

COMPLEX LAGRANGIANS IN A HYPERKÄHLER MANIFOLD AND THE RELATIVE ALBANESE

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ABSTRACT. Let M be the moduli space of complex Lagrangian submanifolds of a hyperKähler manifold X , and let $\varpi : \widehat{\mathcal{A}} \rightarrow M$ be the relative Albanese over M . We prove that $\widehat{\mathcal{A}}$ has a natural holomorphic symplectic structure. The projection ϖ defines a completely integrable structure on the symplectic manifold $\widehat{\mathcal{A}}$. In particular, the fibers of ϖ are complex Lagrangians with respect to the symplectic form on $\widehat{\mathcal{A}}$. We also prove analogous results for the relative Picard over M .

1. INTRODUCTION

A compact Kähler manifold admits a holomorphic symplectic form if and only if it admits a hyperKähler structure [Be], [Ya]. To explain this, let X be a compact manifold equipped with almost complex structures J_1, J_2, J_3 , and let g be a Riemannian metric on X , such that (X, J_1, J_2, J_3, g) is a hyperKähler manifold. Then g defines a C^∞ isomorphism,

$$T^{0,1}X \xrightarrow{g_1} (T^{1,0}X)^*,$$

where $T^{\mathbb{R}}X \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$ is the type decomposition with respect to the almost complex structure J_1 ; also J_2 produces a C^∞ isomorphism

$$T^{1,0}X \xrightarrow{J'_2} T^{0,1}X.$$

The composition of homomorphisms

$$T^{1,0}X \xrightarrow{J'_2} T^{0,1}X \xrightarrow{g_1} (T^{1,0}X)^*,$$

which is a section of $(T^{1,0}X)^* \otimes (T^{1,0}X)^*$, is actually is a holomorphic symplectic form on the compact Kähler manifold (X, J_1, g) . The compact Kähler manifold (X, J_1, g) is Ricci-flat. Conversely, if a compact Kähler manifold admits a holomorphic symplectic form, then its canonical line bundle is holomorphically trivial and hence it admits a Ricci-flat Kähler metric [Ya]. Let (X, J_1, g) be a Ricci-flat compact Kähler manifold equipped with a holomorphic symplectic form. Then we may recover J_2 by reversing the above construction. Finally, we have $J_3 = J_1 \circ J_2$.

Let X be a compact Kähler Ricci-flat manifold admitting a holomorphic symplectic form. Fix a Ricci-flat Kähler form ω on X (such a Kähler form exists [Ya]), and take a holomorphic

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symplectic form Φ on X . It is known that Φ is parallel (meaning, covariant constant) with respect to the Levi–Civita connection on X associated to ω [Be, p. 760, “Principe de Bochner”], [YB, p. 142].

Let M denote the moduli space of compact complex submanifolds of X that are Lagrangian with respect to the symplectic form Φ . Consider the corresponding universal family of Lagrangians

$$\mathcal{Z} \longrightarrow M. \tag{1.1}$$

Let

$$\varpi : \widehat{\mathcal{A}} \longrightarrow M$$

be the relative Albanese over M . So for any Lagrangian $L \in M$, the fiber of $\widehat{\mathcal{A}}$ over L is $\text{Alb}(L) = H^0(L, \Omega_L^1)^*/H_1(L, \mathbb{Z})$. This ϖ is a holomorphic family of compact complex tori over M .

We prove the following (see Theorem 3.3):

The complex manifold $\widehat{\mathcal{A}}$ has a natural holomorphic symplectic form.

The symplectic form on $\widehat{\mathcal{A}}$ is constructed using the canonical Liouville symplectic form on the holomorphic cotangent bundle T^*M of M . The symplectic form Φ on X is implicitly used in the construction of the symplectic form on $\widehat{\mathcal{A}}$. Recall that the Lagrangian submanifolds, and hence M , are defined using Φ .

We prove the following (see Lemma 3.4):

The projection $\varpi : \widehat{\mathcal{A}} \longrightarrow M$ defines a completely integrable structure on the symplectic manifold $\widehat{\mathcal{A}}$. In particular, the fibers of ϖ are Lagrangians with respect to the symplectic form on $\widehat{\mathcal{A}}$.

In Section 4 we consider the relative Picard bundle over M for the family \mathcal{Z} in (1.1). Let

$$\varpi_0 : \mathcal{A} \longrightarrow M$$

be the relative Picard bundle for the family \mathcal{Z} .

We prove that \mathcal{A} is equipped with a natural holomorphic symplectic structure; see Proposition 4.3.

Let $\Theta_{\mathcal{A}}$ denote the above mentioned holomorphic symplectic structure on \mathcal{A} . The following lemma is proved (see Lemma 4.4):

The projection $\varpi_0 : \mathcal{A} \longrightarrow M$ defines a completely integrable structure on \mathcal{A} for the symplectic form $\Theta_{\mathcal{A}}$.

These results are natural generalizations of some known cases of integrable systems, such as the Hitchin system [Hi1] or the Mukai system [Mu] (see also [DEL]). One of our motivations has been mirror symmetry; we hope to come back to the study of \mathcal{A} and $\widehat{\mathcal{A}}$ from the point of view of hyperKähler geometry.

