$\mathbb{Z}/2$-Godeaux surfaces

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Abstract

We compute explicit equations for all (universal coverings of) Godeaux surfaces with torsion group $\mathbb{Z}/2$. We show that their moduli space is irreducible and rational of dimension 8.
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1 Introduction

Let S be a smooth minimal complex algebraic surface. Its topological invariants are the geometric genus p_g, the irregularity q and the self-intersection K^2 of a canonical divisor. The holomorphic Euler characteristic is $\chi = 1 + p_g - q$. Gieseker [Gie77] has shown that for each pair (χ, K^2) and S of general type, there exists a coarse moduli space $\mathcal{M}_{\chi,K^2}$ that is a quasi-projective variey. Naturally geometers want to understand which of these families are non-empty, and then if possible to classify them. It is frustrating that this has not been achieved even for the first case in the list, the one with $\chi = K^2 = 1$.

For these surfaces $p_g = q = 0$, and they are known to exist since Godeaux' construction in 1931 [God31]. Nowadays surfaces of general type with $p_g = q = 0, K^2 = 1$ are called numerical Godeaux surfaces. Miyaoka [Miy76] showed that the order of their torsion group is at most 5, and Reid [Rei78] excluded the case $(\mathbb{Z}/2)^2$, so their possible torsion groups are $\mathbb{Z}/n$ with $1 \le n \le 5$. Reid constructed the moduli space for the cases $n = 5, 4, 3$, and it follows from his work that the topological fundamental group coincides with the torsion group for $n = 5, 4$. Coughlan and Urzúa [CU18] showed that the same happens for $n = 3$. In those three cases the moduli space is irreducible of dimension 8.

Coughlan [Cou16] has obtained a family of $\mathbb{Z}/2$-Godeaux surfaces (i.e. with torsion group $\mathbb{Z}/2$) depending on 8 parameters. More recently, we have studied with Urzúa [DRU20] all possible degenerations of $\mathbb{Z}/2$-Godeaux surfaces into stable surfaces with one Wahl singularity, which produces many boundary divisors of dimension 7 in the KSBA compactification of the moduli space of these surfaces. This is done by means of abstract constructions (i.e. showing the existence of particular singular surfaces with no obstructions in deformations), and computational constructions based on Coughlan's family. We have ended up proving in [DRU20] that Coughlan's family is at most 7 dimensional. For the case of Godeaux surfaces with trivial torsion, we know the examples due to Barlow [Bar85], Craighero-Gattazzo [CG94] (see also [RTU17]), and Lee-Park type of constructions (cf. [LP07]).

Besides the work of several other authors, these cases $n = 2, 1$ are still open (at the time of submitting this paper, Schreyer and Stenger put out a preprint

[SS20] claiming the construction of an 8-dimensional family of simply connected Godeaux surfaces, but without obtaining a full classification).

Catanese and Debarre [CD89] showed that the étale double covers of $\mathbb{Z}/2$-Godeaux surfaces have hyperelliptic canonical curve and birational bicanonical map onto an octic in $\mathbb{P}^3$, and they did a general study of its canonical ring. That octic is given by the determinant of a certain matrix α.

In this paper we continue their work. Using an idea from Miles Reid [Rei90], we get more precise information about α by looking first to its restriction to the case of the canonical curve, then extending to the surface. Then we give an algorithm for the computation of all such matrices, from which we obtain equations for the étale double covers of all $\mathbb{Z}/2$-Godeaux surfaces. We show that their moduli space is irreducible of dimension 8, which implies that the topological fundamental group of $\mathbb{Z}/2$-Godeaux surfaces is also $\mathbb{Z}/2$.

We note that our method is not brute force computation: for the main algorithm, the calculations used only 32 MB of RAM memory, and took 85 seconds on a low-end computer.

Recently two special $\mathbb{Z}/2$-Godeaux surfaces have appeared in the literature: a $(\mathbb{Z}/3)^2$-quotient of a fake projective plane, constructed by Borisov and Fatighenti [BF20], which has 4 cusp singularities; a degree 6 quotient of the so-called Cartwright-Steger surface, given by Borisov-Yeung [BY20], which has 3 cusp singularities and a certain configuration of rational curves. As an exercise, we give the coordinates of these surfaces in our family.

All computations are implemented with Magma [BCP97], and can be found in some arXiv ancillary files. In particular, using the files
`5_Verifications_alpha_i_c_j.txt` one can choose any surface in the family and compute its invariants and singular set.

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2 Results from Catanese-Debarre

We collect here some results from the paper [CD89] that will be used throughout the text.

