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Polynomial cubic decomposition

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Resumo

Neste trabalho é estabelecida a decomposição cúbica geral (DC) de uma qualquer sucessão de polinómios, assim como a correspondente construção a partir dos coeficientes de estrutura da sucessão considerada. A DC diagonal e a DC da sucessão canónica são caracterizadas. Uma atenção particular é dada à DC de uma sucessão ortogonal, analisando a ortogonalidade das componentes principais e descrevendo completamente as sucessões ortogonais que admitem DCs específicas. A DC de uma sucessão 2-orthogonal é também estudada de maneira geral, sendo ainda analisado, neste contexto, o caso diagonal.

Para cada tipo de sucessão polinomial considerada e DC específica escolhida, são descritos os coeficientes de estrutura da sucessão, as sucessões componentes, assim como as relações entre os funcionais lineares das sucessões duais da sucessão polinomial dada e das sucessões componentes da DC.

Finalmente, é apresentada uma implementação computacional da DC usando a linguagem simbólica *Mathematica*, que é aplicável a qualquer sucessão de polinómios. Neste âmbito, são construídos alguns procedimentos que visam testar certas propriedades das sucessões componentes. Desta experimentação, surgem resultados e conjecturas relativas à simetria das sucessões componentes.

Abstract

We establish the general cubic decomposition (CD) of any polynomial sequence and the correspondent construction based on its structure coefficients. The diagonal CD and the CD of the canonical sequence are characterized. Particular attention is given to the CD of an orthogonal sequence, where the principal components orthogonality is inquired and several particular CDs are developed and fully described. The CD of a 2-orthogonal sequence is also studied in general, and, also, taken into consideration the diagonal case.

For each type of polynomial sequence considered and specific CD chosen, we describe the structure coefficients of the sequence, the component sequences, and, also, the relations between the linear functionals of the dual sequences of the given polynomial sequence and component sequences of the CD.

Finally, we present a computational approach to the CD using symbolic *Mathematica* language, which is applicable to any polynomial sequence. Some procedures are constructed in order to test certain properties of the component sequences of the CD. The performed tests allowed us to advance with some results and conjectures with respect to the symmetry of the component sequences.

Résumé

On établit la décomposition cubique générale (DC) d′une suite de polynômes quelconque et la correspondante construction basée sur les coefficients de structure de la suite donnée. La DC diagonale et la DC de la suite canonique sont caractérisées. Une attention spéciale est donnée à la DC d′une suite orthogonale, en analysant l′orthogonalité des suites composantes principales et en décrivant des DCs particulières. On traite aussi la DC d′une suite 2-orthogonale, en général, et aussi, le cas diagonal.

Pour chaque suite polynomiale considérée et chaque DC particulière choisie, on décrit les coefficients de structure de la suite, les suites composantes, et, aussi, les relations entre les formes linéaires des suites duales de la suite polynomiale donnée et des suites composantes de la DC.

Finalement, on développe la DC d′une suite de polynômes quelconque du point de vue formel en utilisant la langage symbolique *Mathematica*. On construit quelques procédures pour tester certaines propriétés des suites composantes de la DC. Les tests réalisés ont permis d′énoncer des résultats et des conjectures relatives à la symétrie des suites composantes.

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Contents

Introduction

To a monic polynomial sequence $\{W_n\}_{n\geq 0}$ we can associate two monic polynomial sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ such that

$$W_{2n}(x) = P_n(x^2) + xa_{n-1}(x^2),$$
$$W_{2n+1}(x) = b_n(x^2) + xR_n(x^2),$$

where $a_n(x)$ and $b_n(x)$ are polynomials with degrees that do not exceed n and $a_{-1}(x) = 0$. We call it a quadratic decomposition of $\{W_n\}_{n\geq 0}$.

In 1990 [28] and 1993 [30], Maroni established several results concerning the quadratic decomposition, namely necessary and sufficient conditions for the orthogonality of the sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, when the given sequence $\{W_n\}_{n\geq 0}$ is orthogonal.

The general quadratic decomposition, defined by:

$$W_{2n}(x) = P_n(\varpi(x)) + (x-a)a_{n-1}(\varpi(x)),$$

$$W_{2n+1}(x) = b_n(\varpi(x)) + (x-a)R_n(\varpi(x)),\ n \geq 0,$$

where the degrees of $a_n(x)$ and $b_n(x)$ do not exceed n, $a_{-1}(x) = 0$ and $\varpi(x) = x^2 + px + q,\ p, q \in \mathbb{C}$, appears only in 2004, in the Ph.D. thesis of Macedo [21, 22], where some results of Maroni are generalized.

The cubic decomposition (CD) of a polynomial sequence was, until the present day, only treated in special cases, presented in the following references hereby.