2. COTANGENT BUNDLE OF FAMILY OF LAGRANGIANS

Let X be a compact Kähler Ricci–flat manifold of complex dimension $2d$ equipped with a Kähler form ω . Let Φ be a holomorphic symplectic form on X which is parallel with respect to the Levi–Civita connection on X given by the Kähler metric on X associated to ω .

A complex Lagrangian submanifold of X is a compact complex submanifold $L \subset X$ of complex dimension d such that $\iota^*\Phi = 0$, where

$$\iota : L \hookrightarrow X \tag{2.1}$$

is the inclusion map.

It is known that the infinitesimal deformations of a complex Lagrangian submanifold of X are unobstructed [Mc], [Vo], [Hi2]. Furthermore, the moduli space of complex Lagrangian submanifolds of X is a special Kähler manifold [Hi2, p. 84, Theorem 3].

Let M be the moduli space of complex Lagrangian submanifolds of X . Let

$$L \subset X$$

be a complex Lagrangian submanifold. The point of M representing L will also be denoted by L . Let $N_L \rightarrow L$ be the normal bundle of $L \subset X$; it is a quotient bundle of ι^*TX of rank d , where ι is the map in (2.1). The infinitesimal deformations of the complex submanifold L are parametrized by $H^0(L, N_L)$. Since L is complex Lagrangian, the holomorphic symplectic form Φ on X produces a holomorphic isomorphism

$$N_L \xrightarrow{\sim} (TL)^* = \Omega_L^1,$$

where TL (respectively, Ω_L^1) is the holomorphic tangent (respectively, cotangent) bundle of L . Using this isomorphism we have

$$H^0(L, N_L) = H^0(L, \Omega_L^1).$$

Among the infinitesimal deformations of the complex submanifold L , there are those which arise from deformations within the category of complex Lagrangian submanifolds, meaning those arise from deformations of complex Lagrangian submanifolds as Lagrangian submanifolds. An infinitesimal deformation

$$\alpha \in H^0(L, N_L) = H^0(L, \Omega_L^1)$$

of L lies in this subclass if and only if the holomorphic 1–form α on L is closed [Hi2, pp. 78–79]. But any holomorphic 1–form on L is closed because L is Kähler. Therefore, M is in fact an open subset of the corresponding Douady space for X .

Consider the Ricci–flat Kähler form ω on X . The form

$$\omega_L := \iota^*\omega \tag{2.2}$$

on L is also Kähler, where ι is the map in (2.1). Therefore, the pairing

$$\phi_L : H^0(L, \Omega_L^1) \otimes H^1(L, \mathcal{O}_L) \longrightarrow \mathbb{C}, \quad w \otimes c \longmapsto \int_L w \wedge c \wedge \omega_L^{d-1} \tag{2.3}$$

is nondegenerate. We shall identify $H^1(L, \mathcal{O}_L)$ with $H^0(L, \Omega_L^1)^*$ using this nondegenerate pairing.

It was noted above that

$$T_L M = H^0(L, N_L) = H^0(L, \Omega_L^1). \quad (2.4)$$

Using the pairing in (2.3) and (2.4), we have

$$T_L^* M = H^0(L, \Omega_L^1)^* = H^1(L, \mathcal{O}_L). \quad (2.5)$$

Let

$$\mathcal{Z} \subset X \times M \quad (2.6)$$

be the universal family of complex Lagrangians over M . So \mathcal{Z} is the locus of all $(x, L') \in X \times M$ such that $x \in L' \subset X$. Consider the natural projections $p_X : X \times M \rightarrow X$ and $p_M : X \times M \rightarrow M$. Let

$$p : \mathcal{Z} \rightarrow X \quad \text{and} \quad q : \mathcal{Z} \rightarrow M \quad (2.7)$$

be the restrictions of p_X and p_M respectively to the submanifold $\mathcal{Z} \subset X \times M$. So $p(q^{-1}(L')) \subset X$ for every $L' \in M$ is the Lagrangian L' itself.

Let $\Omega_{\mathcal{Z}/M}^1 \rightarrow \mathcal{Z}$ be the relative cotangent bundle for the projection q to M in (2.7). It fits in the short exact sequence of holomorphic vector bundles

$$0 \rightarrow q^* \Omega_M^1 \rightarrow \Omega_{\mathcal{Z}}^1 \rightarrow \Omega_{\mathcal{Z}/M}^1 \rightarrow 0$$

over M . The direct image $R^1 q_* \mathcal{O}_{\mathcal{Z}}$ fits in the short exact sequence of sheaves on M

$$0 \rightarrow R^0 q_* \Omega_{\mathcal{Z}/M}^1 \rightarrow R^1 q_* \underline{\mathbb{C}} \rightarrow R^1 q_* \mathcal{O}_{\mathcal{Z}} \rightarrow 0, \quad (2.8)$$

