On uniformly continuous endomorphisms of hyperbolic groups

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Abstract

We prove a generalization of the fellow traveller property for a certain type of quasi-geodesics and use it to present three equivalent geometric formulations of the bounded reduction property. We then provide an affirmative answer to a question from Araújo and Silva as to whether every nontrivial uniformly continuous endomorphism of a hyperbolic group with respect to a visual metric satisfies a Hölder condition. We remark that these results combined with the work done by Paulin prove that every endomorphism admitting a continuous extension to the completion has a finitely generated fixed point subgroup.

1 Introduction

The dynamical study of endomorphisms of groups started with the (independent) work of Gersten [12] and Cooper [8], using respectively graph-theoretic and topological approaches. They proved that the subgroup of fixed points $\text{Fix}(\varphi)$ of some fixed automorphism φ of F_n is always finitely generated, and Cooper succeeded on classifying from the dynamical viewpoint the fixed points of the continuous extension of φ to the boundary of F_n. Bestvina and Handel subsequently developed the theory of train tracks to prove that $\text{Fix}(\varphi)$ has rank at most n in [2]. The problem of computing a basis for $\text{Fix}(\varphi)$ had a tribulated history and was finally settled by Bogopolski and Maslakova in 2016 in [3].

This line of research extended early to wider classes of groups. For instance, Paulin proved in 1989 that the subgroup of fixed points of an automorphism of a hyperbolic group is finitely generated [16]. Fixed points were also studied for right-angled Artin groups [17] and lamplighter groups [15].

Regarding the continuous extension of an endomorphism to the completion, infinite fixed points of automorphisms of free groups were also discussed by Bestvina and Handel in [2] and Gaboriau, Jaeger, Levitt and Lustig in [11]. The dynamics of free groups automorphisms is proved to be asymptotically periodic in [14]. In [7], the dynamical study of infinite fixed points was performed for monoids defined by special confluent rewriting systems (which contain free groups as a particular case). This was also achieved in [19] for virtually injective endomorphisms of virtually free groups. Endomorphisms of free-abelian times free groups $\mathbb{Z}^m \times F_n$ have been studied

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in [9] and their continuous extension to the completion is studied in [5] and the case of $F_n \times F_m$ with $m, n \geq 2$ is dealt with in [6]

For natural reasons, to study the dynamics of the continuous extension to the completion, in case the topology is metrizable (as it is in the case of hyperbolic groups via visual metrics) it is of utmost importance to describe uniformly continuous endomorphisms for a certain class of groups. Also, it is important to notice that one of the essential tools used in proving these results is the bounded reduction property (also known as the bounded cancellation lemma) introduced in [8] and followed by many others.

Motivated by the possibility of defining new pseudometrics in the group of automorphisms of a hyperbolic group, the authors in [1] studied endomorphisms for which a Hölder condition holds.

In this paper, we will describe uniformly continuous endomorphisms of hyperbolic groups and present some geometric versions of the bounded reduction property with hope that these techniques could be used to study the dynamics of infinite points for arbitrary uniformly continuous endomorphisms of hyperbolic groups, for which not much is known so far. We will also show that Hölder conditions are satisfied by every uniformly continuous endomorphism with respect to a visual metric, answering a question from [1] and highlight that this result, combined with previous work from Paulin suffices to show that every uniformly continuous endomorphism of a hyperbolic group has finitely generated fixed point subgroup.

The paper is organized as follows. In Section 2, we present some preliminaries on hyperbolic metric spaces and hyperbolic groups. In Section 3, we present a generalization of the fellow traveller property for $(1, k)$-quasi-geodesics, which will be seen to arise naturally. We formulate the bounded reduction property in geometric terms throughout Section 4. In Section 5, we show that every nontrivial uniformly continuous endomorphism of a hyperbolic group must satisfy a Hölder condition and observe that these results combined with the work in [16] yield that every endomorphism of a finitely generated hyperbolic group admitting a continuous extension to the completion has a finitely generated fixed point subgroup.

2 Preliminaries

We now introduce some well-known results on hyperbolic groups. For more details, the reader is referred to [13] and [4].

A mapping $\varphi : (X, d) \to (X', d')$ between metric spaces is called an *isometric embedding* if $d'(x\varphi, y\varphi) = d(x, y)$, for all $x, y \in X$. A surjective isometric embedding is an *isometry*.

A metric space (X, d) is said to be *geodesic* if, for all $x, y \in X$, there exists an isometric embedding $\xi : [0, s] \to X$ such that $0\xi = x$ and $s\xi = y$, where $[0, s] \subset \mathbb{R}$ is endowed with the usual metric of $\mathbb{R}$. We call ξ a geodesic of (X, d). We shall often call $\mathrm{Im}(\xi)$ a geodesic as well. In this second sense, we may use the notation $[x, y]$ to denote an arbitrary geodesic connecting x to y. When the endpoint of a geodesic α coincides with the starting point of a geodesic β, we denote the concatenation of both geodesics by $[\alpha\beta]$. Note that a geodesic metric space is always (path) connected. A *quasi-isometric embedding* of metric spaces is a mapping $\varphi : (X, d) \to (X', d')$ such

that there exist constants $\lambda \geq 1$ and $K \geq 0$ satisfying

$$\frac{1}{\lambda}d(x,y) - K \leq d'(x\varphi, y\varphi) \leq \lambda d(x,y) + K$$

for all $x, y \in X$. We may call it a (λ, K)-quasi-isometric embedding if we want to stress the constants.

If in addition

$$\forall x' \in X 0 \exists x \in X : d'(x', x\varphi) \leq K,$$

we say that φ is a *quasi-isometry*. Two metric spaces (X, d) and (X', d') are said to be *quasi-isometric* if there exists a quasi-isometry $\varphi : (X, d) \to (X', d')$. Quasi-isometry turns out to be an equivalence relation on the class of metric spaces. A (λ, K)-*quasi-geodesic* of (X, d) is a (λ, K)-quasi-isometric embedding $\xi : [0, s] \to X$ such that $0\xi = x$ and $s\xi = y$, where $[0, s] \subset \mathbb{R}$ is endowed with the usual metric of $\mathbb{R}$.

Given $x, y, z \in X$, a *geodesic triangle* $[[x, y, z]]$ is a collection of three geodesics $[x, y]$, $[y, z]$ and $[z, x]$ in X. Given $\delta \geq 0$, we say that X is δ-*hyperbolic* if

$$\forall w \in [x, y] \, d(w, [y, z] \cup [z, x]) \leq \delta$$

holds for every geodesic triangle $[[x, y, z]]$.

