{a}(n)$, $\Gamma(s)\,\mathscr{Z}_2(s;\,a;\,\lambda)$ is a real valued function on the line $\mathrm{Re}(s)=r$. From this observation and combining (4.43) with (4.41), we obtain that the inequality

$$\begin{aligned}\left|\Theta_2\left(e^{i\left(\frac{\pi}{2}-\varepsilon\right)};\,a;\,\lambda\right)\right|&=\frac{1}{2\pi}\int_{|y|>2T_0}|\Gamma(r+iy)\;\mathscr{Z}_2(r+iy;\,a;\,\lambda)|\;e^{\left(\frac{\pi}{2}-\varepsilon\right)y}dy+O(1)\\&\geq\frac{|a(c)|^2(2\lambda_c)^{-r}}{\pi}\int_{2T_0}^{\infty}y^r\,e^{-\varepsilon y}\,dy+O\left(\int_{2T_0}^{\infty}y^{\sigma_a+\frac{\max\{1,r\}}{2}-1}e^{-\varepsilon y}dy\right)\\&=\frac{|a(c)|^2(2\lambda_c)^{-r}\,\Gamma(r+1)}{\pi}\,\varepsilon^{-(r+1)}+O\left(\varepsilon^{-\sigma_a-\frac{\max\{1,r\}}{2}}\right),\end{aligned} \tag{4.44}$$

is valid for every positive ε. However, since

$$\Theta_2(z;a;\lambda)=\sum_{m,n\neq 0}^{\infty} a(m)\,a(n)\,e^{-(\lambda_m+\lambda_n)z}=2a(0)\,\Theta(z;a;\lambda)+\Theta^2(z;a;\lambda), \tag{4.45}$$

it follows from (4.40) that $\Theta_2\left(e^{i\left(\frac{\pi}{2}-\varepsilon\right)};a;\lambda\right)=o\left(\varepsilon^{-r-1}\right)$ as $\varepsilon\to 0^+$, which contradicts (4.44). □

Remark 4.5. Note that, when restricted to the class $\mathscr{A}$, the previous corollary extends Berndt's result [[18], p. 679, Theorem 1]. Note also that (4.40), when restricted to the class of Dirichlet series satisfying Theorem 4.1, improves Hecke's condition (see [[52], p. 74, Satz 1]),

$$\Theta\left(e^{i\left(\frac{\pi}{2}-\varepsilon\right)};a;\lambda\right)=O(\varepsilon^{-\rho}), \tag{4.46}$$

for some $0\leq\rho<\frac{r+1}{2}$. This condition implies also the infinitude of zeros at the critical line of Hecke Dirichlet series with signature. A generalization of Hecke's theorem to other classes of Dirichlet series is given in [[18], p.680, Thm 3].

Corollary 4.3. *Let $\phi(s)$ be a Dirichlet series with $r<1$ and satisfying the conditions of Theorem 4.1. Assume also that $a(n)$ satisfies $\arg\{a(n)\}_{n\in\mathbb{N}}=$ constant. Then $\phi(s)$ has infinitely many zeros on the critical line $Re(s)=\frac{r}{2}$. Moreover, if $r=1$ and if $a(n)$ satisfies the condition*

$$|a(c)|>\sqrt{2\pi\lambda_c}\,|\rho|, \tag{4.47}$$

where ρ denotes the residue of ϕ at $s=r=1$, then $\phi(s)$ has infinitely many zeros at the critical line $Re(s)=\frac{1}{2}$.

Proof. Assume first that $0<r<1$ and that $\phi(s)$ does not have infinitely many zeros on the critical line $\text{Re}(s)=\frac{r}{2}$. Then an application of (4.44) gives

$$\varepsilon^{r+1}\left|\Theta_2\left(e^{i\left(\frac{\pi}{2}-\varepsilon\right)};a;\lambda\right)\right|\geq C, \tag{4.48}$$

for some positive constant C. On the other hand, using Bochner's modular relation for $\Theta_2(z;a;\lambda)$ (2.37) and the hypothesis of Theorem 4.1, $b(n)=\overline{a}(n)$, we know that

$$\Theta_2(z;a;\lambda)=z^{-2r}\sum_{m,n=1}^{\infty}\overline{a}(m)\,\overline{a}(n)\,e^{-(\lambda_m+\lambda_n)/z}+\Gamma^2(r)\,\rho^2 z^{-2r}+\mathscr{Z}_2(0,a,\lambda). \tag{4.49}$$

Using the fact that $\arg_{n\in\mathbb{N}}\{a(n)\}$ is constant, we have from the previous relation,

$$\begin{aligned}\left|\Theta_2\left(e^{i\left(\frac{\pi}{2}-\varepsilon\right)};a;\lambda\right)\right| &\leq \sum_{m,n\neq 0}^{\infty}\left|a(m)\,a(n)\,e^{-(\lambda_m+\lambda_n)\sin(\varepsilon)}\right|=\left|\sum_{m,n=1}^{\infty}a(m)\,a(n)\,e^{-(\lambda_m+\lambda_n)\sin(\varepsilon)}\right|\\ &=\left|\sin^{-2r}(\varepsilon)\sum_{m,n\neq 0}^{\infty}\overline{a}(m)\,\overline{a}(n)\,e^{-(\lambda_m+\lambda_n)/\sin(\varepsilon)}+\Gamma^2(r)\,\rho^2\sin^{-2r}(\varepsilon)+\mathscr{Z}_2(0,a,\lambda)\right|\\ &=\Gamma^2(r)\,|\rho|^2\sin^{-2r}(\varepsilon)+O(1).\end{aligned} \tag{4.50}$$

Since, by hypothesis, (4.48) holds, then we must have

$$0 < C \leq \varepsilon^{r+1} \left| \Theta_2 \left(e^{i\left(\frac{\pi}{2}-\varepsilon\right)}; a; \lambda \right) \right| = \varepsilon^{r+1}\, \Gamma^2(r)\, |\rho|^2 \sin^{-2r}(\varepsilon) + O(\varepsilon^{r+1}) \tag{4.51}$$

for every $\varepsilon > 0$. However, since $r < 1$, we see that the right-hand side of (4.51) tends to zero as $\varepsilon \to 0^+$, contradicting (4.48). This concludes the proof for the first case.

Assume now that $r = 1$: from (4.44) with $r = 1$, we know that the constant C given above can be explicitly written as $C = |a(c)|^2/2\pi\lambda_c$, and so (4.51) contradicts (4.47). □

Remark 4.6. By a Theorem of Fekete [39], which in its turn extends a classical Theorem of Landau on Dirichlet series, the condition $\arg_{n\in\mathbb{N}}\{a(n)\} = \text{const.}$ implies that $\phi(s)$ cannot be entire. Moreover, if we assume that $\phi \in \mathscr{A}$, then the previous Corollary is only valid for $\sigma_a = r$.

In any case it is not strictly necessary that $\arg_{n\in\mathbb{N}}\{a(n)\} = \text{const.}$ to arrive at the conclusion of Corollary 4.2. In fact, it suffices to suppose that $a(n)$ satisfies $|a(n)| \leq \mathfrak{c}(n)$ for every $n \in \mathbb{N}$, with $\mathfrak{c}(n)$ being such that the Dirichlet series $\varphi(s) = \sum \mathfrak{c}(n)\, \lambda_n^{-s}$ satisfies Hecke's functional equation with $r \leq 1$, as well as the conditions of Theorem 4.1 For example, Dirichlet L-functions attached to primitive Dirichlet characters satisfy this condition.

Remark 4.7. In the previous two corollaries we gave conditions assuring the infinitude of zeros at the critical line of a large class of Dirichlet series. As particular cases, these include large subclasses of the Epstein zeta function and the Dedekind zeta function attached to an imaginary quadratic field [28, 62]. We may apply the previous result for the non-diagonal Epstein zeta function studied in the previous section, which satisfies Hecke's functional equation with $r = 1$.

Assume that $\phi(s)$ is a Bochner Dirichlet series satisfying the conditions of the previous corollary with the additional assumption that $\sigma_a < \frac{3}{2}$. Assume also that its pole at $s = 1$ has residue ρ (which must be nonzero by the previous remark). Moreover, let $Q := ax^2 + bxy + cy^2$ be a positive definite real quadratic form and q be its minimum over the range of the sequence $(\lambda_m, \lambda_n)_{m,n\in\mathbb{N}}$, i.e., $q = \min_{(m,n)\neq(0,0)} Q(\lambda_m, \lambda_n)$.