Motivated by previous investigations on generalizations of a quadratic transformation relating Hermite and Laguerre polynomials, Chihara posed the following problem in 1964 [9]: to find a pair of orthogonal polynomial sequences $\{W_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ such that $W_{3n}(x) = P_n(z)$, where z is a cubic in x. Having the purpose of finding concrete examples, a specific cubic decomposition is treated by Barrucand and Dickinson in 1966 [1]. In that paper, the authors search two orthogonal polynomial sequences, symmetric and positive definite, such that $W_{3n}(x) = P_n(x^3 + qx)$. The main result of that study gives sufficient conditions to establish the above relation between the two polynomial sequences, in terms of the corresponding structure coefficients (see below (1.1) and (1.2)). The authors goal consists in obtaining a CD for $W_{3n}(x)$, not involving, *a priori*, the other two kinds of elements: $W_{3n+1}(x)$ and $W_{3n+2}(x)$. Afterwards, they express these latest as rational fractions involving $P_n(x^3 + qx)$ and $P_{n+1}(x^3 + qx)$. A similar treatment is given by Marcellán and

Sansigre in 1993 [23], to orthogonal sequences neither necessarily symmetric nor definite positive, and considering the cubic transformation x^3.

This problem is also approached in several references, concerning polynomial transformations of measures, sieved polynomials, polynomial mappings and positive-definite linear functionals [2, 6, 7, 15, 19]. In particular, Marcellán and Petronilho developed in 2000 [24] the following three separate cases P1, P2 and P3: given an orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$, to find necessary and sufficient conditions in order that $\{W_n\}_{n\geq 0}$, defined by (P1) $W_{3n}(x) = P_n(\pi_3(x))$, (P2) $W_{3n+1}(x) = (x-a)P_n(\pi_3(x))$, or (P3) $W_{3n+2}(x) = (x-a)(x-b)P_n(\pi_3(x))$, be an orthogonal polynomial sequence, for fixed constants a, b and a monic polynomial of degree 3, $\pi_3(x)$. For each problem, the remaining subsequences of $\{W_n\}_{n\geq 0}$ (for instance, in P1, $W_{3n+1}(x)$ and $W_{3n+2}(x)$) are written as rational fractions involving elements of the sequence $\{P_n\}_{n\geq 0}$. See below theorem 3.17 that brings a better answer, for P1, to the problem of determining these subsequences of $\{W_n\}_{n\geq 0}$. In 2001 [25], the two authors deal with these problems for the positive-definite case.

In this work, we generalize the CD approach presented by González and Montes in 1989 [17], where the three elements $W_{3n}(x)$, $W_{3n+1}(x)$ and $W_{3n+2}(x)$, of a given orthogonal sequence $\{W_n\}_{n\geq 0}$, are decomposed by nine polynomial sequences, considering the cubic transformation x^3, and thus, it is achieved a full (polynomial) CD of the sequence $\{W_n\}_{n\geq 0}$. Furthermore, here, we consider the complete cubic transformation given by $\varpi(x) = x^3 + px^2 + qx + r$.

Let us notice that, in 1992 [11], Douak and Maroni obtained a diagonal CD of a 2-symmetric and 2-orthogonal sequence, considering the CD announced by González and Montes. Latter on, in 1997 [13, 14], the two authors describe the component sequences of the CD of a 2-symmetric and 2-classical sequence $\{W_n\}_{n\geq 0}$ fulfilling one of the following identities: $W_n^{(1)} = W_n,\ n \geq 0$, where $\{W_n^{(1)}\}_{n\geq 0}$ is the associated sequence of $\{W_n\}_{n\geq 0}$, or, $W_n^{(1)} = Q_n,\ n \geq 0$, where $Q_n(x) = (n+1)^{-1}W'_{n+1}(x),\ n \geq 0$.

The main goal of this thesis is to establish the general environment of the CD and to describe all the component sequences involved, for any given polynomial sequence, not necessarily orthogonal or 2-orthogonal. Consequently, the fundamental results presented here allow further developments related to symmetry, orthogonality, d-orthogonality, classical sequences, among others, and their CD.

The present work consists of six chapters. In the first chapter we present the basic definitions and results needed in the sequel.

The second chapter is reserved to the proof of existence and uniqueness of the CD of a monic polynomial sequence, and to the constructive characterization of the nine component sequences that constitute a CD. Furthermore, the diagonal CD, which is the most simple CD, is characterized in several manners, namely through the forms of the dual sequence of $\{W_n\}_{n\geq 0}$. We also describe the component sequences of the CD of the canonical sequence.

The third chapter is dedicated to the study of the CD of an orthogonal sequence. We begin to give two equivalent characterizations of the CD of an orthogonal sequence, from which we can establish sufficient conditions for the orthogonality of the principal components. Some CDs where specific secondary components vanish are described, in particular, we approach the case where $W_{3n}(x) = P_n(\varpi(x))$, and we exemplify it with symmetric sequences, giving new insights to the main result of [1]. In fact, such sequences belong to a particular class of orthogonal sequences, for which we give the complete CD.

In the fourth chapter we study the CD of a 2-orthogonal sequence, beginning with two equivalent characterizations of the CD of a 2-orthogonal sequence, from which we can, also, announce sufficient conditions for the 2-orthogonality of the principal components. The structure coefficients of the 2-orthogonal sequences which admit a diagonal CD are described and we prove that the correspondent principal components are also 2-orthogonal. This particular case is also analysed for a 2-symmetric and 2-orthogonal sequence.