Let (X, d) be a metric space and Y, Z nonempty subsets of X. We call the ε-neighbourhood of Y in X and we denote by $\mathcal{V}_\varepsilon(Y)$ the set $\{x \in X \mid d(x, Y) \leq \varepsilon\}$. We call the *Hausdorff distance* between Y and Z and we denote by $\text{Haus}(Y, Z)$, the number defined by

$$\inf\{\varepsilon > 0 \mid Y \subseteq \mathcal{V}_\varepsilon(Z) \text{ and } Z \subseteq \mathcal{V}_\varepsilon(Y)\}$$

if it exists. If it doesn't, we say that $\text{Haus}(Y, Z) = \infty$.

Given a group $H = \langle A \rangle$, consider its Cayley graph $\Gamma_A(H)$ with respect to A endowed with the *geodesic metric* d_A, defined by letting $d_A(x, y)$ to be the length of the shortest path in $\Gamma_A(H)$ connecting x to y. This is not a geodesic metric space, since d_A only takes integral values. However, we can define the *geometric realization* $\bar{\Gamma}_A(H)$ of its Cayley graph $\Gamma_A(H)$ by embedding (H, d_A) isometrically into it. Then, edges of the Cayley graph become segments of length 1. With the metric induced by d_A, which we will also denote by d_A, $\bar{\Gamma}_A(H)$ becomes a geodesic metric space.

We say that a group H is *hyperbolic* if the metric space $(\bar{\Gamma}_A(H), d_A)$ is hyperbolic. We will simply write d instead of d_A when no confusion arises. Also, for $x \in H$ we will often denote $d_A(1, u)$ by $|u|$.

From now on, H will denote a finitely generated hyperbolic group generated by a finite set A and $\pi : \widetilde{A}^* \to H$ will be a matched epimorphism.

An important property of the class of automatic groups, for which the class of hyperbolic groups is a subclass, is the *fellow traveller property*. Given a word $u \in \widetilde{A}^*$, we denote by $u^{[n]}$, the prefix of u with n letters. If $n > |u|$, then we consider $u^{[n]} = u$. We say that the fellow traveller property holds for $L \subseteq \widetilde{A}^*$ if, for every $M \in \mathbb{N}$, there is some $N \in \mathbb{N}$ such that, for every $u, v \in L$,

$$d_A(u\pi, v\pi) \leq M \Rightarrow d_A(u^{[n]}\pi, v^{[n]}\pi) \leq N,$$

for every $n \in \mathbb{N}$.

Given $g, h, p \in H$, we define the *Gromov product of g and h* taking p as basepoint by

$$(g|h)_p^A = \frac{1}{2}(d_A(p, g) + d_A(p, h) - d_A(g, h)).$$

We will often write $(g|h)$ to denote $(g|h)_1^A$. Notice that, in the free group case, we have that $(g|h) = |g \wedge h|$.

Let G be a subgroup of a hyperbolic group $H = \langle A \rangle$ and let $q \geq 0$. We say that G is *q-quasiconvex* with respect to A if

$$\forall x \in [g, g'] \quad d_A(x, G) \leq q$$

holds for every geodesic $[g, g']$ in $\overline{\Gamma}_A(H)$ with endpoints in G. We say that G is *quasiconvex* if it is q-quasiconvex for some $q \geq 0$.

There are many ways to describe the *Gromov boundary* ∂H of H, such as being the equivalence classes of geodesic rays, when two rays are considered equivalent if the Hausdorff distance between them is finite. Another model for ∂H can be defined using *Gromov sequences*. We say that a sequence of points $(x_i)_{i \in \mathbb{N}}$ in H is a Gromov sequence if $(x_i|x_j) \to \infty$ as $i \to \infty$ and $j \to \infty$. Two such sequences $(x_i)_{i \in \mathbb{N}}$ and $(y_j)_{j \in \mathbb{N}}$ are *equivalent* if

$$\lim_{i,j \to \infty} (x_i|y_j) = \infty.$$

The set of all the equivalence classes is another standard model for the boundary ∂H. Identifying an element h in H with the constant sequence $(h)_i$, we can extend the Gromov product to the completion by putting

$$(\alpha|\beta)_p^A = \sup \left\{ \liminf_{i,j \to \infty} (x_i|y_j)_p^A \, \big| \, (x_i)_{i \in \mathbb{N}} \in \alpha, (y_j)_{j \in \mathbb{N}} \in \beta \right\}$$

We define

$$\rho_{p,\gamma}^A(g, h) = \begin{cases} e^{-\gamma(g|h)_p^A} & \text{if } g \neq h \\ 0 & \text{otherwise} \end{cases}$$

for all $p, g, h \in H$.

Given $p \in H$, $\gamma > 0$ and $T \geq 1$, we denote by $V^A(p, \gamma, T)$ the set of all metrics d on H such that

$$\frac{1}{T} \rho_{p,\gamma}^A(g, h) \leq d(g, h) \leq T \rho_{p,\gamma}^A(g, h) \tag{1}$$

We refer to the metrics in some $V^A(p, \gamma, T)$ as the *visual metrics* on H. Let $d \in V^A(p, \gamma, T)$. The metric space (H, d) is not complete in general. However, its completion $(\widehat{H}, \hat{d})$ is a compact space and can be obtained by considering $\widehat{H} = H \cup \partial H$. It is a well-known fact that the topology induced by $\hat{d}$ on ∂H is the *Gromov topology*.

Considering the extension of $\rho_{p,\gamma}^A$ to the boundary, we define, for $\alpha, \beta \in \widehat{H}$

$$\hat{\rho}_{p,\gamma}^A(\alpha, \beta) = \begin{cases} e^{-\gamma(\alpha|\beta)_p^A} & \text{if } \alpha \neq \beta \\ 0 & \text{otherwise} \end{cases}$$

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By continuity, for every $\alpha, \beta \in \hat{H}$, the inequalities

$$\frac{1}{T}\hat{\rho}_{p,\gamma}^{A}(\alpha, \beta) \leq \hat{d}(\alpha, \beta) \leq T\hat{\rho}_{p,\gamma}^{A}(\alpha, \beta)$$

hold [4, Section III.H.3].

3   Fellow Traveller Property for $(1,k)$-quasi-geodesics

The goal of this section is to present a slightly more general formulation of the fellow traveller property. While we usually state the property considering geodesics, we can see that we get essentially the same result when the paths considered are $(1,k)$-quasi-geodesics for some $k \in \mathbb{N}$. We will present the result in a different version, considering quasi-geodesics with endpoints at distance at most one from one another, but we remark that the result holds as long as the distance between the endpoints is bounded by some constant.