The previous Corollary has the consequence that, if $\mathfrak{U}_Q(c)$ given in (1.15) satisfies the condition

$$|\mathfrak{U}_Q(c)|^2 > \frac{4\pi^3 q}{\sqrt{|d|}}\, |\rho|^4, \tag{4.52}$$

then $\mathscr{Z}_2(s; a; Q; \lambda)$ has infinitely many zeros at the critical line $\text{Re}(s) = \frac{1}{2}$. A condition analogous to (4.52) and holding for the classical Epstein zeta function (1.1) has been indicated by Kober [[62], p. 5].

It is also of interest to note that (4.36) contradicts any bound of the type $\mathscr{Z}_2(r + it; a; \lambda) = o\left(|t|^{\frac{1}{2}}\right)$, which may be obtained independently via more sophisticated methods employing exponential sums, as those in [56].

Under our general setting, however, it is not simple to assure these subconvex estimates for the diagonal Epstein zeta functions $\mathscr{Z}_{2^k}$ on their critical lines. In general, all the information about the behavior of $\mathscr{Z}_{2^k}$ on the line $\mathrm{Re}(s) = 2^{k-1}r$ comes exclusively from the Phragmén-Lindelöf principle.

Based solely on these classical estimates, in the next result we impose a condition which implies that, for a sufficiently large k, the order of the dyadic Epstein zeta function $\mathscr{Z}_{2^{k+1}}$ at the critical line $\mathrm{Re}(s) = 2^k r$ will eventually contradict (4.37).

We remark that the next result was already proved by Berndt [17] under a different setting. Since our proof is drastically different from his and it has the advantage of avoiding the exponential integrals typical in most part of the proofs of Hardy's Theorem (see, for instance, [28, 66, 69, 73], where exponential integrals are used to deduce analogues of Hardy's Theorem), we present our alternative short proof. We also remark that the foregoing corollary can be extended for higher classes of Dirichlet series, not necessarily belonging to the class $\mathscr{A}$.

Corollary 4.4. *Let $\phi(s)$ be a Dirichlet series satisfying the conditions of Theorem 4.1. Assume that its abcissa of absolute convergence satisfies*

$$\sigma_a < \frac{r+1}{2}. \tag{4.53}$$

Then ϕ has infinitely many zeros on the critical line $Re(s) = \frac{r}{2}$.

Proof. By the Phragmén-Lindelöf principle applied to the Dirichlet series $\mathscr{Z}_{2^{k+1}}(s;\, a;\, \lambda)$, we have that, for every positive δ,

$$\mathscr{Z}_{2^{k+1}}(2^k r + 2it;\, a;\, \lambda) = O\left(|t|^{2^{k+1}\sigma_a - 2^k r + \delta}\right), \quad |t| \to \infty. \tag{4.54}$$

However, if $\phi(s)$ has finitely many zeros on the line $\mathrm{Re}(s) = \frac{r}{2}$, then there is some $T_0 > 0$ such that, for every $k \in \mathbb{N}_0$ (see claim 4.1)

$$|\mathscr{Z}_{2^{k+1}}(2^k r + 2it;\, a;\, \lambda)| > B\, |t|^{\frac{2^{k+1}-1}{2}}, \quad |t| > T_k := 2^k T_0, \tag{4.55}$$

for some $B > 0$ depending on k. After a direct comparison between (4.54) and (4.55), and letting $|t|$ tend to infinity, we deduce that the inequality $\sigma_a \geq \frac{r+1}{2} - \frac{1}{2^{k+2}} - \delta$ must hold for every $k \in \mathbb{N}_0$ and every $\delta > 0$. This contradicts (4.53) and the corollary follows. □

Due to its "fairly elementary nature", whose construction was only based on the classical Theory of Dirichlet series and the classical Fourier Bessel expansion for generalized Dirichlet series, the method of proving Theorem 4.1 and its corollaries does not yield better estimates than the ones already known for the familiar particular cases. It may be possible, however, to invoke more sophisticated methods involving exponential sums and transformation of Dirichlet Polynomials [56] in order to study the order at the critical line of a class of Epstein zeta functions, at least when $\lambda_n = \frac{2\pi n}{\lambda}$.

As an application of the method described in Theorem 4.1 and Corollary 4.4, we now give a quantitative result regarding the distribution of the zeros of $\phi(s)$ on the critical line, i.e., for a sufficiently large T we find $H = H(T)$ such that the interval $[T, T+H]$ contains the ordinate of a zero of $\phi(s)$ located at the critical line $\mathrm{Re}(s) = \frac{r}{2}$.

Corollary 4.5. *Let $\phi(s)$ be a Dirichlet series satisfying the conditions of Corollary 4.4. Then, for any fixed $\varepsilon > 0$, there exists a positive number $T_0(\varepsilon)$, such that, for all $T \geq T_0(\varepsilon)$, there exists a zero of $\phi(s)$ of the form $s = \frac{r}{2} + i\tau$, with τ satisfying*

$$|\tau - T| \leq T^{\sigma_a + \frac{1-r}{2} + \varepsilon}. \tag{4.56}$$

Proof. We just indicate the main steps: instead of taking a fixed parameter T_0 as in the proof of Theorem 4.1, we will take a sufficiently large T. If one assumes that $\phi(\frac{r}{2} + iy)$ does not possess a zero whenever $y \in [T, T+H]$, then the integrand on (4.19) does not possess a zero when $y \in [t-T-H, t-T]$. If we let t vary on the interval $(T + \frac{H}{4}, T + \frac{H}{2}]$ and take a partition of the integral exactly as (4.21), we will now obtain a partition of the form

$$\left\{\int_{\alpha H}^{\infty} + \int_{-\infty}^{-\alpha H} + \int_{-\alpha H}^{\alpha H}\right\} R_{\psi}\left(\frac{r}{2} + i(y-t)\right) R_{\phi}\left(\frac{r}{2} + i(y+t)\right) dy,$$

where $\frac{1}{4} < \alpha \leq \frac{1}{2}$. Estimates of the first two integrals will be similar to the already given (4.22) and by the contradiction hypothesis, we are allowed to take the modulus inside the third integral as in (4.20) and then, just as in (4.28), we can use Cauchy's Theorem to arrive at a lower bound. At this point, since the amplitude of the interval considered for $\mathscr{A}_3(t)$ is $2\alpha H$, the quantity $H(T)$ needs to be chosen so that it must be greater than the estimates for the integrals along the horizontal segments. An immediate adaptation of the steps leading to (4.30) will give a lower bound for $\mathscr{Z}_2(r+2it; a; \lambda)$ of the form

$$|\mathscr{Z}_2(r+2it; a; \lambda)| > B' T^{-\frac{1}{2}} H, \quad T + \frac{H}{4} < t \leq T + \frac{H}{2},$$

for some positive B'. An inductive reasoning similar to the one given in Claim 4.1 will give the lower bound

$$|\mathscr{Z}_{2^{k+1}}\left(2^k r + 2it; a; \lambda\right)| > B' T^{-\frac{2^{k+1}-1}{2}} H^{2^{k+1}-1}, \quad T + \frac{H}{2^{k+2}} < t \leq T + \frac{H}{2^{k+1}} \tag{4.57}$$

valid for all $k \in \mathbb{N}_0$. If our Dirichlet series is such that (4.53) holds, the choice $H(T) = T^{\sigma_a + \frac{1-r}{2} + \varepsilon}$ satisfies all of our requirements. In fact, under the hypothesis (4.53), (4.57) will contradict the Phragmén-Lindelöf bound (4.54) for $\mathscr{Z}_{2^{k+1}}$. □

Remark 4.8. Although (4.56) is valid in the general conditions of Corollary 4.4, in most of the particular cases direct adaptations of the method described in Theorem 4.1 and Corollary 4.5 yield far better estimates than (4.56) (see Example 5.4 below). This happens when we have enough information regarding the Epstein zeta functions associated with $\phi(s)$.

