On the dynamics of extensions of free-abelian times free groups endomorphisms to the completion

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Abstract

We obtain conditions of uniform continuity for endomorphisms of free-abelian times free groups for the product metric defined by taking the prefix metric in each component. Considering the extension of an endomorphism to the completion we count the number of orbits for the action of the subgroup of fixed points (resp. periodic) points on the set of infinite fixed (resp. periodic) points. Finally, we study the dynamics of infinite points: for type II endomorphisms we prove that every infinite point is either periodic or wandering, which implies that the dynamics is asymptotically periodic. We also prove the latter for the case of automorphisms.

1 Introduction

The dynamical study of endomorphisms of groups started with the (independent) work of Gersten [7] and Cooper [4], using respectively graph-theoretic and topological approaches. They proved that the subgroup of fixed points $\text{Fix}(\varphi)$ of some fixed automorphism φ of F_n is always finitely generated, and Cooper succeeded on classifying from the dynamical viewpoint the fixed points of the continuous extension of φ to the boundary of F_n. Bestvina and Handel subsequently developed the theory of train tracks to prove that $\text{Fix}(\varphi)$ has rank at most n in [1]. The problem of computing a basis for $\text{Fix}(\varphi)$ had a tribulated history and was finally settled by Bogopolski and Maslakova in 2016 in [2].

This line of research extended early to wider classes of groups. For instance, Paulin proved in 1989 that the subgroup of fixed points of an automorphism of a hyperbolic group is finitely generated [10]. Fixed points were also studied for right-angled Artin groups [11] and lamplighter groups [9].

Regarding the extension of an endomorphism to the completion, infinite fixed points of automorphisms of free groups were also discussed by Bestvina and Handel in [1] and Gaboriau, Jaeger, Levitt and Lustig in [6]. The dynamics of free groups automorphisms is proved to be asymptotically periodic in [8]. In [3], the dynamical study of infinite fixed points was performed for monoids defined by special confluent rewriting systems (which contain free groups as a particular case). This was also achieved in [12] for virtually injective endomorphisms of virtually free groups.

In this paper we will study the dynamics of infinite points of free-abelian times free groups which is a subclass of the well-known right-angled Artin groups. This class of groups have been thoroughly studied both algebraically and algorithmically in [5]. Although the class of free-abelian groups is very well known and the class of free groups, while being much more complex than the first, has also been deeply studied, some problems in the product $\mathbb{Z}^m \times F_n$ are not easily reduced to problems in each factor. In particular, when endomorphisms (and automorphisms) are considered, we have that many endomorphisms are not obtained by

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applying an endomorphism of $\mathbb{Z}^m$ to the first component and one of F_n to the second, so some problems arise when the dynamics of an endomorphism is considered. Also, it is worth pointing out that, while the study of the dynamics of infinite points have been done for some classes of groups, such as the free groups and the virtually free groups, it is the first time this problem is tackled for non hyperbolic groups.

The paper is organized as follows. In section 2, we present some preliminaries on groups and endomorphisms. We find equivalent conditions to uniform continuity of endomorphisms in Section 3. In Section 4 we count the orbits of the action of finite fixed (resp. periodic) points on inifinite fixed (resp. periodic) points and classify the infinite fixed points of automorphisms. In Section 5 we study some dynamical aspects of inifinite points of endomorphisms and automorphisms, proving that for type II endomorphisms every point is either periodic or wandering and that the dynamics of an automorphism is asymptotically periodic. Finally, in section 6 we include some open questions

2 Preliminaries

We denote by F_n the free group of rank n and its alphabet by $A = \{a_1, \ldots, a_n\}$. Given two words u and v on a free group, we write $u \wedge v$ to denote the longest common prefix of u and v. The prefix metric on a free group is defined by

$$d(u,v) = \begin{cases} 2^{-|u\wedge v|} \text{ if } u \neq v \\ 0 \text{ otherwise} \end{cases} .$$

The prefix metric on a free group is in fact an ultrametric and its completion $(\hat{F}_n, \hat{d})$ is a compact space which can be described as the set of all finite and infinite reduced words on the alphabet $A \cup A^{-1}$. We will denote by ∂F_n the set consisting of only the infinite words and call it the *boundary* of F_n.

A reduced word $z = z_1 \cdots z_n$, with $z_i \in A \cup A^{-1}$, is said to be *cyclically reduced* if $z_1 \neq z_n^{-1}$. Every word admits a decomposition of the form $z = w\tilde{z}w^{-1}$, where $\tilde{z}$ is cyclically reduced. The word $\tilde{z}$ is called the *cyclically reduced core* of z.

A free-abelian times free group is of the form $\mathbb{Z}^m \times F_n$ and we consider them endowed with the product metric given by taking the prefix metric in each (free) component, i.e.,

$$d((a,u),(b,v)) = \max\{d(a_1,b_1), \ldots, d(a_m,b_m), d(u,v)\},$$

where a_i and b_i denote the i-th component of a and b, respectively. This metric is also an ultrametric and $\widehat{\mathbb{Z}^m \times F_n}$ is homeomorphic to $\hat{\mathbb{Z}^m} \times \hat{F_n}$ by uniqueness of the completion (Theorem 24.4 in [13]).

It is proved in [5] that for $G = \mathbb{Z}^m \times F_n$, with $n \neq 1$, all endomorphisms of G are of one of the following forms:

(I) $\Psi_{\Phi,Q,P} = (a,u) \mapsto (aQ + uP, u\Phi)$, where $\Phi \in End(F_n)$, $Q \in \mathcal{M}_m(\mathbb{Z})$, and $P \in \mathcal{M}_{n\times m}(\mathbb{Z})$.

(II) $\Psi_{z,\ell,h,Q,P} = (a,u) \mapsto (aQ + \mathbf{u}P, z^{a\ell^T + \mathbf{u}h^T})$, where $1 \neq z \in F_n$ is not a proper power, $Q \in \mathcal{M}_m(\mathbb{Z})$, $P \in \mathcal{M}_{n\times m}(\mathbb{Z})$, $\mathbf{0} \neq \ell \in \mathbb{Z}^m$, and $h \in \mathbb{Z}^n$,

where $\mathbf{u} \in \mathbb{Z}^n$ denotes the abelianization of the word $u \in F_n$.

For free groups, it is well known that an endomorphism $\varphi \in End(F_n)$ is uniformly continuous if and only if it is either trivial or injective.

3 Uniform continuity of endomorphisms

Since we are interested in the study of the continuous extension of endomorphisms to the completion, we will obtain conditions for an endomorphism of a free-abelian group to be uniformly continuous. We present a proof of the following trivial lemma for sake of completeness.

Lemma 3.1. *Let $\varphi_1 : G_1 \to G_3$ and $\varphi_2 : G_2 \to G_4$ be homomorphisms of groups. The homomorphism $\varphi : G_1 \times G_2 \to G_3 \times G_4$ given by $(x,y)\varphi = (x\varphi_1, y\varphi_2)$ is uniformly continuous for the product metric if and only if φ_1 and φ_2 are uniformly continuous for the prefix metric.*

Proof. Consider the homomorphism $\varphi : G_1 \times G_2 \to G_3 \times G_4$ given by $(x,y)\varphi = (x\varphi_1, y\varphi_2)$ and suppose it is uniformly continuous. Let $\varepsilon > 0$ and take δ such that for every $(x_1,x_2),(y_1,y_2) \in G_1 \times G_2$ such that $d((x_1,x_2),(y_1,y_2)) < \delta$, we have that $d((x_1\varphi_1, x_2\varphi_2),(y_1\varphi_1, y_2\varphi_2))$. We know that for every $x_1, y_1 \in G_1$ such that $d(x_1,y_1) < \delta$ we have $d(x_1\varphi_1, y_1\varphi_1) < \varepsilon$, since $d((x_1,1),(y_1,1)) = d(x_1,y_1) < \delta$ and so

$$d(x_1\varphi_1, y_1\varphi_1) = d((x_1\varphi_1, 1\varphi_2),(y_1\varphi_1, 1\varphi_2)) < \varepsilon.$$

Conversely, if both φ_1, φ_2 are uniformly continuous, then taking $\varepsilon > 0$, there are δ_i such that for every $x_i, y_i \in G_i$ such that $d(x,y) < \delta_i$, we have $d(x\varphi_i, y\varphi_i) < \varepsilon$. Taking $\delta = \min \delta_i$, we know that for every $(x_1,y_1),(x_2,y_2) \in G_1 \times G_2$ such that $d((x_1,x_2),(y_1,y_2)) < \delta$, then $d(x_1,y_1) < \delta \leq \delta_1$ and $d(x_2,y_2) < \delta \leq \delta_2$, thus $d(x\varphi_i, y\varphi_i) < \varepsilon$ and

$$d((x_1,x_2)\varphi,(y_1,y_2)\varphi) = d((x_1\varphi_1, x_2\varphi_2),(y_1\varphi_1, y_2\varphi_2)) < \varepsilon.$$

□

Proposition 3.2. *Let $u \in \mathbb{Z}^m$ and $\varphi : \mathbb{Z}^m \to \mathbb{Z}$ be a homomorphism given by $v \mapsto vu^T$. Then φ is uniformly continuous if and only if u has at most one nonzero entry.*

Proof. If u=0, then φ is uniformly continuous. It is clear that if u has a single nonzero entry, then φ is uniformly continuous. Indeed, take $u \in \mathbb{Z}^m$ such that $u_k = \lambda \neq 0$ for some $k \in [m]$ and $u_j = 0$ for all $j \in [m] \setminus \{k\}$. Take $\varepsilon > 0$. Set $\delta = \varepsilon$ and take $a, b \in \mathbb{Z}^m$ such that $d(a,b) < \delta$. Notice that $au^T = \lambda a_k$ and $bu^T = \lambda b_k$. If $a_k = b_k$, then $d(au^T, bu^T) = 0 < \varepsilon$. If not, since $d(a,b) < \delta$ then $d(a_i,b_i) < \delta$ for all $i \in [m]$. In particular $d(a_k,b_k) < \delta$. This means that $|a_k \wedge b_k| > log_2(\frac{1}{\delta})$, i.e. $a_kb_k > 0$ and $|a_k|,|b_k| > log_2(\frac{1}{\delta})$. But then, $au^Tbu^T = \lambda^2 a_kb_k > 0$ and $|\lambda a_k| = |\lambda||a_k| \geq |a_k| > log_2(\frac{1}{\delta})$. Similarly, $|\lambda b_k| > log_2(\frac{1}{\delta})$. This means that $d(au^T, bu^T) \leq d(a,b) < \delta = \varepsilon$

Suppose now that u has at least one positive and one negative entry. Let $u_{i_1}, \ldots, u_{i_r}$ be the nonnegative entries and $u_{j_1}, \ldots, u_{j_s}$ be the negative entries and suppose w.l.o.g. that $\sum_{x=1}^{r} u_{i_x} \geq \sum_{x=1}^{s} u_{j_x}$. We will show that for every $\delta > 0$, there are $v, w \in \mathbb{Z}^m$ such that $d(v,w) < \delta$ and $d(vu^T, wu^T) = 1$ and so φ is not uniformly continuous. Take $\delta > 0$, v such that

$$v_i = 1 + \lceil log_2(\frac{1}{\delta})\rceil, \text{ for every } i \in [m]$$

and w such that

$$w_{i_k} = 1 + \left\lceil log_2\left(\frac{1}{\delta}\right)\right\rceil, \text{ for all } k \in [r]$$

and

$$w_{j_k} = \sum_{x=1}^{r} u_{i_x}\left(1 + \left\lceil log_2\left(\frac{1}{\delta}\right)\right\rceil\right), \text{ for all } k \in [s].$$

Then $d(v,w) < \delta$ since, for every $i \in [m]$, $v_iw_i > 0$ and $|v_i|,|w_i| > log_2(\frac{1}{\delta})$ (notice that $\sum_{x=1}^{r} u_{i_x} \geq 1$). Also, $vu^T = \sum v_iu_i \geq 0$, since we are assuming that $\sum_{x=1}^{r} u_{i_x} \geq \sum_{x=1}^{s} u_{j_x}$. We have that

$$wu^T = \sum_{x=1}^{r} u_{i_x}\left(1 + \left\lceil log_2\left(\frac{1}{\delta}\right)\right\rceil\right)\left(1 + \sum_{x=1}^{s} u_{j_x}\right) \leq 0.$$

Thus, $d(vu^T, wu^T) = 1$.