Proposition 3.1. *Let u be a $(1,r)$-quasi-geodesic and v be a $(1,s)$-quasi-geodesic with the same starting point. Then, there is a constant N depending on r, s, δ such that, for all $n \in \mathbb{N}$, we have that*

$$d_A(u\pi, v\pi) \leq 1 \Rightarrow d_A(u^{[n]}\pi, v^{[n]}\pi) \leq N.$$

Proof. Since the Cayley graph of H with respect to A is vertex-transitive, we can assume that 1 is the starting point of both u and v. Let p and q be the endpoints of u and v, and consider a geodesic w from q to p.
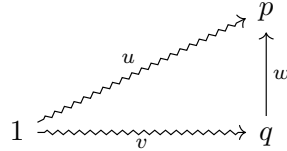
Let $k = \max\{r, s\}$. Then u, v and w are all $(1, k)$-quasi-geodesics. By Corollary 1.8, Chapter III.H in [4], there is a constant δ', depending only on r, s, δ, such that the triangle (u, v, w) is δ'-thin. Let $n \in \mathbb{N}$. We will prove that

$$d_A(u^{[n]}\pi, v^{[n]}\pi) \leq 3k + 2\delta' + 4.$$

Consider the factorizations of u and v given by

$$1 \xrightarrow{\;\;u^{[n]}\;\;} p_n \xrightarrow{\;\;u_n\;\;} p, \qquad 1 \xrightarrow{\;\;v^{[n]}\;\;} q_n \xrightarrow{\;\;v_n\;\;} q, \;\;.$$

Suppose that $v_n = 1$. This means that $q_n = q$, $d_A(1, q) \leq n$ and $d_A(1, p) \leq n + 1$. Also, notice that, since u is a $(1, k)$-quasi-geodesic, we have that

$$|u| - k \leq d_A(1, p) \leq |u|.$$

We have that

$$d_A(u^{[n]}\pi, p) \leq |u| - n \leq d_A(1, p) - n + k \leq k + 1 \leq 3k + 2\delta' + 4.$$

The case $u_n = 1$ is analogous, so we assume that $u_n, v_n \neq 1$.

Since (u, v, w) is a δ'-thin triangle, there is some $m \leq |u|$, such that

$$d_A(v^{[n]}\pi, u^{[m]}\pi) \leq \delta' + 1.$$

Again, since u and v are $(1, k)$-quasi-geodesics, we have that

$$d_A(1, q) - n - k \leq |v| - n - k \leq d_A(q_n, q) \leq |v| - n \leq d_A(1, q) - n + k$$

and

$$d_A(1, p) - m - k \leq |u| - m - k \leq d_A(p_m, p) \leq |u| - m \leq d_A(1, p) - m + k.$$

So,

$$d_A(1, q) - n - k \leq d(q_n, q) \leq d(q_n, p_m) + d(p_m, p) + d(p, q)$$
$$\leq \delta' + 1 + d_A(1, p) - m + k + 1,$$

thus, $m - n \leq \delta' + 1 + d_A(1, p) - d_A(1, q) + 2k + 1 \leq \delta' + 2k + 3$. Similarly, we have that

$$d_A(1, p) - m - k \leq d(p_m, p) \leq d(p_m, q_n) + d(q_n, q) + d(q, p)$$
$$\leq \delta' + 1 + d_A(1, q) - n + 2k + 1,$$

thus, $n - m \leq \delta' + 1 + d_A(1, q) - d_A(1, p) + 2k + 1 \leq \delta' + 2k + 3$.

Hence, we have that

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n) \leq |m - n| + k + \delta' + 1 \leq \delta' + 2k + 3 + k + \delta' + 1 = 3k + 2\delta' + 4.$$

$\square$

4   Bounded Reduction Property

In this section, we will present three (equivalent) geometric versions of the Bounded Reduction Property (also known as the Bounded Cancellation Lemma) for hyperbolic groups. The bounded reduction property has proved itself to be a most useful tool in studying the dynamics of (virtually) free group endomorphisms (see, for example, [2], [8],[11],[18] [19]). We will later use it to prove the main result of this paper.

For free groups, the *bounded cancellation lemma* is said to hold for an endomorphism $\varphi \in \mathrm{End}(F_n)$ if there is some constant $N \in \mathbb{N}$ such that

$$|u^{-1} \wedge v| = 0 \Rightarrow |u^{-1}\varphi \wedge v\varphi| < N$$

holds for all $u, v \in F_n$, where $u \wedge v$ denotes the longest common prefix between u and v.

Since, for $u, v \in F_n$, the Gromov product $(u|v)$ coincides with $u \wedge v$, a natural generalization for the hyperbolic case is

$$\exists N \in \mathbb{N} \left((u \mid v) = 0 \Rightarrow (u\varphi \mid v\varphi) \leq N \right).$$

In Proposition 15, Chapter 5 in [13], it is shown that if φ is a (λ, K)-quasi-isometric embedding, then there exists a constant A depending on λ, K and δ, with the property that

$$\frac{1}{\lambda}(u|v) - A \leq (u\varphi|v\varphi) \leq \lambda(u|v) + A. \tag{2}$$

So, if H is a hyperbolic group and $\varphi : H \to H$ is a quasi-isometric embedding, then for every $p \geq 0$, there exists $q \geq 0$ so that

$$(u \mid v) \leq p \Rightarrow (u\varphi \mid v\varphi) \leq q.$$

We will now present a geometric formulation of the inequality $(u|v) \leq p$.

Lemma 4.1. *Let $u, v \in H$ and $p \in \mathbb{N}$. Then $(u|v) \leq p$ if and only if for any geodesics α and β from 1 to u^{-1} and v, respectively, we have that the concatenation*

$$1 \xrightarrow{\quad \alpha \quad} u^{-1} \xrightarrow{\quad \beta \quad} u^{-1}v$$

is a $(1, 2p)$-quasi-geodesic.

Proof. Let $u, v \in H$ and $p \in \mathbb{N}$. Suppose that $(u|v) \leq p$ and take geodesics α and β from 1 to u^{-1} and v, respectively. Take the concatenation $\zeta : [0, |u| + |v|] \to H$, where $0\zeta = 1$, $(|u|)\zeta = u^{-1}$ and $(|u| + |v|)\zeta = u^{-1}v$.

We will prove that ζ is a $(1, 2p)$-quasi-geodesic, i.e., for $0 \leq i \leq j \leq |u| + |v|$, we have that

$$j - i - 2p \leq d(i\zeta, j\zeta) \leq j - i + 2p.$$

We can assume that $0 \leq i \leq |u| \leq j \leq |u| + |v|$. Clearly, we have that $d(i\zeta, j\zeta) \leq j - i \leq j - i + 2p$.