Let S be the étale double cover of a numerical Godeaux surface with torsion group $\mathbb{Z}/2$, and denote the corresponding involution by σ. The invariants of S are $K^2 = 2$, $p_g = 1$, $q = 0$. Define the canonical ring of S as

$$\mathcal{R} = \bigoplus_{n=0}^{\infty} H^0(S, nK_S),$$

and let $\mathcal{A} = \mathbb{C}[x, y_1, y_2, y_3]$ be the $\mathbb{C}$-graded algebra with

$$\deg(x) = 1, \quad \deg(y_i) = 2.$$

The involution σ acts on $\mathcal{R}$, the canonical ring of S, splitting it into eigenspaces $\mathcal{R} = \mathcal{R}^+ \oplus \mathcal{R}^-$. Denoting the Godeaux surface by T, we have

$$\mathcal{R}^+ \cong \bigoplus_{n\geq 0} H^0(T, nK_T), \qquad \mathcal{R}^- \cong \bigoplus_{n\geq 0} H^0(T, nK_T + \eta),$$

where $\eta \in \mathrm{Pic}(T)$ is a 2-torsion element. Furthermore, by Riemann-Roch,

$$\dim \mathcal{R}_n^+ = \dim \mathcal{R}_n^- = 1 + \binom{n}{2}, \text{ for } m \geq 2.$$

Throughout the paper we denote the set of generators of $\mathcal{R}$ by

$$\begin{array}{c|c} & x \in \mathcal{R}_1^- \\ x^2, y_2 \in \mathcal{R}_2^+ & y_1, y_3 \in \mathcal{R}_2^- \\ xy_1, xy_3, z_3, z_4 \in \mathcal{R}_3^+ & x^3, xy_2, z_1, z_2 \in \mathcal{R}_3^- \\ & t \in \mathcal{R}_4^-. \end{array}$$

Notice that the vector space $\mathcal{R}_4^+$ is 7 dimensional and contains

$$\{x^4, x^2y_2, y_1^2, y_1y_3, y_2^2, y_3^2, xz_1, xz_3\}.$$

Then from [CD89, Lemma 4.5], and possibly doing a change of variables, there is a (unique) relation that can be written as

$$Q^+(y_1, y_2, y_3) + \lambda x z_1.$$

For the next 5 items see Proposition 1.1 and Theorem 6.1, Proposition 4.2 and Theorem 4.3, Theorem 4.6, and the proof of Theorem 4.6 of [CD89].

(1) The bicanonical map of S is a birational morphism and its canonical curve is hyperelliptic.

(2) $\mathcal{R}$ is a Cohen-Macaulay $\mathcal{A}$-module, which implies that $\mathcal{R}$ admits a length one free resolution of $\mathcal{A}$-modules that can be written as

$$\mathcal{R} \leftarrow \mathcal{A} \oplus \mathcal{A}(-3)^4 \oplus \mathcal{A}(-4) \overset{\alpha}{\longleftarrow} \mathcal{A}(-4) \oplus \mathcal{A}(-5)^4 \oplus \mathcal{A}(-8) \leftarrow 0,$$

where α is a matrix with homogeneous entries in $\mathcal{A}$.

(3) The matrix α can be chosen symmetric of the form

$$\alpha = \left(\begin{array}{c|cccc|c} x^2G & xq_1 & xq_2 & xq_3 & xq_4 & Q \\ \hline xq_1 & a_{11} & a_{12} & a_{13} & a_{14} & x \\ xq_2 & a_{12} & a_{22} & a_{23} & a_{24} & 0 \\ xq_3 & a_{13} & a_{23} & a_{33} & a_{34} & 0 \\ xq_4 & a_{14} & a_{24} & a_{34} & a_{44} & 0 \\ \hline Q & x & 0 & 0 & 0 & 0 \end{array}\right)$$

where G, q_i, a_{ij} are of degrees $3, 2, 1$ in $(y_0 = x^2, y_1, y_2, y_3)$, respectively. The 3×3-minors of (a_{ij}) are in the ideal (x, Q), and $\det(a_{ij})$ is in (x, Q^2).

(4) The matrix α satisfies the following *rank condition*:

(RC) For each cofactor β_{ij} of α there exist $l_{ij}^k \in \mathcal{A}$ such that

$$\beta_{ij} = \sum_{k=1}^{6} l_{ij}^k \beta_{1k}.$$

(5) Conversely, for any matrix α belonging to an open subset of the set of matrices as in (3) and (4), it is possible to define a ring structure on the $\mathcal{A}$-module $\mathcal{R}$ which α defines. The surface $X = \mathrm{Proj}(\mathcal{R})$ is the canonical model of a minimal surface S with $K^2 = 2$, $p_g = 1$, $q = 0$ for which the bicanonical map is birational onto an octic in $\mathbb{P}^3$ with equation $\det(\alpha)$.