where $\underline{\mathbb{C}}$ is the constant sheaf on \mathcal{Z} with stalk \mathbb{C} . The direct image $R^1 q_* \underline{\mathbb{C}}$ is a flat complex vector bundle, equipped with the Gauss–Manin connection. We briefly recall the construction of the Gauss–Manin connection. For any point $L' = y \in M$, let $U_y \subset M$ be a contractible open neighborhood of y . Since U_y is contractible, the inverse image $q^{-1}(U_y)$ is diffeomorphic to $U_y \times L'$ such that the diffeomorphism between $q^{-1}(U_y)$ and $U_y \times L'$ takes q to the natural projection from $U_y \times L'$ to U_y . Using this diffeomorphism, the restriction of $R^1 q_* \underline{\mathbb{C}}$ to U_y coincides with the trivial vector bundle

$$U_y \times H^1(L', \mathbb{C}) \rightarrow U_y \quad (2.9)$$

with fiber $H^1(L', \mathbb{C})$. Using this isomorphism between $(R^1 q_* \underline{\mathbb{C}})|_{U_y}$ and the trivial vector bundle $U_y \times H^1(L', \mathbb{C})$ in (2.9), the trivial connection on the trivial vector bundle in (2.9) produces a flat connection on $(R^1 q_* \underline{\mathbb{C}})|_{U_y}$. This connection on $(R^1 q_* \underline{\mathbb{C}})|_{U_y}$ does not depend on the choice of the diffeomorphism between $q^{-1}(U_y)$ and $U_y \times L'$. Consequently, these locally defined flat connections on $R^1 q_* \underline{\mathbb{C}}$ patch together compatibly to define a flat connection on $R^1 q_* \underline{\mathbb{C}}$. This flat connection is the Gauss–Manin connection mentioned above.

Since the Gauss–Manin connection on $R^1 q_* \underline{\mathbb{C}}$ is flat, and M is a complex manifold, the Gauss–Manin connection produces a natural holomorphic structure on the C^∞ vector bundle $R^1 q_* \underline{\mathbb{C}}$. The direct image $R^0 q_* \Omega_{\mathcal{Z}/M}^1$ is a holomorphic subbundle of $R^1 q_* \underline{\mathbb{C}}$, but it is not preserved by the flat connection in general. The holomorphic structure on the quotient

$R^1q_*\mathcal{O}_{\mathcal{Z}}$ induced by that of $R^1q_*\underline{\mathbb{C}}$ coincides with its own holomorphic structure; the fiber of $R^1q_*\mathcal{O}_{\mathcal{Z}}$ over any $L' \in M$ is $H^1(L', \mathcal{O}_{L'})$.

Since ω_L in (2.2) is the restriction of a global Kähler form on X , the section of $R^2q_*\underline{\mathbb{C}}$

$$M \longrightarrow R^2q_*\underline{\mathbb{C}}, \quad L' \longmapsto [\omega_{L'}] = [\omega|_{L'}] \in H^2(L', \mathbb{C})$$

is covariant constant with respect to the Gauss–Manin connection on $R^2q_*\underline{\mathbb{C}}$. Consequently, the homomorphism

$$(R^1q_*\underline{\mathbb{C}}) \otimes (R^1q_*\underline{\mathbb{C}}) \longrightarrow \underline{\mathbb{C}}, \quad (2.10)$$

that sends any $v \otimes w \in (R^1q_*\underline{\mathbb{C}})_t \otimes (R^1q_*\underline{\mathbb{C}})_t$, $t \in M$, to

$$\int_{q^{-1}(t)} v \wedge w \wedge (\omega|_{q^{-1}(t)})^{d-1} \in \mathbb{C}$$

is also covariant constant with respect to the connection on $(R^1q_*\underline{\mathbb{C}}) \otimes (R^1q_*\underline{\mathbb{C}})$ induced by the Gauss–Manin connection on $R^1q_*\underline{\mathbb{C}}$. Hence the pairing in (2.10) produces a holomorphic isomorphism of vector bundles

$$(q_*\Omega_{\mathcal{Z}/M}^1)^* \xrightarrow{\sim} R^1q_*\mathcal{O}_{\mathcal{Z}} \quad (2.11)$$

on M . The restriction of this isomorphism to any point $L \in M$ coincides with the isomorphism $H^0(L, \Omega_L^1)^* = H^1(L, \mathcal{O}_L)$ in (2.5).

On the other hand, the pointwise isomorphisms in (2.4) combine together to produce a holomorphic isomorphism of vector bundles

$$\Omega_M^1 \xrightarrow{\sim} (q_*\Omega_{\mathcal{Z}/M}^1)^* \quad (2.12)$$

on M . Composing the isomorphisms in (2.11) and (2.12), we obtain a holomorphic isomorphism of vector bundles

$$\chi : \Omega_M^1 \longrightarrow R^1q_*\mathcal{O}_{\mathcal{Z}} \quad (2.13)$$

over M .