Suppose that $|j\zeta| < j - 2(u|v)$ and consider a geodesic $\gamma : [0, |j\zeta|]$ from 1 to $j\zeta$ and concatenate it with $\zeta|_{[j,|u|+|v|]}$, which gives us a path of length

$$
\begin{aligned}
|u| + |v| - j + |j\zeta| &< |u| + |v| - j + j - 2(u|v) \\
&= |u| + |v| - 2(u|v) \\
&= |u| + |v| - |u| - |v| + |u^{-1}v| \\
&= |u^{-1}v|
\end{aligned}
$$

from 1 to $u^{-1}v$ and that contradicts the definition of d. Thus, $|j\zeta| \geq j - 2(u|v)$.
So,

$$
\begin{aligned}
&\phantom{\leq{}} d(u^{-1}, j\zeta) + |u^{-1}v| - |v| - d(i\zeta, j\zeta) - d(1, i\zeta) \\
&\leq d(u^{-1}, j\zeta) + |u^{-1}v| - |v| - |j\zeta| \\
&= j - |u| + d(u, v) - |v| - |j\zeta| \\
&= j - 2(u|v) - |j\zeta| \\
&\leq 0,
\end{aligned}
$$

thus,

$$
\begin{aligned}
d(i\zeta, j\zeta) &\geq d(u^{-1}, j\zeta) + |u^{-1}v| - |v| - d(1, i\zeta) \\
&= j - |u| + |u^{-1}v| - |v| - i \\
&= j - i - 2(u|v) \\
&\geq j - i - 2p
\end{aligned}
$$

To prove the converse, suppose that, for any geodesics α and β from 1 to u^{-1} and v, respectively, we have that the concatenation

$$
1 \xrightarrow{\quad \alpha \quad} u^{-1} \xrightarrow{\quad \beta \quad} u^{-1}v
$$

is a $(1, 2p)$-quasi-geodesic. Then, take any geodesics α and β in the conditions above and consider the $(1, 2p)$-quasi-geodesic ζ obtained by concatenating α and β. Then

$$
d(u, v) = d(1, u^{-1}v) = d(0\zeta, (|u| + |v|)\zeta) \geq |u| + |v| - 2p,
$$

so $2(u|v) = |u| + |v| - d(u, v) \leq 2p$ and we are done. $\qquad\square$

Let $\varphi : H \to H$ be a map. We say that *the BRP holds for* φ if, for every $p \geq 0$ there is some $q \geq 0$ such that: given two geodesics u and v such that

$$
1 \xrightarrow{\quad u \quad} u \xrightarrow{\quad v \quad} uv
$$

is a $(1, p)$-quasi-geodesic, we have that given any two geodesics α, β, from 1 to $u\varphi$ and from $u\varphi$ to $(uv)\varphi$, respectively, the path

$$
1 \xrightarrow{\quad \alpha \quad} u\varphi \xrightarrow{\quad \beta \quad} (uv)\varphi
$$

is a $(1, q)$-quasi-geodesic.

Proposition 4.2. *Let H be a hyperbolic group and $\varphi : H \to H$ be a mapping. Then, the BRP holds for φ if and only if for every $p \geq 0$ there is some $q \geq 0$ such that*

$$
(u \mid v) \leq p \Rightarrow (u\varphi \mid v\varphi) \leq q. \tag{3}
$$

holds.

Proof. Let H be a hyperbolic group, $\varphi : H \to H$ be a mapping and p be a non-negative integer. Suppose that the BRP holds for φ and take $u, v \in H$ such that $(u|v) \leq p$. Then, by Lemma 4.1 and the BRP, there is some $q \geq 0$ such that, given geodesics α and β from 1 to $u^{-1}\varphi$ and from $u^{-1}\varphi$ to $(u^{-1}v)\varphi$, respectively, the concatenation $\zeta : [0, |u^{-1}\varphi| + |v\varphi|] \to H$ is a $(1, 2q)$-quasi-geodesic. Using Lemma 4.1 again, we have that $(u\varphi|v\varphi) \leq q$.

Now, suppose that for every $p \geq 0$, there is some $q \geq 0$ such that (3) holds and take geodesics α, β such that the concatenation

$$
1 \xrightarrow{\quad \alpha \quad} u \xrightarrow{\quad \beta \quad} uv
$$

is a $(1, p)$-quasi-geodesic. In particular, it is also a $(1, 2p)$-quasi-geodesic. Then by Lemma 4.1, we have that $(u^{-1}|v) \leq p$, so, using (3), we have that $(u^{-1}\varphi|v\varphi) \leq q$, so by Lemma 4.1, we have that the path

$$1 \xrightarrow{\quad \alpha \quad} u\varphi \xrightarrow{\quad \beta \quad} (uv)\varphi$$

is a $(1, 2q)$-quasi-geodesic for every geodesics α, β as above. □

The following proposition is an immediate consequence of (3) and Proposition 4.2.

Proposition 4.3. *If $\varphi : H \to H$ is a quasi-isometric embedding, then the BRP holds for φ.* □

The next proposition shows that the bounded reduction property can be reduced to the case where $p = 0$.

Proposition 4.4. *Let H be a hyperbolic group and $\varphi \in End(H)$. If the BRP holds for φ for $p = 0$, then it holds for every $p \in \mathbb{N}$.*

Proof. Let H be a hyperbolic group and $\varphi \in End(H)$ and assume that the BRP holds when $p = 0$. Let $p \in \mathbb{N}$, $u, v \in H$, and take two geodesics α and β from 1 to u and from u to uv, respectively, so that the concatenation $[\alpha\beta]$ is a $(1, p)$-quasi-geodesic. By Proposition 3.1, there is some $N \in \mathbb{N}$ such that for a $(1, p)$-quasi-geodesic ξ starting in 1 and ending in uv, we have that

$$d(\xi^{[n]}, [\alpha\beta]^{[n]}) < N, \tag{4}$$

for every $n \in \mathbb{N}$. We will prove that, there is some $q \in \mathbb{N}$ such that, given any two geodesics ξ_1 and ξ_2 from 1 to $u\varphi$ and from $u\varphi$ to $(uv)\varphi$, respectively, their concatenation is a $(1, q)$-quasi-geodesic.

Take γ to be a geodesic from 1 to uv and consider the factorization

$$1 \xrightarrow{\quad \gamma^{[|u|]} \quad} x \xrightarrow{\quad \gamma_2 \quad} uv.$$

Notice that both $\gamma^{[|u|]}$ and γ_2 are geodesics and their concatenation, γ is also a geodesic. In particular γ is a $(1, p)$-quasi-geodesic with the same starting and ending points as the concatenation of α and β. So, by, (4) we have that $d(x, u) < N$.