Now, suppose that $u \in (\mathbb{Z}_0^+)^m$ has at least two nonzero entries (the nonpositive case is analogous). Let u_k be a minimal nonzero entry. As above, we will show that for every $\delta > 0$, there are $v, w \in \mathbb{Z}^m$ such that $d(v, w) < \delta$ and $d(vu^T, wu^T) = 1$ and so φ is not uniformly continuous. Take $\delta > 0$, v such that

$$v_i = 1 + \left\lceil log_2\left(\frac{1}{\delta}\right)\right\rceil \text{ for every } i \in [m] \setminus \{k\} \quad \text{and} \quad v_k = -1 - \left\lceil log_2\left(\frac{1}{\delta}\right)\right\rceil$$

and w such that

$$w_i = 1 + \left\lceil log_2\left(\frac{1}{\delta}\right)\right\rceil \text{ for every } i \in [m] \setminus \{k\}$$

and

$$w_k = -\sum_{i \in [m]} u_i \left(1 + \left\lceil log_2\left(\frac{1}{\delta}\right)\right\rceil\right).$$

Now, $w_i v_i > 0$ and $|w_i|, |v_i| > log_2(\frac{1}{\delta})$ for every $i \in [m]$, so $d(v, w) < \delta$. Also,

$$vu^T = \left(1 + \left\lceil log_2\left(\frac{1}{\delta}\right)\right\rceil\right)\left(\sum_{i \neq k} u_i - u_k\right) \geq 0$$

by minimality of u_k. We have that

$$wu^T = \left(1 + \left\lceil log_2\left(\frac{1}{\delta}\right)\right\rceil\right)\left(\sum_{i \neq k} u_i - u_k \sum u_i\right) \leq 0.$$

Thus, $d(vu^T, wu^T) = 1$. □

Corollary 3.3. *Let $Q \in M_m(\mathbb{Z})$ and $\varphi \in End(\mathbb{Z}^m)$ to be given by $u \mapsto uQ$. Then φ is uniformly continuous if and only if every column Q_i of Q has at most one nonzero entry.*

Proof. Consider the homomorphisms $\varphi_i : \mathbb{Z}^m \to \mathbb{Z}$ defined by $u \mapsto uQ_i$. Then $\varphi(u) = (\varphi_1(u), \cdots, \varphi_m(u))$. □

We are now capable of obtaining conditions of uniform continuity for endomorphisms of type I.

Proposition 3.4. *Let $G = \mathbb{Z}^m \times F_n$, with $n > 1$ and consider an endomorphism φ of type I, mapping (a, u) to $(aQ + \mathbf{u}P, u\Phi)$. Denote by ψ the endomorphism of $\mathbb{Z}^m$ defined by $a \mapsto aQ$. Then the following conditions are equivalent:*

i. φ is uniformly continuous.

ii. $P = 0$, ψ is uniformly continuous and Φ is either trivial or injective.

Proof. *i.* $\Rightarrow$ *ii.* Consider the alphabet of F_n to be $\{x_1 \dots x_n\}$. Suppose $P \neq 0$ and pick entries $p_{rs} \neq 0$ and p_{ts} with $r \neq t$. We will prove that φ is not uniformly continuous, by showing that $\forall \delta > 0$ there exists $X, Y \in G$ such that $d(X, Y) < \delta$ and $d(X\varphi, Y\varphi) = 1$. We may assume $\delta \leq 1$, so pick such δ and, as usual, set $q = 1 + \lceil log_2(\frac{1}{\delta})\rceil$. Take $\beta \in \mathbb{Z}$ to be such that $sgn(p_{ts}) \neq sgn(p_{ts}q + \beta p_{rs})$ (if $p_{ts} = 0$, put $\beta = 1$). Let $X = (0, x_t^q)$ and $Y = (0, x_t^q x_r^\beta)$. To simplify notation, write u and v for the free parts of X and Y, respectively, so $u = x_t^q$ and $v = x_t^q x_r^\beta$.

Since the free abelian parts coincide, $d(X, Y) = d(u, v) = 2^{-q} < \delta$.

We have that $d((\mathbf{u}P, u\Phi), (vP, v\Phi)) = \max\{d(u\Phi, v\Phi), d(\mathbf{u}P, vP)\} \geq d(\mathbf{u}P, vP)$. But

$$\mathbf{u}P = \begin{bmatrix} 0 & \cdots & q & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & & \\ p_{n1} & \cdots & p_{nm} \end{bmatrix} = \begin{bmatrix} p_{ti} q \end{bmatrix}_{i \in [m]}$$

and

$$\mathbf{v}P = \begin{bmatrix} 0 & \cdots & q & \cdots & \beta & \cdots & 0 \end{bmatrix} \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & & \\ p_{n1} & \cdots & p_{nm} \end{bmatrix} = \begin{bmatrix} p_{ti} q + \beta p_{ri} \end{bmatrix}_{i \in [m]}$$

thus, $d(\mathbf{u}P, \mathbf{v}P) = \max\{d((\mathbf{u}P)_i, (\mathbf{v}P)_i)\} \geq d((\mathbf{u}P)_s, (\mathbf{v}P)_s) = d(p_{ts}q, p_{ts}q + \beta p_{rs}) = 1$, by definition of β.

The remaining conditions follow by Lemma 3.1.

ii. $\Rightarrow$ *i.* This implication is obvious by Lemma 3.1 since both $a \mapsto aQ$ and $u \mapsto u\Phi$ are uniformly continuous. $\square$

Now we deal with the type II endomorphisms.

Proposition 3.5. *Let $G = \mathbb{Z}^m \times F_n$, with $n > 1$ and consider an endomorphism φ of type II, mapping (a, u) to $(aQ + \mathbf{u}P, z^{a\ell^T + \mathbf{u}h^T})$. Denote by ψ_1 the endomorphism of $\mathbb{Z}^m$ defined by $a \mapsto aQ$ and $\psi_2 : \mathbb{Z}^m \to \mathbb{Z}$ the homomorphism defined by $a \mapsto a\ell^T$. Then the following conditions are equivalent:*

1. *φ is uniformly continuous.*

2. *$\mathbf{P} = 0$, $\mathbf{h} = 0$ and both ψ_1 and ψ_2 are uniformly continuous.*

Proof. i. $\Rightarrow$ *ii.* The proof that $\mathbf{P} = 0$ is the same as in the previous proposition.

Now, we will prove that if $h \neq 0$, then for all $\delta > 0$, there are $X, Y \in G$ such that $d(X, Y) < \delta$ and $d(X\varphi, Y\varphi) = 1$. Suppose then $h \neq 0$ and pick entries $h_k \neq 0$ and h_t, with $t \neq k$ and some $\delta > 0$. Set $q = 1 + \lceil log_2(\frac{1}{\delta}) \rceil$ and take $X = (0, x_t^q)$ and $Y = (0, x_t^q x_k^\alpha)$ for some $\alpha \in \mathbb{Z}$ such that $sgn(h_t q + \alpha h_k) \neq sgn(h_t q)$ (if $h_t = 0$ put $\alpha = 1$). Then $d(X, Y) < \delta$ and $1 \geq d(X\varphi, Y\varphi) \geq d(z^{h_t q}, z^{\alpha h_k + h_t q}) = 1$.

The proof that ψ_1 is uniformly continuous is analogous the the one in the previous proposition.

Now, suppose ψ_2 is not uniformly continuous. There exists $\varepsilon > 0$ such that for every $\delta > 0$, there are $a, b \in \mathbb{Z}^m$ such that $d(a, b) < \delta$ and

$$d\left(\sum_{i \in [m]} a_i \ell_i, \sum_{i \in [m]} b_i \ell_i\right) \geq \varepsilon. \qquad (1)$$

We now show that for every $\delta > 0$, there are $X, Y \in G$ such that $d(X, Y) < \delta$ and $d(X\varphi, Y\varphi) \geq 2^{-|w| - |\tilde{z}| \lceil log_2(\frac{1}{\varepsilon}) \rceil}$, where $\tilde{z}$ is the cyclically reduced core of z and w is such that $z = w\tilde{z}w^{-1}$. Notice that $\tilde{z}$ and w^{-1} don't share a prefix. Take $\delta > 0$ and take $a, b \in \mathbb{Z}^m$ such that $d(a, b) < \delta$ satisfying (1). Now, consider $X = (a, 1)$ and $Y = (b, 1)$. Clearly $d(X, Y) = d(a, b) < \delta$ and

$$d(X\varphi, Y\varphi) = d\left(\left(aQ, z^{a\ell^T}\right), \left(bQ, z^{b\ell^T}\right)\right) \geq d\left(z^{a\ell^T}, z^{b\ell^T}\right)$$

We know that (1) holds, so, either

$$a\ell^T b\ell^T = \sum_{i \in [m]} a_i \ell_i \sum_{i \in [m]} b_i \ell_i \leq 0$$

and in that case $d\left(z^{a\ell^T}, z^{b\ell^T}\right) = 1$, or

$$\sum_{i\in[m]} a_i\ell_i \sum_{i\in[m]} b_i\ell_i > 0 \text{ and } 2^{-\min\{|a\ell^T|,|b\ell^T|\}} \geq \varepsilon,$$

which means that

$$\min\left\{\left|a\ell^T\right|, \left|b\ell^T\right|\right\} \leq log_2\left(\frac{1}{\varepsilon}\right).$$

In this case, we have that

$$\begin{aligned} d(z^{a\ell^T}, z^{b\ell^T}) &= 2^{-|z^{a\ell^T}\wedge z^{b\ell^T}|} = 2^{-|w\tilde{z}^{a\ell^T}w^{-1}\wedge w\tilde{z}^{b\ell^T}w^{-1}|} \\ &= 2^{-|w|-|\tilde{z}|\min\{|a\ell^T|,|b\ell^T|\}} \geq 2^{-|w|-|\tilde{z}|\lceil log_2(\frac{1}{\varepsilon})\rceil}. \end{aligned}$$

ii. $\Rightarrow$ *i.* Straightforward. □

So, type I uniformly continuous endomorphisms are of the form $(a, u) \mapsto (aQ, u\phi)$ where $Q \in \mathcal{M}_m(\mathbb{Z})$ has at most one nonzero entry in each column and $\phi \in End(F_n)$ is either trivial or injective. Type II endomorphisms are uniformly continuous if they map (a, u) to $(aQ, z^{\lambda a_k})$ where $Q \in \mathcal{M}_m(\mathbb{Z})$ has at most one nonzero entry in each column, $k \in [m]$, $1 \neq z \in F_n$ is not a proper power and $0 \neq \lambda \in \mathbb{Z}$. Notice that $\lambda \neq 0$, since by definition of a type II endomorphism we have that $\ell \neq 0$.