For notational convenience, the total space of $R^1q_*\mathcal{O}_{\mathcal{Z}}$ will be denoted by \mathcal{Y} . Let

$$\gamma : \mathcal{Y} \longrightarrow M \quad (2.14)$$

be the natural projection. Consider the canonical Liouville 1-form on Ω_M^1 . Using the isomorphism χ in (2.13) this Liouville 1-form on Ω_M^1 gives a holomorphic 1-form on \mathcal{Y} . Let

$$\theta \in H^0(\mathcal{Y}, \Omega_{\mathcal{Y}}^1) \quad (2.15)$$

be the holomorphic 1-form given by the Liouville 1-form on Ω_M^1 . Note that for any $L \in M$, and any $v \in \gamma^{-1}(L)$, the form

$$\theta(v) : T_v\mathcal{Y} \longrightarrow \mathbb{C}$$

coincides with the composition of homomorphisms

$$T_v\mathcal{Y} \xrightarrow{d\gamma} T_LM = H^0(L, \Omega_L^1) \xrightarrow{\phi_L(-, v)} \mathbb{C},$$

where ϕ_L is the bilinear pairing constructed in (2.3), and $d\gamma$ is the differential of the projection γ in (2.14); the above identification

$$T_LM = H^0(L, \Omega_L^1)$$

is the one constructed in (2.4).

It is straight-forward to check that the 2-form

$$d\theta \in H^0(\mathcal{Y}, \Omega_{\mathcal{Y}}^2) \quad (2.16)$$

is a holomorphic symplectic form on the manifold \mathcal{Y} in (2.14). Indeed, $d\theta$ evidently coincides with the 2-form on \mathcal{Y} given by the Liouville symplectic form on Ω_M^1 via the isomorphism χ in (2.13).

3. THE FAMILY OF ALBANESE TORI

Take a compact complex Lagrangian submanifold $L \subset X$ represented by a point of M . We know that $T_L^*M \cong H^0(L, \Omega_L^1)^*$ (see (2.4)). Note that the non-degenerate pairing

$$H^{1,0}(L) \otimes H^{d-1,d}(L) \longrightarrow \mathbb{C}, \quad \alpha \otimes \beta \longmapsto \int_L \alpha \wedge \beta,$$

which is also the Serre duality pairing, yields an isomorphism

$$H^0(L, \Omega_L^1)^* \cong H^d(L, \Omega_L^{d-1}).$$

For a fixed L , we have the Hodge decomposition

$$H^{2d-1}(L, \mathbb{C}) = H^{d,d-1}(L) \oplus H^{d-1,d}(L);$$

but if we move L in the family M , meaning if we consider the universal family

$$q : \mathcal{Z} \longrightarrow M$$

in (2.6), then only $R^{d-1}q_*\Omega_{\mathcal{Z}/M}^d$ is a holomorphic subbundle of $R^{2d-1}q_*\underline{\mathbb{C}}$, and we have the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow R^{d-1}q_*\Omega_{\mathcal{Z}/M}^d \longrightarrow R^{2d-1}q_*\underline{\mathbb{C}} \longrightarrow R^dq_*\Omega_{\mathcal{Z}/M}^{d-1} \longrightarrow 0, \quad (3.1)$$

on M .

The holomorphic vector bundle $R^{2d-1}q_*\underline{\mathbb{C}}$ is equipped with the Gauss–Manin connection, which is an integrable connection. The quotient $R^dq_*\Omega_{\mathcal{Z}/M}^{d-1}$, in (3.1), of $R^{2d-1}q_*\underline{\mathbb{C}}$ is a holomorphic vector bundle on M with fiber $H^d(L', \Omega_{L'}^{d-1})$ over any $L' \in M$.

The homomorphism

$$(R^1q_*\underline{\mathbb{C}}) \otimes (R^{2d-1}q_*\underline{\mathbb{C}}) \longrightarrow \underline{\mathbb{C}}, \quad (3.2)$$

that sends any $\alpha \otimes \beta \in (R^1q_*\underline{\mathbb{C}})_t \otimes (R^{2d-1}q_*\underline{\mathbb{C}})_t$, $t \in M$, to

$$\int_{q^{-1}(t)} v \wedge w \in \mathbb{C}$$

is covariant constant with respect to the connection on $(R^1q_*\underline{\mathbb{C}}) \otimes (R^{2d-1}q_*\underline{\mathbb{C}})$ induced by the Gauss–Manin connections on $R^1q_*\underline{\mathbb{C}}$ and $R^{2d-1}q_*\underline{\mathbb{C}}$. Consequently, the pairing in (3.2) yields a holomorphic isomorphism of vector bundles

$$(q_*\Omega_{\mathcal{Z}/M}^1)^* \xrightarrow{\sim} R^dq_*\Omega_{\mathcal{Z}/M}^{d-1}$$

over M . Combining this isomorphism with the isomorphism in (2.12) we get a holomorphic isomorphism of vector bundles

$$\widehat{\chi} : \Omega_M^1 \xrightarrow{\sim} R^d q_* \Omega_{Z/M}^{d-1} \quad (3.3)$$

over M .

We shall denote the total space of the holomorphic vector bundle $R^d q_* \Omega_{Z/M}^{d-1}$ by \mathcal{W} , so

$$\mathcal{W} := R^d q_* \Omega_{Z/M}^{d-1} \xrightarrow{\widehat{\chi}} M \quad (3.4)$$

is a holomorphic fiber bundle.