Set $B_\varphi = \max\{|a\varphi| \mid a \in \widetilde{A}\}$. Then, we have that $d(x\varphi, u\varphi) \leq B_\varphi d(x, u) < B_\varphi N$. Let ζ_1 and ζ_2 be geodesics from 1 to $x\varphi$ and from $x\varphi$ to $(uv)\varphi$, respectively. Since the BRP holds for φ when $p = 0$, we have that there is some constant p_0 such that the concatenation $[\zeta_1\zeta_2]$ is a $(1, p_0)$-quasi-geodesic and that constant is independent from the choice of γ. So, we have that

$$d(1, (uv)\varphi) \geq d(1, x\varphi) + d(x\varphi, (uv)\varphi) - p_0. \tag{5}$$

Since $d(u\varphi, (uv)\varphi) \leq d(u\varphi, x\varphi) + d(x\varphi, (uv)\varphi) \leq NB_\varphi + d(x\varphi, (uv)\varphi)$, we have that

$$d(x\varphi, (uv)\varphi) \geq d(u\varphi, (uv)\varphi) - NB_\varphi. \tag{6}$$

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Similarly, we have that $d(1, u\varphi) \leq d(1, x\varphi) + d(x\varphi, u\varphi) \leq d(1, x\varphi) + NB_\varphi$, and so

$$d(1, x\varphi) \geq d(1, u\varphi) - NB_\varphi. \tag{7}$$

Combining (5) with (6) and (7), we have that

$$d(1, (uv)\varphi) \geq d(1, u\varphi) - NB_\varphi + d(u\varphi, (uv)\varphi) - NB_\varphi - p_0$$
$$= d(1, u\varphi) + d(1, v\varphi) - 2NB_\varphi - p_0.$$

Hence

$$2\left((u\varphi)^{-1} | v\varphi\right) = d(1, u\varphi) + d(1, v\varphi) - d((u\varphi)^{-1}, v\varphi)$$
$$= d(1, u\varphi) + d(1, v\varphi) - d(1, (uv)\varphi)$$
$$\leq 2NB_\varphi + p_0.$$

By Lemma 4.1, we have that the concatenation $[\xi_1 \xi_2]$ is a $(1, 2NB_\varphi + p_0)$-quasi-geodesic. $\qquad\square$

If (X, d) is δ-hyperbolic and $\lambda \geq 1$, $K \geq 0$, it follows from [4, Thm 1.7, Section III.H.3] that there exists a constant $R(\delta, \lambda, K)$, depending only on δ, λ, K, such that any geodesic and (λ, K)-quasi-geodesic in X having the same initial and terminal points lie at Hausdorff distance $\leq R(\delta, \lambda, K)$ from each other. This constant will be used in the proof of the next result.

We recall that for a geodesic $\alpha : [0, n] \to H$, we will often denote its image by α as well. We are now ready to present two more (equivalent) geometric formulations of the BRP.

Theorem 4.5. *Let $\varphi \in End(H)$. The following conditions are equivalent:*

(i) *the BRP holds for φ.*

(ii) *there is some $N \in \mathbb{N}$ such that, for all $x, y \in H$ and every geodesic $\alpha = [x, y]$, we have that $\alpha\varphi$ is at bounded Hausdorff distance to every geodesic $[x\varphi, y\varphi]$.*

(iii) *there is some $N \in \mathbb{N}$ such that, for all $x, y \in H$ and every geodesic $\alpha = [x, y]$, we have that $\alpha\varphi \subseteq \mathcal{V}_N(\xi)$ for every geodesic $\xi = [x\varphi, y\varphi]$.*

Proof. Clearly (ii) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (ii) : Let $x, y \in H$ and $N \in \mathbb{N}$ given by the BRP when $p = 0$. Consider geodesics $\alpha = [x, y]$ and $\xi = [x\varphi, y\varphi]$. Let $u \in \xi$ and $k = d(x\varphi, u)$. So, clearly, $d(x\varphi, y\varphi) \geq k$. Put $B_\varphi = \max\{|a\varphi| \,|\, a \in \widetilde{A}\}$. We may assume that $k > B_\varphi$ since, otherwise $d(u, \alpha\varphi) \leq B_\varphi$. Since $d(x\varphi, x\varphi) = 0 < k$ and $d(y\varphi, x\varphi) \geq k$, there is some $n_k > 0$ such that $d(x\varphi, (x(\alpha^{[n_k]}\pi))\varphi) < k$ and $d(x\varphi, (x(\alpha^{[n_k+1]}\pi))\varphi) \geq k$ (notice that $n_k > 0$ since $d(x\varphi, (x(\alpha^{[1]}\pi))\varphi) \leq B_\varphi < k$). Consider the following factorization of α

$$x \xrightarrow{\;\;\alpha^{[n_k]}\;\;} x_k \xrightarrow{\;\;\alpha_k\;\;} y \;.$$

Using the BRP, we have that, given geodesics β, γ from $x\varphi$ to $x_k\varphi$ and from $x_k\varphi$ to $y\varphi$, respectively, the concatenation
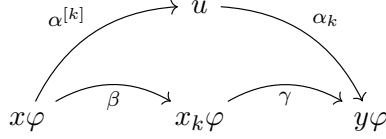
$$x\varphi \xrightarrow{\;\;\beta\;\;} x_k\varphi \xrightarrow{\;\;\gamma\;\;} y\varphi$$

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is a $(1, N)$-quasi-geodesic. Set $x_{k+1} = x(\alpha^{[n_k+1]}\pi)$. We know that $d(x_k\varphi, x_{k+1}\varphi) \leq B_\varphi$ and so

$$k \leq d(x\varphi, x_{k+1}\varphi) \leq d(x\varphi, x_k\varphi) + B_\varphi.$$

Thus, we have that $d(x_k\varphi, x\varphi) \geq k - B_\varphi$.

Let $z = [\beta\gamma]^{[k]}$. Notice that, since $d(x_k\varphi, x\varphi) < k$, then $z \in \gamma$.

Using the fellow traveller property for $(1, N)$-quasi-geodesics, there is some constant M depending only on N and δ such that $d(u, z) < M$. Since γ is a geodesic, then $d(x_k\varphi, z) = k - d(x\varphi, x_k\varphi) \leq k - (k - B_\varphi) = B_\varphi$. Thus,

$$d(u, x_k\varphi) \leq d(u, z) + d(z, x_k\varphi) < M + B_\varphi$$

and $u \in \mathcal{V}_{M+B_\varphi}(\alpha\varphi)$. Since u is an arbitrary element of ξ such that $d(u, \alpha\varphi) > B_\varphi$, we have that $\xi \subseteq \mathcal{V}_{M+B_\varphi}(\alpha\varphi)$

Now, let $v \in \alpha$. Since the BRP holds, then, taking any geodesics, β', γ' from $x\varphi$ to $v\varphi$ and from $v\varphi$ to $y\varphi$, the concatenation

$$x\varphi \xrightarrow{\quad \beta' \quad} v\varphi \xrightarrow{\quad \gamma' \quad} y\varphi$$

is a $(1, N)$-quasi-geodesic, so $\mathrm{Haus}([\beta'\gamma'], \xi) < R(\delta, 1, N)$. In particular, $d(v\varphi, \xi) \leq R(\delta, 1, N)$. Since v is arbitrary, we have that $\alpha\varphi \subseteq \mathcal{V}_{R(\delta,1,N)}(\xi)$.

