

A REMARK ON FIRST INTEGRALS OF VECTOR FIELDS

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ABSTRACT. We provide examples of vector fields on $(\mathbb{C}^3, 0)$ admitting a formal first integral but no holomorphic first integral. These examples are related to a question raised by D. Cerveau and motivated by the celebrated theorems of Malgrange [7] and Mattei-Moussu [9].

1. INTRODUCTION

A celebrated theorem due to Mattei and Moussu [9] states that a holomorphic codimension 1 foliation admitting a formal first integral necessarily possesses a holomorphic first integral as well. The theorem and its proof completely clarify the relationship between formal and holomorphic first integrals for codimension 1 foliations, whereas the general investigation of the existence of these first integrals also includes an influential work of Malgrange [6]. For higher codimension foliations, the relationship between formal and holomorphic first integrals remains quite mysterious. In this context, D. Cerveau naturally asked whether a holomorphic vector field X defined on a neighborhood of the origin of \mathbb{C}^3 and admitting one - or two - formal first integral must possess holomorphic first integrals as well. The goal of this paper is to show that the existence of a single formal first integral is not enough to guarantee the existence of holomorphic ones and this will be done by means of the following theorem:

Theorem 1.1. *Consider the family $X_{a,b,c}$ of vector fields on \mathbb{C}^3 defined by*

$$(1) \quad X_{a,b,c} = x^2 \frac{\partial}{\partial x} + (1 + ax) \left[y_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial y_2} \right] + bxy_2 \frac{\partial}{\partial y_1} + cxy_1 \frac{\partial}{\partial y_2},$$

where a , b , and c are complex parameters. Assume that the parameters are such that

$$\cos(2\pi a) \neq \cos(2\pi \sqrt{a^2 + bc}).$$

Then the vector field $X_{a,b,c}$ possesses no (non-constant) holomorphic first integrals, albeit it does possess formal first integrals.

In particular, the vector field $X_{1,1,1}$ obtained by setting $a = b = c = 1$ admits a formal first integral but no holomorphic one. The existence of these examples was certainly expected, given the transcendental behavior of singular foliations, but we highlight the simplicity of its expression which suggests that this might be a fairly common phenomena in applications. The issue is therefore also related to Malgrange's theorem in [7] in that they confirm that some (strong) additional assumptions are, in fact, needed (see below for further information). As a side note, the simple nature of the examples provided by Theorem 1.1 also bears some similarities with certain results quoted in the survey article [11] by Stolovitch: many normalization results for "simple" vector fields having formulas not too different from $X_{a,b,c}$ are presented under some additional geometric condition (for example "volume-preserving" or "hamiltonian"). We might wonder what is the influence of these conditions on the problem discussed here. Conversely, it

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is also fair to wonder if the (potential) ability to turn formal first integrals into holomorphic ones may shed new light in more general normal form problems.

As a matter of fact, our observation of the vector field (1) is a by-product of our study of the global dynamics of the Airy and Painlevé I and II equations [3]. Our original motivation was the local analysis of the saddle-node singularity associated with the vector-field

$$Y_A = -\frac{1}{2}x^4\frac{\partial}{\partial x} + \left(z - \frac{1}{2}x^3y\right)\frac{\partial}{\partial y} + (y - x^3z)\frac{\partial}{\partial z},$$

which appears in a convenient birational model for the compactified Airy equation. The formal normal form of Y_A as well as the corresponding Stokes phenomenon can accurately be computed with the same technique detailed in Section 2 for the vector field $X_{a,b,c}$. In doing so, there follows that Y_A admits a first integral in the field of fractions of formal power series, i.e., there is a formal first integral of the form F/G with $F, G \in \mathbb{C}[[x, y, z]]$. Yet Y_A has no holomorphic or meromorphic first integral. Basically, the difference between the example provided by Y_A and Cerveau's general questions lies in the fact the "formal first integral" of the vector field Y_A has a "meromorphic" nature rather than a more standard power series representation without negative terms. In turn, there are deep differences between first integrals of "holomorphic" and of "meromorphic" natures as already underlined in the topological context. In fact, in codimension 1, Mattei-Moussu [9] theorem asserts that first integrals are topological invariants and the existence of formal first integrals implies the existence of holomorphic ones. On the other hand, the existence of meromorphic first integrals is not a topological invariant already in the two-dimensional ambient case, cf. [4, 8, 10]. Similarly, in codimension 2 complete integrability in the holomorphic sense is not a topological invariant either [10]. From this point of view, the vector-field Y_A falls genuinely short of shedding light into Cerveau's questions due to the nature of its formal first integral.