4 Infinite fixed and periodic points

4.1 Finiteness Conditions on Infinite Fixed Points

Let φ be a type I uniformly continuous endomorphism of $\mathbb{Z}^m \times F_n$, with $n > 1$. Then $\varphi : \mathbb{Z}^m \times F_n \to \mathbb{Z}^m \times F_n$ is given by $(a, u) \mapsto (aQ, u\phi)$, for some $Q \in \mathcal{M}_m(\mathbb{Z})$ such that every column of Q contains at most one nonzero entry and some either trivial or injective $\phi \in End(F_n)$. Consider $\varphi_1 \in End(\mathbb{Z}^m)$ to be defined as $a \mapsto aQ$. Clearly, $\text{Fix}(\varphi) = \text{Fix}(\varphi_1) \times \text{Fix}(\phi)$ and it is finitely generated.

Let $\hat{\varphi} : \widehat{\mathbb{Z}^m} \times \widehat{F_n} \to \widehat{\mathbb{Z}^m} \times \widehat{F_n}$ be its continuous extension to the completion. By uniqueness of the extension, we have that $\hat{\varphi}$ is given by $(a, u) \mapsto (a\widehat{\varphi}_1, u\widehat{\phi})$. Then, we have that $\text{Fix}(\hat{\varphi}) = \text{Fix}(\hat{\varphi}_1) \times \text{Fix}(\hat{\phi})$; $\text{Sing}(\hat{\varphi}) = \text{Sing}(\hat{\varphi}_1) \times \text{Sing}(\hat{\phi})$ and $\text{Reg}(\hat{\varphi}) = \text{Fix}(\hat{\varphi}_1) \times \text{Reg}(\hat{\phi}) \cup \text{Reg}(\hat{\varphi}_1) \times \text{Fix}(\hat{\phi})$. There is no hope of finding a finiteness condition in this case that holds in general since, if $n = 2$ and ϕ is the identity mapping, which is injective, then $\text{Sing}(\hat{\phi})$ is uncountable, thus so are both $\text{Reg}(\hat{\varphi})$ and $\text{Sing}(\hat{\varphi})$.

However, we will see that this is not the case when dealing with type II endomorphisms.

Let $\varphi \in End(\mathbb{Z}^m)$ defined by $a \mapsto aQ$ be a uniformly continuous endomorphism. From Corollary 3.3 we know that Q has at most one nonzero entry in each column. Given a column Q_j, if $Q_j \neq 0$, we call λ_j its nonzero entry and denote by α_j the corresponding row, so that $q_{ij} = \lambda_i$ if $i = \alpha_j$ and $q_{ij} = 0$ otherwise. If $Q_j = 0$ we put $\lambda_j = 0$ and $\alpha_j = 1$. Then, we have that $[a_i]_{i\in[m]}$ is mapped to $[\lambda_i a_{\alpha_i}]_{i\in[m]}$ and $a \in \text{Fix}(\varphi)$ if $a_i = \lambda_i a_{\alpha_i}$ for every $i \in [m]$. Take $\hat{\varphi}_1 : \widehat{\mathbb{Z}^m} \to \widehat{\mathbb{Z}^m}$ to be the continuous extension of φ_1 to the completion.

Given $a \in \widehat{\mathbb{Z}^m}$, we set $i_1, \ldots, i_r$ to be the indices such that $a_{i_1} = \ldots = a_{i_r} = +\infty$; $j_1, \ldots, j_s$ to be the indices such that $a_{j_1} = \ldots = a_{j_s} = -\infty$ and $k_1, \ldots, k_t$ to be the indices such that $a_{k_1}, \ldots a_{k_t} \notin \{-\infty, +\infty\}$. Define, for every $n \in \mathbb{N}$, $a_n \in \mathbb{Z}^m$ such that

$$a_{n_{i_l}} = n, \text{ for } l \text{ in } [r]; \quad a_{n_{j_l}} = -n, \text{ for } l \text{ in } [s] \quad \text{and} \quad a_{n_{k_l}} = a_{k_l} \text{ for } l \text{ in } [t].$$

We have that, given $\varepsilon > 0$, for every $n > log_2(\frac{1}{\varepsilon})$, $d(a_n, a) < \varepsilon$, so $(a_n) \to a$. Thus, $a\hat{\varphi} = (\lim a_n)\hat{\varphi} = \lim(a_n\varphi)$. Since $(a_n\varphi)$ is such that $(a_n\varphi)_i = \lambda_i a_{\alpha_i}$ if $a_{\alpha_i} \notin \{+\infty, -\infty\}$,

$(a_n\varphi)_i = n\lambda_i$ if $a_{\alpha_i} = +\infty$, and $(a_n\varphi)_i = -n\lambda_i$ if $a_{\alpha_i} = -\infty$, we have that $a\hat{\varphi} = [\lambda_i a_{\alpha_i}]$, assuming that $0 \times \infty = 0$. So, $a \in \text{Fix}(\hat{\varphi})$ if and only if $a_i = \lambda_i a_{\alpha_i}$ for every $i \in [m]$.

Defining the sum of an integer with infinite in the natural way, we have that the subgroup $\text{Fix}(\varphi) \leq \mathbb{Z}^m \times F_n$ acts on $\text{Fix}(\hat{\varphi})$ by left multiplication. Given $a \in \text{Fix}(\varphi)$ and $b \in \text{Fix}(\hat{\varphi})$, then $(a + b)\hat{\varphi} = a\varphi + b\hat{\varphi} = a + b \in \text{Fix}(\hat{\varphi})$. We now count the orbit of this action.

Proposition 4.1. *Fix*$(\hat{\varphi})$ *has* $\sum_{i=0}^{m} 2^i \binom{m}{i}$ *Fix*(φ)*-orbits.*

Proof. Let $a \in \text{Fix}(\hat{\varphi})$ and define $r, s, t \in \{0, \dots, m\}$, and i_l, j_l, k_l as above. We will prove that, for $b \in \text{Fix}(\hat{\varphi})$, we have that $b \in (\text{Fix}\varphi)a$ if and only if $b_{i_l} = a_{i_l}$ for every $l \in [r]$, $b_{j_l} = a_{j_l}$, for every $l \in [s]$ and $b_{k_l} \notin \{+\infty, -\infty\}$, for every $l \in [t]$, i.e., if their infinite entries coincide. If that is the case, then every orbit is defined by the position and the signal of their infinite entries. Obviously, for $i \in \{0, \dots, m\}$, there are $\binom{m}{i}$ choices for i infinite entries and each of them can be $+\infty$ or $-\infty$, hence the 2^i factor.

Start by supposing that $b \in \text{Fix}(\hat{\varphi})$ is such that $b \in (\text{Fix}\varphi)a$. Then, there is some $c \in \text{Fix}\varphi$ such that $b = c + a$. This means that for every $l \in [r]$, we have that $b_{i_l} = c_{i_l} + a_{i_l} = c_{i_l} + (+\infty) = +\infty$, since $c \in \mathbb{Z}^m$. Similarly, we have that $b_{j_l} = a_{j_l}$, for every $j \in [s]$ and $b_{k_l} \notin \{+\infty, -\infty\}$. It is clear that $|b_{k_l}| < \infty$ for $l \in [t]$.

Now, suppose $b \in \text{Fix}(\hat{\varphi})$ is such that $b_{i_l} = +\infty$, for $l \in [r]$, $b_{j_l} = -\infty$, for $l \in [s]$ and $b_{k_l} \notin \{+\infty, -\infty\}$, for $l \in [t]$. Consider $c \in \mathbb{Z}^m$ defined by $c_{k_l} = b_{k_l} - a_{k_l}$ and all other entries are 0. Clearly $b = c + a$. We only have to check that $c \in \text{Fix}(\varphi)$, i.e., $\lambda_i c_{\alpha_i} = c_i$ for every $i \in [m]$. For i such that $a_i = \pm\infty$, we have that $c_i = 0$ and $a_{\alpha_i} = sgn(\lambda_i) a_i = \pm\infty$, which implies that $c_{\alpha_i} = 0$. If not, $c_i = b_i - a_i = \lambda_i(b_{\alpha_i} - a_{\alpha_i}) = \lambda_i c_{\alpha_i}$ and we are done. □

Now, let φ be a type II uniformly continuous endomorphism of $\mathbb{Z}^m \times F_n$, with $n > 1$. Then $\varphi : \mathbb{Z}^m \times F_n \to \mathbb{Z}^m \times F_n$ is given by $(a, u) \mapsto (aQ, z^{\lambda a_k})$, for some $Q \in \mathcal{M}_m(\mathbb{Z})$ such that every column of Q contains at most one nonzero entry, $0 \neq \lambda \in \mathbb{Z}$ and $k \in [m]$. Consider $\varphi_1 \in End(\mathbb{Z}^m)$ to be defined as $a \mapsto aQ$ and $\varphi_2 : \mathbb{Z}^m \to F_n$ that maps a to $z^{\lambda a_k}$, which are both uniformly continuous. Observe that $\text{Fix}(\varphi) = \{(a, a\varphi_2) \mid a \in \text{Fix}(\varphi_1)\}$ and it is finitely generated (see Proposition 6.2 in [5]).

By uniqueness of extension, we have that $\hat{\varphi} : \widehat{\mathbb{Z}^m} \times \widehat{F_n} \to \widehat{\mathbb{Z}^m} \times \widehat{F_n}$ is defined by $(a, u) \mapsto (a\hat{\varphi}_1, a\hat{\varphi}_2)$, thus $\text{Fix}(\hat{\varphi}) = \{(a, a\hat{\varphi}_2) \mid a \in \text{Fix}(\hat{\varphi}_1)\}$.