5 A class of examples and identities arising from the Selberg-Chowla formula

Example 5.1. (A Generalized Watson formula) On Taylor's paper regarding the functional for the Epstein zeta function and its Selberg-Chowla representation, formula (1.2) was attributed to Kober [[89], p. 182]. Although Kober [63] hadn't the explicit purpose of studying the analytic continuation of the Epstein zeta function, he did study a generalization of a previous formula due to Watson [[95], eq. (4)]. By using the Poisson summation formula, G. N. Watson proved that, for $\text{Re}(s)>0$ and $x>0$, the following identity holds

$$\sum_{n=1}^{\infty} n^{s}K_{s}(2\pi nx)+\frac{1}{4}(\pi x)^{-s}\Gamma(s)-\frac{\sqrt{\pi}}{4}(\pi x)^{-s-1}\Gamma\left(s+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2x}\left(\frac{x}{\pi}\right)^{s+1}\Gamma\left(s+\frac{1}{2}\right)\sum_{n=1}^{\infty}\frac{1}{(n^{2}+x^{2})^{s+\frac{1}{2}}}. \tag{5.1}$$

In this first example, we look at generalizations of (5.1). These generalizations also include Kober's formulas given in [63]. Looking at our derivation of formulas (3.11) and (3.12) on Theorem 3.1, it is not hard to write generalizations of Watson's formula for Bochner Dirichlet series. Let $\phi(s)=\sum a(n)\lambda_{n}^{-s}$, $\text{Re}(s)>\sigma_{a}$, be a Bochner Dirichlet series belonging to the class $\mathscr{B}$. If ν is a complex number satisfying $\text{Re}(\nu)>\frac{\sigma_{a}}{2}$, consider the infinite series

$$\mathscr{S}(\nu):=\sum_{n\in\mathbb{Z}}\frac{a(n)}{\left(a\lambda_{n}^{2}+b\lambda_{n}+c\right)^{\nu}},$$

where the denominator is positive definite, i.e., $-d=4ac-b^{2}>0$, $a>0$. By looking at the proof of Theorem 3.1, if we perform a summation with respect to only one variable, we obtain the following formulas

$$\begin{aligned}-2\phi(0)\,c^{-\nu}+\sum_{n\neq0}\frac{a(n)}{\left(a\lambda_{n}^{2}+b\lambda_{n}+c\right)^{\nu}}&=\rho\sqrt{\pi}\,\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma(\nu)}a^{-\nu}k^{1-2\nu}\\&+\frac{4k^{\frac{1}{2}-\nu}a^{-\nu}}{\Gamma(\nu)}\sum_{n=1}^{\infty}b(n)\cos\left(\frac{b}{a}\mu_{n}\right)\mu_{n}^{\nu-\frac{1}{2}}K_{\nu-\frac{1}{2}}(2k\mu_{n}),\quad a(n)\text{ even},\end{aligned} \tag{5.2}$$

$$\sum_{n\neq0}\frac{a(n)}{\left(a\lambda_{n}^{2}+b\lambda_{n}+c\right)^{\nu}}=-\frac{4k^{\frac{1}{2}-\nu}a^{-\nu}}{\Gamma(\nu)}\sum_{n=1}^{\infty}b(n)\sin\left(\frac{b}{a}\mu_{n}\right)\mu_{n}^{\nu-\frac{1}{2}}K_{\nu-\frac{1}{2}}(2k\mu_{n}),\quad a(n)\text{ odd}, \tag{5.3}$$

The first of these identities generalize Watson's formula and its extension by Kober [[63], p. 614, eq. (2b)] . Furthermore, the Selberg-Chowla formula (3.21) given in Remark 3.5 gives another formula of Watson-type

$$\begin{aligned}\sum_{n\neq0}\frac{a(n)\,\lambda_{n}}{\left(a\lambda_{n}^{2}+b\lambda_{n}+c\right)^{\nu}}&=\frac{4k^{\frac{3}{2}-\nu}a^{-\nu}}{\Gamma(\nu)}\sum_{n=1}^{\infty}b(n)\mu_{n}^{\nu-\frac{1}{2}}\\&\times\left\{\cos\left(\frac{b}{a}\mu_{n}\right)K_{\frac{3}{2}-\nu}(2k\mu_{n})+\frac{2b}{\sqrt{|d|}}\sin\left(\frac{b}{a}\mu_{n}\right)K_{\frac{1}{2}-\nu}(2k\mu_{n})\right\},\quad a(n)\text{ odd}.\end{aligned} \tag{5.4}$$

Let χ be a non-principal and primitive Dirichlet character modulo ℓ: since $\zeta(s)$ and $L(s,\chi)$ both satisfy functional equations of Bochner-type, we can replace the sequences appearing in (5.2) and (5.4) by those of these Dirichlet series. For example, if $\phi(s)=\zeta(s)$ and $b=0$, (5.2) reduces to an equivalent form of Watson's formula (5.1). Under the same hypothesis of diagonal $Q(x,y)$, if we replace, respectively, $a(n)$ by $\chi(n)$ in (5.2) and (5.4) we obtain, respectively, formulas (2.9) and (2.10) given in [[20], p. 3, Thm 2.1].

When $\phi(s)=\zeta(s)$, the continuation of (5.2) $\nu=0$ was derived by Watson [[95], p. 300]. It is also worthy to note that Guinand [[43], p. 600] derived this continuation by appealing to a L_2 analogue of the Poisson summation formula.

Example 5.2 (Generalization of Ramanujan-Guinand formula and direct corollaries)**.** In his Lost Notebook [23], Ramanujan recorded the following formula. Let $s\in\mathbb{C}$ and assume that $\alpha,\beta>0$ are such that $\alpha\beta=\pi^2$. Then the following identity holds

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\beta)=$$
$$\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\left\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\right\}+\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\left\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\right\}. \tag{5.5}$$

This formula was rediscovered by Guinand in 1955 [46], who employed Watson's formula (5.1) to obtain it.

In this example, we give a generalization of the Ramanujan-Guinand formula using a similar idea. To do so, for $\xi>0$, recall the analogue of the Epstein zeta function $\mathscr{Z}_2(s;a_1,a_2;\lambda,\xi\lambda')$ given in (4.1). By using the first Selberg-Chowla representation (2.28), we have deduced (4.2). By using the same substitutions there mentioned and using the second Selberg-Chowla formula (2.29), we can analogously derive

$$\Gamma(s)\sum_{m,n\neq0}^{\infty}\frac{a_1(m)\,a_2(n)}{(\lambda_m+\lambda_n'\xi)^s}=-\xi^{-s}\,\phi_1(0)\,\Gamma(s)\,\phi_2(s)+\xi^{-r_2}\rho_2\,\Gamma(r_2)\,\Gamma(s-r_2)\,\phi_1(s-r_2)$$
$$+\,2\,\xi^{-\frac{r_2+s}{2}}\sum_{m,n=1}^{\infty}b_2(m)\,a_1(n)\left(\frac{\mu_m'}{\lambda_n}\right)^{\frac{s-r_2}{2}}K_{r_2-s}\left(2\sqrt{\frac{\mu_m'\lambda_n}{\xi}}\right). \tag{5.6}$$

Viewed in this way, the Selberg-Chowla formula not only provides the analytic continuation of a Dirichlet series $\mathscr{Z}_2(s,;a_1,a_2;\lambda,\xi\lambda')$ depending on $\xi>0$ but it also gives a reformulation of this continuation in terms of a formula of a modular type. A comparison between (4.2) and (5.6) yields a generalization of the Ramanujan-Guinand formula (5.5), valid for all $s\in\mathbb{C}$ by Proposition 2.1 and the assumptions on the class $\mathscr{A}$.

Stated in a clear way, let $\phi_i(s)$, $i=1,2$ represent the pair of Dirichlet series given in (1.9) which also satisfy Hecke's functional equation (1.7) and belong to the class $\mathscr{A}$. Denote also by ρ_i the residue that ϕ_i

has at $s=r_i$. Then, for all $s\in\mathbb{C}$ and $x>0$, the following generalization of Guinand's formula holds

$$2x^{r_1-s}\sum_{j=1}^{\infty}\sigma_{s-r_1}\left(\nu_j;b_1,a_2\right)\nu_j^{\frac{r_1-s}{2}}K_{r_1-s}\left(2x\sqrt{\nu_j}\right)-\frac{2}{x^{r_2+s}}\sum_{j=1}^{\infty}\sigma_{s-r_2}\left(\nu_j';b_2,a_1\right)\nu_j'^{\frac{r_2-s}{2}}K_{r_2-s}\left(\frac{2\sqrt{\nu_j'}}{x}\right)$$
$$=\Gamma(s)\left\{\phi_2(0)\phi_1(s)-\phi_1(0)\phi_2(s)x^{-2s}\right\}+\rho_2x^{-2r_2}\Gamma(r_2)\Gamma(s-r_2)\phi_1(s-r_2)-\rho_1x^{2r_1-2s}\Gamma(r_1)\Gamma(s-r_1)\phi_2(s-r_1),\tag{5.7}$$

where σ_z denotes the generalized weighted divisor function described by (2.23).