It is now interesting to investigate whether a holomorphic vector field X defined on a neighborhood of the origin of \mathbb{C}^3 and admitting two formal first integrals F_1, F_2 such that $dF_1 \wedge dF_2 \neq 0$, necessarily admits at least one holomorphic first integral. The best result in this direction, as far as we are aware of, remains the previously mentioned theorem of Malgrange [7] concerning Pfaffian systems in arbitrary dimensions. More precisely, given a codimension r foliation defined in some open set of \mathbb{C}^n and generated by r one-forms $\Omega = \{\omega_1, \dots, \omega_r\}$, denote by $S(\Omega)$ the singular locus of Ω , that is, the set of points where the r -form $\omega_1 \wedge \dots \wedge \omega_r$ is identically zero. We say that Ω is integrable (respectively formally integrable) at $x \in \mathbb{C}^n$, if there exists r holomorphic function germs $f_1, \dots, f_r \in \mathcal{O}_x$ (respectively r formal power series in $\widehat{\mathcal{O}}_x$) such that the module generated by $\{df_1, \dots, df_r\}$ coincides with $\Omega \cdot \mathcal{O}_x$ (respectively, with $\Omega \cdot \widehat{\mathcal{O}}_x$). In [7], Malgrange shows that if $S(\Omega)$ has codimension 3, or if Ω is formally integrable and $S(\Omega)$ has codimension 2, then Ω is integrable. As mentioned, these hypotheses are generally quite strong when we consider a Pfaffian system obtained as the dual of a vector field.

The proof of Theorem 1.1 relies on the standard theory of linear systems (normal forms and Stokes phenomena among others). We refer the reader to [5, §16 and 20] and references there-within (or [1, 12]) for an introduction to the methods used in this work.

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2. PROOF OF THEOREM 1.1

Let us begin our approach to Theorem 1.1 by noticing that the vector field $X_{a,b,c}$ is associated with the time-dependent linear differential system

$$(2) \quad x^2 \frac{dy}{dx} = \begin{bmatrix} 1 + ax & bx \\ cx & -1 - ax \end{bmatrix} y, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Following classical terminology of linear systems, the system above has a non-resonant irregular singular point of Poincaré rank 1 at $x = 0$, see for example [5, § 20]. Note that this differential system is in the so-called Birkhoff normal form: the system is well defined for all $x \in \mathbb{C}\mathbb{P}^1$ and has only two singular points, namely $x = 0$ and $x = \infty$, see e.g. [5, § 20B]. In turn, the singularity at $x = \infty$ is a *Fuchsian one*. In other words, the system has a simple pole at $x = \infty$, see e.g. [5, Definition 16.9]. In addition, since the linear system (2) is non-resonant, it can formally be transformed into a diagonal linear system by means of the standard Poincaré-Dulac method [5, Theorem 20.7]. Whereas the resulting (formal) power series is divergent, Sibuya's Theorem asserts that it is Borel 1-summable in all directions $x \in e^{i\alpha}\mathbb{R}_{>0}$ with exception of the singular directions, namely the directions corresponding to $\alpha \in \pi\mathbb{Z}$. The preceding is made accurate by the lemma below:

Lemma 2.1. *There exists a formal linear change of coordinates having the form $y = \hat{T}(x)u$, with $\hat{T}(0) = I$, which conjugates system (2) to the (diagonal) linear system*

$$(3) \quad x^2 \frac{du}{dx} = \begin{bmatrix} 1 + ax & 0 \\ 0 & -1 - ax \end{bmatrix} u, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Moreover, for every $\alpha \in]0, \pi[\cup]\pi, 2\pi[$, there exists a holomorphic transformation $y = T_\alpha(x)u$ conjugating systems (2) and (3) and satisfying the following conditions:

- (a) $T_\alpha(x)$ is analytic on the open sector of opening angle π bisected by the half-line $e^{i\alpha}\mathbb{R}_{>0}$.
- (b) $T_\alpha(x)$ and $T_\beta(x)$, with $\alpha < \beta$, coincide on the intersection of the corresponding half planes provided that the interval $]\alpha, \beta[$ does not contain an integral multiple of π .
- (c) $T_\alpha(x)$ is asymptotic to $\hat{T}(x)$.