Furthermore, from [Cat84]:

(6) The equations of S are given by

$$v_i v_j = \sum_{k=1}^{6} l_{ij}^k v_k, \quad i, j = 2, \ldots, 6,$$

$$\sum_{j=1}^{6} \alpha_{ij} v_j, \quad i = 1, \ldots, 6,$$

with

$$v_1 = 1, v_2 = z_1, v_3 = z_2, v_4 = z_3, v_5 = z_4, v_6 = t.$$

3 The matrix α

Proposition 1. *The matrix α can be written in the form*

$$\alpha = \left(\begin{array}{c|cccc|c} x^2G^- & xq_1^- & xq_2^- & xq_3^+ & xq_4^+ & Q^+ \\ \hline xq_1^- & a_{11}^- & a_{12}^- & a_{13}^+ & a_{14}^+ & x \\ xq_2^- & a_{12}^- & a_{22}^- & a_{23}^+ & a_{24}^+ & 0 \\ xq_3^+ & a_{13}^+ & a_{23}^+ & a_{33}^- & a_{34}^- & 0 \\ xq_4^+ & a_{14}^+ & a_{24}^+ & a_{34}^- & a_{44}^- & 0 \\ \hline Q^+ & x & 0 & 0 & 0 & 0 \end{array}\right)$$

with

$$Q = y_1^2 - y_2^2 - d^2 y_3^2,$$

and where the superscript signs mean σ-invariant $(+)$ or σ-anti-invariant $(-)$.

Proof. The bicanonical map of S sends its (hyperelliptic) canonical curve C onto the plane conic $Q = 0$, which is contained in the octic surface $\det(\alpha) = 0$ in $\mathbb{P}^3$. Suppose that this conic is a double line. Then $C = 2D + Z$, where Z is supported on a union of (-2)-curves. These curves must be preserved by the Godeaux involution σ, giving rise to curves $C' = 2D' + Z'$, with Z' also a union of (-2)-curves. This contradicts the fact $C'^2 = 1$. Therefore there is a change of variables that allow us to write $Q = y_1^2 - y_2^2 - d^2 y_3^2$, for some constant d.

Using Riemann-Roch and the local basis of $\mathcal{R}$, one sees that there are two σ-invariant relations of degree 5 and two anti-invariant ones. Since x, z_1, z_2, t

are anti-invariant, z_3, z_4 are invariant, and we are assuming Q invariant, the relations

$$\alpha \cdot (1, z_1, \ldots, z_4, t)^T = 0$$

from Section 2 (6) imply that the superscript signs must be as claimed. □

Lemma 2. *The matrix $\alpha|_{x=0}$ cannot be of the type*

$$\left(\begin{array}{c|ccccc|c} 0 & 0 & 0 & 0 & 0 & Q \\ \hline 0 & y_1+dy_3 & 0 & y_2 & 0 & 0 \\ 0 & 0 & y_1+dy_3 & 0 & y_2 & 0 \\ 0 & y_2 & 0 & y_1-dy_3 & 0 & 0 \\ 0 & 0 & y_2 & 0 & y_1-dy_3 & 0 \\ \hline Q & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

with $Q = y_1^2 - y_2^2 - d^2y_3^2$.

Proof. Recall that the cofactors $\beta_{ij}|_{x=0}$ of the matrix satisfy the rank condition (RC). Since $\beta_{1k}|_{x=0} = 0$ for $k = 1, \ldots, 5$, it is not difficult to compute the polynomials $l_{ij}^k|_{x=0}$, and then the equations of the effective canonical divisor of the corresponding surface S. We get that this curve is a double conic (the detailed computations are available in the arXiv ancillary file Lemma2.txt). As in the proof of Proposition 1, this gives a contradiction.

□

Proposition 3. *The matrix $\alpha|_{x=0}$ can be written as*

$$\mathbb{M} := \left(\begin{array}{c|ccccc|c} 0 & 0 & 0 & 0 & 0 & Q \\ \hline 0 & d^2y_3 & y_1 & y_2 & 0 & 0 \\ 0 & y_1 & y_3 & 0 & y_2 & 0 \\ 0 & y_2 & 0 & -y_3 & y_1 & 0 \\ 0 & 0 & y_2 & y_1 & -d^2y_3 & 0 \\ \hline Q & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

with $Q = y_1^2 - y_2^2 - d^2y_3^2$.

Proof. From Proposition 1, the matrix $\alpha|_{x=0}$ can be written as

$$M := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & Q \\ 0 & m_1 & m_2 & r_1y_2 & r_2y_2 & 0 \\ 0 & m_2 & m_3 & r_3y_2 & r_4y_2 & 0 \\ 0 & r_1y_2 & r_3y_2 & m_4 & m_5 & 0 \\ 0 & r_2y_2 & r_4y_2 & m_5 & m_6 & 0 \\ Q & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with $m_i = a_iy_1 + b_iy_3$. Denote its lines, columns by l_i, c_i, respectively.

If $m_2 \neq 0$, then we can assume $m_3 \neq 0$, by possibly doing the elementary operations $l_3 \to l_3 + \alpha l_2$, $c_3 \to c_3 + \alpha c_2$, for some constant α. Now operations of the type $l_2 \to l_2 + \beta l_3$, $c_2 \to c_2 + \beta c_3$ take us to one of the cases $m_2 = 0$, $m_2 \doteq y_1$ or $m_2 \doteq y_3$, where here the notation $a \doteq b$ means that $a = \tau b$ for some constant $\tau \neq 0$.