In the study of particular CDs of orthogonal and 2-orthogonal sequences, we obtain conditions for the structure coefficients of the given sequence, the correspondent component sequences, relations between the linear functionals of the dual sequences of the given sequence and the principal component sequences of the CD, and, in some cases, properties of quasi-orthogonality of the secondary components.

In certain calculations performed in the proofs presented in the second, third and fourth chapters, we adopted the *Mathematica* software [39], although with the only purpose of confirm or simplify some of the algebraic computations (see, for example, the final part of the proof of theorem 3.2). We remark that the strategies taken during these proofs were developed by the author and cannot be replaced by simple commands of the mentioned software or other. For instance, in the arguments used, it is often necessary to obtain different expressions, with adequate and manageable coefficients, for a given polynomial component. In these processes, some of the hand work could only be replaced by typing it in the keyboard without any gain of time, or even having to spend more time confirming the correctness of the inserted expressions, and obtaining more extensive expressions (and thus, not suitable) due to the presence of six parameters in this CD.

In the fifth chapter, we present procedures constructed in the *Mathematica* language, which aim is to investigate properties of each component sequence, only using symbolic computations. Some conjectures are advanced and a few results established concerning the symmetry character of the principal components and the parity of some secondary components. For this matter, we refer the CAOP [20], a package for calculating formulas for orthogonal polynomials belonging to the Askey scheme in *Maple*, one approach based on special functions available on internet.

The last chapter is reserved to global considerations regarding the techniques used in this thesis and forthcoming developments.

Chapter 1

Preliminaries and notations

In this first chapter, we present basic definitions and results, concerning the polynomial sequences, needed in the sequel. The notation is established and a few simple results are proved.

1.1 Polynomial sequences

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $< u, f >$ the action of the form or linear functional $u \in \mathcal{P}'$ on $f \in \mathcal{P}$.

Remark 1.1. *Except mentioned otherwise, we consider n, m and k integers.*

In particular, $(u)_n =< u, x^n >$, $n \geq 0$, are called the moments of u. A form u is equivalent to the sequence $\{(u)_n\}_{n\geq 0}$, and u is called normalized if and only if $(u)_0 = 1$.

The form δ_c defined by

$$< \delta_c, f(x) >:= f(c), \quad \forall f \in \mathcal{P},$$

is called the Dirac form applied on $c \in \mathbb{C}$.

Remark 1.2. $\delta := \delta_0$.

We will only use polynomial sequences $\{W_n\}_{n\geq 0}$ such that $\deg W_n \leq n, \;\; W_n \in \mathcal{P}, \;\; n \geq 0$. Such a sequence is free if and only if $\deg W_n = n$, $\forall n \geq 0$.

In the following, we will call polynomial sequence (PS) to any free polynomial sequence. We will also call monic polynomial sequence (MPS) to any free sequence of monic polynomials, meaning that in each polynomial the leading coefficient is equal to one ($W_n(x) = x^n + \sum_{\nu=0}^{n-1} a_\nu x^\nu$).

Given a MPS $\{W_n\}_{n\geq 0}$, by euclidean division, there are complex sequences, $\{\beta_n\}_{n\geq 0}$ and $\{\chi_{n,\nu}\}_{0\leq \nu\leq n,\; n\geq 0}$, such that

$$W_0(x) = 1, \;\; W_1(x) = x - \beta_0, \tag{1.1}$$

$$W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{n,\nu} W_\nu(x). \tag{1.2}$$

This relation is called the structure relation of $\{W_n\}_{n\geq 0}$, and $\{\beta_n\}_{n\geq 0}$ and $\{\chi_{n,\nu}\}_{0\leq\nu\leq n,\ n\geq 0}$ are called the structure coefficients.

Also, there exists a unique sequence $\{w_n\}_{n\geq 0}$, $w_n \in \mathcal{P}'$, called the dual sequence of $\{W_n\}_{n\geq 0}$, and defined by the biorthogonal condition

$$< w_n, W_m >= \delta_{n,m}, \quad m \geq 0,$$

where $\delta_{n,m}$ denotes the Kronecker symbol (see [3]).

The first element of the dual sequence, w_0, is called the canonical form of $\{W_n\}_{n\geq 0}$.

Consequently, [31]

$$\beta_n =< w_n, xW_n(x) >, \ \ n \geq 0, \tag{1.3}$$

$$\chi_{n,\nu} =< w_\nu, xW_{n+1}(x) >, \ \ 0 \leq \nu \leq n, \ \ n \geq 0. \tag{1.4}$$

Lemma 1.3. *Given a form $u \in \mathcal{P}$ and a PS $\{W_n\}_{n\geq 0}$, if $< u, W_n >= 0$ for all $n \geq 0$, then $u = 0$.*

Proof. $x^n = \sum_{\nu=0}^{n} a_{n,\nu} W_\nu$, for some constants $a_{n,\nu}$. By linearity, $< u, x^n >= \sum_{\nu=0}^{n} a_{n,\nu} < u, W_\nu >= 0$, for all $n \geq 0$. □

Lemma 1.4. *Let $\{W_n\}_{n\geq 0}$ be a MPS and let $\{w_n\}_{n\geq 0}$ be its dual sequence. Given a polynomial $p(x)$, if $< w_n, p(x) >= 0$ for all $n \geq 0$, then $p(x) = 0$.*