Proposition 4.2. *Let* φ *be a type II uniformly continuous endomorphism of* $\mathbb{Z}^m \times F_n$*, with* $n > 1$*. Then, Sing*$(\hat{\varphi}) = \{(a, a\hat{\varphi}_2) \mid a \in \textit{Sing}(\hat{\varphi}_1)\}$*. Consequentely, Reg*$(\hat{\varphi}) = \{(a, a\hat{\varphi}_2) \mid a \in \textit{Reg}(\hat{\varphi}_1)\}$*.*

Proof. We start by showing that $\text{Sing}(\hat{\varphi}) \subseteq \{(a, a\hat{\varphi}_2) \mid a \in \text{Sing}(\hat{\varphi}_1)\}$. Take some $(a, a\hat{\varphi}_2) \in (\text{Fix}(\varphi))^c$ with $a \in \text{Fix}(\hat{\varphi}_1)$. Then, for every $\varepsilon > 0$, the open ball of radius ε centered in $(a, a\hat{\varphi}_2)$ contains an element $(b_\varepsilon, b_\varepsilon\varphi_2) \in \text{Fix}(\varphi)$, with $b_\varepsilon \in \text{Fix}(\varphi_1)$. Notice that $d(a, b_\varepsilon) \leq d((a, a\hat{\varphi}_2), (b_\varepsilon, b_\varepsilon\varphi_2)) < \varepsilon$, thus $a \in (\text{Fix}(\varphi_1))^c$.

For the reverse inclusion, take some $a \in \text{Sing}(\hat{\varphi}_1)$. As above, we know that for every $\varepsilon > 0$, there is some $b_\varepsilon \in B(a; \varepsilon) \cap \text{Fix}(\varphi_1)$. Notice that, since $\hat{\varphi}_2$ is uniformly continuous, for every $\varepsilon > 0$, there is some δ_ε such that, for all $a, b \in \widehat{\mathbb{Z}^m}$ such that $d(a, b) < \delta_\varepsilon$, we have that $d(a\hat{\varphi}_2, b\hat{\varphi}_2) < \varepsilon$. We want to prove that $(a, a\hat{\varphi}_2) \in (\text{Fix}(\varphi))^c$, by showing that, for every $\varepsilon > 0$, the ball centered in $(a, a\hat{\varphi}_2)$ contains a fixed point of φ. So, let $\varepsilon > 0$ and consider $\delta = \min\{\delta_\varepsilon, \varepsilon\}$. We have that $(b_\delta, b_\delta\hat{\varphi}_2) \in B((a, a\hat{\varphi}_2); \varepsilon)$ since, by definition of b_δ, we have that $d(a, b_\delta) < \delta \leq \varepsilon$ and also, $d(a, b_\delta) < \delta_\varepsilon$ means that $d(a\hat{\varphi}_2, b_\delta\hat{\varphi}_2) < \varepsilon$. □

As a corollary of Proposition 4.1 we get that $\text{Fix}(\hat{\varphi})$ has $\sum_{i=0}^{m} 2^i \binom{m}{i}$ $(\text{Fix}\varphi)$-orbits, since, for $(a, a\hat{\varphi}_2), (b, b\hat{\varphi}_2) \in \text{Fix}(\hat{\varphi})$, we have that $(a, a\hat{\varphi}_2)$ belongs to $(\text{Fix}\varphi)(b, b\hat{\varphi}_2)$ if and only if there is some $(c, c\varphi_2) \in \text{Fix}\varphi$ such that $(a, a\hat{\varphi}_2) = (c, c\varphi_2)(b, b\hat{\varphi}_2)$, i.e., a and b belong to the same orbit of $\text{Fix}(\varphi_1)$.

4.2 Finiteness Conditions on Infinite Periodic Points

We proceed in a similar way to the case of fixed points regarding periodic points.

We know that $\mathrm{Per}(\varphi)$ acts on $\mathrm{Per}(\hat{\varphi})$ on the left, since $a\varphi^p = a$ and $b\hat{\varphi}^q = b$ implies that $(a+b)\hat{\varphi}^{pq} = a\hat{\varphi}^{pq} + b\hat{\varphi}^{pq} = a + b$ and we want to count the orbits of such action.

When we consider a type I endomorphism, we have the same issue we had in the fixed points case, in the sense that we have $\mathrm{Per}\hat{\varphi} = \mathrm{Per}\hat{\varphi}_1 \times \mathrm{Per}\hat{\varphi}$, which might also be uncountable.

To obtain a result for type II endomorphisms, we start as above, by dealing with the free-abelian part first. Let $\varphi \in End(\mathbb{Z}^m)$ defined by $a \mapsto aQ$ be a uniformly continuous endomorphism. As above, given a column Q_j, if $Q_j \neq 0$, we call λ_j to its nonzero entry of column and α_j to its row, and if $Q_j = 0$ we put $\lambda_j = 0$ and $\alpha_j = 1$. Also, we will define the mapping $\psi : [m] \to [m]$ mapping i to α_i. Take $\hat{\varphi} : \widehat{\mathbb{Z}^m} \to \widehat{\mathbb{Z}^m}$ to be its continuous extension to the completion.

This way, we have that $aQ = [\lambda_i a_{\alpha_i}]_{i\in[m]}$ and

$$aQ^r = \left[\left(\prod_{j=1}^{r} \lambda_{i\psi^{j-1}}\right) a_{i\psi^r}\right]_{i\in[m]}.$$

To lighten notation, for $i \in [m]$ and $r \in \mathbb{N}$ we will write

$$\pi_i^{(r)} := \prod_{j=1}^{r} \lambda_{i\psi^{j-1}} \tag{2}$$

This notation will be used throughout this paper.

Proposition 4.3. *$Per(\hat{\varphi})$ has $\sum\limits_{i=0}^{m} 2^i\binom{m}{i}$ $Per(\varphi)$-orbits.*

Proof. Let $a \in \mathrm{Per}(\hat{\varphi})$. As done in the fixed point case, we will prove that, for $b \in \mathrm{Per}(\hat{\varphi})$, we have that $b \in (\mathrm{Per}\varphi)a$ if and only if their infinite entries coincide. and that suffices.

Clearly, if $b \in \mathrm{Per}(\hat{\varphi})$ is such that $b = c + a$ for some $c \in \mathrm{Per}(\varphi)$, then the infinite entries of a and b coincide.

Now, suppose a and b are two infinite periodic points whose infinite entries coincide. Then, there are $p, q \in \mathbb{N}$ such that $a\hat{\varphi}^p = a$ and $b\hat{\varphi}^q = b$, so $a\hat{\varphi}^{pq} = a$ and $b\hat{\varphi}^{pq} = b$. Consider $c \in \mathbb{Z}^m$ defined by $c_i = 0$ if $a_i, b_i \in \{+\infty, -\infty\}$ and $c_i = b_i - a_i$ otherwise. Clearly, $b = c + a$. We only have to check that $c \in \mathrm{Per}(\varphi)$ and for that, we will show that $c\varphi^{pq} = c$. We have that, for $r > 0$, the mapping φ^r is defined by

$$[c_i]_{i\in[m]} \mapsto \left[\pi_1^{(r)} c_{i\psi^r}\right]_{i\in[m]}.$$

Now, we only have to see that, for every $i \in [m]$, we have that $c_i = \pi_i^{(pq)} c_{i\psi^{pq}}$. Let $i \in [m]$ such that $a_i, b_i \in \{+\infty, -\infty\}$ and so $c_i = 0$. Since $|a_i| = \infty$, $a_i = (a\hat{\varphi}^{pq})_i = \pi_i^{(pq)} a_{i\psi^{pq}}$ and all λ_k's are finite, then we have that $a_{i\psi^{pq}} \in \{+\infty, -\infty\}$ (and the same holds for $b_{i\psi^{pq}}$), thus, $c_{i\psi^{pq}} = 0$. Then, we have that $\pi_i^{(pq)} c_{i\psi^{pq}} = 0 = c_i$. Now, take $i \in [m]$ such that $a_i, b_i \notin \{+\infty, -\infty\}$. Then $c_i = b_i - a_i = (b\hat{\varphi}^{pq})_i - (a\hat{\varphi}^{pq})_i = \pi_i^{(pq)} b_{i\psi^{pq}} - \pi_i^{(pq)} a_{i\psi^{pq}} = \pi_i^{(pq)}(b_{i\psi^{pq}} - a_{i\psi^{pq}}) = \pi_i^{(pq)} c_{i\psi^{pq}}$, since $a_{i\psi^{pq}}$ and $b_{i\psi^{pq}}$ are both finite. □

4.3 Finiteness Conditions on Type II endomorphisms

This case is also similar to the fixed point case. Let φ be a type II uniformly continuous endomorphism of $\mathbb{Z}^m \times F_n$, with $n > 1$. Then $\varphi : \mathbb{Z}^m \times F_n \to \mathbb{Z}^m \times F_n$ is given by

$(a,u) \mapsto (aQ, z^{\lambda a_k})$, for some $Q \in \mathcal{M}_m(\mathbb{Z})$ such that every column of Q contains at most one nonzero entry and $0 \neq \lambda \in \mathbb{Z}$, $k \in [m]$. Consider $\varphi_1 \in End(\mathbb{Z}^m)$ to be defined as $a \mapsto aQ$ and $\varphi_2 : \mathbb{Z}^m \to F_n$ that maps a to $z^{\lambda a_k}$, which are both uniformly continuous.

By uniqueness of extension, we have that $\hat{\varphi} : \widehat{\mathbb{Z}^m} \times \widehat{F_n} \to \widehat{\mathbb{Z}^m} \times \widehat{F_n}$ is defined by $(a,u) \mapsto (a\hat{\varphi}_1, a\hat{\varphi}_2)$. Hence, if $(a,u) \in \text{Per}(\hat{\varphi})$, then $a \in \text{Per}(\hat{\varphi}_1)$.

It is easy to see, by induction on r that so for every $r > 0$, we have that $(a,u)\hat{\varphi}^r = (a\hat{\varphi}_1^r, a\hat{\varphi}_1^{r-1}\hat{\varphi}_2)$. Indeed, it is true for $r = 1$ and if we have that $(a,u)\hat{\varphi}^r = (a\hat{\varphi}_1^r, a\hat{\varphi}_1^{r-1}\hat{\varphi}_2)$, then $(a,u)\hat{\varphi}^{r+1} = (a\hat{\varphi}_1^r, a\hat{\varphi}_1^{r-1}\hat{\varphi}_2)\hat{\varphi} = (a\hat{\varphi}_1^{r+1}, a\hat{\varphi}_1^r\hat{\varphi}_2)$. So we have that

$$(a,u)\hat{\varphi}^r = \left([\pi_i^r a_{i\psi^r}]_{i\in[m]}, z^{\lambda \pi_k^{r-1} a_{k\psi^{r-1}}} \right). \tag{3}$$

Let $(a,u),(b,v) \in \text{Per}(\hat{\varphi})$. We have that $(b,v) \in (\text{Per}\varphi)(a,u)$ if and only if $b \in (\text{Per}(\varphi_1))a$. Indeed, if $b \in (Per(\varphi_1))a$, let $c \in \text{Per}(\varphi_1)$ defined as in the proof of Proposition 4.3, such that $b = c + a$ and denote by p, q the periods of a and b, respectively.