As in Section 2, consider the canonical Liouville holomorphic 1-form on the total space of Ω_M^1 . Using the isomorphism in (3.3), this Liouville 1-form on the total space of Ω_M^1 produces a holomorphic 1-form on \mathcal{W} in (3.4). Let

$$\theta' \in H^0(\mathcal{W}, \Omega_{\mathcal{W}}^1)$$

be this holomorphic 1-form on \mathcal{W} . We note that

$$d\theta' \in H^0(\mathcal{W}, \Omega_{\mathcal{W}}^2) \quad (3.5)$$

is a holomorphic symplectic form on \mathcal{W} . Indeed, the isomorphism in (3.3) takes $d\theta'$ to the Liouville symplectic form on the total space of Ω_M^1 .

Remark 3.1. Note that while the Kähler form ω on X was used in the construction of the symplectic form $d\theta$ on \mathcal{Y} in (2.16) (see the pairing in (2.10)), the construction of $d\theta'$ in (3.5) does not use the Kähler form ω on X . We recall that the isomorphism in (2.12) is constructed from the pointwise isomorphisms in (2.4). Note that the isomorphism in (2.4) does not depend on the Kähler form ω .

The Albanese $\text{Alb}(Y)$ of a compact Kähler manifold Y is defined to be

$$\text{Alb}(Y) = H^0(Y, \Omega_Y^1)^* / H_1(Y, \mathbb{Z}) = H^n(Y, \Omega_Y^{n-1}) / H_1(Y, \mathbb{Z}),$$

where $n = \dim_{\mathbb{C}} Y$ (see [GH, p. 331]). It is a compact complex torus.

For each point $L \in M$, consider the composition of homomorphisms

$$H^{2d-1}(L, \mathbb{Z}) \longrightarrow H^{2d-1}(L, \mathbb{C}) \longrightarrow H^{2d-1}(L, \mathbb{C}) / H^{d-1}(L, \Omega_L^d) = H^d(L, \Omega_L^{d-1})$$

(see (3.1)). It produces a homomorphism

$$R^{2d-1} q_* \underline{\mathbb{Z}} \longrightarrow \mathcal{W}, \quad (3.6)$$

where \mathcal{W} is defined in (3.4).

Remark 3.2. The Gauss–Manin connection on $R^{2d-1} q_* \underline{\mathbb{C}} \longrightarrow M$ evidently preserves the subbundle of lattices

$$R^{2d-1} q_* \underline{\mathbb{Z}} \subset R^{2d-1} q_* \underline{\mathbb{C}}.$$

From this it follows immediately that the C^∞ submanifold $R^{2d-1} q_* \underline{\mathbb{Z}} \subset \mathcal{W}$ in (3.6) is in fact a complex submanifold.

The quotient

$$\widehat{\mathcal{A}} := \mathcal{W}/(R^{2d-1}q_*\underline{\mathbb{Z}}) \longrightarrow M \quad (3.7)$$

for the homomorphism in (3.6) is in fact a holomorphic family of compact complex tori over M . Note that the fiber of $\widehat{\mathcal{A}}$ over each $L \in M$ is the Albanese torus

$$\text{Alb}(L) = H^d(L, \Omega_L^{d-1})/H^{2d-1}(L, \mathbb{Z}).$$

For any complex Lagrangian $L \in M$, using Serre duality,

$$H^d(L, \Omega_L^{d-1}) = H^0(L, \Omega_L^1)^*,$$

and the underlying real vector space for $H^0(L, \Omega_L^1)^*$ is identified with

$$H^1(L, \mathbb{R})^* = H_1(L, \mathbb{R}).$$

Using Poincaré duality for L , we have $H^{2d-1}(L, \mathbb{Z})/\text{Torsion} = H_1(L, \mathbb{Z})/\text{Torsion}$. Consequently, $\widehat{\mathcal{A}}$ in (3.7) admits the following isomorphism:

$$\widehat{\mathcal{A}} = \mathcal{W}/(R^{2d-1}q_*\underline{\mathbb{Z}}) = \mathcal{W}/\widetilde{H}_1(\mathbb{Z}) = (q_*\Omega_{\mathbb{Z}/M}^1)^*/\widetilde{H}_1(\mathbb{Z}), \quad (3.8)$$

where $\widetilde{H}_1(\mathbb{Z})$ is the local system on M whose stalk over any $L \in M$ is $H_1(L, \mathbb{Z})/\text{Torsion}$. Also we have the isomorphism of real tori

$$\widehat{\mathcal{A}} = (R^1q_*\underline{\mathbb{R}})^*/\widetilde{H}_1(\mathbb{Z}) = \widetilde{H}_1(\mathbb{R})/\widetilde{H}_1(\mathbb{Z}), \quad (3.9)$$

where $\widetilde{H}_1(\mathbb{R})$ is the local system on M whose stalk over any $L \in M$ is $H_1(L, \mathbb{R})$.

Theorem 3.3. *The 2-form $d\theta'$ in (3.5) on \mathcal{W} descends to the quotient torus $\widehat{\mathcal{A}}$ in (3.7).*

Proof. Take a point $L_0 \in M$. In [Hi2] Hitchin constructed a C^∞ coordinate function on M defined around the point $L_0 \in M$ that takes values in $H^1(L_0, \mathbb{R})$ [Hi2, p. 79, Theorem 2]; we will briefly recall this construction.