Proof. Let m denote $p(x)$ degree, then, $p(x) = \sum_{\nu=0}^{m} a_\nu W_\nu(x)$, for some constants a_ν. For each integer $0 \leq n \leq m$, we obtain, by linearity, $< w_n, p(x) >= \sum_{\nu=0}^{n} a_\nu < w_n, W_\nu >= \sum_{\nu=0}^{n} a_\nu \delta_{n,\nu} = a_n$. Hence, $a_n = 0$, for all $0 \leq n \leq m$. □

Lemma 1.5. *[31] For each $u \in \mathcal{P}'$ and each $m \geq 1$, the two following propositions are equivalent.*

a) $< u, W_{m-1} >\neq 0, \ < u, W_n >= 0, \ n \geq m.$

b) $\exists \lambda_\nu \in \mathbb{C}, \ 0 \leq \nu \leq m-1, \ \lambda_{m-1} \neq 0$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu w_\nu$.

Furthermore, let $\{W_n\}_{n\geq 0}$ be a MPS and let $p(x)$ be a polynomial with degree m,

$$p(x) = \sum_{\nu=0}^{m} a_\nu W_\nu(x).$$

Applying the form w_k, from the dual sequence, we obtain

$$< w_k, p(x) >= \sum_{\nu=0}^{m} a_\nu < w_k, W_\nu >= \sum_{\nu=0}^{m} a_\nu \delta_{k,\nu} = a_k.$$

Hence, for each integer $0 \leq \nu \leq m$, $a_\nu =< w_\nu, p(x) >$, which proves the following lemma.

Lemma 1.6. *Let $\{W_n\}_{n\geq 0}$ be a MPS and let $\{w_n\}_{n\geq 0}$ be its dual sequence. Given a PS $\{\widetilde{W}_n\}_{n\geq 0}$ we may write:*

$$\widetilde{W}_n(x) = \sum_{\nu=0}^{n} < w_\nu, \widetilde{W}_n > W_\nu(x).$$

A linear operator $T : \mathcal{P} \to \mathcal{P}$ has a transpose ${}^tT : \mathcal{P}' \to \mathcal{P}'$ defined by

$$<^t T(u), p >=< u, T(p) >, \quad u \in \mathcal{P}', \ p \in \mathcal{P}.$$

Therefore, given $\varpi \in \mathcal{P}$ and $u \in \mathcal{P}'$, the form ϖu, called the left-multiplication of u by the polynomial ϖ, is defined by $< \varpi u, p >=< u, \varpi p >, \quad \forall p \in \mathcal{P}$.

We also obtain, for every $\varpi \in \mathcal{P}$, the transpose operator of $\sigma_\varpi : \mathcal{P} \to \mathcal{P}$, $p(x) \mapsto \sigma_\varpi p(x) = p(\varpi(x))$, defined by

$$< \sigma_\varpi u, p >=< u, \sigma_\varpi p >, \quad \forall p \in \mathcal{P}. \tag{1.5}$$

Let us also consider the following operators on $\mathcal{P}$ (see [29]).

$$p \to (\tau_b p)(x) = p(x-b), \ \ b \in \mathbb{C},$$

$$p \to (h_a p)(x) = p(ax), \ \ a \in \mathbb{C}\backslash\{0\},$$
$$p \to (Dp)(x) = p'(x).$$

Transposing, we obtain the corresponding operators on $\mathcal{P}'$, respectively.

$$u \to \tau_b u : \ < \tau_b u, p >=< u, \tau_{-b} p >=< u, p(x+b) >, \quad \forall p \in \mathcal{P},$$

$$u \to h_a u : \ < h_a u, p >=< u, h_a p >=< u, p(ax) >, \quad \forall p \in \mathcal{P},$$
$$u \to Du : \ < Du, p >= - < u, p' >, \quad \forall p \in \mathcal{P}.$$

Let $\{W_n\}_{n\geq 0}$ be a MPS. Given $A \in \mathbb{C}\backslash\{0\}$ and $B \in \mathbb{C}$, let us define the shifted MPS

$$\tilde{W}_n(x) = A^{-n} W_n(Ax+B), \ \ n \geq 0. \tag{1.6}$$

If $\{\tilde{w}_n\}_{n\geq 0}$ denotes the dual sequence of $\{\tilde{W}_n\}_{n\geq 0}$, then we have [31]:

$$\tilde{w}_n = A^n (h_{A^{-1}} \circ \tau_{-B}) w_n.$$

Definition 1.7. *[8] A form u is positive definite if $< u, \pi(x) >> 0$ for every polynomial $\pi(x)$ that is not identically zero and is non-negative for all real x.*

Definition 1.8. *[8] Let u be positive definite. Then, u has real moments.*

1.2 Regular orthogonality

Definition 1.9. *[4, 31] A sequence $\{W_n\}_{n\geq 0}$ is regularly orthogonal with respect to the form u if and only if it fulfils*

$$< u, W_n W_m >= 0,\ n \neq m, \quad n, m \geq 0, \tag{1.7}$$

$$< u, W_n^2 >\neq 0,\ n \geq 0. \tag{1.8}$$

Then, the form u is said regular and $\{W_n\}_{n\geq 0}$ is an orthogonal polynomial sequence (OPS). The conditions (1.7) are called the orthogonality conditions and the conditions (1.8) are called the regularity conditions.