Hence $\mathrm{Haus}(\xi, \alpha\varphi) \leq \max\{R(\delta, 1, N), M + B_\varphi\}$.

(iii)$\Rightarrow$ (i) Take N such that (iii) holds and $u, v \in H$. Let $\alpha = [1, u]$, $\beta = [u, uv]$ be such that $[\alpha\beta]$ is a geodesic and consider $\gamma = [1, u\varphi]$ and $\zeta = [u\varphi, (uv)\varphi]$. We want to prove that there is some $M \in \mathbb{N}$ such that $[\gamma\zeta]$ is a $(1, M)$-quasi-geodesic and that suffices by Proposition 4.4. From (iii), we have that $\alpha\varphi \subseteq \mathcal{V}_N(\gamma)$, $\beta\varphi \subseteq \mathcal{V}_N(\zeta)$ and $[\alpha\beta]\varphi \subseteq \mathcal{V}_N([\gamma\zeta])$. Since $[\alpha\beta]$ is a geodesic, then, given a geodesic $\xi = [1, (uv)\varphi]$, we have that $([\alpha\beta]\varphi) \subseteq \mathcal{V}_N(\xi)$ too.
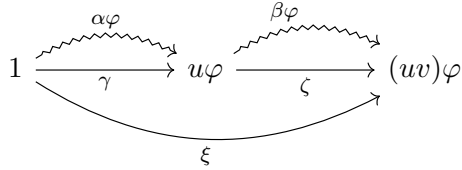
Now, we have that there is some $x_u \in \xi$ such that $d(u\varphi, x_u) < N$. Suppose that $d(1, x_u) \geq |\gamma|$ and denote $\xi^{[|\gamma|]}$ by y. So, γ and $\xi_u = \xi^{[|x_u|]}$ are two geodesics with the same starting point that end at bounded distance. By the fellow traveller property, there is some $K \in \mathbb{N}$ such that $d(u\varphi, y)) < K$. So, we have that,

$$|\zeta| = d(u\varphi, (uv)\varphi) \leq d(u\varphi, y) + d(y, (uv)\varphi) \leq K + d(y, (uv)\varphi).$$

Now,

$$|\xi| = d(1, y) + d(y, (uv)\varphi) = |\gamma| + d(y, (uv)\varphi) \geq |\gamma| + |\zeta| - K.$$

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If $d(1, x_u) < |\gamma|$, the same inequality can be obtained analogously, considering the geodesics ζ^{-1} and ξ^{-1}, since $|\gamma| + |\zeta| \geq |\xi|$.

So, we have that

$$\begin{aligned}
(u^{-1}\varphi|v\varphi) &= \frac{1}{2}(d(1, u\varphi) + d(1, v\varphi) - d(u^{-1}\varphi, v\varphi)) \\
&= \frac{1}{2}(|\gamma| + |\zeta| - d(1, (uv)\varphi)) = \frac{1}{2}(|\gamma| + |\zeta| - |\xi|) \\
&\leq \frac{K}{2}.
\end{aligned}$$

By Lemma 4.1, we have that $[\gamma\zeta]$ is a $(1, K)$-quasi-geodesic. $\qquad\square$

So, combining Propositions 4.2 and 4.4 with Theorem 4.5, we have proved the following result:

Theorem 4.6. *Let $\varphi \in End(H)$. The following conditions are equivalent:*

  i. The BRP holds for φ.

  ii. The BRP holds for φ when $p = 0$.

  iii. $\forall\, p > 0 \,\exists\, q > 0 \,\forall\, u, v \in H\, ((u|v) \leq p \Rightarrow (u\varphi|v\varphi) \leq q)$.

  iv. $\exists\, q > 0 \,\forall\, u, v \in H\, ((u|v) = 0 \Rightarrow (u\varphi|v\varphi) \leq q)$.

  v. there is some $N \in \mathbb{N}$ such that, for all $x, y \in H$ and every geodesic $\alpha = [x, y]$, we have that $\alpha\varphi$ is at bounded Hausdorff distance to every geodesic $[x\varphi, y\varphi]$.

  vi. there is some $N \in \mathbb{N}$ such that, for all $x, y \in H$ and every geodesic $\alpha = [x, y]$, we have that $\alpha\varphi \subseteq \mathcal{V}_N(\xi)$ for every geodesic $\xi = [x\varphi, y\varphi]$.

5   Uniformly continuous endomorphisms

We will start by describing when an endomorphism of a hyperbolic group admits a continuous extension to the completion. It is well know by a general topology result [10, Section XIV.6] that every uniformly continuous mapping $\varphi : H \to H'$ admits a continuous extension to the completion. So, we have the following lemma:

Lemma 5.1. *Let $\varphi : H \to H'$ be a mapping of hyperbolic groups, and let d and d' be visual metrics on G and G', respectively. Then, the following conditions are equivalent:*

  1. φ is uniformly continuous with respect to d and d';

  2. φ admits a continuous extension $\hat{\varphi} : (\widehat{H}, \hat{d}) \to (\widehat{H}', \hat{d}')$.

A mapping $\varphi : (X, d) \to (X', d')$ between metric spaces satisfies a *Hölder condition* of exponent $r > 0$ if there exists a constant $K > 0$ such that

$$d'(x\varphi, y\varphi) \leq K(d(x, y))^r$$

for all $x, y \in X$. It clearly implies uniform continuity. We will show that in case of hyperbolic groups, we have equivalence.

In [1], the authors thoroughly study endomorphisms of hyperbolic groups satisfying a Hölder condition. In particular, they find several properties equivalent to satisfying a Hölder condition. An endomorphism φ of H is *virtually injective* if its kernel is finite.