Note that, if $\phi_1(s)=\phi_2(s)=\pi^{-s}\zeta(2s)$, then $r_1=r_2=\frac{1}{2}$ and an application of (5.7) gives (5.5) after one takes $x=\frac{\alpha}{\pi}$.

Berndt [[10], p. 343, eq. (9.1)] established other analogues of Guinand's formula. But these formulas of Berndt are only valid for arithmetical functions whose Dirichlet series satisfy Hecke's functional equation, while in general the arithmetical function of divisor type $c_z(j):=\sigma_z(\nu_j;b_1,a_2)$ does not give rise to a Dirichlet series satisfying (1.7). In fact, the Dirichlet series attached to this general divisor function is actually $\varphi_z(s):=\psi_1(s-z)\,\phi_2(s)$ and the assumption that ψ_1 and ϕ_2 satisfy Hecke's functional equation (1.7) allows to write a functional equation (involving a product of two $\Gamma-$functions) for $\varphi_z(s)$. Thus, (5.7) is actually equivalent to this functional equation that $\varphi_z(s)$ possesses (see also [[10], p. 324]).

From (5.7) it is possible to derive generalizations of the classical Koshliakov formula [64], by mimicking the steps given in [23]. Writing the Laurent series for $\phi_i(s)$ around $s=r_i=r$ as

$$\phi_i(s)=\frac{\rho_i}{s-r}+\rho_{0,i}+O\left(s-r\right),\quad i=1,2,\tag{5.8}$$

assuming the same condition as the ones given in Ramanujan-Guinand's formula (5.7) and taking $r_1=r_2=r$ we have, for any $x>0$, that the following formula of Koshliakov type holds

$$2\sum_{j=1}^{\infty}d\left(\nu_j;\,b_1,\,a_2\right)K_0\left(2x\sqrt{\nu_j}\right)-2x^{-2r}\sum_{j=1}^{\infty}d\left(\nu_j';\,b_2,\,a_1\right)K_0\left(\frac{2}{x}\sqrt{\nu_j'}\right)$$
$$=\Gamma(r)\left\{\phi_2(0)\rho_{0,1}+\phi_2(0)\,\rho_1\,\frac{\Gamma'(r)}{\Gamma(r)}-\rho_1\,\left\{\rho_2^{\star}\Gamma'(r)+\rho_{0,2}^{\star}\Gamma(r)\right\}+2\,\log(x)\,\phi_2(0)\,\rho_1\right\}+\tag{5.9}$$
$$+x^{-2r}\Gamma(r)\left\{\left\{\rho_1^{\star}\Gamma'(r)+\rho_{0,1}^{\star}\Gamma(r)\right\}\rho_2-\phi_1(0)\,\rho_{0,2}-\phi_1(0)\,\rho_2\,\frac{\Gamma'(r)}{\Gamma(r)}+2\log(x)\,\phi_1(0)\,\rho_2\right\},$$

where $\rho_i^{\star}$ and $\rho_{0,i}^{\star}$ denote the coefficients of the meromorphic expansion (5.8) for $\psi_i(s)$ and $d(\nu_j;b_1,a_2)$ is defined by (2.25). We obtain (5.9) by letting $s\to r$ in (5.7) and using the meromorphic expansions for $\Gamma(s)$ and $\phi_i(s)$ around $s=0$, (4.4) and (4.5), as well as (5.8).

Special cases of (5.7) and (5.9) are also possible to obtain for Dirichlet series in the Bochner class. Another important corollary of Koshliakov's formula (or, in fact, a reformulation of it for $r=\frac{1}{2}$ [71, 87]) is due to K. Soni. According to [23], Soni's formula already appeared in Ramanujan's lost notebook (see also a multidimensional analogue of Soni's formula in [[101], p. 813, eq. (3.12)]).

Indeed, taking $r=\frac{1}{2},\frac{3}{2}$ in (5.9) and mimicking the steps in [23, 87], it is possible to generalize Soni's formula [[87], p. 543, eq. (4)] and some character analogues appearing in [20]. For the case where $r=\frac{1}{2}$ and assuming convergence of all the infinite series involved, we have the following formula

$$
\begin{aligned}
2\sum_{j=1}^{\infty} d\left(\nu_j;b_1,a_2\right)\frac{\log\left(\alpha/2\sqrt{\nu_j}\right)}{\alpha^2-4\nu_j}-\frac{2\pi}{\alpha}\sum_{j=1}^{\infty} d\left(\nu_j';b_2,a_1\right)K_0\left(2\sqrt{2}\nu_j'^{\frac{1}{4}}\alpha^{\frac{1}{2}}\right)& \\
=\frac{\sqrt{\pi}}{\alpha^2}\left\{\phi_2(0)\,\rho_{0,1}-\sqrt{\pi}\rho_1\rho_{0,2}^{\star}-2\rho_1\phi_2(0)\,(2\gamma+\log(2\alpha))\right\}& \qquad (5.10)\\
+\frac{\pi^{\frac{3}{2}}}{2\alpha}\left\{-\phi_1(0)\rho_{0,2}+\sqrt{\pi}\rho_2\rho_{0,1}^{\star}+2\rho_2\phi_1(0)\log\left(\frac{2}{\alpha}\right)\right\},&
\end{aligned}
$$

where $\alpha\in\mathbb{R}_+\setminus\{2\sqrt{\nu_j}\}_{j\in\mathbb{N}}$.

Example 5.3 (A character analogue of the Epstein zeta function)**.** In this example we study two particular cases of the non-diagonal Epstein zeta function considered in section 3. Let Q denote an integral, positive definite and binary Quadratic form and let χ_1 and χ_2 be two non-principal, primitive Dirichlet characters (with the same parity) having moduli ℓ_1 and ℓ_2. We introduce the character analogue of the Epstein zeta function in the following form

$$Z_2(s,Q,\chi_1,\chi_2)=\sum_{(m,n)\neq(0,0)}\frac{\chi_1(m)\,\chi_2(n)}{Q(m,n)^s},\ \ \mathrm{Re}(s)>1. \qquad (5.11)$$

Let us assume first that χ_1 and χ_2 are even Dirichlet characters and let us take in Theorem 3.1 $\phi_1(s)=\pi^{-\frac{s}{2}}L(s,\chi_1)$ and $\phi_2(s)=\pi^{-\frac{s}{2}}L(s,\chi_2)$. The functional equation for $L(s,\chi)$ is given by [29]

$$\left(\frac{\pi}{\ell}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi)=\frac{G(\chi)}{\sqrt{\ell}}\left(\frac{\pi}{\ell}\right)^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)L(1-s,\overline{\chi}), \qquad (5.12)$$

where $G(\chi)$ denotes the Gauss sum associated to the character χ,

$$G(\chi)=\sum_{r=1}^{\ell-1}\chi(r)\,e^{2\pi ir/\ell}. \qquad (5.13)$$

Thus $\phi_1(s),\phi_2(s)\in\mathscr{B}$ with $\delta=0$ and in order to apply our Theorem 3.1 we need to perform the following substitutions in (3.11):

$$a_j(n)=\chi_j(n),\ \ b_j(n)=\frac{G(\chi_j)}{\ell_j}\overline{\chi}_j(n),\ \ \lambda_{n,j}=\sqrt{\pi}n,\ \ \mu_{n,j}=\frac{\sqrt{\pi}n}{\ell_j}. \qquad (5.14)$$

Analogously, if χ_1 and χ_2 are odd and primitive Dirichlet characters having moduli ℓ_1 and ℓ_2, let us take $\phi_1(s)=\pi^{-\frac{s}{2}}L(s,\chi_1)$ and $\phi_2(s)=\pi^{-\frac{s}{2}}L(s,\chi_2)$. The functional equation for a Dirichlet $L-$function attached to an odd and primitive character χ modulo ℓ reads [29]

$$\left(\frac{\pi}{\ell}\right)^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi)=-\frac{iG(\chi)}{\sqrt{\ell}}\left(\frac{\pi}{\ell}\right)^{\frac{s-2}{2}}\Gamma\left(\frac{2-s}{2}\right)L(1-s,\overline{\chi}), \qquad (5.15)$$

which implies that $\phi_1, \phi_2 \in \mathscr{B}$ with $\delta = 1$. In order to use Theorem 3.1, we also take the substitutions

$$a_j(n) = \chi_j(n), \;\; b_j(n) = -\frac{iG(\chi_j)}{\ell_j}\overline{\chi}_j(n), \;\; \lambda_{n,j} = \sqrt{\pi}n, \;\; \mu_{n,j} = \frac{\sqrt{\pi}n}{\ell_j}. \tag{5.16}$$

After a simple application of Theorem 3.1 and Corollary 3.1 and using the substitutions outlined in (5.14) and (5.16), we see that $Z_2(s, Q, \chi_1, \chi_2)$ has a continuation to the complex plane as an entire function.