We can prove that $\deg W_n = n$; furthermore we can normalize $\{W_n\}_{n\geq 0}$ in order that it becomes monic, then it is unique and we note it as a MOPS. Considering $\{w_n\}_{n\geq 0}$ the corresponding dual sequence, it holds $u = \lambda w_0,\ \lambda = (u)_0 \neq 0$.

Example 1.10. *The PS $\{x^n\}_{n\geq 0}$, called canonical sequence, is orthogonal with respect to the form δ, but is not regularly orthogonal.*

Theorem 1.11. *[31]*
Let $\{W_n\}_{n\geq 0}$ be a MPS and $\{w_n\}_{n\geq 0}$ its dual sequence. The following statements are equivalent:

a) *The sequence $\{W_n\}_{n\geq 0}$ is orthogonal (with respect to w_0);*

b) $\chi_{n,k} = 0,\ 0 \leq k \leq n-1, \quad n \geq 1; \quad \chi_{n,n} \neq 0, \ \ n \geq 0;$

c) $x w_n = w_{n-1} + \beta_n w_n + \chi_{n,n} w_{n+1},\ \chi_{n,n} \neq 0, \quad n \geq 0,\ w_{-1} = 0;$

d) *For each $n \geq 0$, there is a polynomial ϕ_n with $\deg(\phi_n) = n$ such that $w_n = \phi_n w_0$;*

e) $w_n = \Big(< w_0, W_n^2 > \Big)^{-1} W_n w_0,\ n \geq 0;$

where $\chi_{n,k}$ and β_n are defined by (1.1-1.2).

Let $\{W_n\}_{n\geq 0}$ be a MOPS. From statement b) of theorem 1.11 we obtain the following second order recurrence relation

$$W_0(x) = 1, \qquad W_1(x) = x - \beta_0, \tag{1.9}$$

$$W_{n+2}(x) = (x - \beta_{n+1}) W_{n+1}(x) - \gamma_{n+1} W_n(x), \quad n \geq 0, \tag{1.10}$$

where $\gamma_{n+1} = \chi_{n,n} \neq 0, \ \ n \geq 0$, and also by e), we have:

$$\beta_n = \frac{< w_0, x W_n^2(x) >}{< w_0, W_n^2(x) >}, \qquad \gamma_{n+1} = \frac{< w_0, W_{n+1}^2(x) >}{< w_0, W_n^2(x) >}.$$

In the case of orthogonality, β_n and γ_{n+1} are called recurrence coefficients instead of structure coefficients.

Remark 1.12. *Regularity conditions (1.8) are fulfilled if and only if* $\gamma_{n+1} \neq 0,\ n \geq 0$.

A form w_0 *is positive definite if and only if* β_n *is real and* $\gamma_{n+1} > 0$, *for* $n \geq 0$, *and* $(w_0)_0 > 0$ *(see [8], p.21).*

If $\{W_n\}_{n\geq 0}$ is a MOPS, then the shifted MPS defined by (1.6) is orthogonal and its recurrence coefficients are [8, 31]

$$\widetilde{\beta}_n = \frac{\beta_n - B}{A}, \quad \widetilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{A^2}, \quad n \geq 0.$$

The isomorphism $J = s(h_A \circ \tau_{-B})$, $s \in \mathbb{C}\backslash\{0\}$

$$p(x) \rightarrow Jp(x) = s\, p(Ax + B), \ \ p \in \mathcal{P}$$

is the only one that transforms every MOPS in another MOPS [29].

Definition 1.13. *A MPS* $\{W_n\}_{n\geq 0}$ *is called classical, if and only if it satisfies the Hahn´s property [18], that is to say, the derivative MPS* $\{W_n^{[1]}\}_{n\geq 0}$, $W_n^{[1]}(x) := (n+1)^{-1}W'_{n+1}(x)$, *is also orthogonal. The classical character is preserved by a shift, which defines an equivalence relation between orthogonal sequences.*

The classical polynomials are divided in four (equivalence) classes: Hermite, Laguerre, Bessel and Jacobi [8, 31], and characterized by the functional equation [31]

$$D(\phi u) + \psi u = 0,$$

where ψ *and* ϕ *are two polynomials such that:* $\deg \psi = 1$, $\deg \phi \leq 2$ *and* $\psi' - \frac{1}{2}\phi'' n \neq 0,\ n \geq 1$.

A form u is said to be symmetric if and only if $(u)_{2n+1} = 0,\ n \geq 0$. A polynomial sequence $\{W_n(x)\}_{n\geq 0}$ is said to be symmetric if and only if $W_n(-x) = (-1)^n W_n(x),\ n \geq 0$.

Theorem 1.14. *[29] For each MPS* $\{W_n\}_{n\geq 0}$, *the following statements are equivalent:*

a) $\{W_n\}_{n\geq 0}$ *is symmetric;*

b) $\beta_n = 0$; $\chi_{2n+1,2\nu} = 0,\ 0 \leq \nu \leq n,\ n \geq 0$; $\quad \chi_{2n,2\nu+1} = 0,\ 0 \leq \nu \leq n-1,\ n \geq 1$.