Then $(c, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2)\varphi^{pq} = (c, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2)$, so $(c, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2) \in \text{Per}(\varphi)$ and

$$\begin{aligned}
(b,v) &= (b\hat{\varphi}_1^{pq}, b\hat{\varphi}_1^{pq-1}\hat{\varphi}_2) = ((c+a)\hat{\varphi}_1^{pq}, (c+a)\hat{\varphi}_1^{pq-1}\hat{\varphi}_2) \\
&= (c\hat{\varphi}_1^{pq} + a\hat{\varphi}_1^{pq}, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2 a\hat{\varphi}_1^{pq-1}\hat{\varphi}_2) \\
&= (c\hat{\varphi}_1^{pq}, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2)(a\hat{\varphi}_1^{pq}, a\hat{\varphi}_1^{pq-1}\hat{\varphi}_2) \\
&= (c, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2)(a,u).
\end{aligned}$$

Hence, we have that $\text{Per}(\hat{\varphi})$ has $\sum\limits_{i=0}^{m} 2^i \binom{m}{i}$ $\text{Per}(\varphi)$-orbits.

4.4 Classification of the Infinite Fixed Points for Automorphisms

In [5], the authors prove that an endomorphism $\varphi \in End(\mathbb{Z}^m \times F_n)$ is an automorphism if it is of type I and of the form $(a,u) \mapsto (aQ + \mathbf{u}P, u\phi)$, with $\phi \in Aut(F_n)$ and $Q \in GL_m(\mathbb{Z})$. In the case where φ is uniformly continuous, then every entry of Q is either 1 or -1 and $Q = AD$, where D is diagonal and A is a permutation matrix. There are $2^m m!$ of such matrices, which we call *uniform*. So, a uniformly continuous automorphism of $\mathbb{Z}^m \times F_n$ is defined as $(a,u) \mapsto (aQ, u\phi)$, where Q is a uniform matrix and $\phi \in Aut(F_n)$. As above, we define $\varphi_1 : \mathbb{Z}^m \to \mathbb{Z}^m$ that maps a to aQ and we have that $\text{Fix}(\hat{\varphi}) = \text{Fix}(\hat{\varphi}_1) \times \text{Fix}(\hat{\phi})$

We are interested in classifying infinite fixed points. We start by presenting some standard definitions.

Definition 4.4. An infinite fixed point $\alpha \in \text{Fix}(\hat{\varphi})$ is

- an *attractor* if

$$\exists \varepsilon > 0\ \forall \beta \in \widehat{\mathbb{Z}^m \times F_n}(d(\alpha,\beta) < \varepsilon \Rightarrow \lim_{n\to+\infty} \beta\hat{\varphi}^n = \alpha)$$

- a *repeller* if

$$\exists \varepsilon > 0\ \forall \beta \in \widehat{\mathbb{Z}^m \times F_n}(d(\alpha,\beta) < \varepsilon \Rightarrow \lim_{n\to+\infty} \beta\hat{\varphi}^{-n} = \alpha)$$

Proposition 4.5. *Let $\varphi \in End(\mathbb{Z}^m)$ be an endomorphism defined by $a \mapsto aQ$, where Q is a uniform matrix. Then $Sing(\hat{\varphi}) = Fix(\hat{\varphi})$.*

Proof. By definition, $\text{Sing}(\hat{\varphi}) \subseteq \text{Fix}(\hat{\varphi})$.

Let $\pi \in S_m$ be a permutation such that φ maps $[a_i]_{i\in[m]}$ to $[\lambda_i a_{\pi(i)}]_{i\in[m]}$, and $\lambda_i = \pm 1$. Then $\text{Fix}(\varphi) = \{a \in \mathbb{Z}^m \mid \forall i \in [m],\, a_i = \lambda_i a_{\pi(i)}\}$ and $\text{Fix}(\hat{\varphi}) = \{a \in \widehat{\mathbb{Z}^m} \mid \forall i \in [m],\, a_i = \lambda_i a_{\pi(i)}\}$. Given $a \in \text{Fix}(\hat{\varphi})$ and $\varepsilon > 0$, choosing some

$$n > \max_{a_i \in \mathbb{Z}} \left\{ |a_i| \,, \left\lceil log_2 \left(\frac{1}{\varepsilon} \right) \right\rceil \right\},$$

consider $b \in \mathbb{Z}^m$ such that $b_i = n$ if $a_i = +\infty$; $b_i = -n$ if $a_i = -\infty$ and $b_i = a_i$, otherwise. Then $b_i \in \text{Fix}(\varphi)$ and $d(a,b) < \varepsilon$. Thus, b is a point of closure of $\text{Fix}(\varphi)$ and we are done. □

Proposition 4.6. *An infinite fixed point $\alpha = (a,u)$, where $a \in Fix(\hat{\varphi}_1)$ and $u \in Fix(\hat{\phi})$ is an attractor (resp. repeller) if and only if a and u are attractors (resp. repellers) for $\hat{\varphi}_1$ and $\hat{\phi}$, respectively.*

Proof. Let $\alpha = (a,u)$ be an infinite fixed point, where $a \in \text{Fix}(\hat{\varphi}_1)$ and $u \in \text{Fix}(\hat{\phi})$. Clearly if $a \in \text{Fix}(\hat{\varphi}_1)$ and $u \in \text{Fix}(\hat{\phi})$ are attractors, then, $(a,u) \in \text{Fix}(\hat{\varphi})$ is an attractor. Indeed, in that case, there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\forall b \in \widehat{\mathbb{Z}^m}, \left(d(a,b) < \varepsilon_1 \Rightarrow \lim_{n\to+\infty} b\hat{\varphi}_1^n = a \right)$$

and

$$\forall v \in \widehat{F_n}, \left(d(u,v) < \varepsilon_2 \Rightarrow \lim_{n\to+\infty} v\hat{\phi}^n = u \right).$$

Thus, taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have that

$$\begin{aligned} \forall (b,v) \in \widehat{\mathbb{Z}^m \times F}_n (d((a,u),(b,v)) < \varepsilon &\Rightarrow d(a,b) < \varepsilon \wedge d(u,v) < \varepsilon \\ &\Rightarrow \lim_{n\to+\infty} b\hat{\varphi}_1^n = a \wedge \lim_{n\to+\infty} v\hat{\phi}^n = u \\ &\Rightarrow \lim_{n\to+\infty} (b,v)\hat{\varphi}^n. \end{aligned}$$

Conversely, suppose w.l.o.g that a is not an attractor for $\hat{\varphi}_1$. Then, for every $\varepsilon > 0$, there is some $b_\varepsilon \in \widehat{\mathbb{Z}^m}$ such that $d(a,b) < \varepsilon$ but $b_\varepsilon \hat{\varphi}_1 \not\to a$. In this case, we have that, for every $\varepsilon > 0$, $d((a,u),(b_\varepsilon,u)) < \varepsilon$ and $(b_\varepsilon,u)\hat{\varphi}^n = (b_\varepsilon \hat{\varphi}_1^n, u) \not\to (a,u)$. □

Since there are no infinite fixed points of $\hat{\phi}$ both singular and attractors or repellers (see Proposition 1.1 in [6]), then we have that there is not an infinite fixed point $\alpha \in \text{Sing}(\hat{\varphi})$ that is an attractor or a repeller. We are now able to classify the infinite fixed points in this case.

Proposition 4.7. *Let $\varphi \in End(\mathbb{Z}^m)$ be defined by $a \mapsto aQ$, where Q is a uniform matrix. Then an infinite fixed point $\alpha \in Fix(\hat{\varphi}) \setminus Fix(\varphi)$ is neither an attractor nor a repeller.*

Proof. Let $a \in \text{Fix}\hat{\varphi} \setminus \text{Fix}\varphi$. Let $\varepsilon > 0$ and define $q = \lceil log_2(\frac{1}{\varepsilon}) \rceil$. Take $p = \max_{a_i \in \mathbb{Z}}\{q, |a_i|\}$. Take $b \in \mathbb{Z}^m$ such that $b_i = p$ for every i such that $a_i = +\infty$; $b_i = -p$ for every i such that $a_i = -\infty$ and $b_i = a_i$ otherwise. Then $d(a,b) < \varepsilon$ but $b\hat{\varphi}^n \not\to a$ since $\max_{i\in[m]} |b_i| = p$ and applying $\hat{\varphi}$ simply changes order and signal of the entries, so for every $n \in \mathbb{N}$, we have that $\max_{i\in[m]}\{|(b\hat{\varphi}^n)_i|\} = n$. Hence $d(a, b\hat{\varphi}^n) \geq 2^{-p}$ since there is some k such that $a_k \in \{+\infty, -\infty\}$ and $d(a_k, b_k) \geq d(a_k, sgn(a_k)|p|) = 2^{-p}$. The repeller case is analogous, since the inverse of a uniform matrix is uniform. □

Corollary 4.8. *Let $\varphi \in Aut(\mathbb{Z}^m \times F_n)$ be a uniformly continuous automorphism such that $(a,u)\hat{\varphi} = (a\hat{\varphi}_1, b\hat{\phi})$, where φ_1 is given by a uniform matrix and $\phi \in Aut(F_n)$. Then an infinite fixed point $(a,u) \in Fix(\hat{\varphi}) \setminus Fix(\varphi)$ is an attractor (resp. repeller) if and only if $a \in Fix(\varphi)$ and u is an attractor (resp. repeller) for $\hat{\phi}$.*

Notice that, given an infinite attractor (resp. repeller) $u \in \hat{F}_n$ and denoting by S_u the set of points attracted (resp. repelled) to it, then the set of points attracted (resp. repelled) by (a,u) is given by $T_{(a,u)} = \{(a,y) \mid y \in S_u\}$.

5 Dynamics of infinite points

5.1 The automorphism case

We have that a uniformly continuous automorphism of $\mathbb{Z}^m \times F_n$ is defined as $(a,u) \mapsto (aQ, u\phi)$, where Q is a uniform matrix and $\phi \in Aut(F_n)$.

We start by observing that, for the abelian part, the dynamics is simple in the sense that every point is periodic.

Proposition 5.1. *Let $\varphi \in Aut(\mathbb{Z}^m)$ be defined by $a \mapsto aQ$, where Q is a uniform matrix and consider $\hat{\varphi}$ to be its continuous extension to the completion. Then, there is some constant $p \leq 2^m m!$ such that $\hat{\varphi}^p = Id$. Hence, $Per(\hat{\varphi}) = \widehat{\mathbb{Z}^m}$ and the period of every element divides p.*

Proof. There are only $2^m m!$ distinct uniform $m \times m$ matrices, so there are $0 < p < q \leq 2^m m! + 1$ such that $Q^p = Q^q$, thus $I_m = Q^{q-p}$. □

We now present some standard dynamical definitions, that will be useful to the classification of infinite points.

Definition 5.2. Let G be a group and $\varphi \in End(G)$. A point $x \in G$ is said to be a φ-*wandering point* if there is a neighbourhood U of x and a positive integer N such that for all $n > N$, we have that $U\varphi^n \cap U = \emptyset$. When it is clear, we simply say x is a wandering point.

Definition 5.3. Let G be a group and $\varphi \in End(G)$. A point $x \in G$ is said to be a φ-*recurrent point* if, for every neighbourhood U of x, there exists $n > 0$ such that $x\varphi^n \in U$. When it is clear, we simply say x is a recurrent point.

Definition 5.4. Let f be a homeomorphism of a compact space K. Given $y \in K$, the ω-*limit set* $\omega(y,f)$, or simply $\omega(y)$, is the set of limit points of the sequence $f^n(y)$ as $n \to +\infty$.