Proof. As previously stated, the existence of a formal change of variables conjugating systems (2) and (3), as well as its analytic nature on the indicated sectors, goes back to classical results by Birkhoff and Malmquist (or more general versions by Hukuhara, Turittin, and Sibuya, see [5, Theorems 20.7 and 20.16]). Therefore it only remains to check that the diagonal matrix appearing in (3) has the indicated form. To do this, note that the formal invariants of the initial system (2) can be read off a suitable finite jet of the eigenvalue functions associated with the matrix

$$\begin{bmatrix} 1 + ax & bx \\ cx & -1 - ax \end{bmatrix}.$$

Clearly these eigenvalue functions are equal to $\pm\sqrt{(1+ax)^2 + bcx^2}$. Now, since the Poincaré rank of the singularity is 1, only the 1-jet of the eigenvalue function is a formal invariant, c.f. [5, Proposition 20.2]. Therefore $\pm(1+ax)$ are the only formal invariants of the system. This completes the proof of the lemma. \square

Next, note that the system (3) clearly admits

$$(4) \quad U(x) = \begin{bmatrix} x^a e^{-1/x} & 0 \\ 0 & x^{-a} e^{1/x} \end{bmatrix}$$

as fundamental (matrix) solution. Consider one of the two singular directions, namely $\beta = 0$ or $\beta = \pi$. Let $T_{\beta+}$ and $T_{\beta-}$ denote, respectively, the Borel sums on the “left” and on the “right” of the fixed singular direction β . Then, there is a constant matrix S_β satisfying

$$T_{\beta-}(x) = T_{\beta+}(x)U(x)S_\beta U(x)^{-1},$$

for $x \in e^{i\beta}\mathbb{R}_{>0}$. The matrices S_0 and S_π are called the *Stokes matrices* and they have the general forms

$$S_0 = \begin{bmatrix} 1 & s_0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S_\pi = \begin{bmatrix} 1 & 0 \\ s_\pi & 1 \end{bmatrix},$$

for suitable constants $s_0, s_\pi \in \mathbb{C}$, see e.g. [5, § 20G]. In the particular case in question, explicit formulas for s_0 and s_π are known, see [2, pages 86 and 87]. However, for our purposes, it suffices to prove that:

Lemma 2.2. *The product $s_0 s_\pi \neq 0$ if and only if $\cos(2\pi a) \neq \cos(2\pi\sqrt{a^2 + bc})$.*

Proof. The lemma will be proved by explicitly computing the monodromy matrix M associated with the system (2) around $x = 0$ in two different ways: first we compute the matrix directly around $x = 0$ by using the Stokes matrices and then we will compute the monodromy (holonomy) around $x = \infty$ which is a Fuchsian singular point. The monodromy around $x = \infty$ is the inverse of the monodromy matrix M since the system in question has only two singular points (corresponding to $x = 0$ and to $x = \infty$). The result will then easily follow by computing the trace of M in each situation.

Claim. The monodromy matrix around the origin is conjugate to $M = S_0 N S_\pi$, where

$$N = \begin{bmatrix} e^{2\pi i a} & 0 \\ 0 & e^{-2\pi i a} \end{bmatrix}$$

is the “formal monodromy” of the fundamental matrix solution $U(x)$ introduced in Equation (4).

Proof of the claim. The statement follows from the sequence of equations

$$\begin{aligned} T_{0+}(e^{2\pi i x})U(e^{2\pi i x}) &= T_{\pi-}(e^{2\pi i x})U(e^{2\pi i x}) = T_{\pi+}(e^{2\pi i x})U(e^{2\pi i x})S_\pi \\ &= T_{2\pi-}(e^{2\pi i x})U(e^{2\pi i x})S_\pi = T_{0-}(x)U(x)NS_\pi = T_{0+}(x)U(x)S_0NS_\pi, \end{aligned}$$