Take any

$$c \in H_1(L_0, \mathbb{Z})/\text{Torsion}. \quad (3.10)$$

Let $U \subset M$ be a contractible neighborhood of the point L_0 . Choose a S^1 -subbundle of the fiber bundle \mathcal{Z} (see (2.6)) over U

$$B \xrightarrow{\iota_U} \mathcal{Z}|_U \xrightarrow{q} U \quad (3.11)$$

such that the fiber of the S^1 -bundle B over L_0 represents the homology class c in (3.10); recall that the fiber of \mathcal{Z} over the point $L_0 \in M$ is the Lagrangian L_0 itself.

Consider the symplectic form Φ on X . Integrating $\iota_U^*p^*\text{Re}(\Phi)$ along the fibers of B , where ι_U and p are the maps in (3.11) and (2.7) respectively, we get a closed 1-form ξ_c on U . Let f_c be the unique function on U such that $f_c(L_0) = 0$ and $df_c = \xi_c$. Now, let

$$\mu : U \longrightarrow H^1(L_0, \mathbb{R})$$

be the function uniquely determined by the condition that $\phi_{L_x}(c \otimes \mu(x)) = f_c(x)$ for all $x \in U$ and $c \in H_1(L_0, \mathbb{Z})/\text{Torsion}$, where ϕ_{L_x} is the pairing constructed as in (2.3) for the complex Lagrangian $L_x = q^{-1}(x) \subset X$, where q is the projection in (2.7). This μ is a local diffeomorphism [Hi2, p. 79, Theorem 2].

Using the Kähler form $\omega_{L_0} := \omega|_{L_0}$ on L_0 (see (2.2)), we identify $H^1(L_0, \mathbb{R})$ with $H_1(L_0, \mathbb{R})$ as follows. Since the pairing

$$H^1(L_0, \mathbb{R}) \otimes H^1(L_0, \mathbb{R}) \longrightarrow \mathbb{R}, \quad v \otimes w \longmapsto \int_L v \wedge w \wedge \omega_{L_0}^{d-1}$$

is nondegenerate, it produces an isomorphism

$$H^1(L_0, \mathbb{R}) \xrightarrow{\sim} H^1(L_0, \mathbb{R})^* = H_1(L_0, \mathbb{R}). \quad (3.12)$$

On the other hand, there is the natural homomorphism $H^1(L_0, \mathbb{Z}) \longrightarrow H^1(L_0, \mathbb{R})$. Let

$$\Gamma \subset H_1(L_0, \mathbb{R}) \quad (3.13)$$

be the subgroup that corresponds to $H^1(L_0, \mathbb{Z})$ by the isomorphism in (3.12).

Using the above coordinate function μ on U , we have

$$\mathcal{T}^*U \xrightarrow{\sim} \mathcal{T}^*\mu(U) = \mu(U) \times H^1(L_0, \mathbb{R})^* = \mu(U) \times H_1(L_0, \mathbb{R}),$$

where \mathcal{T}^* denotes the real cotangent bundle.

The Liouville symplectic form on $\mathcal{T}^*\mu(U)$ is clearly the constant 2-form on

$$H^1(L_0, \mathbb{R}) \times H_1(L_0, \mathbb{R})$$

given by the natural isomorphism of $H_1(L_0, \mathbb{R})$ with $H^1(L_0, \mathbb{R})^*$. From this it follows immediately that

$$\mu(U) \times \Gamma \subset \mu(U) \times H_1(L_0, \mathbb{R})$$

is a Lagrangian submanifold with respect to the Liouville symplectic form on $\mathcal{T}^*\mu(U)$, where Γ defined in (3.13).

Since $\mu(U) \times \Gamma \subset \mu(U) \times H_1(L_0, \mathbb{R})$ is a Lagrangian submanifold, it follows that for $\mathcal{W} = (R^1q_*\mathbb{R})^*$ (see (3.9) and (3.4)), the image of the natural map

$$\tilde{H}_1(\mathbb{Z}) \longrightarrow (R^1q_*\mathbb{R})^* = \mathcal{W}$$

(see (3.8)) is Lagrangian with respect to the real symplectic form $\text{Re}(d\theta')$ on \mathcal{W} , where $d\theta'$ is constructed in (3.5).

It was noted in Remark 3.2 that $R^{2d-1}q_*\mathbb{Z}$ is a complex submanifold of \mathcal{W} . Consequently, the 2-form on $R^{2d-1}q_*\mathbb{Z}$ obtained by restricting the holomorphic 2-form $d\theta'$ on \mathcal{W} is also holomorphic. Since the real part of the holomorphic 2-form on $R^{2d-1}q_*\mathbb{Z}$ given by $d\theta'$ vanishes identically, we conclude that the holomorphic 2-form on $R^{2d-1}q_*\mathbb{Z}$, given by $d\theta'$, itself vanishes identically. Therefore, $R^{2d-1}q_*\mathbb{Z}$ is a Lagrangian submanifold of the holomorphic symplectic manifold \mathcal{W} equipped with the holomorphic symplectic form $d\theta'$.