Definition 5.5. Let G be a group. A uniformly continuous endomorphism $\varphi \in End(G)$ has *asymptotically periodic dynamics* on $\hat{G}$ if there exists $q \geq 1$ such that, for every $x \in \hat{G}$, the sequence $x\hat{\varphi}^{qn}$ converges to a fixed point of $\hat{\varphi}^q$.

The next proposition shows how being $\hat{\varphi}$-recurrent (resp. wandering) relates with being $\hat{\phi}$-recurrent (resp. wandering).

Proposition 5.6. *Let $\varphi \in End(\mathbb{Z}^m \times F_n)$ be a uniformly continuous endomorphism defined by $(a,u) \mapsto (a\varphi_1, u\phi)$, where $\varphi_1 \in End(\mathbb{Z}^m)$ and $\phi \in End(F_n)$ and $(a,u) \in \widehat{\mathbb{Z}^m \times F_n}$. We have the following:*

1. *(a,u) is $\hat{\varphi}$-periodic $\Rightarrow$ u is $\hat{\phi}$-periodic.*
2. *(a,u) is $\hat{\varphi}$-wandering $\Leftarrow$ u is $\hat{\phi}$-wandering.*
3. *(a,u) is $\hat{\varphi}$-recurrent $\Rightarrow$ u is $\hat{\phi}$-recurrent.*

Proof.

1. This is obvious. Observe that if Q is uniform, which is the case when we deal with automorphisms then the reverse implication holds as well, by Propostion 5.1.

2. Suppose u is $\hat{\phi}$-wandering. Then, take $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, we have that $B(u;\varepsilon)\hat{\phi}^n \cap B(u;\varepsilon) = \emptyset$. Let $n > N$ and take $V = B((a,u);\varepsilon)$ and $(b,v) \in V$. Then $v \in B(u;\varepsilon)$, thus $v\hat{\phi}^n \notin B(u;\varepsilon)$ and $(b,v)\hat{\varphi}^n = (b\hat{\varphi}_1^n, v\hat{\phi}^n) \notin V$. Since (b,v) is an arbitrary point of V, we have that (a,u) is $\hat{\phi}$-wandering.

3. Suppose (a,u) is $\hat{\varphi}$-recurrent. Take $\varepsilon > 0$. There is some $n \in \mathbb{N}$ such that $(a,u)\hat{\varphi}^n \in B((a,u);\varepsilon)$ and so $u\hat{\phi}^n \in B(u;\varepsilon)$. □

Notice that, in case $u \in F_n$ and $\phi \in End(F_n)$, if u is nonwandering, then it must be periodic, since we can take taking $U = \{u\}$, we have that there is some n such that $U\hat{\phi}^n \cap U \neq 0$ and so, it is periodic. So, in case $\phi \in Aut(\mathbb{Z}^m \times F_n)$, we have that a point $(a,u) \in \hat{\mathbb{Z}^m} \times F_n$ must be periodic or wandering, by Proposition 5.6.

We now present two lemmas from [8] regarding free groups automorphisms, that will be very useful in this case.

Lemma 5.7. *[8] Let f be a homeomorphism of a compact space K. Given $y \in K$ and $q \geq 1$, the following conditions are equivalent:*

1. *$\omega(y)$ is finite and has q elements.*

2. *$\omega(y)$ is a periodic orbit of order q.*

3. *The sequence $f^{qn}(y)$ converges as $n \to +\infty$, and q is minimal for this property.*

Given $p \geq 2$, the set $\omega(y, f^p)$ is finite if and only if $\omega(y, f)$ is finite.

If these equivalent conditions hold, we say that the point y is *asymptotically periodic*. If every point is asymptotically periodic, then the endomorphism has asymptotically periodic dynamics (this definition is equivalent to Definition 5.5).

Theorem 5.8. *[8] Every automorphism $\alpha \in Aut(F_n)$ has asymptotically periodic dynamics.*

Using these lemmas, we obtain the same result for free-abelian times free groups.

Proposition 5.9. *Every uniformly continuous automorphism $\varphi \in Aut(\mathbb{Z}^m \times F_n)$ has asymptotically periodic dynamics on $\widehat{\mathbb{Z}^m \times F_n}$.*

Proof. Let φ be a uniformly continuous automorphism defined by $(a,u) \mapsto (a\varphi_1, u\phi)$, where $\varphi_1 \in Aut(\mathbb{Z}^m)$ is given by a uniform matrix and $\phi \in Aut(F_n)$. We will prove that for every $(a,u) \in \widehat{\mathbb{Z}^m \times F_n}$, we have that $\omega((a,u),\varphi)$ is a periodic orbit. Take $(b,v) \in \omega((a,u),\varphi)$. Then, there is an increasing sequence $\{n_k\}$ such that $(a,u)\hat{\varphi}^{n_k} \to (b,v)$. This means that $a\hat{\varphi}_1^{n_k} \to b$ and so $b \in \omega(a,\varphi_1)$ and $u\hat{\phi}^{n_k} \to v$ and so $v \in \omega(u,\phi)$ and (b,v) must be a periodic point belonging to the orbit of (a,x), for $x \in \omega(u,\phi)$, since $\omega(a,\hat{\varphi}_1)$ is the orbit of a. Obviously, the entire orbit must belong to $\omega((a,u),\varphi)$ since $(a,u)\hat{\varphi}^{n_k+p} \to (b,v)\hat{\varphi}^p$, for every $p \geq 0$. □

Definition 5.10. Let φ be an endomorphism of F_n. We say that φ is *eventually length-nondecreasing* if $\exists p \in \mathbb{N}\, \forall u \in F_n\, (|u| \geq p \Rightarrow |u\varphi| \geq |u|)$.

We can prove that the dichotomy periodic vs wandering always holds when we deal with an eventually length-nondecreasing automorphism.

Proposition 5.11. *Let φ be an eventually length-nondecreasing endomorphism of F_n. Then a point $u \in \hat{F_n}$ is either recurrent or wandering. If φ is an automorphism, then every point is either periodic or wandering.*

Proof. Let ϕ be an eventually length-nondecreasing endomorphism of F_n. In particular, it is uniformly continuous (see lemma 6.2 in [3]). Take $p \in \mathbb{N}$ such that

$$\forall u \in F_n \, (|u| \geq p \Rightarrow |u\varphi| \geq |u|) \tag{4}$$

and $u \in \hat{F_n}$. Suppose u is not wandering and take $\varepsilon < \frac{1}{2^p}$. Then, for every $N \in \mathbb{N}$ there is some $r > N$ such that $B(u;\varepsilon)\hat{\phi}^r \cap B(u;\varepsilon) \neq \emptyset$ and so there is some $v_r \in \hat{F_n}$ such that $d(u, v_r) < \varepsilon$ and $d(u, v_r\hat{\phi}^r) < \varepsilon$. So, $|u \wedge v| > p$ and $|(u \wedge v)\hat{\phi}^r| > p$, by (4). Since $|u \wedge v\hat{\phi}^r| > p$, we have that $|u \wedge u\phi^r| > p$, hence $u\hat{\phi}^r \in B(u;\varepsilon)$ and u is recurrent.

In case φ is an automorphism, we have that every recurrent point must be periodic since such a point must belong to its limit set, which is a periodic orbit. □

Notice that this implies that if $\varphi = (\varphi_1, \phi) \in Aut(\mathbb{Z}^m \times F_n)$ is such that ϕ is length-nondecreasing, then every point (a, u) is either wandering or periodic.

5.2 Type II Endomorphisms

The purpose of this subsection is to establish the dichotomy periodic vs wandering for Type II endomorphisms. Recall the notation introduced in 4.2. Also, recall (3) and the decomposition $z = w\tilde{z}w^{-1}$where $\tilde{z}$ is the cyclically reduced core of z and the definition of B_i for $i \in [m]$.

Remark 5.12. *Assume $n > 1$. The set of periodic points of the extension of a uniformly continuous type II endomorphism to the completion is not dense in the entire space, even when we restrict ourselves to the boundary. Indeed, if we take a point (a, u) such that u does not share a prefix with z and z^{-1}, then $B\left((a, u); \frac{1}{2}\right)$ does not contain a periodic point. Also, the system does not admit the existence of a dense orbit: Indeed, given $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$, choosing a point $(b, v) \in \widehat{\mathbb{Z}^m \times F_n}$ such that $b \neq a$ and v doesn't share a prefix of size with neither z nor z^{-1}, we have that $B((b, v); \frac{1}{2})$ does not contain any point in the orbit.*

Also, in the automorphism case, we have that the first component is always periodic, so there is not a dense orbit, even when restricted to the boundary.

We will now prove two technical lemmas that will be very useful for proving the main result.

Lemma 5.13. *Consider a uniformly continuous endomorphism φ of a free-abelian group $\mathbb{Z}^m$ and take $i \in [m]$ and some positive integer $r > m$. Then, the following conditions are equivalent:*

(i) $\exists N \in \mathbb{N} \, \forall p > N \; |\lambda_{i\psi^p}| = 1$

(ii) $\exists N \leq m \, \forall p > N \; |\lambda_{i\psi^p}| = 1$

(iii) $|\pi^{(r)}_{i\psi^{tr}}| = 1$, *for every positive integer t*

(iv) $|\pi^{(r)}_{i\psi^{tr}}| = 1$, *for some positive integer t*

Proof. It is obvious that (ii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv). We will prove that (i) $\Rightarrow$ (ii) and that (iv)$\Rightarrow$ (i).

(i)⇒ (ii): Suppose that there is some $N > m$ such that for every $p > N$ we have that $|\lambda_{i\psi^p}| = 1$. We have that ψ maps $[m]$ to a subset of $[m]$, and so, for every $i \in [m]$, there is some $k_i \leq m$ such that $i\psi^{m+1} = i\psi^{k_i}$. This way, we have a periodic orbit (can be fixed) of ψ given by $\{i\psi^{k_i}, \dots i\psi^m\}$. So, for every $p \geq N$, we define $j_p \in \{k_i, \dots, m\}$ to be such that $i\psi^{j_p} = i\psi^p$. Also, if for some $j > j_N$, we had $|\lambda_{i\psi^j}| > 1$, we could obtain p arbitrarily large such that $|\lambda_{i\psi^p}| > 1$, which is absurd. So, we have that,

$$\forall p > j_N \ |\lambda_{i\psi^p}| = 1$$

and $j_N \leq m$.