Suppose that $m_2 \doteq y_1$. If $m_3 \doteq y_1$, we can go to the case $m_2 = 0$. If not, we can take $m_3 \doteq y_3$. Analogously if $m_2 \doteq y_3$, we can take $m_2 = 0$ or $m_3 \doteq y_1$.

Now by multiplying l_3, c_3 and l_2, c_2 by constants, we can assume that $m_2 = 0$, $m_2 = y_1$ and $m_3 = y_3$, or $m_2 = y_3$ and $m_3 = y_1$.

Let D be the determinant of the 4×4 central matrix of M. One can check that the coefficient of y_2^4 in D is $(r_1 r_4 - r_2 r_3)^2$. Since D is a multiple of Q^2 (from Section 2 (3)), we must have $r_1 r_4 - r_2 r_3 \neq 0$. Then elementary operations over the lines l_4, l_5 and the columns c_4, c_5 allow us to assume that $r_1 = r_4 = 1$ and $r_2 = r_3 = 0$.

Summing up, we have three possible cases:

1) $m_2 = y_1$, $m_3 = y_3$, $r_1 = r_4 = 1$, $r_2 = r_3 = 0$;

2) $m_2 = y_3$, $m_3 = y_1$, $r_1 = r_4 = 1$, $r_2 = r_3 = 0$;

3) $m_2 = 0$, $r_1 = r_4 = 1$, $r_2 = r_3 = 0$.

Let N be the 4×4 central matrix of M. Notice that the rank condition (RC) implies that each cofactor C_{ij} of N is divisible by the quadric Q. We show below that this is enough to conclude the proof.

The computational details for the following three cases are available in the arXiv ancillary file Proposition3.txt.

Case 1)
We have

$$\begin{aligned} -C_{1,3}/y_2 &= a_5 y_1^2 + (b_5 + a_6) y_1 y_3 - y_2^2 + b_6 y_3^2, \\ C_{1,4}/y_2 &= a_4 y_1^2 + (b_4 + a_5) y_1 y_3 + b_5 y_3^2, \\ C_{2,3}/y_2 &= (a_1 a_5 + a_6) y_1^2 + (a_1 b_5 + b_1 a_5 + b_6) y_1 y_3 + b_1 b_5 y_3^2. \end{aligned}$$

The only possibility then is that the first one is equal to Q, and the other two are zero. This implies

$$a_1 = 0, b_1 = d^2, a_4 = 0, b_4 = -1, a_5 = 1, b_5 = 0, a_6 = 0, b_6 = -d^2.$$

Case 2)
We have

$$\begin{aligned} -C_{1,3}/y_2 &= a_6 y_1^2 + (a_5 + b_6) y_1 y_3 - y_2^2 + b_5 y_3^2, \\ C_{1,4}/y_2 &= a_5 y_1^2 + (a_4 + b_5) y_1 y_3 + b_4 y_3^2, \\ -C_{2,3}/y_2 &= a_1 a_4 y_1^2 + (a_1 b_4 + b_1 a_4 + a_5) y_1 y_3 - y_2^2 + (b_1 b_4 + b_5) y_3^2. \end{aligned}$$

The only possibility is that the second one is zero, and the other two are equal to Q. This implies that $d \neq 0$ and

$$a_1 = d^{-2}, b_1 = 0, a_4 = d^2, b_4 = 0, a_5 = 0, b_5 = -d^2, a_6 = 1, b_6 = 0.$$

Now let

$$P := \mathrm{Diag}\left(1, r^3, r^3 d^{-2}, -r d^{-2}, -r, 1\right)$$

with $r^4 + d^2 = 0$. The Product PMP^T shows that we can send M to the matrix $\mathbb{M}$ above by a change of variables, more precisely

$$PMP^T = \mathbb{M}\left(-r^2 y_3, y_2, -r^2 d^{-2} y_1\right).$$

Case 3)

In this case we have

$$-C_{12}/y_2 = y_2(a_5y_1 + b_5y_3),$$
$$-C_{13}/y_2 = \left(a_3a_6y_1^2 + (b_3a_6 + a_3b_6)y_1y_3 - y_2^2 + b_3b_6y_3^2\right),$$
$$-C_{24}/y_2 = \left(a_1a_4y_1^2 + (b_1a_4 + a_1b_4)y_1y_3 - y_2^2 + b_1b_4y_3^2\right).$$

This implies $a_5 = b_5 = b_3a_6 + a_3b_6 = b_1a_4 + a_1b_4 = 0$, $a_3a_6 = a_1a_4 = 1$ and $b_3b_6 = b_1b_4 = -d$. Then $a_1a_3 \neq 0$ and we can assume $a_1 = a_3 = 1$. This way we obtain 4 matrices which, by changing y_3 to $-y_3$, reduce to

$$M_j := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & Q \\ 0 & y_1 + dy_3 & 0 & y_2 & 0 & 0 \\ 0 & 0 & y_1 - (-1)^i dy_3 & 0 & y_2 & 0 \\ 0 & y_2 & 0 & y_1 - dy_3 & 0 & 0 \\ 0 & 0 & y_2 & 0 & y_1 + (-1)^i dy_3 & 0 \\ Q & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $j = 1, 2$.