where item (b) of Lemma 2.1 has implicitly been used. \square

Since $M = S_0 N S_\pi$, it immediately follows that

$$\text{tr } M = 2 \cos(2\pi a) + e^{-2\pi i a} s_0 s_\pi.$$

Let us now compute the matrix M by looking at the singular point $x = \infty$. Let $v = 1/x$ so that the system (2) becomes

$$(5) \quad v \frac{dy}{dv} = \begin{bmatrix} a + v & b \\ c & -a - v \end{bmatrix} y = A(v)y,$$

and note that $v = 0$ corresponds to $x = \infty$. Denote by λ_1 and λ_2 the eigenvalues of the matrix $A(0)$. Naturally the matrix $A(0)$ is the so-called *residue matrix* of system (5). Clearly these two eigenvalues are symmetric and, up to relabeling, we set $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$ where $\lambda = \sqrt{a^2 + bc}$. In this case, the system is non-resonant if $2\lambda \notin \mathbb{Z}$, see e.g. [5, Definition 16.12]. In turn, provided that there is no resonance, the system is locally holomorphically equivalent to the Euler system $tv' = A(0)v$, see e.g. [5, Theorem 16.16]. In turn, the monodromy matrix around $v = 0$ is conjugate to the exponential of $2\pi i A(0)$ and the latter matrix is conjugate to the inverse of the initial monodromy matrix M . Since traces of matrices remain invariant under conjugations, the preceding finally yields

$$\text{tr } M = 2 \cos(2\pi\lambda).$$

In fact, this last formula holds whether or not the system (5) is resonant as it immediately follows from the continuity of $\text{tr} M$ with respect to the parameters a , b , and c (the set of non-resonant systems is open and dense). Lemma 2.2 promptly follows. \square

Next, note that the the diagonal differential system (3) is naturally equivalent to the following family of vector fields on \mathbb{C}^3 :

$$(6) \quad X_a = x^2 \frac{\partial}{\partial x} + (1 + ax) \left[u_1 \frac{\partial}{\partial u_1} - u_2 \frac{\partial}{\partial u_2} \right],$$

where $a \in \mathbb{C}$. Clearly vector fields in the family X_a admits the function $h(u_1, u_2) = u_1 u_2$ as a holomorphic first integral. Furthermore, we have:

Lemma 2.3. *The function $h(u) = u_1 u_2$ is a primitive first integral of X_a in the following sense: if $\hat{F} = \hat{F}(x, u_1, u_2) \in \mathbb{C}[[x, u]]$ is a formal first integral of X_a , then there exists a formal power series $\hat{G} \in \mathbb{C}[[z]]$ such that $\hat{F} = \hat{G} \circ h$.*

Proof. Assume that $\hat{F} = \hat{F}(x, u_1, u_2)$ is a formal first integral of X_a and consider a Taylor expansion of the form:

$$(7) \quad \hat{F}(x, u_1, u_2) = \sum_{j=0}^{\infty} x^j \hat{f}_j(u_1, u_2) = \sum_{j=0}^{\infty} x^j \sum_{k \in \mathbb{Z}} u_1^k \hat{f}_{j,k}(u_1 u_2).$$

Claim. We have $\hat{F}(x, u_1, u_2) = \hat{f}_{0,0}(u_1 u_2) + o(x^n)$ for all $n \in \mathbb{N}$.

Clearly the lemma is an immediate consequence of the claim so that it suffices to prove the claim.

Proof of the Claim. We argue by induction. Assume the claim holds for $n = n_0$ (where the possibility of having $n_0 = 0$ is not excluded). Since F is a formal first integral of X_a , a direct computation yields

$$0 = d_0 \hat{F} \cdot X_a = x^{n_0} \left(\sum_{k \in \mathbb{Z}} k u_1^k \hat{f}_{n_0,k}(u_1 u_2) \right) + o(x^{n_0+1}).$$

By comparing monomial degrees, there follows that all the functions $\hat{f}_{n_0,k}(\cdot)$ must vanish identically provided that $k \neq 0$. Thus the power series expansion (7) of \hat{F} takes on the form

$$\hat{F} = \hat{f}_{0,0}(u_1 u_2) + x^{n_0} \hat{f}_{n_0,0}(u_1 u_2) + \sum_{j=n_0+1}^{\infty} x^j \sum_{k \in \mathbb{Z}} u_1^k \hat{f}_{j,k}(u_1 u_2).$$