(iv)⇒ (i) If, for some $i \in [m]$, we have that $|\pi^{(r)}_{i\psi^{tr}}| = 1$, for some $t \in \mathbb{N}$, then for every $j \in \{tr, \dots, (t+1)r - 1\}$, we have that $|\lambda_{i\psi^j}| = 1$. In this case, for some $s \geq (t+1)r$, we have that $i\psi^s = i\psi^j$ for some $j \in \{tr, \dots, (t+1)r - 1\}$ and so $|\lambda_{i\psi^s}| = 1$. Thus, (i) holds for $N = r$. □

Lemma 5.14. *Consider a nonwandering point $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$ such that a has finite entries. Let $\delta = \max\limits_{a_i \in \mathbb{Z}}\{|a_i|\}$ and $U = B\left((a, z^{+\infty}); \frac{1}{2^\delta}\right)$. Consider a point $(b, v) \in U$ and a positive integer $r > m$ such that $(b, v)\hat{\varphi}^r \in U$. Then the conditions from Lemma 5.13 hold for every index $i \in [m]$ such that $a_i \neq \pi_i^r a_{i\psi^r}$.*

Moreover, if $u = z^{+\infty}$ and $|a_{k\psi^{r-1}}| < \infty$, then the conditions from Lemma 5.13 hold when $i = k$.

Proof. Consider a nonwandering point $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$ such that a has finite entries, let $\delta = \max\limits_{a_i \in \mathbb{Z}}\{|a_i|\}$ and $U = B((a, u); \frac{1}{2^\delta})$. Consider a point $(b, v) \in U$ and a positive integer $r > m$ such that $(b, v)\hat{\varphi}^r \in U$. So, for $i \in [m]$, we have that:

$$\text{if } |a_i| < \infty, \text{ then } a_i = b_i = \pi_i^{(r)} b_{i\psi^r} \tag{5}$$

and

$$\text{if } |a_i| = \infty, \text{ then } sgn(a_i) = sgn(b_i) = sgn(\pi_i^{(r)} b_{i\psi^r}) \text{ and } |b_i|, |\pi_i^{(r)} b_{i\psi^r}| > \delta. \tag{6}$$

If $a\hat{\varphi}_1^r = a$, then $a_i = \pi_i^{(r)} a_{i\psi^r}$ for every $i \in [m]$ and the first part of the lemma trivially holds.

If not, take $q \in [m]$ such that $a_q \neq \pi_q^{(r)} a_{q\psi^r}$. We start by observing that a_q must be infinite since if that is not the case, then, by (5), we have that $a_q = b_q = \pi_q^{(r)} b_{q\psi^r}$. If $\pi_q^{(r)} = 0$, then $a_q = \pi_q^{(r)} b_{q\psi^r} = 0 = \pi_q^{(r)} a_{q\psi^r}$. If not, then $b_{q\psi^r} \leq b_q$, which means that $b_{q\psi^r} \leq \delta$ and by (6), we have that $a_{q\psi^r}$ is finite and by (5), it follows that $a_{q\psi^r} = b_{q\psi^r}$, so $a_q = \pi_q^{(r)} b_{q\psi^r} = \pi_q^{(r)} a_{q\psi^r}$. Also, $a_{q\psi^r}$ must be finite, since, if it is infinite, then by (6) we have that $sgn(a_{q\psi^r}) = sgn(b_{q\psi^r})$ and $sgn(a_q) = sgn(\pi_q^{(r)} b_{q\psi^r})$, thus $sgn(a_q) = sgn(\pi_q^{(r)} a_{q\psi^r})$ and that implies that $a_q = \pi_q^{(r)} a_{q\psi^r}$, since $\pi_q^{(r)} \neq 0$.

So, using (5) with $i = q\psi^r$, we have that

$$a_{q\psi^r} = b_{q\psi^r} = \pi^{(r)}_{q\psi^r} b_{q\psi^r\psi^r} = \pi^{(r)}_{q\psi^r} b_{q\psi^{2r}}. \tag{7}$$

We have that a_q is infinite, so, by (6), we know that $\pi_q^{(r)} b_{q\psi^r} > \delta$. This means in particular that $a_{q\psi^r} \neq 0$, because otherwise we would have $b_{q\psi^r} = 0$, by (5).

Suppose now that for every positive integer t, we have that $|\pi^{(r)}_{q\psi^{tr}}| \neq 1$. If $\pi^{(r)}_{q\psi^r} = 0$, then $a_{q\psi^r} = 0$, which is absurd. Then, we have that $|\pi^{(r)}_{q\psi^r}| > 1$, and $|b_{q\psi^{2r}}| < |b_{q\psi^r}| < \infty$ and again, using (5) with $i = q\psi^{2r}$, we get that $a_{q\psi^{2r}} = b_{q\psi^{2r}} = \pi^{(r)}_{q\psi^{2r}} b_{q\psi^{3r}}$. Again, if $\pi^{(r)}_{q\psi^{2r}} = 0$, then $a_{q\psi^{2r}} = 0$ and by (5), it follows that $b_{q\psi^{2r}} = 0$. From (7), we reach a contradiction. So, we must have $|\pi^{(r)}_{q\psi^r}| > 1$ and $|b_{q\psi^{3r}}| < |b_{q\psi^{2r}}| < \infty$. Proceeding like this, since the value of $|b_{q\psi^{pr}}|$, for $p \in \mathbb{N}$ cannot decrease indefinitely, then we must have that $b_{q\psi^r} = 0$ and so $a_{q\psi^r} = 0$, which is absurd.

So, the conditions from Lemma 5.13 hold when $i = q$.

If we have that $u = z^{+\infty}$ and $|a_{k\psi^{r-1}}| < \infty$, then by (5), we have that $a_{k\psi^{r-1}} = b_{k\psi^{r-1}} = \pi^{(r)}_{k\psi^{r-1}} b_{k\psi^{2r-1}}$. If the conditions from Lemma 5.13 do not hold when $i = k$, then using the same argument as above, we obtain $a_{k\psi^{r-1}} = b_{k\psi^{r-1}} = 0$, which is absurd since $\left|z^{+\infty} \wedge z^{\lambda\pi_k^{r-1} b_{k\psi^{r-1}}}\right| > \delta$. □

Notice that for every i for which the conditions from Lemma 5.13 hold, there is some constant $B_i \geq 1$ such that any product of the form $\prod_{j=s}^{t} \lambda_{i\psi^j}$, with $t \geq s$ is bounded above by B_i. Also, we remark that if follows from the proof that for q such that $a_q \neq \pi^{(r)}_q a_{q\psi^r}$ we must have $\pi^{(r)}_q \neq 0$, a_q is infinite and $a_{q\psi^r}$ is finite.

Proposition 5.15. *Let $\varphi \in End(\mathbb{Z}^m \times F_n)$ be a type II uniformly continuous endomorphism defined by $(a, u) \mapsto (aQ, z^{\lambda a_k})$, for some $k \in [m]$, $1 \neq z \in F_n$, which is not a proper power and Q such that $a \mapsto aQ$ is uniformly continuous. Consider $\hat{\varphi}$, its continuous extension to the completion. Then every point $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$ is either wandering or periodic.*

Proof. Let $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$. Clearly, if (a, u) is wandering, it is not periodic. To prove the reverse inclusion, we will consider several cases.

Case 1: $u \in F_n$. Start by supposing that every entry in a is infinite. In this case, (a, u) is never periodic, so we will prove it is wandering. Take $U = B\left((a, u); \frac{1}{2^{|u|}}\right)$, $r \in \mathbb{N}$ and $(b, v) \in U$. We have that, if $(b, v) \in U$, then $d(u, v) < \frac{1}{2^{|u|}}$, which means $u = v$ and for every $i \in [m]$, we have that $a_i = b_i$, or $a_i b_i > 0$ and $|a_i|, |b_i| > |u|$.

Then, we have that $(b, v)\hat{\varphi}^r = \left(\left[\pi^{(r)}_i b_{i\psi^r}\right]_{i\in[m]}, z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}}\right)$. This means that $\pi^{(r)}_k \geq 1$, because, if $\pi^{(r)}_k = 0$, then $\pi^{(r)}_k b_{k\psi^r} = 0 < |u|$, which is absurd. Since $|b_{k\psi^{r-1}}| > |u|$, then

$$\left|z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}}\right| = 2|w| + \left|\lambda\pi^{(r-1)}_k b_{k\psi^{r-1}}\right| |\tilde{z}| > |u|$$

and $(b, v)\hat{\varphi}^r \notin U$. So, in this case, (a, u) is wandering.

Now, we deal with the case where a has finite entries. Suppose (a, u) is not wandering. Then, for every neighbourhood U of (a, u), we have that $U\varphi^r \cap U \neq \emptyset$ for arbitrarily large r. Set $\delta = \max_{a_i \in \mathbb{Z}}\{|a_i|, |u|\}$ and consider $U = B\left((a, u); \frac{1}{2^\delta}\right)$. We have that, if $(b, v) \in U$, then $u = v$ and for $i \in [m]$, if a_i is finite we have (5) and if a_i is infinite, then we have (6). Take $r > m$, $(b, v) \in U$ such that $(b, v)\varphi^r \in U$. If $a\hat{\varphi}^r_1 = a$, then if $\pi^{(r-1)}_k = 0$, we have that

$$u = z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}} = 1 = z^{\lambda\pi_k^{(r-1)} a_{k\psi^{r-1}}} = a\hat{\varphi}^r_2.$$

If $\pi^{(r-1)}_k \geq 1$, then, since $u = z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}}$, we have that $|b_{k\psi^{r-1}}| < |u| \leq \delta$ and so $a_{k\psi^{r-1}}$ must be finite by (6). Thus, by (5), we have that $a_{k\psi^{r-1}} = b_{k\psi^{r-1}}$ and

$$u = z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}} = z^{\lambda\pi_k^{(r-1)} a_{k\psi^{r-1}}} = a\hat{\varphi}^r_2.$$

So, we have that if $a\hat{\varphi}^r_1 = a$, then (a, u) is periodic. If not, then by Lemma 5.14, we have that the conditions from Lemma 5.13 hold for every i such that $a_i \neq \pi^{(r)}_i a_{i\psi^r}$. Thus, the set

$$X = \{j \in [m] \mid \text{the conditions from Lemma 5.13 hold for } i = j\}$$

is nonempty.

Now, take $\tau = \max\{B_q \mid q \in X\}$ and $U' = B\left((a,u); \frac{1}{2^{\tau\delta}}\right)$. Notice that $U' \subseteq U$ and so Lemma 5.14 can be applied. Since (a,u) is nonwandering, there is some $r' > m$ and $(b',v') \in U'$ such that $(b',v')\hat{\varphi}^{r'} \in U'$. We will prove that $a\hat{\varphi}_1^{r'} = a$. Suppose not and take $q \in [m]$ such that $a_q \neq \pi_q^{(r')} a_{q\psi^{r'}}$. So $q \in X$ and from the proof of the Lemma 5.14 it follows that $\pi_q^{(r')} \neq 0$, a_q is infinite and $a_{q\psi^r}$ is finite. But then, since $(b',v')\hat{\varphi}^{r'} \in U'$, we must have $\pi_q^{(r')} b'_{q\psi^{r'}} > \delta\tau$, which is absurd since $\pi_q^{(r')} \leq \tau$ and $b'_{q\psi^{r'}} = a_{q\psi^{r'}} \leq \delta$.

As done above, we can check that $a\hat{\varphi}_2^{r'} = u$ and so (a,u) is periodic.