Theorem 5.2. *[1, Thm 4.3] Let φ be a nontrivial endomorphism of a hyperbolic group G and let $d \in V^A(p, \gamma, T)$ be a visual metric on G. Then the following conditions are equivalent:*

 (i) *φ satisfies a Hölder condition with respect to d;*

 (ii) *φ admits an extension to $\widehat{G}$ satisfying a Hölder condition with respect to $\widehat{d}$;*

 (iii) *there exist constants $P > 0$ and $Q \in \mathbb{R}$ such that*

$$P(g\varphi | h\varphi)_p^A + Q \geq (g|h)_p^A$$

 for all $g, h \in G$;

 (iv) *φ is a quasi-isometric embedding of (G, d_A) into itself;*

 (v) *φ is virtually injective and $G\varphi$ is a quasiconvex subgroup of G.*

The authors in [1] conjecture that every uniformly continuous endomorphism satisfies a Hölder condition. We will give a positive answer to that problem later in this section.

We now present a natural result following from Theorem 4.5.

Corollary 5.3. *Let $\varphi \in End(H)$ such that the BRP holds for φ. Then $H\varphi$ is quasiconvex.*

Proof. Let $x, y \in H$ and take $N \in \mathbb{N}$ given by condition (ii) of Theorem 4.5. Consider geodesics $\alpha = [x, y]$ and $\xi = [x\varphi, y\varphi]$. We have that $\mathrm{Haus}(\xi, \alpha\varphi) \leq N$. Let $u \in \xi$. Then

$$d(u, H\varphi) \leq d(u, \alpha\varphi) \leq \mathrm{Haus}(\xi, \alpha\varphi) \leq N.$$

$\square$

Lemma 4.1 in [1] states that a uniformly continuous endomorphism is virtually injective. So, next we will prove that uniform continuity implies the BRP. In that case, it follows that uniformly continuous endomorphisms are precisely the ones satisfying a Hölder condition.

Let $p, q, x, y \in H$. We have that

$$\begin{aligned}
(x|y)_p &= \frac{1}{2}(d_A(p, x) + d_A(p, y) - d_A(x, y)) \\
&\leq \frac{1}{2}(d_A(q, x) + d_A(q, y) - d_A(x, y) + 2d_A(p, q)) \\
&= (x|y)_q + d_A(p, q)
\end{aligned}$$

Similarly, $(x|y)_q \leq (x|y)_p + d_A(p, q)$, so

$$(x|y)_q - d_A(p, q) \leq (x|y)_p \leq (x|y)_q + d_A(p, q) \tag{8}$$

Proposition 5.4. *Let $G = \langle A \rangle$ and $H = \langle B \rangle$ be hyperbolic groups and consider visual metrics $d_1 \in V^A(p, \gamma, T)$ and $d_2 \in V^B(p', \gamma', T')$ on G and H, respectively. Let $\varphi : (G, d_1) \to (H, d_2)$ be an injective uniformly continuous homomorphism. Then, for every $M \geq 0$, there is some $N \geq 0$ such that*

$$(u|v)^A \leq M \Rightarrow (u\varphi|v\varphi)^B \leq N$$

holds for every $u, v \in G$.

Proof. Since φ is uniformly continuous, by a general topology result it admits a continuous extension $\hat{\varphi} : (\hat{G}, \hat{d}_1) \to (\hat{H}, \hat{d}_2)$. Since $\hat{\varphi}$ is a continuous map between compact spaces, then it is closed, and so it has a closed (thus compact) image.

Now, restricting the codomain of $\hat{\varphi}$ to the image, we have a continuous bijection between compact spaces, and so it is a homeomorphism. Its inverse, $\psi : \text{Im}(\hat{\varphi}) \to \hat{G}$ is a continuous map between compact spaces, hence uniformly continuous. So, the restriction $\psi' : (\text{Im}(\varphi), d_2) \to (G, d_1)$ is also uniformly continuous, i.e.,

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, (d_2(x, y) < \delta \Rightarrow d_1(x\psi', y\psi') < \varepsilon),$$

which, by construction of ψ', means that

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, (d_2(x\varphi, y\varphi) < \delta \Rightarrow d_1(x, y) < \varepsilon). \tag{9}$$

Using (1), we have that (9) is equivalent to

$$\forall M \in \mathbb{N} \, \exists N \in \mathbb{N} \, ((x\varphi|y\varphi)^B_{p'} > N \Rightarrow (x|y)^A_p > M). \tag{10}$$

Since p and p' are fixed, we can change the basepoint to 1 using (8). So, (10) becomes equivalent to

$$\forall M \in \mathbb{N} \, \exists N \in \mathbb{N} \, ((x|y)^A \leq M \Rightarrow (x\varphi|y\varphi)^B \leq N).$$

$\square$

Proposition 5.5. *Let $d \in V^A(p, \gamma, T)$ be a visual metric on H and let φ be a uniformly continuous endomorphism of H with respect to d. Then, the BRP holds for φ.*

Proof. If φ is injective it follows from Proposition 5.4. Now, in case φ is not injective, by Lemma 4.1 in [1], it must have finite kernel K. Consider $\pi : H \to {}^H\!/_K$ to be the projection and the geodesic metric $d_{A\pi}$ on the quotient. Let $\varphi' : \left({}^H\!/_K, d_{A\pi}\right) \to (H, d_A)$ be the injective homomorphism induced by φ.

Let

$$L = \max\{d_A(1, x) \mid x \in K\}.$$

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Let $g, h \in H$. We claim that

$$d_A(g, h) - L \leq d_{A\pi}(g\pi, h\pi) \leq d_A(g, h). \tag{11}$$

Since $h = ga_1 \ldots a_n$ implies $h\pi = (ga_1 \ldots a_n)\pi$ for all $a_1, \ldots, a_n \in \widetilde{A}$, we have $d_{A\pi}(g\pi, h\pi) \leq d_A(g, h)$.

Write $h\pi = (gw)\pi$, where w is a word on $\widetilde{A}$ of minimum length. Then $h = gwx$ for some $x \in K$ and so

$$d_A(g, h) \leq d_A(g, gw) + d_A(gw, gwx) = d_A(1, w) + d_A(1, x) \leq |w| + L.$$

By minimality of w, we have actually $|w| = d_{A\pi}(g\pi, h\pi)$ and thus (11) holds.

This means that (H, d_A) and $\left(^{H}\!/_{K}, d_{A\pi}\right)$ are quasi-isometric. In particular, it yields that $^{H}\!/_{K}$ is hyperbolic.