Indeed, it possesses the following Selberg-Chowla formulas:

1. If χ_1 and χ_2 are even, then an application of the Selberg-Chowla formulas (3.11) and (3.13) yields

$$Z_2(s,Q,\chi_1,\chi_2) = \frac{8\pi^s a^{-s}}{\Gamma(s)}G(\chi_1)k^{\frac{1}{2}-s}\ell_1^{-(s+\frac{1}{2})}\sum_{n=1}^{\infty}\sigma_{1-2s}(n,\overline{\chi}_1,\chi_2)n^{s-\frac{1}{2}}\cos\left(\pi\frac{b}{a}\frac{n}{\ell_1}\right)K_{\frac{1}{2}-s}\left(\frac{2\pi kn}{\ell_1}\right), \tag{5.17}$$

as well as

$$Z_2(s,Q,\chi_1,\chi_2) = \frac{8\pi^s c^{-s}}{\Gamma(s)}G(\chi_2)k'^{\frac{1}{2}-s}\ell_2^{-(s+\frac{1}{2})}\sum_{n=1}^{\infty}\sigma_{1-2s}(n,\overline{\chi}_2,\chi_1)n^{s-\frac{1}{2}}\cos\left(\pi\frac{b}{c}\frac{n}{\ell_2}\right)K_{\frac{1}{2}-s}\left(\frac{2\pi k'n}{\ell_2}\right). \tag{5.18}$$

2. If χ_1 and χ_2 are odd and primitive, then we equivalently have the Selberg-Chowla formula (from (3.12))

$$Z_2(s,Q,\chi_1,\chi_2) = -\frac{8i\pi^s a^{-s}}{\Gamma(s)}G(\chi_1)k^{\frac{1}{2}-s}\ell_1^{-(s+\frac{1}{2})}\sum_{n=1}^{\infty}\sigma_{1-2s}(n,\overline{\chi}_1,\chi_2)n^{s-\frac{1}{2}}\sin\left(\pi\frac{b}{a}\frac{n}{\ell_1}\right)K_{\frac{1}{2}-s}\left(\frac{2\pi kn}{\ell_1}\right) \tag{5.19}$$

with an analogous second Selberg-Chowla formula taking place.

In all of the previous representations, $\sigma_z(n, \chi_a, \chi_b)$ is a character analogue of the divisor function, defined by

$$\sigma_z(n, \chi_a, \chi_b) = \sum_{d|n}\chi_a(d)\,\chi_b\left(\frac{n}{d}\right)d^z. \tag{5.20}$$

Moreover, it follows from Corollary 3.1 that $Z_2(s; Q; \chi_1, \chi_2)$ satisfies the functional equation

$$G(\overline{\chi}_1)\,G(\overline{\chi}_2)\left(\frac{2\pi}{\sqrt{|d|}}\right)^{-s}\Gamma(s)\,Z_2(s, Q, \chi_1, \chi_2) = \left(\frac{2\pi}{\sqrt{|d|}}\right)^{s-1}\Gamma(1-s)\,Z_2\left(1-s, Q^{-1}_{\ell_1,\ell_2}, \overline{\chi}_1, \overline{\chi}_2\right), \tag{5.21}$$

where $Q^{-1}_{\ell_1,\ell_2}$ denotes the quadratic form $Q^{-1}\left(\frac{x}{\ell_1}, \frac{y}{\ell_2}\right)$.

By using the previous example, we can establish analogues of the Ramanujan-Guinand formula for the character analogue of the divisor function (5.20). We only do this for the first case, i.e., assuming that the characters χ_1 and χ_2 are even. It is effortless to see that the comparison of (5.17) and (5.18) and the substitution $a = c = 1$, $b = 0$, yield the double-weighted Ramanujan-Guinand's formula

$$\sqrt{\alpha}\, G(\chi_1)\, \ell_1^{-\frac{s}{2}-1} \sum_{n=1}^{\infty} \sigma_{-s}(n, \overline{\chi}_1, \chi_2)\, n^{s/2} K_{\frac{s}{2}}\left(\frac{2\alpha n}{\ell_1}\right) = \sqrt{\beta}\, G(\chi_2)\, \ell_2^{-\frac{s}{2}-1} \sum_{n=1}^{\infty} \sigma_{-s}(n, \overline{\chi}_2, \chi_1)\, n^{s/2} K_{\frac{s}{2}}\left(\frac{2\beta n}{\ell_2}\right), \tag{5.22}$$

where $\alpha, \beta > 0$ are such that $\alpha\beta = \pi^2$. Note that (5.22) can also be obtained from (5.7) and by using the fact that $\pi^{-s} L(2s, \chi) \in \mathscr{A}$ and satisfies Hecke's functional equation with $r = \frac{1}{2}$.

Taking the limit $s \to 0$ in (5.22) yields Koshliakov's formula for double characters. To write it in the form (5.9), take the substitution $\alpha = \pi\sqrt{\frac{\ell_1}{\ell_2}}\, x$ and replace $\overline{\chi_1}$ by χ_1. Use also the relation for Gauss sums [29],

$$G(\chi)\, G(\overline{\chi}) = \chi(-1)\, \ell. \tag{5.23}$$

Then a particular case of (5.9) is formulated as

$$\sum_{n=1}^{\infty} d_{\chi_1,\chi_2}(n)\, K_0\left(\frac{2\pi n x}{\sqrt{\ell_1 \ell_2}}\right) = \frac{G(\chi_1)\, G(\chi_2)}{x\sqrt{\ell_1 \ell_2}} \sum_{n=1}^{\infty} d_{\overline{\chi}_1,\overline{\chi}_2}(n)\, K_0\left(\frac{2\pi n}{x\sqrt{\ell_1 \ell_2}}\right), \quad \chi_1, \chi_2 \text{ even}, \tag{5.24}$$

where $d_{\chi_1,\chi_2}(n)$ represents (5.20) at $z = 0$, this is, the double-weighted divisor function, $\sum_{d|n} \chi_1(d)\, \chi_2(n/d)$.

Identity (5.24) is given in [[22], p. 45, eq. (3.7.)]. From (5.20), one can also derive character analogues of Soni's formula: applying (5.10) gives (after replacing also α by $2\pi\sqrt{\ell_1 \ell_2}\, \alpha$)

$$\sum_{n=1}^{\infty} \frac{d_{\chi_1,\chi_2}(n)\, \log\left(\frac{\ell_1 \ell_2 \alpha}{n}\right)}{(\ell_1 \ell_2 \alpha)^2 - n^2} = \frac{2\pi^2 G(\chi_1)\, G(\chi_2)}{\alpha\, \ell_1^2\, \ell_2^2} \sum_{n=1}^{\infty} d_{\overline{\chi}_1,\overline{\chi}_2}(n)\, K_0\left(4\pi\sqrt{\alpha n}\right), \tag{5.25}$$

which is valid for every $\alpha \neq n/\ell_1 \ell_2$, $n \in \mathbb{N}$. A particular case of (5.25) where $\chi_1(n) = \chi_2(n) = \chi(n)$, $\ell_1 = \ell_2 = \ell$, has been observed in [[20], p. 5, eq. (3.8)] and is equivalent to

$$\sum_{n=1}^{\infty} \frac{\chi(n)\, d(n)\, \log\left(\frac{\alpha \ell^2}{n}\right)}{\alpha^2 \ell^4 - n^2} = \frac{2\pi^2 G(\chi)^2}{\alpha \ell^4} \sum_{n=1}^{\infty} \overline{\chi}(n)\, d(n)\, K_0(4\pi\sqrt{\alpha n}).$$

Extensions of other formulas given in [20] are also possible. For instance, it is not hard to derive generalizations of [[20], eq. (3.15), (3.16) and (3.19)] for the character analogue of the divisor function $d_{\chi_1,\chi_2}(n)$.