Theorem 1.15. *[8] Let* $\{W_n(x)\}_{n\geq 0}$ *be a MOPS with respect to* u. *The following statements are equivalent.*

a) u *is symmetric. b)* $\{W_n(x)\}_{n\geq 0}$ *is symmetric. c)* $\beta_n = 0,\ n \geq 0$.

Definition 1.16. *[29] Let* $\{W_n\}_{n\geq 0}$ *be a MOPS, with structure relation*

$$W_0(x) = 1, \qquad W_1(x) = x - \beta_0,$$

$$W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x), \quad n \geq 0, \tag{1.11}$$

where $\gamma_{n+1} \neq 0, \quad n \geq 0$. *Given* $\mu \in \mathbb{C}$, *the MOPS* $\{W_n(\mu)\}_{n\geq 0}$ *defined by the initial conditions*

$$W_0(\mu)(x) = 1, \qquad W_1(\mu)(x) = x - \beta_0 - \mu,$$

and fulfilling the recurrence (1.11) is called a co-recursive sequence of $\{W_n\}_{n\geq 0}$.

Definition 1.17. *[29] Let* $\{W_n\}_{n\geq 0}$ *be a MOPS, with structure relation given by (1.11) and* $W_0(x) = 1,\ W_1(x) = x - \beta_0$, *and let* $r \geq 1$.

Given $\mu_0 \in \mathbb{C}$, $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{C}^r$, $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Big(\mathbb{C}\backslash\{0\}\Big)^r$, *where either* $\mu_r \neq 0$ *or* $\lambda_r \neq 1$, *the MOPS* $\{\tilde{W}_n(x)\}_{n\geq 0}$ *defined by*

$$\tilde{W}_0(x) = 1, \qquad \tilde{W}_1(x) = x - \tilde{\beta}_0,$$

$$\tilde{W}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{W}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{W}_n(x), \quad n \geq 0,$$

where

$$\tilde{\beta}_0 = \beta_0 + \mu_0,$$

$$\tilde{\beta}_n = \beta_n + \mu_n, \quad \tilde{\gamma}_n = \lambda_n\gamma_n, \quad 1 \leq n \leq r,$$

$$\tilde{\beta}_n = \beta_n, \quad \tilde{\gamma}_n = \gamma_n, \quad n \geq r+1,$$

is called a r-perturbed sequence of $\{W_n\}_{n\geq 0}$ *and denoted by* $\Big\{W_n\Big(\mu_0;\ \begin{matrix}\mu\\ \lambda\end{matrix}\ ; r; x\Big)\Big\}_{n\geq 0}$.

1.3 d-orthogonal sequences and d-symmetric sequences

Definition 1.18. *[27, 11, 13] Given* $\Gamma^1, \Gamma^2, \ldots, \Gamma^d \in \mathcal{P}'$, $d \geq 1$, *the polynomial sequence* $\{W_n\}_{n\geq 0}$ *is called regularly d-orthogonal polynomial sequence (d-OPS) with respect to* $\Gamma = (\Gamma^1, \ldots, \Gamma^d)$ *if it fulfils*

$$< \Gamma^\alpha, W_m W_n >= 0, \ \ n \geq md + \alpha, \ \ m \geq 0, \tag{1.12}$$

$$< \Gamma^\alpha, W_m W_{md+\alpha-1} >\neq 0, \ \ m \geq 0, \tag{1.13}$$

for each integer $\alpha = 1, \ldots, d$.

The conditions (1.12) are called the d-orthogonality conditions and the conditions (1.13) are called the regularity conditions.

In this case, the functional Γ, *of dimension* d, *is said regular.*

Remark 1.19. • *In the above circumstances, the sequence* $\{W_n\}_{n\geq 0}$ *is a free polynomial sequence and each* W_n *can be normalized, rewriting, if necessary,* $\{W_n\}_{n\geq 0}$ *as a monic polynomial sequence. Thus, a d-MOPS is unique.*

- *The presented definition of d-orthogonality is equivalent to the following:* $\{W_n\}_{n\geq 0}$ *is d-orthogonal with respect to* $\Gamma = (\Gamma^1, \ldots, \Gamma^d)$ *if*

$$< \Gamma^\alpha, x^m W_n >= 0, \ \ n \geq md + \alpha, \ \ m \geq 0,$$

$$< \Gamma^\alpha, x^m W_{md+\alpha-1} > \neq 0, \ \ m \geq 0,$$

for each integer $\alpha = 1, \ldots, d$.

- *If* $d = 1$*, then we meet again the notion of regular orthogonality (see definition 1.9).*

Such a functional Γ, of dimension d, is not unique. In fact, using lemma (1.5) we may prove the following result.