In turn, this refined formula for \hat{F} yields

$$0 = d_0 \hat{F} \cdot X_a = x^{n_0+1} \left(n_0 \hat{f}_{n_0,0}(u_1 u_2) + \sum_{k \in \mathbb{Z}} k u_1^k \hat{f}_{n_0+1,k}(u_1 u_2) \right) + o(x^{n_0+2}).$$

Therefore also $\hat{f}_{n_0,0}(\cdot)$ must vanish identically unless $n_0 = 0$. Hence \hat{F} is actually of the form $\hat{F}(x, u_1, u_2) = \hat{f}_{0,0}(u_1 u_2) + o(x^{n_0+1})$ which establishes the induction step. The proof of the claim is complete and so is the proof of the lemma. \square

Remark 2.4. The computation carried out in the proof of Lemma 2.3 is related to a qualitative issue that is worth pointing out. For this, note first that the general solution of the diagonal system (3) has the form

$$\begin{cases} u_1(x) = c_1 e^{-1/x} x^a \\ u_2(x) = c_2 e^{1/x} x^{-a} \end{cases}$$

for suitable constants $c_1, c_2 \in \mathbb{C}$. It follows that for any formal first integral $\hat{F} = \hat{F}(x, u_1, u_2)$ of X_a , the composition $\hat{F}(x, c_1 e^{-\frac{1}{x}} x^a, c_2 e^{\frac{1}{x}} x^{-a})$ must be a constant, and hence must factor through h due to the presence of the essential singular point arising from $e^{\frac{1}{x}}$.

We are now able to provide the proof of Theorem 1.1.

Proof of Theorem 1.1. Owing to Lemma 2.1, the vector field $X_{a,b,c}$ has a *formal first integral* $\hat{f} = \hat{f}(x, y_1, y_2)$ which is obtained out of the first integral $h(u_1, u_2)$ of X_a by means of the equation

$$\hat{f}(x, \hat{T}(x)u) = h(u).$$

Furthermore, according to Lemma 2.3, every other formal first integral of $X_{a,b,c}$ must formally factor through \hat{f} . The proof of Theorem 1.1 is then reduced to showing that if a (non-constant) first integral of $X_{a,b,c}$ is holomorphic then we necessarily have $\cos(2\pi a) = \cos(2\pi\sqrt{a^2 + bc})$.

Let us then assume there is a (non-constant) holomorphic first integral f for the vector field $X_{a,b,c}$ defined as in (1). It follows from Lemmas 2.1 and 2.3 the existence of a formal series $\hat{G} \in \mathbb{C}[[z]]$ such that $f(x, \hat{T}(x)u) = \hat{G} \circ h$, where $u = (u_1, u_2)$. Since the formal series on the left side is 1-Borel summable in the variable x , while the right hand side is independent of x , we conclude that $\hat{G} \circ h$ is an analytic function on $u = (u_1, u_2)$. We set $\hat{G} \circ h = g(u)$. In particular, there follows that $f_\alpha(x, T_\alpha(x)u) = g(u)$, where $f_\alpha(x, y)$ denotes a sectorial Borel sum. We thus obtain that $f_{\beta+} = f_{\beta-}$ for both singular directions $\beta = 0$ and $\beta = \pi$, where $f_{\beta+}$ (resp. $f_{\beta-}$) stands as usual for the Borel sum on the “left” (resp. “right”) of the fixed singular direction β . Therefore we have

$$\begin{aligned} g(u) &= f_{\beta-}(x, T_{\beta-}(x)u) = f_{\beta+}(x, T_{\beta+}(x)U(x)S_\beta U(x)^{-1}u) \\ &= g(U(x)S_\beta U(x)^{-1}u). \end{aligned}$$

In other words, the function $g(u)$ is invariant by the Stokes operators

$$S : u \mapsto U(x)S_\beta U(x)^{-1}u.$$

However, it follows from direct computation that the function $g(u)$, which factors through $h(u) = u_1 u_2$, is invariant by this operator only if $s_\beta = 0$. Hence, the existence of the holomorphic first integral f implies that the product $s_0 s_\pi$ equals zero so that Lemma 2.2 ensures that we must have $\cos(2\pi a) = \cos(2\pi\sqrt{a^2 + bc})$. This ends the proof of Theorem 1.1. \square

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