Now, take $d' \in V^{A\pi}(p'\pi, \gamma', T')$ to be a visual metric on $^{H}\!/_{K}$. For every $u, v, p \in H$, we have that

$$
\begin{aligned}
(u\pi | v\pi)_{p\pi}^{A\pi} &= \frac{1}{2}(d_{A\pi}(p\pi, u\pi) + d_{A\pi}(p\pi, v\pi) - d_{A\pi}(u\pi, v\pi)) \\
&\leq \frac{1}{2}(d_A(p, u) + d_A(p, v) - d_A(u, v) + L) \\
&= (u|v)_p^A + \frac{L}{2}
\end{aligned}
$$

From uniform continuity of φ with respect to d, we get that

$$\forall M \in \mathbb{N} \, \exists N \in \mathbb{N} \, ((x|y)_p^A > N \Rightarrow (x\varphi|y\varphi)_p^A > M).$$

It follows that

$$\forall M \in \mathbb{N} \, \exists N \in \mathbb{N} \, ((x\pi|y\pi)_{p\pi}^{A\pi} > N \Rightarrow (x\varphi|y\varphi)_p^A > M),$$

and so φ' is uniformly continuous with respect to d' and d. It follows from Proposition 5.4 that for every $p \geq 0$, there is some $q \geq 0$ such that

$$(u\pi|v\pi)^{A\pi} \leq p \Rightarrow (u\pi\varphi'|v\pi\varphi')^A \leq q \tag{12}$$

holds for all $u\pi, v\pi \in {}^{H}\!/_{K}$.

Take $u, v \in H$ such that $(u|v)^A = 0$. Then

$$(u\pi|v\pi)^{A\pi} \leq (u|v)^A + \frac{L}{2} = \frac{L}{2}.$$

So, by (12), there is some q which does not depend on u, v such that $(u\varphi|v\varphi)^A \leq q$. By Proposition 4.4, the BRP holds for φ. $\qquad\square$

We can now answer Problem 6.1 left by the authors in [1].

Theorem 5.6. *Let $d \in V^A(p, \gamma, T)$ be a visual metric on H and let φ be an endomorphism of H. Then φ is uniformly continuous with respect to d if and only if the conditions from Theorem 5.2 hold.*

Proof. It is straightforward to see that condition (i) from Theorem 5.2 implies uniform continuity. Now, if φ is uniformly continuous, by Lemma 4.1 in [1] it must be virtually injective and combining Proposition 5.5 with Corollary 5.3, we have that $H\varphi$ is quasiconvex, so condition (v) of Theorem 5.2 holds. □

We now present a visual representation of these properties, where the shaded region represents the nontrivial uniformly continuous endomorphisms. Indeed, we have proved that every nontrivial uniformly continuous endomorphism satisfies the BRP (Proposition 5.5) and that every endomorphism satisfying the BRP must have quasiconvex image (Corollary 5.3). Also, every virtually injective endomorphism with quasiconvex image must be uniformly continuous by Theorem 5.2. In [1], the authors give an example of an injective endomorphism of a torsion-free hyperbolic group with non quasiconvex image. So, unlike the case of virtually free groups, the BRP does not hold in general for injective endomorphisms of hyperbolic groups, not even when restricted to torsion-free hyperbolic groups. In the virtually free groups case, that does not happen, as every virtually injective endomorphism is uniformly continuous [19].

Taking $\varphi : F_3 \to F_3$ defined by $a \mapsto a$, $b \mapsto b$ and $c \mapsto 1$, we have that $F_3\varphi = \langle a, b \rangle$. Since $F_3\varphi$ is finitely generated and the standard embedding $\langle a, b \rangle \hookrightarrow F_3$ is a quasi-isometric embedding, then φ has quasiconvex image. But the BRP does not hold for φ since $|cb^n \wedge b^n| = 0$ and $|(cb^n)\varphi \wedge b^n\varphi| = |b^n \wedge b^n| = n$, which can be arbitrarily large.

Defining an endomorphism with finite image it is easy to find examples of endomorphisms for which the BRP holds that are not virtually injective, even for virtually free groups. For example, take $H = \mathbb{Z} \times \mathbb{Z}_2$ and φ defined by $(n, 0) \mapsto (0, 0)$ and $(n, 1) \mapsto (0, 1)$. Then, the BRP holds for φ and its kernel is infinite (in particular, it can't be uniformly continuous).
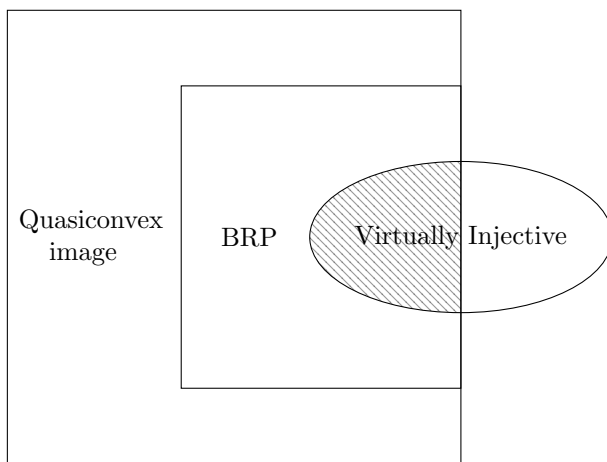

Figure 1: Nontrivial uniformly continuous endomorphisms of hyperbolic groups

So, for hyperbolic groups, we have the figure above in which every region is nonempty. Notice that, for virtually free groups, the only difference is that nontrivial uniformly continuous endomorphisms are precisely the virtually injective ones and

the BRP holds for all of them.

In the case of free groups, it is even simpler as, for every nontrivial endomorphism, the properties of being injective, uniformly continuous and satisfying the BRP are equivalent. Indeed, it is well-known that injective endomorphisms coincide with uniformly continuous ones and that the BRP holds for this class. It is easy to see that the converse also holds. For $n \geq 2$, let $X = \{x_1, \ldots, x_n\}$ be a finite alphabet and $F_n = \langle X \rangle$ be a free group of rank n. If a nontrivial endomorphism $\varphi \in \text{End}(F_n)$ is not injective, then there is some $w \in \text{Ker}(\varphi)$ such that $w \neq 1$ and some letter a such that $a\varphi \neq 1$. Let $p = |w|$. We have that w is not a proper power of a since, in that case we would have that $w\varphi$ would be a proper power of $a\varphi$ and so, nontrivial. So, we have that, for arbitrarily large $m \in \mathbb{N}$, $|wa^m \wedge a^m| < |w|$, but $|(wa^m)\varphi \wedge a^m\varphi| = |a^m\varphi|$, which is arbitrarily large. So, nontrivial endomorphisms for which the BRP holds in a free group of finite rank are precisely the injective ones.

In [16], Paulin proved that $\text{Fix}(\varphi)$ is finitely generated if $\varphi \in \text{Aut}(H)$. We remark that its proof also yields the result for quasi-isometric embeddings. So, we have proved the following result.

Theorem 5.7. *Let $\varphi \in End(H)$ be an endomorphism admitting a continuous extension $\hat{\varphi} : \widehat{H} \to \widehat{H}$ to the completion of H. Then, $Fix(\varphi)$ is finitely generated.* $\qquad \square$

Acknowledgements

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