Actions of the additive group G_a on Ore extensions

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Abstract

We connect the theorems of Rentschler [18] and Dixmier [11] on locally nilpotent derivations and automorphisms of the polynomial ring A_0 and of the Weyl algebra A_1, both over a field of characteristic zero, by establishing the same type of results for the family of algebras $\mathsf{A}_h = \langle x, y \mid yx - xy = h(x) \rangle$, where h is an arbitrary polynomial in x. On the second part of the paper we consider a field $\mathbb{F}$ of prime characteristic and study $\mathbb{F}[t]$-comodule algebra structures on A_h. We also compute the Makar-Limanov invariant of absolute constants of A_h over a field of arbitrary characteristic and show how this subalgebra determines the automorphism group of A_h.

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1 Introduction

The purpose of this note is to connect two groundbreaking papers which appeared in 1968: in [18], Rentschler classified the actions of the additive group G_a on the 2-dimensional plane and in [11] Dixmier determined the automorphism group of the Weyl algebra $\mathsf{A}_1 = \langle x, y \mid yx - xy = 1 \rangle$, the algebra of differential operators with polynomial coefficients in one variable, both over a field of characteristic 0. What they have in common is the use of locally nilpotent derivations as a fundamental tool to obtain their respective main results, each related to a corresponding automorphism group. Indeed, a consequence of Rentschler's Theorem is a description of the automorphism group of the polynomial ring in two variables $\mathsf{A}_0 = \langle x, y \mid yx = xy \rangle$.

Although polynomial rings and Weyl algebras can seem to be on opposite sides of the spectrum when it comes to certain algebraic properties (e.g., one is commutative, has plenty

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of prime ideals and can be made into a Hopf algebra in a natural way, while the other is noncommutative and simple, with no Hopf structure), it should not be surprising that they are quite strongly related. A striking connection is the fact that the Jacobian conjecture is equivalent to the (weak) Dixmier conjecture (see [21], [3] and [22] for definitions and details).

One way of explicitly connecting A_0 and A_1 is through a family of algebras A_h, parametrized by a polynomial $h(x)$ in x, which was introduced and studied in [4], [6] and [5]. The algebra A_h can be defined as the algebra with generators x, y satisfying the commutation relation $yx - xy = h(x)$. When $h = 0, 1$ we retrieve the polynomial algebra A_0 and the Weyl algebra A_1, respectively. Other choices of h give algebras like the enveloping algebra of the two-dimensional non-abelian Lie algebra, as A_x, the Jordan plane, as A_{x^2}, and many others. In characteristic 0, one can think of all of these algebras as deformations of the coordinate ring of the 2-dimensional plane, the polynomial ring A_0. This can be made explicit by means of the so-called *Groenewold-Moyal product.* Consider the derivations $\phi = \frac{d}{dy}$ and $\psi = h(x)\frac{d}{dx}$ of A_0. Then the infinitesimal $\phi \wedge \psi$ defines an associative star product on $\mathsf{A}_0[[\hbar]]$, with

$$a \star b = \sum_{n \geq 0} \frac{\phi^n(a)\psi^n(b)}{n!}\hbar^n.$$

It is easy to verify that

$$x \star x = x^2, \qquad y \star y = y^2, \qquad y \star x = yx + h(x)\hbar, \qquad x \star y = xy,$$

so $y \star x - x \star y = h(x)\hbar$. Since ϕ is locally nilpotent, we can specialize at $\hbar = 1$, hence retrieving A_h as a deformation of the commutative polynomial algebra A_0.

We show in Section 2 that, over a field of characteristic 0, the descriptions given in Dixmier and Rentschler's aforementioned papers still hold in general for A_h, for any h, although in a more rigid form, in case h is not a constant polynomial. After describing explicitly the locally nilpotent derivations of A_h, we determine the so-called Makar-Limanov invariant of absolute constants, $\mathsf{ML}(\mathsf{A}_h)$ and use it to give an alternative proof of [1, Prop. 3.6], that the automorphism group of A_h is tame (generated by affine and triangular automorphisms). See [16] for the corresponding results for the free Poisson algebra, [14] for the free generic Poisson algebra and the recent papers [12] and [9] for related results on the free algebra (all cases mentioned are in rank two).