Some particular cases of (5.22) include identities derived by Dixit [33] by an entirely different method [[33], p. 322, Theorem 1.5., eq. (1-15)], employing the symmetric properties of an integral involving Riemann's $\Xi-$function and a character analogue of it. Our formula (5.22) applied to the case $\chi_1 = \chi_2 = \chi$ yields $F(z, \alpha, \chi) = F(-z, \beta, \overline{\chi})$ in formula (1-15) of [33]. Although Dixit's results may seem to be of a different nature, the integral representation involving the Riemann $\Xi-$function appearing in his paper [[33], eq. (1-14)] can be generalized to our class of Dirichlet series. In fact, it is a generalization of Dixit's integral representation that we employ to study the zeros of the Dirichlet series in Theorem 4.1, since we can represent the entire function $H_{r_i}(s;b_i;a_i)$ in a similar form (c.f. eq. (4.12) and (4.15) in the previous section).

Example 5.4 (Hardy's Theorem and the 4-square Theorem)**.** Based upon Theorem 4.1, this example gives a new proof of Hardy's Theorem for $\zeta(s)$, which is curious as its conclusion is derived from presumably independent properties of the arithmetical function $r_4(n)$. As it is well known, $r_4(n)$ counts the number of representations of n as a sum of 4 squares with different signs and different orders of the summands giving distinct representations. Let us invoke Theorem 4.1 to $\phi(s)=\pi^{-s}\zeta(2s)$, which satisfies Hecke's functional equation with $r=\frac{1}{2}$. If $\phi(s)$ does not have infinitely many zeros on the critical line $\text{Re}(s)=\frac{1}{2}$, then all of its dyadic Epstein zeta functions (2.44) will only have finitely many zeros at their critical lines. Consider $\mathscr{Z}_4(s)$: it comes immediately from the definition of multidimensional Epstein zeta function that, for $\text{Re}(s)>2$,

$$\mathscr{Z}_4(s)=\pi^{-s}\sum_{m_1,\ldots,m_4\neq 0}^{\infty}\frac{1}{\left(m_1^2+m_2^2+m_3^2+m_4^2\right)^s}=\frac{\pi^{-s}}{4}\sum_{n=1}^{\infty}\frac{r_4(n)}{n^s}=\frac{\pi^{-s}}{4}\zeta_4(s).$$

Hence, according with our assumption, $\zeta_4(s)$ cannot have infinitely many zeros at its critical line $\text{Re}(s)=1$. However, by Jacobi's four square theorem [54], $\zeta_4(s)$ is described by

$$\zeta_4(s)=8\left(1-2^{2-2s}\right)\zeta(s)\,\zeta(s-1). \tag{5.26}$$

Note that the right-hand side of (5.26) has infinitely many zeros in the line $\text{Re}(s)=1$, all of them being of the form $1\pm\frac{\pi i}{\log(2)}k$, with $k\in\mathbb{N}$. This means that $\mathscr{Z}_4(s)$ has infinitely many zeros on the line $\text{Re}(s)=1$ and a contradiction is derived, proving Hardy's Theorem in its classical form.

The novelty in this proof is the observation that a purely arithmetical property such as (5.26) implies a deep analytic theorem regarding the distribution of the zeros of the Riemann zeta function. One may argue that since (5.26) admits proofs using modular forms and a particular modular form is used in Hardy's proof of Hardy's Theorem [48], the reason should be this. Nonetheless, we should remark that there are elementary proofs of (5.26) available, such as in [53].

This observation can also be used to improve the quantitative result given in Corollary 4.5. For the case where $\phi(s)=\pi^{-s}\zeta(2s)$ or $\pi^{-\frac{s-\delta}{2}}L(2s-\delta,\chi)$, for χ being a primitive Dirichlet character modulo ℓ, the result on Corollary 4.5 gives $H(T)=T^{\frac{3}{4}+\varepsilon}$, which is a result attributed to Landau. The argument invoking the 4-square theorem can actually improve this result for $\zeta(s)$ to $H(T)=T^{\frac{1}{2}+\varepsilon}$ [40]. As it is clear from the brief sketch of the proof, the condition $H(T)=T^{\sigma_a+\frac{1-r}{2}+\varepsilon}$ is only necessary in order to (4.57) contradict the Phragmén-Lindelöf estimates (4.54) combined with the assumption (4.53). In fact, for the case where $\phi(s)=\pi^{-s}\zeta(2s)$ it suffices to assume that $H(T)=T^{\frac{1}{2}+\varepsilon}$ until the point we arrive at the condition (4.57). In this particular case, (4.57) gives the inequality

$$|\zeta_4(1+2it)| > B' \left(\frac{H}{T^{\frac{1}{2}}}\right)^3, \quad B' > 0 \tag{5.27}$$

valid for every $T + \frac{H}{8} < t \leq T + \frac{H}{4}$. This obviously contradicts (5.26), which implies that there is always a zero of $\zeta_4(1+it)$ in any interval of length $\frac{\pi}{\log(2)}$.

Other proofs of Hardy's Theorem for $\zeta(s)$ can be derived from corollaries 4.3 and 4.4, yielding even easier arguments. Note also that the Dirichlet L-function, $L(s,\chi)$, attached to a primitive character modulo ℓ also satisfies the conditions of Remark 4.6.

Kober [62] established the condition given in Corollary 4.2 for a class of Epstein zeta functions attached to binary and positive definite quadratic forms whenever $a/\sqrt{|d|}$ is irrational. For the case where $a/\sqrt{|d|}$ is rational, then by using a set of conditions similar to that given in Corollary 4.3, we are only left with a finite set of Epstein zeta functions for which we need to verify Hardy's Theorem (see [62], p. 8). In all of them, a similar argument mimicking the one using Jacobi's 4-square Theorem would suffice, but we would have to know apriori a formula for the representation of a given integer as the particular quaternary forms involved. The general condition given in (4.52) combined with remark 4.6 can be also used to prove Hardy-type theorems for character analogues of the Epstein zeta function of the form (5.11) with $\chi_1(n) = \chi_2(n) = \chi(n)$ and with the Quadratic form normalized in order to match the conditions of Theorem 4.1. By virtue of their functional equations [[90], p. 481, Thm 2] and remark 4.3, it is also possible to derive Hardy's Theorem for multidimensional Epstein zeta functions satisfying the symmetric conditions given in Theorem 4.1, as subconvex estimates at the critical line of the type (4.18) are already known for these Dirichlet series. See also [84] for very sharp results in this direction.

Example 5.5 (Character analogue for the divisor function and Dedekind zeta function)**.** Let χ be an odd and primitive Dirichlet character modulo ℓ. By the functional equations for $\zeta(s)$ and $L(s,\chi)$ (5.15), one can check that $\zeta(s)L(s,\chi)$ satisfies the functional equation

$$\left(\frac{2\pi}{\sqrt{\ell}}\right)^{-s}\Gamma(s)\,\zeta(s)\,L(s,\chi) = -\frac{iG(\chi)}{\sqrt{\ell}}\left(\frac{2\pi}{\sqrt{\ell}}\right)^{-(1-s)}\Gamma(1-s)\,\zeta(1-s)\,L(1-s,\overline{\chi}). \tag{5.28}$$

Let us write the associated Dirichlet series satisfying (5.28) as

$$\phi(s) = \zeta(s)L(s,\chi) = \sum_{n=1}^{\infty}\frac{d_\chi(n)}{n^s}, \quad \mathrm{Re}(s) > 1,$$

where $d_\chi(n)$ denotes the character analogue of the divisor function [[22], p. 42, eq. (3.1)],

$$d_\chi(n) = \sum_{d|n}\chi(d).$$

If χ_1 and χ_2 denote two odd and primitive Dirichlet characters modulo ℓ_1 and ℓ_2, we know by (5.28) that $\phi_j(s)=\zeta(s)L(s,\chi_j)$ satisfies Hecke's functional equation with $r_j=1,\ j=1,2$. Moreover, each $\phi_i(s)$ belongs to the class $\mathscr{A}$ and their residues at $s=1$ are simply $\rho_j=L(1,\chi_j)$. We define the diagonal Epstein zeta function for $d_{\chi_1}(n)$ and $d_{\chi_2}(n)$ as the double Dirichlet series

$$\mathscr{Z}_2\left(s;d_{\chi_1},d_{\chi_2}\right):=\sum_{m,n\neq0}^{\infty}\frac{d_{\chi_1}(m)\,d_{\chi_2}(n)}{(m+n)^s},\quad \mathrm{Re}(s)>2. \tag{5.29}$$

As in Example 5.3, in order to match the functional equation (5.28) with the formulations of Theorem 2.1 and Corollary 2.1, we need to perform the following substitutions for $\phi_j(s)$,

$$a_j(n)=d_{\chi_j}(n),\ b_j(n)=-\frac{2\pi iG(\chi_j)}{\ell_j}\,d_{\overline{\chi_j}}(n),\ \lambda_{n,j}=n,\ \mu_{n,j}=\frac{4\pi^2}{\ell_j}n.$$