From Lemma 2, only the matrix M_2 with $d \neq 0$ can correspond to a Godeaux surface. Let

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & d/2 & 0 & 0 & 0 \\ 0 & -i/d & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i/2 & 1/d & 0 \\ 0 & 0 & 0 & id/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with $i^2 = -1$. The product PM_2P^T shows that M_2 is equivalent to a matrix of the type $\mathbb{M}$ above.

□

Now we introduce a new degree 2 variable y_4 negative for the involution σ.

Theorem 4. *The matrix α can be written as*

$$\alpha_j = \left(\begin{array}{c|cccc|c} x^2G^- & xq_1^- & xq_2^- & xq_3^+ & xq_4^+ & Q \\ \hline xq_1^- & y_4 & y_1 & y_2 & 0 & x \\ xq_2^- & y_1 & y_3 & cx^2 & y_2 & 0 \\ xq_3^+ & y_2 & cx^2 & -y_3 & y_1 & 0 \\ xq_4^+ & 0 & y_2 & y_1 & -y_4 & 0 \\ \hline Q & x & 0 & 0 & 0 & 0 \end{array}\right)$$

with

$$Q = y_1^2 - y_2^2 - y_3y_4,$$

G, q_i polynomials of degree $3, 2$ in $\left(y_0 = x^2, y_1, y_2, y_3, y_4\right)$, respectively, and

$$y_4 = d^2y_3 \quad (j = 1) \qquad \text{or} \qquad y_1 = -\frac{1}{2d}y_3 \quad (j = 2) \qquad \text{or} \qquad y_3 = 0 \quad (j = 3).$$

(As above the superscript signs mean σ-invariant or σ-anti-invariant.)

Moreover, we can assume $c = 1$ or $c = 0$.

Proof. We want to extend the matrix $\mathbb{M}$ from Proposition 3 by adding polynomials divisible by x. This must respect the signs given in Proposition 1, hence concerning the entries of order 2, we can only add multiples of x^2 to the σ-invariant ones. We get the matrix

$$\alpha = \begin{pmatrix} x^2G^- & xq_1^- & xq_2^- & xq_3^+ & xq_4^+ & Q \\ xq_1^- & d^2y_3 & y_1 & y_2+c_1x^2 & c_2x^2 & c_5x \\ xq_2^- & y_1 & y_3 & c_3x^2 & y_2+c_4x^2 & c_6x \\ xq_3^+ & y_2+c_1x^2 & c_3x^2 & -y_3 & y_1 & 0 \\ xq_4^+ & c_2x^2 & y_2+c_4x^2 & y_1 & -d^2y_3 & 0 \\ Q & c_5x & c_6x & 0 & 0 & 0 \end{pmatrix}.$$

We know that $\det(\alpha)$ defines an irreducible surface in $\mathbb{P}^3$, thus $c_5 = c_6 = 0$ is impossible. If $c_6 = 0$, we can take $c_5 = 1$ from the change of variable $x \to x/c_5$. Then elementary operations using the last line and column give us $c_1 = c_4$ and $c_2 = 0$. We can assume $c_4 = 0$ by doing $y_2 \to y_2 - c_4x^2$ and the result follows.

Now assume that $c_6 \neq 0$. We consider 3 cases. (The computational details are available in the arXiv ancillary file Theorem4.txt.)

Case 1: $d^2 \neq (c_5/c_6)^2$, $c_5 \neq 0$
Let

$$a := \frac{c_5^2 + d^2c_6^2}{c_5^2 - d^2c_6^2}, \qquad b := \frac{2c_5c_6}{c_5^2 - d^2c_6^2}$$

and

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & -d^2rc_6/c_5 & 0 & 0 & 0 \\ 0 & -rc_6/c_5 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & r & rc_6/c_5 & 0 \\ 0 & 0 & 0 & d^2rc_6/c_5 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with

$$r^2 = \frac{c_5^2}{c_5^2 - d^2c_6^2}.$$

The product $P\alpha P^T$ is a matrix of the type

$$\begin{pmatrix} x^2G' & xq_1' & xq_2' & xq_3' & xq_4' & Q \\ xq_1' & d^2Y_3 & Y_1 & y_2+c_1'x^2 & c_2'x^2 & c_5'x \\ xq_2' & Y_1 & Y_3 & c_3'x^2 & y_2+c_4'x^2 & 0 \\ xq_3' & y_2+c_1'x^2 & c_3'x^2 & -Y_3 & Y_1 & 0 \\ xq_4' & c_2'x^2 & y_2+c_4'x^2 & Y_1 & -d^2Y_3 & 0 \\ Q & c_5'x & 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$\begin{pmatrix} Y_1 \\ Y_3 \end{pmatrix} = \begin{pmatrix} a & -d^2b \\ -b & a \end{pmatrix} \begin{pmatrix} y_1 \\ y_3 \end{pmatrix}.$$

Notice that the determinant of this 2×2 matrix is $a^2 - d^2b^2 = 1$.