Case 2: $u \in \partial F_n \setminus \{z^{+\infty}, z^{-\infty}\}$. In this case (a,u) is never periodic, so we will prove it is wandering. Take $\delta = \max\{|z^{-\infty} \wedge u|, |z^{+\infty} \wedge u|\}$ and consider $U = B\left((a,u); \frac{1}{2^\delta}\right)$. Let $(b,v) \in U$. We have that $|v \wedge u| > \delta$ and for every $i \in [m]$, $a_i = b_i$ or $a_i b_i > 0$ and $|a_i|, |b_i| > \delta$. So, for every $r \in \mathbb{N}$, we have that $(b,v)\hat{\varphi}^r \notin U$, since

$$\begin{cases} \left| z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}} \wedge u \right| = |z^{+\infty} \wedge u| & \text{if } \lambda\pi_k^{(r-1)} b_{k\psi^{r-1}} > 0 \\ 0 & \text{if } \lambda\pi_k^{(r-1)} b_{k\psi^{r-1}} = 0 \\ \left| z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}} \wedge u \right| = |z^{-\infty} \wedge u| & \text{if } \lambda\pi_k^{(r-1)} b_{k\psi^{r-1}} < 0. \end{cases}$$

Case 3: $u \in \{z^{+\infty}, z^{-\infty}\}$. Suppose (a,u) is not wandering and assume w.l.o.g. that $u = z^{+\infty}$. Suppose first that every entry of a is infinite and consider $U = B\left((a,u), \frac{1}{2^{|w|}}\right)$. Take $r > m$, $(b,v) \in U$ such that $(b,v)\varphi^r \in U$. Denote the first letter of $\tilde{z}$ by $\tilde{z}_1$. We have that $w\tilde{z}_1$ is a prefix of v and for every $i \in [m]$, either $a_i = b_i$ or $a_i b_i > 0$ and $|a_i|, |b_i| > |w|$.

From $(b,v)\hat{\varphi}^r \in U$ we deduce that $w\tilde{z}_1$ is a prefix of $z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}}$, so $\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}} > 0$ and $\lambda\pi_k^{(r-1)} a_{k\psi^{r-1}} = +\infty$, since it has the same sign and every entry of a is infinite. Also, $a\hat{\varphi}_1^r = a$ because $a_i b_i > 0$ and $a_i (b\hat{\varphi}_1^r)_i > 0$, so $\hat{\varphi}_1^r$ doesn't change the signs of the entries in b, thus it also does not change the ones in a. In that case, $(a,u)\hat{\varphi}^r = (a,u)$ and a is periodic.

To complete the proof, we take $(a, z^{+\infty})$ such that a has finite entries and suppose it is not wandering nor periodic. Take $\delta = \max\limits_{a_i \in \mathbb{Z}}\{|a_i|\}$ and $U = B\left((a, z^{+\infty}); \frac{1}{2^\delta}\right)$. Take $r > m$ and $(b,v) \in U$ such that $(b,v)\hat{\varphi}^r \in U$. We now consider two subcases:

Subcase 3.1: $|a_{k\psi^{r-1}}| = \infty$. We have that

$$\left| z^{+\infty} \wedge z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}} \right| > \delta$$

and so $\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}} > 0$. Thus, using (6) with $i = k\psi^{r-1}$, we have that $\lambda\pi_k^{(r-1)} a_{k\psi^{r-1}} > 0$ and, since $|a_{k\psi^{r-1}}| = \infty$, we have that $z^{\lambda\pi_k^{(r-1)} a_{k\psi^{r-1}}} = z^{+\infty}$. Since (a,u) is not periodic, then we have that $a\hat{\varphi}_1^r = [\pi_i^{(r)} a_{i\psi^r}]_{i\in[m]} \neq a$. Take q such that $a_q \neq \pi_q^{(r)} a_{q\psi^r}$. By Lemma 5.14, we have that the conditions from Lemma 5.13 hold when $i = q$. Thus, the set $X = \{j \in [m] \mid \text{ the conditions from Lemma 5.13 hold for i=j}\}$ is nonempty.

Consider $\delta = \max\limits_{a_i \in \mathbb{Z}}\{|a_i|\}$, $\tau = \max\{B_q \mid q \in X\}$ and $\delta' = 2|w| + \lambda\tau\sigma|\tilde{z}|$ and let $U' = B((a,u); \frac{1}{2^{\delta'}})$. Since (a,u) is nonwandering, there is some $r' > m$ and $(b',v') \in U'$ such that $(b',v')\hat{\varphi}^{r'} \in U'$. Notice that $U' \subseteq U$ and so Lemma 5.14 can be applied. We will prove that $(a,u)\hat{\varphi}^{r'} = (a,u)$, which is absurd.

We start by showing that $z^{\lambda\pi_k^{(r'-1)} a_{k\psi^{r'-1}}} = z^{+\infty}$. Indeed, if $|a_{k\psi^{r'-1}}| < \infty$, then by the last statement of Lemma 5.14, we have that $k \in X$. Also, we know by (5) that $b'_{k\psi^{r'-1}} = a_{k\psi^{r'-1}}$, so

$$\left| z^{\lambda\pi_k^{(r'-1)} b'_{k\psi^{r'-1}}} \right| = 2|w| + |\lambda\pi_k^{(r'-1)} b'_{k\psi^{r'-1}}||\tilde{z}| \leq 2|w| + \lambda\tau\delta|\tilde{z}| = \delta',$$

which is absurd because $(b,v)\hat{\varphi}^r \in U'$ implies that

$$\left| z^{+\infty} \wedge z^{\lambda \pi_k^{(r'-1)} b'_{k\psi^{r'-1}}} \right| > \delta'. \tag{8}$$

So this can never happen and so we must have $|a_{k\psi^{r'-1}}| = \infty$. Since we have (8), it follows that $\lambda\pi_k^{(r'-1)} b'_{k\psi^{r'-1}} > 0$. Thus, by (6), we have that $\lambda\pi_k^{(r'-1)} a_{k\psi^{r'-1}} > 0$ and, since $|a_{k\psi^{r'-1}}| = \infty$, we have that $z^{\lambda\pi_k^{(r'-1)} a_{k\psi^{r'-1}}} = z^{+\infty}$.

We only have to see that $a\varphi_1^{r'} = a$. If that is not the case, then there is some $q \in [m]$ such that $a_q \neq \pi_q^{r'} a_{q\psi^{r'}}$. By Lemma 5.14, we have that $q \in X$, which implies that a_q is infinite and that $a_{q\psi^{r'}}$ is finite. It follows from (6) that $\pi_q^{(r')} b'_{q\psi^{r'}} > \delta'$, which is absurd since $\pi_q^{(r')} \leq \tau$ and, by (5), we have that $b'_{q\psi^{r'}} = a_{q\psi^{r'}} \leq \delta$.

Subcase 3.2: $|a_{k\psi^{r-1}}| < \infty$ By Lemma 5.14, we know that $k \in X$. Consider $\delta'' = 2|w| + \lambda\delta B_k |\tilde{z}|$ and $U'' = B\left((a,u); \frac{1}{2^{\delta''}}\right)$. As usual, take $r'' > m$ and $(b'', v'') \in U''$ such that $(b'',v'')\hat{\varphi}^{r''} \in U''$. We have that

$$\left| z^{+\infty} \wedge z^{\lambda \pi_k^{(r''-1)} b''_{k\psi^{r''-1}}} \right| > \delta''.$$

But that cannot happen since

$$\left| z^{\lambda \pi_k^{(r''-1)} b''_{k\psi^{r''-1}}} \right| = 2|w| + |\lambda \pi_k^{(r''-1)} b''_{k\psi^{r''-1}}||\tilde{z}| \leq \delta''.$$

□

Since $\widehat{\mathbb{Z}^m \times F_n}$ is compact, then every ω-limit set is nonempty. Since such a set cannot contain wandering points, then, for every point, its limit set is a periodic orbit, which means that every point is asymptotically periodic.

We also remark that most of these results should be easily extended to the product of a free group with a finitely generated abelian group. So, consider $P = \mathbb{Z}_{p_1} \times \cdots \mathbb{Z}_{p_r}$ for some $r \in \mathbb{N}$ and powers of (not necessarily distinct) prime numbers p_i, for $i \in [r]$, endowed with the product metric given by taking the discrete metric in each component. This is a complete space. Take $G = F_n \times \mathbb{Z}^m \times P$ and $\varphi \in End(G)$. We have that a point of the form $(1, 0, x_1, \ldots, x_r)$ must be mapped to a point of the same form. Indeed, setting $(1, 0, x_1, \ldots, x_r)\varphi = (w, a, y_1, \ldots l_r)$, we have that $(1, 0, x_1, \ldots, x_r)\varphi = (1, 0, x_1, \ldots, x_r)^{p+1}\varphi$, thus $w = w^{p+1}$ and $a = (p+1)a$, so $w = 1$ and $a = 0$. It follows that $(u,a,p)\varphi = ((u,a)\psi_1, (u,a,p)\psi_2)$, for some $\psi_1 \in End(F_n \times \mathbb{Z}^m)$ and $\psi_2 : F_n \times \mathbb{Z}^m \times P \to P$, i.e., the P-component has no influence in the $F_n \times \mathbb{Z}^m$-component of the image.

If we want φ to be uniformly continuous, we can see that φ must be given by $(u,a,p)\varphi = ((u,a)\psi_1, p\psi_2)$, for some $\psi_1 \in End(F_n \times \mathbb{Z}^m)$ and $\psi_2 \in End(P)$, i.e., the $\mathbb{F}_n \times \mathbb{Z}^m$-component has no influence in the P-component of the image. Indeed, for every element x in the basis of F_n, setting $(x, 0, 0, \ldots, 0)\varphi = (w, a, n_1, \ldots n_r)$, we have that every n_i must be equal to 0 because if that was not the case taking $\varepsilon < 1$, for every delta, choosing q such that $qp_1 \cdots p_r > \log_2(\frac{1}{\delta})$ we would have that

$$d((x, 0, \ldots 0)^{qp_1 \cdots p_r}, (x, 0, \ldots 0)^{qp_1 \cdots p_r + 1}) < \delta$$

but

$$d((x, 0, \ldots 0)^{qp_1 \cdots p_r}\varphi, (x, 0, \ldots 0)^{qp_1 \cdots p_r + 1}\varphi) = 1$$

and the same happens when we consider an element in the basis of $\mathbb{Z}^m$.

Now observe that every point in ψ_2 must be periodic, since P is finite. So, if the dichotomy periodic vs wandering holds for ψ_1 (in particular, if ψ_1 is type II), taking a nonwandering point $(w, a, p) \in G$, we have that (w, a) must be a nonwandering point of ψ_1, thus periodic, and p is periodic and so (w, a, p) is φ-periodic.

6 Further work

Although we were able to establish the dichotomy wandering/periodic for type II endomorphisms, we weren't able to go very far in answering this question for type I endomorphisms. It would be interesting to find conditions on $\varphi \in End(F_n)$ such that a type I endomorphism defined by $(a, u) \mapsto (aQ, u\phi)$ satisfies that property. If we prove that a nonwandering point is always recurrent, then, by the result in [8], we get the result for automorphisms.

Another interesting question could be obtaining conditions on the properties $\phi \in End(F_n)$ must satisfy so that the converse implications of the ones in Proposition 5.6 also hold.

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