In Section 3 we consider the case of fields of positive characteristic p. In this case, locally nilpotent derivations lose some of their properties and they do not capture enough information, as often (although not always) the p-th power of a locally nilpotent derivation will be trivial. The natural analogue in prime characteristic comes from the notion of an action of the additive group of the field, G_a. In algebraic terms, this corresponds to a comodule algebra structure or, equivalently, to a locally nilpotent iterative higher derivation. This point of view fits in naturally with viewing A_h as a deformation of the polynomial ring A_0, allowing for a generalization of the geometric notion of an action on a space, which in this case could be thought of as a noncommutative space. See [19] and [20] for results in this direction in the case of the Weyl algebra A_1. Thus, we define the prime characteristic analogue of the Makar-Limanov invariant, as in [8], and compute it for A_h for any non-constant polynomial h. This again gives sufficient information for computing $\mathsf{Aut}(\mathsf{A}_h)$ over a field of prime characteristic.

2 The locally nilpotent derivations of A_h in characteristic 0

Throughout this section, $\mathbb{F}$ denotes an arbitrary field of characteristic 0. For a unital $\mathbb{F}$-algebra A, we denote by $\mathsf{LND}(\mathsf{A})$ the set of all locally nilpotent derivations of A. In detail, $\mathsf{LND}(\mathsf{A})$ is the set of all linear maps $\partial : \mathsf{A} \longrightarrow \mathsf{A}$ satisfying the Leibniz identity $\partial(ab) = a\partial(b) + \partial(a)b$ and such that the set $\mathsf{N}(\partial, a) = \{n \geq 0 \mid \partial^n(a) \neq 0\}$ is finite, for all $a, b \in \mathsf{A}$. We set $\mathsf{A}^\partial = \mathsf{ker}\partial$, a subalgebra of A. It is well known that every $\partial \in \mathsf{LND}(\mathsf{A})$ induces a degree function on A by setting:

$$\mathsf{deg}_\partial(0) = -\infty, \qquad \mathsf{deg}_\partial(a) = \max \mathsf{N}(\partial, a), \quad \text{for } 0 \neq a \in \mathsf{A}. \tag{2.1}$$

This degree function has especially nice properties in case A is a domain and $\mathsf{char}(\mathbb{F}) = 0$.

Proposition 2.2 ([15]). *Assume that A is a domain and $\mathbb{F}$ is a field of characteristic zero. For any $\partial \in \mathsf{LND}(\mathsf{A})$ and $a, b \in \mathsf{A}$, we have:*

(a) $\mathsf{deg}_\partial(ab) = \mathsf{deg}_\partial(a) + \mathsf{deg}_\partial(b)$;

(b) $\mathsf{deg}_\partial(a+b) \leq \max\{\mathsf{deg}_\partial(a), \mathsf{deg}_\partial(b)\}$, *with equality if* $\mathsf{deg}_\partial(a) \neq \mathsf{deg}_\partial(b)$;

(c) $\mathsf{deg}_\partial(\partial(a)) = \mathsf{deg}_\partial(a) - 1$ *if* $\mathsf{deg}_\partial(a) \neq 0$.

It follows from (a) *above that A^∂ is factorially closed: if $a, b \in \mathsf{A} \setminus \{0\}$ and $ab \in \mathsf{A}^\partial$, then $a, b \in \mathsf{A}^\partial$.*

Remark 2.3. The hypotheses on A and $\mathbb{F}$ in Proposition 2.2 are needed only for part (a); the remaining parts hold in general.

There is a strong connection between locally nilpotent derivations and algebra automorphisms of A. Given $\partial \in \mathsf{LND}(\mathsf{A})$, there is a well-defined map $\exp(\partial) : \mathsf{A} \longrightarrow \mathsf{A}$ with $\exp(\partial)(a) = \sum_{k\geq 0} \frac{\partial^k(a)}{k!}$ and it is easy to see that $\exp(\partial)$ is an algebra automorphism of A. Although the set $\mathsf{LND}(\mathsf{A})$ is not in general closed under sums or commutators, the automorphism group $\mathsf{Aut}(\mathsf{A})$ acts on $\mathsf{LND}(\mathsf{A})$ by conjugation, and it follows that $\{\exp(\partial) \mid \partial \in \mathsf{LND}(\mathsf{A})\}$ generates a normal subgroup of $\mathsf{Aut}(\mathsf{A})$.

Another connection with automorphisms of A is via the so-called Makar-Limanov invariant of absolute constants, $\mathsf{ML}(\mathsf{A})$, introduced in [15]. By definition,

$$\mathsf{ML}(\mathsf{A}) = \bigcap_{\partial \in \mathsf{LND}(\mathsf{A})} \mathsf{A}^\partial \tag{2.4}$$

and clearly the subalgebra $\mathsf{ML}(\mathsf{A})$ is invariant under automorphisms of A.