Proposition 1.20. *[37] Let* $\{W_n\}_{n\geq 0}$ *be a d-OPS with respect to* $\Gamma = (\Gamma^1, \ldots, \Gamma^d)$*, and* $w = (w_0, \ldots, w_{d-1})$*, where* $\{w_n\}_{n\geq 0}$ *is the dual sequence of* $\{W_n\}_{n\geq 0}$*. Then,*

a) $\{W_n\}_{n\geq 0}$ *is a d-OPS with respect to* $\Lambda = (\Lambda_0, \ldots, \Lambda_{d-1})$*, where* $\Lambda_\nu = \sum_{\alpha=1}^{\nu+1} \tau_\alpha^\nu \Gamma^\alpha$*, for all constants* τ_α^ν*, with* $\tau_{\nu+1}^\nu \neq 0$*,* $\nu = 0, \ldots, d-1$.

b) *There are constants* λ_ν^α*, with* $\lambda_{\alpha-1}^\alpha \neq 0$*, such that,*

$$\Gamma^\alpha = \sum_{\nu=0}^{\alpha-1} \lambda_\nu^\alpha w_\nu, \quad \alpha = 1, \ldots, d. \tag{1.14}$$

c) *(1.14) is equivalent to*

$$w_\nu = \sum_{\alpha=1}^{\nu+1} \tau_\alpha^\nu \Gamma^\alpha, \ \tau_{\nu+1}^\nu \neq 0, \ \ \nu = 0, \ldots, d-1. \tag{1.15}$$

d) *In (1.15) and (1.14), for* $\alpha = 1, \ldots, d$

$$\lambda_n^\alpha =< \Gamma^\alpha, W_n > \ \ n = 0, \ldots, \alpha - 1,$$

$$\tau_{n+1}^\nu = \left(\delta_{\nu,n} - \sum_{\alpha=n+2}^{\nu+1} \tau_\alpha^\nu < \Gamma^\alpha, W_n > \right) / < \Gamma^{n+1}, W_n >, \ n = \nu, \ldots, 0, \ \nu = 0, \ldots, d-1.$$

Therefore, since $w = (w_0, \ldots, w_{d-1})$ is unique, from now on, we will only consider the canonical functional of dimension d, $w = (w_0, \ldots, w_{d-1})$, saying that $\{W_n\}_{n\geq 0}$ is d-OPS $(d \geq 1)$ with respect to $w = (w_0, \ldots, w_{d-1})$ if

$$< w_\nu, W_m W_n >= 0, \;\; n \geq md + \nu + 1, \;\; m \geq 0,$$

$$< w_\nu, W_m W_{md+\nu} >\neq 0, \;\; m \geq 0,$$

for each integer $\nu = 0, 1, \ldots, d-1$.

Remark 1.21. *In the following, we will refer that a PS $\{W_n\}_{n\geq 0}$ is d-orthogonal, meaning that is regularly d-orthogonal or d-OPS.*

A d-MOPS satisfies also a recurrence relation of order $(d+1)$ as the following result establishes.

Theorem 1.22. *[27] Let $\{W_n\}_{n\geq 0}$ be a MPS. The following assertions are equivalent:*

a) *$\{W_n\}_{n\geq 0}$ is d-orthogonal with respect to $w = (w_0, \ldots, w_{d-1})$.*

b) *$\{W_n\}_{n\geq 0}$ satisfies a $(d+1)$-order recurrence relation $(d \geq 1)$:*

$$W_{m+d+1}(x) = (x - \beta_{m+d})W_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} W_{m+d-1-\nu}(x), \;\; m \geq 0,$$

with initial conditions

$$W_0(x) = 1, \;\; W_1(x) = x - \beta_0 \;\; \text{and if } d \geq 2:$$

$$W_n(x) = (x - \beta_{n-1})W_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} W_{n-2-\nu}(x), \;\; 2 \leq n \leq d,$$

and regularity conditions: $\gamma_{m+1}^0 \neq 0, \;\; m \geq 0$.

c) *For each (n, ν), $n \geq 0$, $0 \leq \nu \leq d-1$, there are d polynomials $\Lambda^\mu(n, \nu)$, $0 \leq \mu \leq d-1$ such that*

$$w_{nd+\nu} = \sum_{\mu=0}^{d-1} \Lambda^\mu(n, \nu) w_\mu, \; n \geq 0, \; 0 \leq \nu \leq d-1,$$

and also fulfilling

$$\begin{aligned} &\deg \Lambda^\mu(n, \nu) = n, \;\; 0 \leq \nu \leq d-1, \\ &\deg \Lambda^\mu(n, \nu) \leq n, \;\; 0 \leq \mu \leq \nu - 1, \;\; \text{if } 1 \leq \nu \leq d-1, \\ &\deg \Lambda^\mu(n, \nu) \leq n-1, \;\; \nu + 1 \leq \mu \leq d-1, \;\; \text{if } 0 \leq \nu \leq d-2. \end{aligned}$$

We will now show that d-orthogonality is preserved by a shift, as we indicated above for the particular case of regular orthogonality.

Proposition 1.23. *Let $\{W_n\}_{n\geq 0}$ be a MPS d-orthogonal with respect to $w = (w_0, \ldots, w_{d-1})$, with $d \geq 1$. Given $A \in \mathbb{C}\backslash\{0\}$ and $B \in \mathbb{C}$, the MPS defined by (1.6) is d-orthogonal with respect to $\tilde{w} = (\tilde{w}_0, \ldots, \tilde{w}_{d-1})$, where $\tilde{w}_n = A^n(h_{A^{-1}} \circ \tau_{-B})w_n$.*

Proof. If $d = 1$, then $\{W_n\}_{n\geq 0}$ is a MOPS. This case is already treated in [29], as indicated above.