Since $c_5' = c_5/r \neq 0$, we can proceed as before to get $c_5' = 1, c_1' = c_2' = c_4' = 0$. Finally from

$$Y_1^2 - y_2^2 - d^2Y_3^2 = y_1^2 - y_2^2 - d^2y_3^2$$

we see that the matrix $P\alpha P^T$ is in the form of the matrix α_1 above.

Case 2: $d^2 \neq (c_5/c_6)^2,\ c_5 = 0$
Let

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 \\ 0 & 1/d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/d & 0 \\ 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The product $P\alpha P^T$ gives us a matrix of the type α with $c_6 = 0$. We proceed as above to get α_1.

Case 3: $d^2 = (c_5/c_6)^2$
Since $c_6 \neq 0$, we can take $c_6 = 1$, $c_5 = \pm d$ and, as above, $c_1 = c_3 = c_4 = 0$. By looking to $P\alpha P^T$ with $P := \text{Diag}(1, -1, 1, -1, 1, 1)$, we see that we can consider $c_5 = d$.

Let

$$P' := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The product $P'\alpha P'^T$ gives us a matrix of the type α_2 if $d \neq 0$, or of the type α_3 if $d = 0$.

Finally, if $c \neq 0$, we can assume $c = 1$ by taking the product $P\alpha P^T$ with $P := \text{Diag}(1, c, 1/c, 1/c, c, 1)$, followed by the change $x \mapsto x/c$.

□

4 Computation of the equations

Denote by s_n^-, s_n^+ the sequence of monomials of degree n on the variables (x^2, y_1, y_2, y_3) which are anti-invariant, invariant for σ, respectively. We write

$$G = \sum_1^{10} g_i s_3^-[i],$$

$$q_1 = \sum_1^{4} b_i s_2^-[i],\ \ldots,\ q_4 = \sum_{15}^{20} b_i s_2^+[i].$$

Our goal is to compute the set of parameters $g_1, \ldots, g_{10}$ and $b_1, \ldots, b_{20}$ such that the matrices α_i from Theorem 4 satisfy the rank condition (RC).

By doing elementary operations over the lines and columns of the matrix α_i, we can assume that 8 of the b_i are zero. For instance in the case of α_1 :

- We remove multiples of y_1, y_2 from q_4, except the monomial $y_1 y_3$;

- We remove multiples of y_1, y_3 from q_3;
- We remove multiples of x from q_1.

The idea for the computations is the following: we write the polynomials l_{ij}^k depending on some parameters r_m, then we need to compute the parameters g_p, b_n, r_m such that the coefficients of the polynomials from (RC)

$$\beta_{ij} - \sum_{k=1}^{6} l_{ij}^k \beta_{1k}$$

vanish. After this the equations of the surfaces S follow from Section 2 (6).

The polynomials l_{ij}^k depend on 371 parameters. We have a huge system of 876 coefficients depending on $23 + 371 = 394$ parameters, but the parameters r_m appear linearly. We have developed an algorithm for this problem.

4.1 The algorithm

First we define a Magma function **LinElim** that will be used to solve the system of polynomial equations:

Input: f a sequence of polynomials, g a subset of f, var a sequence of variables, and $n \in \mathbb{N}$.

Process: It checks the elements of g one-by-one and whenever it finds one that is of the form $r - h$ with $r \in var$, h not depending on r, and containing at most n variables that are in var, it substitutes r by h in all elements of f, g. It also adds the pair $[r, h]$ to a list that we call *dependencies*.

Output: The new list f, and the *dependencies*.

We can now start.

(1) We work on $R[x, y_1, y_2, y_3, z_1, z_2, z_3, z_4, t]$ with R a polynomial ring with 394 variables (the parameters). Recall that the involution σ is

$$[-x, -y_1, y_2, -y_3, -z_1, -z_2, z_3, z_4, -t].$$

(2) We define the matrix α_j, with G, q_i depending on some parameters g_m, b_n.

(3) We define the polynomials l_{ij}^k depending on parameters r_p. Notice that these must be chosen with the right degree and σ sign.

(4) We write the polynomials that define the rank condition and a sequence f containing their coefficients. Our goal is to compute the parameters g_m, b_n, r_p such that these polynomial coefficients vanish.

(5) We now use the Magma function LinElim defined above to solve the system of equations $f = 0$. We expect to eliminate all variables except some of the g_i and b_j (which are in the first 23 variables of the ring R). So, the function is used for the remaining 371 variables.

(6) After this we see that there is a set $g \subset f$ of polynomials containing only variables g_i or b_j, and such that some of these variables can also be eliminated. So we run LinElim(f, g, var, n) now with $var = [g_1, \dots, g_{10}, b_1, \dots, b_{12}]$.