Example 2.5. For $\alpha \in \mathbb{F}$, let A_α be the unital associative $\mathbb{F}$-algebra generated by elements x, y, subject to the relation $[y, x] = \alpha$, where $[a, b] = ab - ba$ is the commutator.

(a) If $\alpha = 0$, then $\mathsf{A}_0 = \mathbb{F}[x, y]$ is the usual commutative polynomial algebra of rank 2. Then the partial derivatives $\partial_x = \frac{d}{dx}$ and $\partial_y = \frac{d}{dy}$ are locally nilpotent and it is easy to see that $\mathsf{A}_0^{\partial_x} \cap \mathsf{A}_0^{\partial_y} = \mathbb{F}$. Hence, $\mathsf{ML}(\mathsf{A}_0) = \mathbb{F}$.

(b) If $\alpha \neq 0$, then A_α is isomorphic to A_1, the first Weyl algebra (the algebra of differential operators on $\mathbb{F}[x]$ with polynomial coefficients), with defining relation $yx - xy = 1$. It is well known that all derivations of A_1 are inner (see e.g. [10, 4.6.8]) and thus of the form ad_a, for some $a \in \mathsf{A}_1$, where $\mathsf{ad}_a(b) = [a, b]$. Let $\partial_x = \mathsf{ad}_x$ and $\partial_y = \mathsf{ad}_y$. It is easy to see that $\partial_x, \partial_y \in \mathsf{LND}(\mathsf{A}_1)$ and $\mathsf{A}_1^{\partial_x} \cap \mathsf{A}_1^{\partial_y} = \mathbb{F}$. Hence, $\mathsf{ML}(\mathsf{A}_1) = \mathbb{F}$.

Although it was easy to compute $\mathsf{ML}(\mathsf{A}_\alpha)$ without explicitly determining $\mathsf{LND}(\mathsf{A}_\alpha)$, in these two cases the invariant in itself is of no use for computing $\mathsf{Aut}(\mathsf{A}_\alpha)$. However, in [18] and [11] the authors describe the automorphism groups of the polynomial algebra A_0 and of the first Weyl algebra A_1, respectively, using a characterization of the locally nilpotent derivations of the corresponding algebra. Specifically, given $\alpha \in \mathbb{F}$ (up to isomorphism, it can be assumed that either $\alpha = 0$ or $\alpha = 1$), let G_α be the subgroup of $\mathsf{Aut}(\mathsf{A}_\alpha)$ generated by the affine automorphisms (those which leave the 3-dimensional subspace $\mathbb{F}x \oplus \mathbb{F}y \oplus \mathbb{F}1$ invariant) and the triangular automorphisms (those of the form $x \mapsto x, y \mapsto y + p(x)$, with $p(x) \in \mathbb{F}[x]$).

Theorem 2.6 ([11, 8.9] and [18, Thé.])**.** *Assume that* $\mathsf{char}(\mathbb{F}) = 0$ *and let* $\alpha \in \mathbb{F}$. *Then, for any* $\partial \in \mathsf{LND}(\mathsf{A}_\alpha)$, *there exists* $\Delta \in G_\alpha$ *such that* $\Delta \circ \partial \circ \Delta^{-1}(x) = 0$ *and* $\Delta \circ \partial \circ \Delta^{-1}(y) = p(x)$, *for some* $p(x) \in \mathbb{F}[x]$.

Remark 2.7. Using the notation of Theorem 2.6, in case $\alpha = 0$ we have $\Delta \circ \partial \circ \Delta^{-1} = p(x)\frac{d}{dy}$ and in case $\alpha \neq 0$ we have $\Delta \circ \partial \circ \Delta^{-1} = \mathsf{ad}_{f(x)}$, where $p(x) = -f'(x)$.