Using the first Selberg-Chowla formula (2.28), we obtain the following continuation for (5.29)

$$\begin{aligned}\Gamma(s)\,\mathscr{Z}_2\left(s;d_{\chi_1},d_{\chi_2}\right)&=\frac{1}{2}L(0,\chi_2)\Gamma(s)\,\zeta(s)\,L(s,\chi_1)+L(1,\chi_1)\,\Gamma(s-1)\,L(s-1,\chi_2)\\&-\frac{4\pi iG(\chi_1)}{\ell_1^{\frac{s+1}{2}}}(2\pi)^{s-1}\sum_{m,n=1}^{\infty}d_{\overline{\chi_1}}(m)\,d_{\chi_2}(n)\left(\frac{m}{n}\right)^{\frac{s-1}{2}}K_{s-1}\left(\frac{4\pi}{\ell_1}\sqrt{mn}\right).\end{aligned}$$

Analogously, (2.29) gives a second Selberg-Chowla formula of the form

$$\begin{aligned}\Gamma(s)\,\mathscr{Z}_2\left(s;d_{\chi_1},d_{\chi_2}\right)&=\frac{1}{2}L(0,\chi_1)\Gamma(s)\,\zeta(s)\,L(s,\chi_2)+L(1,\chi_2)\,\Gamma(s-1)\,L(s-1,\chi_1)\\&-\frac{4\pi iG(\chi_2)}{\ell_2^{\frac{s+1}{2}}}(2\pi)^{s-1}\sum_{m,n=1}^{\infty}d_{\chi_1}(m)\,d_{\overline{\chi_2}}(n)\left(\frac{m}{n}\right)^{\frac{1-s}{2}}K_{s-1}\left(\frac{4\pi}{\ell_2}\sqrt{mn}\right).\end{aligned}$$

Both formulas given above provide the analytic continuation of (5.29) as a meromorphic function with a simple pole located at $s=2$ with residue $L(1,\chi_1)\cdot L(1,\chi_2)$. The functional equation for (5.29) can be written as

$$\left(\frac{2\pi}{\sqrt{\ell_1\ell_2}}\right)^{-s}\Gamma(s)\,\mathscr{Z}_2\left(s;d_{\chi_1},d_{\chi_2}\right)=-G(\chi_1)\,G(\chi_2)\left(\frac{2\pi}{\sqrt{\ell_1\ell_2}}\right)^{-(2-s)}\Gamma(2-s)\,\mathscr{Z}_2\left(2-s;\mathbb{I}_{\ell_1,\ell_2};d_{\overline{\chi}_1},d_{\overline{\chi}_2}\right), \tag{5.30}$$

where $\mathscr{Z}_2\left(s;\mathbb{I}_{\ell_1,\ell_2};d_{\overline{\chi}_1},d_{\overline{\chi}_2}\right)$ is described as the double Dirichlet series

$$\mathscr{Z}_2\left(s;\mathbb{I}_{\ell_1,\ell_2};d_{\overline{\chi}_1},d_{\overline{\chi}_2}\right):=\sum_{m,n\neq0}^{\infty}\frac{d_{\overline{\chi}_1}(m)\,d_{\overline{\chi}_2}(n)}{(\ell_2m+\ell_1n)^s},\quad \mathrm{Re}(s)>2.$$

It is also possible to construct analogues of Guinand's formula via the above Selberg-Chowla formulas for $\mathscr{Z}_2(s,d_{\chi_1},d_{\chi_2})$. Analogues of Koshliakov's formula can be also obtained, although we need some additional computations of the values of $L'(1,\chi)$ [30].

In the same spirit, for $i=1,2$, let $F_i(n)$ denote, as usual, the number of integral ideals of norm n in an imaginary quadratic number field with discriminant D_i, $K_i=\mathbb{Q}\left(\sqrt{-D_i}\right)$. Then it is well-known [24] that the associated Dirichlet series is the classical Dedekind zeta function

$$\zeta_{K_i}(s)=\sum_{n=1}^{\infty}\frac{F_i(n)}{n^s},\quad \mathrm{Re}(s)>1, \tag{5.31}$$

which satisfies Hecke's functional equation

$$\left(\frac{2\pi}{\sqrt{D_i}}\right)^{-s}\Gamma(s)\,\zeta_{K_i}(s)=\left(\frac{2\pi}{\sqrt{D_i}}\right)^{s-1}\Gamma(1-s)\,\zeta_{K_i}(1-s). \tag{5.32}$$

Since $\zeta_{K_i}(s)$ has a continuation to the entire complex plane as an analytic function except at a simple pole located at $s=1$, we see that, for $i=1,2$, $\zeta_{K_i}(s)\in\mathscr{A}$ with $r_i=1$. Moreover, the residue of $\zeta_{K_i}(s)$ at $s=1$ can be explicitly computed as

$$\rho_i=\frac{2\pi\,h(K_i)R(K_i)}{\sqrt{D_i}\,w(K_i)},$$

where $h(K_i)$, $R(K_i)$ and $w(K_i)$ denote, respectively, the class number of K_i, the regulator of K_i and the number of roots of unity in K_i. We now take $\phi_i(s):=\zeta_{K_i}(s)$: in order to apply Corollary 2.1, we need to take the following substitutions

$$a_i(n)=F_i(n),\ b_i(n)=\frac{2\pi}{\sqrt{D_i}}F_i(n),\ \lambda_{n,i}=n,\ \mu_n=\frac{4\pi^2}{D_i}n.$$

By the conditions on the class $\mathscr{A}$, we have $\phi_i(0)=-h(K_i)\,R(K_i)/w(K_i)$, so that we take $F_i(0):=h(K_i)\,R(K_i)/w(K_i)$. We now construct the diagonal Epstein zeta function attached to K_1 and K_2 as follows

$$\mathscr{Z}_2\left(s;K_1,K_2\right):=\sum_{m,n\neq 0}^{\infty}\frac{F_1(m)\,F_2(n)}{(m+n)^s},\quad \mathrm{Re}(s)>2. \tag{5.33}$$

Under the substitutions above mentioned, we have from the Selberg-Chowla formula (2.28) that the following identity holds for (5.33)

$$\begin{aligned}\Gamma(s)\,\mathscr{Z}_2\left(s;K_1,K_2\right)&=\frac{h(K_2)\,R(K_2)}{w(K_2)}\,\Gamma(s)\,\zeta_{K_1}(s)+\frac{2\pi\,h(K_1)R(K_1)}{\sqrt{D_1}\,w(K_1)}\,\Gamma(s-1)\,\zeta_{K_2}(s-1)\\&+2\left(\frac{2\pi}{\sqrt{D_1}}\right)^{s}\sum_{m,n=1}^{\infty}F_1(m)\,F_2(n)\,\left(\frac{m}{n}\right)^{\frac{s-1}{2}}\,K_{s-1}\left(\frac{4\pi\sqrt{mn}}{D_1}\right),\end{aligned}$$

giving the analytic continuation of $\mathscr{Z}_2(s;K_1,K_2)$ as a meromorphic function with one simple pole located at $s=2$ whose residue is precisely

$$\mathrm{Res}_{s=2}\mathscr{Z}_2(s;K_1,K_2)=\frac{4\pi^2}{\sqrt{D_1D_2}}\,\frac{h(K_1)h(K_2)\,R(K_1)R(K_2)}{w(K_1)w(K_2)}.$$

Moreover, if we write a second Selberg-Chowla for (5.33), we can deduce that it satisfies the following functional equation:

$$\left(\frac{2\pi}{\sqrt{D_1D_2}}\right)^{-s}\Gamma(s)\,\mathscr{Z}_2\left(s;K_1,K_2\right)=\sqrt{D_1D_2}\left(\frac{2\pi}{\sqrt{D_1D_2}}\right)^{-(2-s)}\Gamma(2-s)\,\mathscr{Z}_2\left(s;\mathbb{I}_{D_1,D_2},K_1,K_2\right),$$

where, just as in (5.30), $\mathscr{Z}_2\left(s;\mathbb{I}_{D_1,D_2},K_1,K_2\right)$ denotes the Dirichlet series

$$\mathscr{Z}_2\left(s;\mathbb{I}_{D_1,D_2},K_1,K_2\right):=\sum_{m,n\neq 0}^{\infty}\frac{F_1(m)\,F_2(n)}{(D_2m+D_1n)^s},\quad \mathrm{Re}(s)>2. \tag{5.34}$$