(7) This process is much more efficient if we do it for $n = 1$, then repeat for $n = 2$, etc. The system is solved when the output f is empty.

(8) We can now compute the matrix α_j, by evaluating its polynomial coefficients at the *dependencies*. We do the same for the l_{ij}^k.

(9) We can finally compute the equations given by Section 2 (6), on the variables $x, y_1, y_2, y_3, z_1, z_2, z_3, z_4, t$.

More details can be found in the arXiv ancillary file `1_TheAlgorithm_alpha_1_c_1.pdf`, which contains a Magma implementation of this algorithm for the case α_1 with $c = 1$ of Theorem 4. The equations that we get still depend on some variables $R.i$ with $i > 23$, except for the ones of degree ≤ 5 which depend on nine of the $R.i$ with $i \leq 23$. We show that the coefficients of the $R.i$ with $i > 23$ are contained in the ideal generated by the degree ≤ 5 equations (see the arXiv ancillary file `3_RemovingTheRi_alpha_1_c_1.txt`), so we can consider $R.i = 0$ for $i > 23$. This gives the final equations.

These computations used only 32 MB of RAM memory, and took 85 seconds on a low-end computer.

The matrix $\alpha_1, c = 1$ is given by:

$G =$
$(-2b_9b_6d + 2b_9b_8d + 4b_9d^2 + 2b_6b_{11} - 2b_8b_{11} - 4db_{11})x^4y_1 + (-2b_5b_9d^2 + b_5db_{11} - 2b_9^2d^2 - b_9db_{11} + 2b_6d^2 + b_6b_{12} + b_8^2d + 2b_8d^2 + dg_9 + 2db_{12} + b_{11}^2)x^4y_3 + (-2b_5b_9d - 2b_9^2d - 2b_9b_{11} + 2b_6d + 2g_9 + 4b_{12})x^2y_1y_2 + (-2b_5d^2 - b_5b_{12} - 2b_9b_6d + 2b_9b_8d - b_9b_{12} + b_6b_{11} + 2db_2 - 2db_{11})x^2y_2y_3 + (2b_9d - 2b_{11})y_1^3 + (b_5b_{11} + b_9^2d + b_9b_{11} - 2b_6d - g_9 - 4b_{12})y_1^2y_3 + (-2b_5d - 4b_9d + 4b_2 - 2b_{11})y_1y_2^2 + (b_5b_{12} - 2b_9d^2 + b_9b_{12} + b_6b_{11})y_1y_3^2 + g_9y_2^2y_3 + (-b_9^2d^2 + 2b_6d^2 + b_6b_{12} + dg_9 + 2db_{12})y_3^3,$

$q_1 = b_2xy_2y_3,$
$q_2 = (b_6 - b_8 - 2d)x^3y_1 + (b_5d + b_{11})x^3y_3 + b_5xy_1y_2 + b_6xy_2y_3,$
$q_3 = (-b_9d + b_{11})x^5 + b_8x^3y_2 + b_9xy_2^2,$
$q_4 = (b_8d + d^2 + b_{12})x^5 + b_{11}xy_1y_3 + b_{12}xy_3^2,$

$Q = y_1^2 - y_2^2 - dy_3^2.$

The computations for the other cases $\alpha_i, c = j$ are given in the files `1_The_Algorithm_alpha_i_c_j.txt`.

5 The moduli space

Denote by $\mathcal{M}$ the moduli space of numerical Godeaux surfaces with torsion group $\mathbb{Z}/2$, and let $\mathcal{M}_i^j$ be the subset of $\mathcal{M}$ corresponding to the matrix α_i with $c = j$ of Theorem 4.

Theorem 5. *The space $\mathcal{M}_1^1$ is isomorphic to an open dense subset of the 8-dimensional weighted projective space $\mathbb{P}(1,1,2,2,2,3,3,4,4)$.*

We have $\mathcal{M}_2^0 = \mathcal{M}_3^0 = \emptyset$. The spaces $\mathcal{M}_1^0$, $\mathcal{M}_2^1$ and $\mathcal{M}_3^1$ are at most 7-dimensional, and are contained in the closure of $\mathcal{M}_1^1$.

Proof. The computations of Section 4.1 give a matrix $\alpha_1, c = 1$ whose entries are polynomials on the variables $(y_0 = x^2, y_1, y_2, y_3)$ with coefficients depending on 9 parameters $(p_1, \ldots, p_9)$. Each determinant $D := \det(\alpha_1)$ gives an octic surface in $\mathbb{P}^3$ which, for general values of the parameters, is the image of Y by its (degree 1) bicanonical map, where Y is the surface that has also been computed by the algorithm. We check that, for any nonzero constant u, we have

$$D = D(y_0/u, y_1, y_2, y_3/u, up_1, up_2, u^2p_3, u^2p_4, u^2p_5, u^3p_6, u^3p_7, u^4p_8, u^4p_9)$$

(see the arXiv ancillary file `7_ItIsWeightedProjSpace_alpha_1_c_1.txt`). Therefore, the octics corresponding to $(p_1, \ldots, p_9)$ and

$$(up_1, up_2, u^2p_3, u^2p_4, u^2p_5, u^3p_6, u^3p_7, u^4p_8, u^4p_9)$$

are identified by the change of variables $(y_0/u, y_1, y_2, y_3/u)$. This implies that the above family of octic surfaces is parametrized by $\mathbb{P}(1,1,2,2,2,3,3,4,4)$. Denote it by S_1^1.