From Theorem 2.6 it is easy to deduce that $\mathsf{Aut}(\mathsf{A}_\alpha) = G_\alpha$. For example, in the case of the Weyl algebra A_1 we can argue as follows (compare [11, 8.10]). Let $\phi \in \mathsf{Aut}(\mathsf{A}_1)$ and set $(u, v) = (\phi(x), \phi(y))$. Then $\mathsf{ad}_u = \phi \circ \mathsf{ad}_x \circ \phi^{-1} \in \mathsf{LND}(\mathsf{A}_1)$. By Theorem 2.6, there exists $\Delta \in G_1$ such that $\Delta \circ \mathsf{ad}_u \circ \Delta^{-1} = \mathsf{ad}_{f(x)}$, for some $f(x) \in \mathbb{F}[x]$. Thus, $\mathsf{ad}_{f(x)} = \mathsf{ad}_{\Delta(u)}$ and since A_1 has trivial center when $\mathsf{char}(\mathbb{F}) = 0$, we deduce that $\Delta \circ \phi(x) = g(x)$, where $g(x)$ differs from $f(x)$ by a constant. Moreover, since $\mathsf{C}_{\mathsf{A}_1}(x) = \mathbb{F}[x]$, where $\mathsf{C}_{\mathsf{A}_1}$ stands for the centralizer in A_1, we have that $\mathsf{C}_{\mathsf{A}_1}(g(x)) = \mathbb{F}[g(x)]$. As $g \notin \mathbb{F}$, it is easy to see that $\mathsf{C}_{\mathsf{A}_1}(g(x)) = \mathbb{F}[x]$, so $g(x) = ax + b$ for some $a, b \in \mathbb{F}$ with $a \neq 0$. Now, applying $\Delta \circ \phi$ to the defining relation $[y, x] = 1$, one concludes that $\Delta \circ \phi(y) = a^{-1}y + p(x)$, for some $p(x) \in \mathbb{F}[x]$, which shows that $\Delta \circ \phi \in G_1$ and $\mathsf{Aut}(\mathsf{A}_1) = G_1$. The proof for the polynomial algebra A_0 follows similar reasoning, with a few adaptations.

Our goal in this note is to point out that these ideas apply more generally to a family A_h of algebras parametrized by arbitrary polynomials $h \in \mathbb{F}[x]$. This family was introduced in [6], where the automorphism groups $\mathsf{Aut}(\mathsf{A}_h)$ were studied using different methods.

Definition 2.8. Let $h \in \mathbb{F}[x]$. The algebra A_h is the unital associative algebra over $\mathbb{F}$ with generators x, y and defining relation $[y, x] = h$, where $[y, x] = yx - xy$.

The algebras A_h include the polynomial algebra as A_0, the Weyl algebra as A_1, the enveloping algebra of the two-dimensional non-abelian Lie algebra as A_x, the Jordan plane as A_{x^2}, and many others (see [6, 4, 5] for more details on these algebras).

For a general $h \in \mathbb{F}[x]$, there is a derivation of A_h which is an analogue of the derivations $p(x)\frac{d}{dy}$ of A_0 and $\mathsf{ad}_{f(x)}$ of A_1. Given $p(x) \in \mathbb{F}[x]$, the derivation $D_{p(x)}$ is determined by

$$D_{p(x)} : \mathsf{A}_h \longrightarrow \mathsf{A}_h, \qquad D_{p(x)}(x) = 0, \quad D_{p(x)}(y) = p(x). \tag{2.9}$$

It is easy to see that $D_{p(x)} \in \mathsf{LND}(\mathsf{A}_h)$. Next, we generalize Theorem 2.6 to the algebras A_h. Notice that the result implies that these algebras are more rigid (in the sense of [7]) when $h \notin \mathbb{F}$.

Proposition 2.10. *Assume that* $\mathsf{char}(\mathbb{F}) = 0$ *and let* $h \in \mathbb{F}[x] \setminus \mathbb{F}$. *Then,* $\mathsf{LND}(\mathsf{A}_h) = \{D_{p(x)} \mid p(x) \in \mathbb{F}[x]\}$ *and* $\mathsf{ML}(\mathsf{A}_h) = \mathbb{F}[x]$.

Proof. Let $\partial \in \mathsf{LND}(\mathsf{A}_h)$. Then,

$$\partial(h) = [\partial(y), x] + [y, \partial(x)]. \tag{2.11}$$

In particular, $\partial(h) \in [\mathsf{A}_h, \mathsf{A}_h]$ and by [6, Lem. 6.1], $[\mathsf{A}_h, \mathsf{A}_h] \subseteq h\mathsf{A}_h$, so $\partial(h) = h\theta$, for some $\theta \in \mathsf{A}_h$. If $\partial(h) \neq 0$, then $\deg_\partial(h) - 1 = \deg_\partial(\partial(h)) = \deg_\partial(h) + \deg_\partial(\theta)$, which is a contradiction as $\deg_\partial$ does not take on the value -1. Thus, $\partial(h) = 0$.