If, for the quadratic fields K_1 and K_2, we define an analogue of the divisor function, $\sigma_\nu(n;K_1,K_2)$, in the following way

$$\sigma_\nu(n;K_1,K_2)=\sum_{d|n}F_1(d)\,F_2\left(\frac{n}{d}\right)d^\nu, \tag{5.35}$$

we have that, for any $x>0$, the identity of Ramanujan-Guinand type (5.7) holds

$$\begin{aligned}&\left(\frac{2\pi}{\sqrt{D_1}}\right)^s x^{1-s}\sum_{n=1}^{\infty}\sigma_{s-1}\left(n;K_1,K_2\right)n^{\frac{1-s}{2}}K_{s-1}\left(\frac{4\pi\sqrt{n}}{\sqrt{D_1}}x\right)\\ &-\left(\frac{2\pi}{\sqrt{D_2}}\right)^s x^{-1-s}\sum_{m,n=1}^{\infty}\sigma_{s-1}\left(n;K_2,K_1\right)n^{\frac{1-s}{2}}K_{s-1}\left(\frac{4\pi\sqrt{n}}{\sqrt{D_2}\,x}\right)\\ &=\frac{h(K_1)\,R(K_1)\,x^{-2s}}{2\,w(K_1)}\left\{\Gamma(s)\,\zeta_{K_2}(s)-\frac{2\pi x^2}{\sqrt{D_1}}\Gamma(s-1)\,\zeta_{K_2}(s-1)\right\}\\ &+\frac{h(K_2)\,R(K_2)}{2\,w(K_2)}\left\{\frac{2\pi}{\sqrt{D_2}\,x^2}\Gamma(s-1)\,\zeta_{K_1}(s-1)-\Gamma(s)\,\zeta_{K_1}(s)\right\}.\end{aligned} \tag{5.36}$$

From the Kronecker limit formula for imaginary quadratic fields, it is also possible to take the limit $s\to 1$ on (5.36) in order to derive analogues of Koshliakov's formula for the divisor function (5.35). One dimensional analogues of (5.36) are also given in the very recent preprint [19].

Example 5.6 (Dirichlet series attached to Cusp Forms)**.** Let f be a cusp form with weight $k\geq 12$ (with k always being an even integer) for the full modular group and $L(s,a)$ the associated $L-$function,

$$L(s,a)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}, \tag{5.37}$$

where the arithmetical functions $a(n)$ represent the Fourier coefficients of f. Then (5.37) is known to be absolutely convergent in the half-plane $\mathrm{Re}(s)>\frac{k+1}{2}$ and can be analitically continued to an entire function [[2], Theorem 6.20]. Furthermore, it obeys to Hecke's functional equation

$$(2\pi)^{-s}\Gamma(s)\,L(s,a)=(-1)^{k/2}(2\pi)^{-(k-s)}\Gamma(k-s)\,L(k-s,a). \tag{5.38}$$

Let us now consider two holomorphic cusp forms f_1 and f_2 having weights k_1 and k_2 respectively and the pairs of Dirichlet series $\phi_i(s)=(2\pi)^{-s}L(s,a_i)$ and $\psi_i(s)=(-1)^{k_i/2}(2\pi)^{-s}L(s,a_i)$, $i=1,2$, associated with them. Here, the substitutions analogous to (5.14) are obvious to consider. Since each $\phi_i(s)$ is entire, it follows from Corollary 2.1 that its (diagonal) Epstein zeta function given by

$$\mathscr{Z}_2(s;f_1,f_2)=(2\pi)^{-s}\sum_{m,n\neq 0}^{\infty}\frac{a_1(m)\,a_2(n)}{(m+n)^s}=(2\pi)^{-s}Z_2(s;f_1,f_2),\quad \mathrm{Re}(s)>\max\{k_1,k_2\}+1, \tag{5.39}$$

is also entire and its continuation satisfies the Selberg-Chowla formulas

$$(2\pi)^{-s}\Gamma(s)\,Z_2\,(s;\,f_1,\,f_2) = 2\,(-1)^{\frac{k_1}{2}}\sum_{m,n=1}^{\infty} a_1(m)\,a_2(n)\,\left(\frac{m}{n}\right)^{\frac{s-k_1}{2}} K_{k_1-s}\left(4\pi\sqrt{mn}\right)$$

and

$$(2\pi)^{-s}\Gamma(s)\,Z_2\,(s;\,f_1,\,f_2) = 2\,(-1)^{\frac{k_2}{2}}\sum_{m,n=1}^{\infty} a_1(m)\,a_2(n)\,\left(\frac{m}{n}\right)^{\frac{k_2-s}{2}} K_{k_2-s}\left(4\pi\sqrt{mn}\right).$$

It follows also from Corollary 2.1 that (5.39) satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)\,Z_2\,(s;\,f_1,\,f_2) = (-1)^{\frac{k_1+k_2}{2}}\,(2\pi)^{-(k_1+k_2-s)}\,\Gamma\,(k_1+k_2-s)\,Z_2\,(k_1+k_2-s;\,f_1,\,f_2)\,. \tag{5.40}$$

Due to the fact that ϕ_1 and ϕ_2 are entire, it is now very easy to establish analogues of formulas of Ramanujan-Guinand and Koshliakov type. In particular, for the case $k_1 = k_2 = 12$, $\phi_1(s) = \phi_2(s) = \phi(s)$, we could consider the Dirichlet series associated to the Ramanujan $\tau-$function,

$$\phi(s) := (2\pi)^{-s}L(s,\,\tau) = (2\pi)^{-s}\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}, \quad \mathrm{Re}(s) > \frac{13}{2}$$

and the resulting diagonal Epstein zeta function (5.39) would satisfy Hecke's functional equation with $r := k_1 + k_2 = 24$. For this case, a curious reflection formula of Koshliakov type takes place in the form

$$\sum_{j=1}^{\infty} d_\tau\,(n)\,K_0\left(4\pi\,\sqrt{n}x\right) = x^{-24}\sum_{j=1}^{\infty} d_\tau(n)\,K_0\left(\frac{4\pi\sqrt{n}}{x}\right), \quad x > 0, \tag{5.41}$$

where $d_\tau(n)$ denotes the divisor function associated with $\tau(n)$,

$$d_\tau(n) = \sum_{d|n}\tau(d)\,\tau\left(\frac{n}{d}\right).$$

A different proof of (5.41) is given in [[8], p. 357, Example 2]. This result is obtained from a generalization of Bochner's modular relation (1.22) for Dirichlet series satisfying the functional equation (1.6) with $\Delta(s) = \Gamma^N(s)$.

Now assume that $a_i(n)$, $i = 1,\,2$, is real. By (5.40), we know that $Z_2\,(s;\,f_1,\,f_2)$ is a real and entire Hecke Dirichlet series with signature $\left(1,\,k_1+k_2,\,(-1)^{\frac{k_1+k_2}{2}}\right)$.

From a classical application of Hecke's theory [[52], p. 79, Satz 3], the fact that the signature parameter $\lambda = 1$ of $Z_2\,(s;\,f_1,\,f_2)$ satisfies $0 < \lambda < 2$, then one has that $\Theta_2(e^{i(\frac{\pi}{2}-\varepsilon)};\,f_1,\,f_2) = O(\varepsilon^{-\rho})$, with $\rho < \frac{k_1+k_2+1}{2}$. Thus, according with Remark 4.5, $Z_2\,(s;\,f_1,\,f_2)$ must have infinitely many zeros on the critical line $\mathrm{Re}(s) = \frac{k_1+k_2}{2}$. In particular, this shows that $Z_2(s;\,f,\,f)$ has infinitely many zeros at the critical line $\mathrm{Re}(s) = k$ and so, by Theorem 4.1, $L(s,\,a)$ has infinitely many zeros at the critical line $\mathrm{Re}(s) = \frac{k}{2}$. Since the L-functions (5.37) admit Selberg-Chowla expansions (see [[86], p. 485, eq. (1.24)]), it would be also interesting to study more analogues of Hardy's Theorem for the Dirichlet series composing these.

Acknowledgements. The first author acknowledges the support from FCT (Portugal) through the PhD scholarship 2020.07359.BD. The second author was partially supported by CMUP, which is financed by national funds through FCT (Portugal) under the project with reference UIDB/00144/2020.

The authors are also grateful to Bruce C. Berndt for sending them the preprint [19].

A great amount of the work presented in this paper was revised and discussed with José Carlos Petronilho. To him our eternal gratitude.

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