In order to check that its dimension is 8, we wrote an algorithm that, given an element of S_1^1, computes all other elements that are projectively equivalent to it. We then use it to show that there are octics in S_1^1 which are not equivalent to any other octic in S_1^1. See the file `8_Dimension_alpha_1_c_1.txt`.

It is easy to check that for $c = 0$ the determinant of the matrix α_3 is a square, hence $\mathcal{M}_3^0 = \emptyset$. The computations for the case $\alpha_2, c = 0$ give equations that contain the point $(0 : 0 : 0 : 1 : 0 : 0 : 0 : 0 : 0)$. Since this point would be fixed by the tricanonical map, this is not possible for surfaces with $p_g = 1$, $q = 0$, $K^2 = 2$. Thus $\mathcal{M}_2^0 = \emptyset$.

For the remaining cases, we run the above algorithm, obtaining again families parametrized by some weighted projective space of dimension 8 (for $\mathcal{M}_1^0$, $\mathcal{M}_2^1$) or 7 (for $\mathcal{M}_3^1$). For the first two families, we find that there are octics for which there exists an irreducible 1-dimensional set of octics that are projectively equivalent to it. For each general octic, we give the expression of a 1-dimensional family of variable changes sending it to another octic in the same family. Thus the dimensions are at most 7. See the corresponding arXiv ancillary files.

Finally, by the results of Kuranishi [Kur65] and Wavrik [Wav69] (as explained in [Cat83]), the number of moduli of each $\mathbb{Z}/2$-Godeaux surface is at least 8. Since only the space $\mathcal{M}_1^1$ is of dimension 8, the spaces $\mathcal{M}_1^0$, $\mathcal{M}_2^1$ and $\mathcal{M}_3^1$ must be contained in the closure of $\mathcal{M}_1^1$.

□

Corollary 6. *The moduli space of numerical Godeaux surfaces with torsion group $\mathbb{Z}/2$ is irreducible and rational of dimension 8. The topological fundamental group of these surfaces is also $\mathbb{Z}/2$.*

Proof. The first part is immediate from Theorem 5. For the second part it suffices to note that there exist $\mathbb{Z}/2$-Godeaux surfaces with topological fundamental group $\mathbb{Z}/2$, see [Bar84]. □

6 Two special surfaces

Borisov and Fatighenti [BF20] give the equations of a surface X with an action of $\mathbb{Z}/3$ such that the surface $Y := X/(\mathbb{Z}/3)$ is an étale double covering of a $\mathbb{Z}/2$-Godeaux surface with 4 cusps, which in turn is a $(\mathbb{Z}/3)^2$-quotient of a fake projective plane.

The surface X is embedded in $\mathbb{P}^7$ by its bicanonical map, and the action of $\mathbb{Z}/3$ is

$$(x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7) \mapsto (x_0 : x_2 : x_3 : x_1 : x_5 : x_6 : x_4 : x_7).$$

The map given by $(x_0 : x_1 + x_2 + x_3 : x_4 + x_5 + x_6 : x_7)$ sends X to the bicanonical image of Y, an octic surface in $\mathbb{P}^3$. Magma gives the equation of this surface, and with computations similar to the ones in the arXiv ancillary file `8_Dimension_alpha_1_c_1.txt`, we get its coordinates in our family (case α_1, $c = 1$):

$$(b_5, b_9, b_6, b_8, d, b_2, b_{11}, g_9, b_{12}) =$$

$$(36r + 36, 64, -360r + 1752, 360r + 4392, -30r - 366, -10176r - 45504,$$

$$20976r + 78960, 238008r + 1635576, -383328r + 867744),$$

with $r = \sqrt{-15}$.

Now with computations analogous to the ones in the file `5_Verifications_alpha_1_c_1.txt`, one can see that the surface Y is as claimed.

Borisov and Yeung [BY20] give the equations of a $\mathbb{Z}/3$-quotient Z of the Cartwright-Steger surface, and they show that Z is an étale double covering of a $\mathbb{Z}/2$-Godeaux surface. The surface Z has 6 cusp singularities and contains 3 disjoint (-3)-curves. Proceeding as above, we find the image of Z by its bicanonical map, and then we compute its coordinates in our family (case α_1, $c = 1$):

$$(b_5, b_9, b_6, b_8, d, b_2, b_{11}, g_9, b_{12}) = (-60, 40, -120, -302, 9, 252, 360, 15903, 648).$$

Again with computations analogous to the ones in the file `8_Dimension_alpha_1_c_1.txt`, one can check the invariants of Z and its singularities.

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