Let n be the degree of h as a polynomial in x. By hypothesis, $n \geq 1$ and by Proposition 2.2, $0 = \deg_\partial(h) = n \deg_\partial(x)$. Therefore, $\partial(x) = 0$. Now, using (2.11), we conclude that $\partial(y) \in \mathsf{C}_{\mathsf{A}_h}(x) = \mathbb{F}[x]$, where the last equality comes from [6, Lem. 6.3]. We thus conclude that $\partial = D_{p(x)}$, where $p(x) = \partial(y)$. The final statement follows from the fact that ${\mathsf{A}_h}^{D_{p(x)}} = \mathbb{F}[x]$ for all $0 \neq p(x) \in \mathbb{F}[x]$. □

Each $D_{p(x)} \in \mathsf{LND}(\mathsf{A}_h)$ determines a triangular automorphism $\phi_{p(x)} = \exp\left(D_{p(x)}\right)$ with $\phi_{p(x)}(x) = x$ and $\phi_{p(x)}(y) = y + p(x)$. There are also affine automorphisms $\tau_{(\alpha,\beta)}$ such that $\tau_{(\alpha,\beta)}(x) = \alpha x + \beta$ and $\tau_{(\alpha,\beta)}(y) = \alpha^{n-1} y$, for every $\alpha, \beta \in \mathbb{F}^2$ with $\alpha \neq 0$ and $h(\alpha x + \beta) = \alpha^n h(x)$, where n is the degree of h as a polynomial in x (see [6, Sec. 8] for more details). Let G_h be the subgroup of $\mathsf{Aut}(\mathsf{A}_h)$ generated by the triangular and the affine automorphisms defined above. As a corollary of Proposition 2.10 we get the analogue of Jung's Theorem [13] for the polynomial ring A_0 and of Dixmier's Theorem [11] for the Weyl algebra A_1. This result was obtained in [6] using different methods but here we wish to underline the common features and properties of the locally nilpotent derivations of the algebras A_h as a whole, showing how they fit into the approach used by Dixmier and Rentschler in [11] and [18], respectively, and how their structure under the action of the group G_h allows for the description of their automorphism groups. Another example of this phenomenon occurs in [2], where the authors study the isomorphisms and automorphisms of a family of generalized Weyl algebras over a polynomial algebra of rank one.

Corollary 2.12. *Assume that* $\mathsf{char}(\mathbb{F}) = 0$ *and let* $h \in \mathbb{F}[x] \setminus \mathbb{F}$. *Then* $\mathsf{Aut}(\mathsf{A}_h) = G_h$, *i.e.* $\mathsf{Aut}(\mathsf{A}_h)$ *is generated by the triangular automorphisms* $\phi_{p(x)} = \exp\left(D_{p(x)}\right)$ *and the affine automorphisms* $\tau_{(\alpha,\beta)}$.

Proof. Let $\phi \in \mathsf{Aut}(\mathsf{A}_h)$ with $h \in \mathbb{F}[x]$ of degree n. By Proposition 2.10, $\mathsf{ML}(\mathsf{A}_h) = \mathbb{F}[x]$, so there are $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq 0$ such that $\phi(x) = \alpha x + \beta$. Applying ϕ to the defining relation of A_h we obtain $[\phi(y), x] = \alpha^{-1} h(\alpha x + \beta)$. Let $\partial = \mathsf{ad}_{-x} = D_h$. Then the relation obtained implies that $\deg_\partial(\phi(y)) = 1$. It is not hard to see that the set of $\theta \in \mathsf{A}_h$ with $\deg_\partial(\theta) = 1$ is $(\mathbb{F}[x]y + \mathbb{F}[x]) \setminus \mathbb{F}[x]$, so there are $f, g \in \mathbb{F}[x]$ with $f \neq 0$ such that $\phi(y) = fy + g$. Substituting into $[\phi(y), x] = \alpha^{-1} h(\alpha x + \beta)$ we deduce that $\alpha f h = h(\alpha x + \beta)$. Hence, comparing the coefficients of x^n on both sides, we get $f = \alpha^{n-1} \in \mathbb{F}^*$ and $\alpha^n h = h(\alpha x + \beta)$. Finally, notice that $\phi = \phi_{\alpha^{1-n} g} \circ \tau_{(\alpha,\beta)} \in G_h$. □

3 Higher derivations of A_h

Unless stated otherwise, throughout this section $\mathbb{F}$ denotes a field of arbitrary characteristic. As remarked after Proposition 2.2, the fundamental properties of $\deg_\partial$ hold over fields of arbitrary characteristic, except for the multiplicative property.

Example 3.1. Assume that $\mathsf{char}(\mathbb{F}) = p > 0$. Then the Weyl algebra has non-inner derivations. One such is E_x, defined by $E_x(x) = y^{p-1}$ and $E_x(y) = 0$. This derivation is locally nilpotent and $\deg_{E_x}(x) = 1$, $\deg_{E_x}(y) = 0$. Since A_1 is a domain, we can still deduce that $\deg_{E_x}(x^p) \leq p$, but in fact we have $E_x(x^p) = -1$, so $\deg_{E_x}(x^p) = 1$ (see [5] for more details).