To complete the proof we recall a general property of the Liouville symplectic form.

Let N be a manifold and α a 1-form on N . Let

$$t : T^*N \longrightarrow T^*N$$

be the diffeomorphism that sends any $v \in T_n^*N$ to $v + \alpha(n)$. If ψ is the Liouville symplectic form on T^*N , the

$$t^*\psi = \psi + d\alpha.$$

In particular, the map t preserves ψ if and only if the form α is closed. Also, the image of t is Lagrangian submanifold of T^*N for ψ if and only α is closed.

Since

$$R^{2d-1}q_*\underline{\mathbb{Z}}$$

is a Lagrangian submanifold of the symplectic manifold $(\mathcal{W}, d\theta')$, from the above property of the Liouville symplectic form it follows immediately that the 2-form $d\theta'$ on \mathcal{W} descends to the quotient space $\widehat{\mathcal{A}}$ in (3.7). \square

Let

$$q_{\widehat{\mathcal{A}}} : \mathcal{W} \longrightarrow \widehat{\mathcal{A}} := \mathcal{W}/(R^{2d-1}q_*\underline{\mathbb{Z}}) \quad (3.14)$$

be the quotient map (see (3.7)). From Theorem 3.3 we know that there is a unique 2-form

$$\Theta_{\widehat{\mathcal{A}}} \in H^0(\widehat{\mathcal{A}}, \Omega_{\widehat{\mathcal{A}}}^2) \quad (3.15)$$

such that

$$q_{\widehat{\mathcal{A}}}^*\Theta_{\widehat{\mathcal{A}}} = d\theta', \quad (3.16)$$

where $q_{\mathcal{A}}$ is the map in (3.14). Since $d\theta'$ is a holomorphic symplectic form, it follows immediately that $\Theta_{\widehat{\mathcal{A}}}$ is a holomorphic symplectic form on $\widehat{\mathcal{A}}$.

The projection $\widehat{\gamma} : \mathcal{W} \longrightarrow M$ in (3.4) clearly descends to a map from \mathcal{A} to M . Let

$$\varpi : \widehat{\mathcal{A}} \longrightarrow M \quad (3.17)$$

be the map given by $\widehat{\gamma}$; so we have

$$\widehat{\gamma} = \varpi \circ q_{\widehat{\mathcal{A}}},$$

where $q_{\widehat{\mathcal{A}}}$ is constructed in (3.14).

Lemma 3.4. *The projection ϖ in (3.17) defines a completely integrable structure on $\widehat{\mathcal{A}}$ for the symplectic form $\Theta_{\widehat{\mathcal{A}}}$ constructed in (3.15). In particular, the fibers of ϖ are Lagrangians with respect to the symplectic form $\Theta_{\widehat{\mathcal{A}}}$.*

Proof. Recall that \mathcal{W} is holomorphically identified with Ω_M^1 by the map $\widehat{\chi}$ in (3.3) (see (3.4)). This map $\widehat{\chi}$ takes the Liouville symplectic form on Ω_M^1 to the symplectic form $d\theta'$ on \mathcal{W} . Therefore, from (3.14) and (3.16) we conclude that $\widehat{\mathcal{A}}$ is locally isomorphic to Ω_M^1 such that the projection ϖ is taken to the natural projection $\Omega_M^1 \longrightarrow M$, and the symplectic form $\Theta_{\widehat{\mathcal{A}}}$ on $\widehat{\mathcal{A}}$ is taken to the Liouville symplectic form on Ω_M^1 . The lemma follows immediately from these, because the natural projection $\Omega_M^1 \longrightarrow M$ defines a completely integrable structure on Ω_M^1 for the Liouville symplectic form. \square

4. THE RELATIVE PICARD GROUP

For any $L \in M$, consider the homomorphisms

$$H^1(L, \mathbb{Z}) \longrightarrow H^1(L, \mathbb{C}) \longrightarrow H^1(L, \mathcal{O}_L), \quad (4.1)$$

where $H^1(L, \mathbb{Z}) \rightarrow H^1(L, \mathbb{C})$ is the natural homomorphism given by the inclusion of \mathbb{Z} in \mathbb{C} , and the projection $H^1(L, \mathbb{C}) \rightarrow H^1(L, \mathcal{O}_L)$ corresponds to the isomorphism

$$H^1(L, \mathbb{C})/H^0(L, \Omega_L^1) = H^1(L, \mathcal{O}_L)$$

(see (2.8)). The image of the composition of homomorphisms in (4.1) is actually a cocompact lattice in $H^1(L, \mathcal{O}_L)$; so $H^1(L, \mathcal{O}_L)/H^1(L, \mathbb{Z})$ is a compact complex torus. We note that the composition of homomorphisms in (4.1) is in fact injective. When the compact complex manifold L is a complex projective variety, then $H^1(L, \mathcal{O}_L)/H^1(L, \mathbb{Z})$ is in fact an abelian variety.

As L moves over the family M , these cocompact lattices fit together to produce a C^∞ submanifold of the complex manifold \mathcal{Y} in (2.14).