One way of circumventing this problem is to follow along the generalization introduced in [8], motivated by the more geometric notion of an action of the additive group G_a on a variety V. From the algebraic point of view, the affine group scheme G_a is represented by the Hopf algebra $\mathbb{F}[t]$, with comultiplication $\Delta : t \mapsto t \otimes 1 + 1 \otimes t$, counit $\epsilon : t \mapsto 0$ and antipode $S : t \mapsto -t$. The action of G_a on V then corresponds to a $\mathbb{F}[t]$-comodule algebra structure on the coordinate ring of V. This is the setting of Rentschler's Theorem in [18], where his result is phrased in terms of actions of the additive group G_a on the affine plane, represented by the polynomial ring A_0.

Let us very briefly explain the connection between this algebraic setting and derivations. Let A be a (not necessarily commutative) algebra. Then a (right) $\mathbb{F}[t]$-comodule algebra structure on A is a map $\delta : \mathsf{A} \longrightarrow \mathsf{A} \otimes \mathbb{F}[t]$ satisfying the following axioms (dualizing the axioms for an action):

(i) δ is an algebra homomorphism;

(ii) $(\mathrm{Id}_{\mathsf{A}} \otimes \Delta) \circ \delta = \left(\delta \otimes \mathrm{Id}_{\mathbb{F}[t]}\right) \circ \delta$;

(iii) $(\mathrm{Id}_{\mathsf{A}} \otimes \epsilon) \circ \delta = \mu$;

where $\mu : \mathsf{A} \longrightarrow \mathsf{A} \otimes \mathbb{F}$ is the canonical isomorphism. Given such a map δ, write

$$\delta(a) = \sum_{k \geq 0} \partial_k(a) \otimes t^k, \tag{3.2}$$

where, for each $a \in \mathsf{A}$, the sum is finite and $\partial_k : \mathsf{A} \longrightarrow \mathsf{A}$. Then the above axioms are equivalent to the following properties ($a, b \in \mathsf{A}$ and $k \geq 0$):

(i) ∂_k is a linear map;

(ii) $\partial_0 = \mathrm{Id}_{\mathsf{A}}$;

(iii) $\partial_k(ab) = \sum_{i=0}^{k} \partial_i(a)\partial_{k-i}(b)$;

(iv) $\partial_k \circ \partial_j = \binom{k+j}{k}\partial_{k+j}$;

(v) $\{k \geq 0 \mid \partial_k(a) \neq 0\}$ is finite.

A sequence $\partial = \{\partial_k\}_{k \geq 0}$ satisfying properties (i)–(iii) above is called a higher derivation of A. If in addition ∂ satisfies (iv), we say that it is iterative, and if it satisfies (v) then we say that it is locally nilpotent. We have thus encoded G_a group actions and, more generally, $\mathbb{F}[t]$-comodule algebra structures, using locally nilpotent iterative higher derivations.

Let $\partial = \{\partial_k\}_{k \geq 0}$ be a locally nilpotent iterative higher derivation of A. Notice that, in particular, ∂_1 is a derivation of A. In case the base field $\mathbb{F}$ has characteristic 0, it is easy to show that $\partial_k = \frac{\partial_1^k}{k!}$, for all $k \geq 0$, and it follows that ∂_1 is locally nilpotent as a derivation of A. Conversely (still in characteristic 0), a locally nilpotent derivation ∂_1 of A determines the locally nilpotent iterative higher derivation $\left\{\frac{\partial_1^k}{k!}\right\}_{k \geq 0}$. So, over fields of characteristic 0 these two concepts coincide, but it is not longer so in characteristic $p > 0$, as the maps ∂_{p^k}, for

$k \geq 0$, are in a sense independent. Specifically, it can be proved that, writing $k = \sum_{i=0}^{m} k_i p^i$, with $0 \leq k_i < p$, then

$$\partial_k = \frac{\partial_{p^0}^{k_0} \circ \partial_{p^1}^{k_1} \circ \cdots \circ \partial_{p^m}^{k_m}}{k_0! k_1! \cdots k_m!}.$$

Therefore, the natural generalization of the Makar-Limanov invariant over fields of positive characteristic uses locally nilpotent iterative higher derivations. We denote the set of such higher derivations of an algebra A by $\mathsf{LNIHD}(\mathsf{A})$ and set

$$\mathsf{ML}(\mathsf{A}) = \bigcap_{\partial \in \mathsf{LNIHD}(\mathsf{A})} \mathsf{A}^{\partial}, \tag{3.3}$$

where $\mathsf{A}^{\partial} = \{a \in \mathsf{A} \mid \partial_k(a) = 0, \text{ for all } k \geq 1\}$. We pursue this further by setting $\mathsf{N}(\partial, a) = \{k \geq 0 \mid \partial_k(a) \neq 0\}$ and

$$\mathsf{deg}_{\,\partial}(0) = -\infty, \qquad \mathsf{deg}_{\,\partial}(a) = \max \mathsf{N}(\partial, a), \quad \text{for } 0 \neq a \in \mathsf{A}. \tag{3.4}$$

Since $\mathsf{Aut}(A)$ acts on $\mathsf{LNIHD}(\mathsf{A})$ by conjugation, the subalgebra $\mathsf{ML}(\mathsf{A})$ is invariant under automorphisms.

With these definitions, the new notions coincide with the existing ones when the base field has characteristic 0. It is easy to see that this extended notion of $\mathsf{deg}_{\,\partial}$, for $\partial \in \mathsf{LNIHD}(\mathsf{A})$, still satisfies the additive property from Proposition 2.2 (b) and, as long as A is a domain, it satisfies the multiplicative property (a) as well. See [8] for more details, especially in the case that A is commutative.

Now we return to the algebras A_h and compute the invariant $\mathsf{ML}(\mathsf{A}_h)$ over a field of arbitrary characteristic.

Lemma 3.5. *Let $h \in \mathbb{F}[x] \setminus \mathbb{F}$. Then, for any $\partial \in \mathsf{LNIHD}(\mathsf{A}_h)$, $x \in \mathsf{A}_h^{\partial}$ and $\partial_k(y)$ commutes with x, for every $k \geq 1$.*

Proof. Fix $\partial = \{\partial_k\}_{k \geq 0} \in \mathsf{LNIHD}(\mathsf{A}_h)$ and recall that ∂ defines an algebra homomorphism $\delta : \mathsf{A}_h \longrightarrow \mathsf{A}_h \otimes \mathbb{F}[t]$ as in (3.2). Then, applying δ to the relation $h = [y, x]$ we see that $\delta(h) \in [\mathsf{A}_h \otimes \mathbb{F}[t], \mathsf{A}_h \otimes \mathbb{F}[t]] \subseteq [\mathsf{A}_h, \mathsf{A}_h] \otimes \mathbb{F}[t] \subseteq h\mathsf{A}_h \otimes \mathbb{F}[t]$. Thus, for every $k \geq 0$, $\partial_k(h) = h\theta_k$, for some $\theta_k \in \mathsf{A}_h$.

Let $k \geq 1$ and $\ell = \mathsf{deg}_{\,\partial}(h)$. For any $j > \ell - k$, we have $\partial_j(\partial_k(h)) = \binom{k+j}{k} \partial_{k+j}(h) = 0$. Thus,

$$\ell + \mathsf{deg}_{\,\partial}(\theta_k) = \mathsf{deg}_{\,\partial}(h\theta_k) = \mathsf{deg}_{\,\partial}(\partial_k(h)) \leq \ell - k.$$

It follows that $\mathsf{deg}_{\,\partial}(\theta_k) = -\infty$, so $\theta_k = 0 = \partial_k(h)$. This shows that $\mathsf{deg}_{\,\partial}(h) = 0$. Now, as in the proof of Proposition 2.10, the latter implies that $\mathsf{deg}_{\,\partial}(x) = 0$, so $x \in \mathsf{A}_h^{\partial}$.

Applying δ once again to the defining relation of A_h gives:

$$h \otimes 1 = \delta(h) = \sum_{k \geq 0} [\partial_k(y), x] \otimes t^k,$$

whence the final claim. □

Guided by the above result, below we construct locally nilpotent iterative higher derivations of A_h which in positive characteristic take the role of the locally nilpotent derivations $p(x)\frac{d}{dy}$ of A_0, $\mathsf{ad}_{f(x)}$ of A_1 and $D_{p(x)}$ of A_h, as defined in (2.9). For related results on the polynomial algebra A_0 and on the Weyl algebra A_1 see [17] and [20], respectively.

Lemma 3.6. *Assume that* $\mathsf{char}(\mathbb{F}) = p > 0$ *and fix* $h \in \mathbb{F}[x]$. *Let* $P = \{P_i(x)\}_{i\geq 0}$ *be a family of elements of* $\mathbb{F}[x]$ *such that* $P_i = 0$ *for* $i \gg 0$. *Then there is a locally nilpotent iterative higher derivation of* A_h, *denoted by* $\partial_P = \{(\partial_P)_k\}_{k\geq 0}$, *such that:*

(a) $(\partial_P)_0 = \mathrm{Id}_{\mathsf{A}_h}$,

(b) $(\partial_P)_k(x) = 0$, *for all* $k \geq 1$,

(c) $(\partial_P)_k(y) = P_i(x)$, *if* $k = p^i$ *for some* $i \geq 0$,

(d) $(\partial_P)_k(y) = 0$ *for all other* $k \geq 1$.