The Gauss–Manin connection on $R^1q_*\underline{\mathbb{C}} \rightarrow M$ evidently preserves the above bundle of cocompact lattices $R^1q_*\underline{\mathbb{Z}} \subset R^1q_*\underline{\mathbb{C}}$. From this it follows immediately that the above C^∞ submanifold $R^1q_*\underline{\mathbb{Z}} \subset \mathcal{Y}$ is in fact a complex submanifold.

Taking fiber-wise quotients, we conclude that

$$\mathcal{A} := \mathcal{Y}/(R^1q_*\underline{\mathbb{Z}}) \rightarrow M \tag{4.2}$$

is a holomorphic family of compact complex tori over M .

Remark 4.1. Let Y be a compact Kähler manifold. Consider the short exact sequence of sheaves on Y given by the exponential map

$$0 \rightarrow \underline{\mathbb{Z}} \rightarrow \mathcal{O}_Y \xrightarrow{\lambda \mapsto \exp(2\pi\sqrt{-1}\lambda)} \mathcal{O}_Y^* \rightarrow 0,$$

where \mathcal{O}_Y^* is a multiplicative sheaf of holomorphic functions with values in $\mathbb{C} \setminus \{0\}$. For the corresponding long exact sequence of cohomologies

$$H^1(Y, \underline{\mathbb{Z}}) \rightarrow H^1(Y, \mathcal{O}_Y) \rightarrow H^1(Y, \mathcal{O}_Y^*) \xrightarrow{c_1} H^2(Y, \underline{\mathbb{Z}}),$$

where the connecting homomorphism c_1 sends any holomorphic line bundle $\xi \in H^1(Y, \mathcal{O}_Y^*)$ to $c_1(\xi)$, the quotient

$$H^1(Y, \mathcal{O}_Y)/H^1(Y, \underline{\mathbb{Z}})$$

gets identified with the Picard group $\text{Pic}^0(Y)$ that parametrizes the topologically trivial holomorphic line bundles on Y .

Consequently, the quotient \mathcal{A} in (4.2) is naturally identified with the moduli space $\text{Pic}_{\mathcal{Z}/M}^0$ of topologically trivial holomorphic line bundles on the fibers of $q : \mathcal{Z} \rightarrow M$. In other words, \mathcal{A} parametrizes all pairs of the form (L, ξ) , where $L \in M$, and ξ is a topologically trivial holomorphic line bundle on L .

Recall that we have the holomorphic symplectic form $d\theta$ on \mathcal{Y} , where θ is constructed in (2.15).

Remark 4.2. Recall from Remark 3.1 that $d\theta$ does depend on the Kähler form ω on X .

Proposition 4.3. *The 2-form $d\theta$ on \mathcal{Y} descends to the quotient space \mathcal{A} in (4.2), or in other words, $d\theta$ is the pullback of a 2-form on \mathcal{A} .*

Proof. The proof of the proposition is very similar to the proof of Theorem 3.3. As before, the local coordinate functions on M constructed in [Hi2] play a crucial role. We omit the details of the proof. \square

Let

$$q_{\mathcal{A}} : \mathcal{Y} \longrightarrow \mathcal{A} := \mathcal{Y}/(R^1q_*\mathbb{Z}) \quad (4.3)$$

be the quotient map (see (4.2)). Let $\Theta_{\mathcal{A}} \in H^0(\mathcal{A}, \Omega_{\mathcal{A}}^2)$ be the holomorphic symplectic form given by Proposition 4.3, so

$$q_{\mathcal{A}}^*\Theta_{\mathcal{A}} = d\theta, \quad (4.4)$$

where $q_{\mathcal{A}}$ is the map in (4.3).

The projection γ in (2.14) clearly descends to a map

$$\varpi_0 : \mathcal{A} \longrightarrow M. \quad (4.5)$$

Note that the isomorphism between \mathcal{A} and $\text{Pic}_{\mathbb{Z}/M}^0$ in Remark 4.1 takes ϖ_0 to the forgetful map

$$\text{Pic}_{\mathbb{Z}/M}^0 \longrightarrow M$$

that forgets the line bundle, or in other words, it sends any $(L, \xi) \in \text{Pic}_{\mathbb{Z}/M}^0$ to the complex Lagrangian L forgetting the line bundle ξ .

Lemma 4.4. *The projection ϖ_0 in (4.5) defines a completely integrable structure on \mathcal{A} for the symplectic form $\Theta_{\mathcal{A}}$.*

Proof. Recall that \mathcal{Y} is holomorphically identified with Ω_M^1 by the map χ in (2.13), and χ takes the Liouville symplectic form on Ω_M^1 to the symplectic form $d\theta$ on \mathcal{Y} . Therefore, from (3.14) and (4.4) we conclude that \mathcal{A} is locally isomorphic to Ω_M^1 such that the projection ϖ_0 is taken to the natural projection $\Omega_M^1 \longrightarrow M$, and the symplectic form $\Theta_{\mathcal{A}}$ on \mathcal{A} is taken to the Liouville symplectic form on Ω_M^1 . The lemma follows immediately from these. \square

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