Proof. As seen in (3.2), the claim is tantamount to the statement that $\delta = \sum_{k\geq 0}(\partial_P)_k \otimes t^k$ defines a $\mathbb{F}[t]$-comodule algebra structure on A_h, which is what will be proved.

Let $c_k = (\partial_P)_k(y)$, so that $c_0 = y$, $c_{p^i} = P_i(x)$ and $c_k = 0$ for all other values of k. Then, as

$$\left[\sum_{k\geq 0} c_k \otimes t^k, x \otimes 1\right] = \sum_{k\geq 0} [c_k, x] \otimes t^k = [y, x] \otimes 1 = h \otimes 1,$$

it follows that there is a unique algebra homomorphism $\delta : \mathsf{A}_h \longrightarrow \mathsf{A}_h \otimes \mathbb{F}[t]$ such that $\delta(x) = x \otimes 1$ and $\delta(y) = \sum_{k\geq 0} c_k \otimes t^k$. This already shows that ∂_P is a higher derivation of A_h and it is locally nilpotent by the finiteness assumption on $P = \{P_i(x)\}_{i\geq 0}$. Hence it remains to prove the iterative property, which is equivalent to the following equality:

$$\delta(c_k) = \sum_{j\geq 0} \binom{k+j}{k} c_{k+j} \otimes t^j, \quad \text{for all } k \geq 0. \tag{3.7}$$

If $k = 0$, then (3.7) reduces to $\delta(y) = \delta(c_0)$, which clearly holds. Otherwise, there is a unique $i \geq 0$ such that $p^i \leq k < p^{i+1}$. Then either $c_k = P_i(x)$ or $c_k = 0$; regardless, $\delta(c_k) = c_k \otimes 1$. Now consider the right-hand side of (3.7). If $j = 0$, the corresponding summand is $c_k \otimes 1$, thus we need to show that for $j > 0$, $\binom{k+j}{k} c_{k+j} = 0$. Suppose that $c_{k+j} \neq 0$. Then, as $k, j > 0$, there is $\ell > i$ such that $k + j = p^\ell$ and by Lucas' Theorem,

$$\binom{k+j}{k} = \binom{p^\ell}{k} \equiv \prod_{m=0}^{\ell} \binom{a_m}{b_m} \pmod{p},$$

where $p^\ell = \sum_m a_m p^m$ and $k = \sum_m b_m p^m$ are the p-adic expansions. Since $a_m = 0$ for all $m < \ell$ and $p^i \leq k < p^{i+1}$, it follows that $a_i = 0$ and $b_i > 0$, so $\binom{a_i}{b_i} = 0$ and $\binom{k+j}{k} \equiv 0 \pmod{p}$. This proves that $\binom{k+j}{k} c_{k+j} = 0$ for all $j > 0$, so ∂_P is iterative. □

Now we are ready to compute the Makar-Limanov invariant $\mathsf{ML}(A_h)$ in case $\mathsf{char}(\mathbb{F}) = p > 0$ and see that this information is enough to describe the automorphism group of A_h, as in Corollary 2.12.

Corollary 3.8. *Let* $h \in \mathbb{F}[x] \setminus \mathbb{F}$. *Then,* $\mathsf{ML}(\mathsf{A}_h) = \mathbb{F}[x]$ *and* $\mathsf{Aut}(\mathsf{A}_h) = G_h$.

Proof. In view of Proposition 2.10 and Corollary 2.12, we can assume that $\mathsf{char}(\mathbb{F}) = p > 0$. Then, by Lemma 3.5, $\mathbb{F}[x] \subseteq \mathsf{ML}(\mathsf{A}_h)$. Now, by Lemma 3.6, for every $\ell \geq 0$, there is $\partial_P \in \mathsf{LNIHD}(\mathsf{A}_h)$ such that $\deg_{\partial_P}(y) = p^\ell \geq 1$. For any such higher derivation, $\mathsf{A}_h^{\partial_P} = \mathbb{F}[x]$, proving equality. Now, as in Corollary 2.12, this implies that any automorphism of A_h sends x to $\alpha x + \beta$ for some $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq 0$. This result and its analogue for the inverse automorphism imply that y is sent to $\alpha^{n-1} y + g$, as in the proof of Corollary 2.12. □

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