Let us suppose $d \geq 2$. By theorem 1.22, the sequence $\{W_n\}_{n\geq 0}$ fulfils

$$W_{m+d+1}(x) = (x - \beta_{m+d})W_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} W_{m+d-1-\nu}(x), \;\; m \geq 0,$$

with initial conditions

$$W_0(x) = 1, \;\; W_1(x) = x - \beta_0,$$

$$W_n(x) = (x - \beta_{n-1})W_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} W_{n-2-\nu}(x), \;\; 2 \leq n \leq d,$$

and regularity conditions: $\gamma_{m+1}^0 \neq 0, \;\; m \geq 0$. With respect to the sequence $\{\tilde{W}_n\}_{n\geq 0}$, we have

$$\tilde{W}_0 = 1, \;\; \tilde{W}_1 = A^{-1}W_1(Ax + B) = x - A^{-1}(\beta_0 - B),$$

for $2 \leq n \leq d$:

$$\begin{aligned}
\tilde{W}_n(x) &= A^{-n}W_n(Ax + B) \\
&= A^{-n}(Ax + B - \beta_{n-1})W_{n-1}(Ax + B) - \sum_{\nu=0}^{n-2} A^{-n}\gamma_{n-1-\nu}^{d-1-\nu} W_{n-2-\nu}(Ax + B) \\
&= \big(x - A^{-1}(\beta_{n-1} - B)\big)\tilde{W}_{n-1}(x) - \sum_{\nu=0}^{n-2} A^{-(2+\nu)}\gamma_{n-1-\nu}^{d-1-\nu} \tilde{W}_{n-2-\nu}(x),
\end{aligned}$$

and for $m \geq 0$:

$$\begin{aligned}
\tilde{W}_{m+d+1}(x) &= A^{-(m+d+1)}W_{m+d+1}(Ax + B) \\
&= A^{-(m+d+1)}(Ax + B - \beta_{m+d})W_{m+d}(Ax + B) \\
&\qquad - \sum_{\nu=0}^{d-1} A^{-(m+d+1)}\gamma_{m+d-\nu}^{d-1-\nu} W_{m+d-1-\nu}(Ax + B) \\
&= \big(x - A^{-1}(\beta_{m+d} - B)\big)\tilde{W}_{m+d}(x) - \sum_{\nu=0}^{d-1} A^{-2-\nu}\gamma_{m+d-\nu}^{d-1-\nu} \tilde{W}_{m+d-1-\nu}(x).
\end{aligned}$$

Since $A \neq 0$, the regularity conditions, for $\{\tilde{W}_n\}_{n\geq 0}$, are assured. As mentioned before, the dual sequence of $\{\tilde{W}_n\}_{n\geq 0}$, given by $\tilde{w}_n = A^n(h_{A^{-1}} \circ \tau_{-B})w_n$, can be found in [31]. By theorem 1.22 the proof is complete. □

Definition 1.24. *[11, 12, 13] Let $\{P_n\}_{n\geq 0}$ be a MPS. The sequence $\{P_n\}_{n\geq 0}$ is called classical d-OPS, or simply d-classical, if it satisfies the Hahn´s property [18], that is to say, the MPS $\{Q_n\}_{n\geq 0}$ is also d-orthogonal, where $Q_n(x) := (n+1)^{-1}P'_{n+1}(x)$ is the monic derivative.*

When $d = 1$, then we meet the definition of a classical sequence (see definition 1.13).

Definition 1.25. *[11] A PS $\{W_n\}_{n\geq 0}$ is d-symmetric if it fulfils*

$$W_n(\xi_k x) = \xi_k^n W_n(x), \;\; n \geq 0, \;\; k = 1, 2, \ldots, d,$$

where $\xi_k = \exp\left(\frac{2ik\pi}{d+1}\right)$, $k = 1, \ldots, d$, $\xi_k^{d+1} = 1$.

If $d = 1$, then $\xi_1 = -1$ and we meet the definition of a symmetric PS in which we have the following property

$$W_n(-x) = (-1)^n W_n(x), \;\; n \geq 0.$$

Definition 1.26. *[11] The functional $\Gamma = (\Gamma^1, \ldots, \Gamma^d)$ is d-symmetric if*

$$(\Gamma^\nu)_{(d+1)n+\mu-1} =< \Gamma^\nu, x^{(d+1)n+\mu-1} >= 0,$$

$$1 \leq \nu \leq d, \;\; 1 \leq \mu \leq d+1, \;\; \nu \neq \mu, \; n \geq 0.$$

In particular, the functional $w = (w_0, \ldots, w_{d-1})$ is d-symmetric if for each integer $0 \leq j \leq d-1$

$$(w_j)_{(d+1)n+i} = 0, \; i = 0, 1, \ldots, d, \;\; i \neq j, \; n \geq 0.$$

If $d = 1$, then Γ is reduced to a single form and we meet the definition of a symmetric form in which we have the following property

$$(\Gamma)_{(2n+1)} = 0, \;\; n \geq 0.$$

Theorem 1.27. *[11] Let $\{W_n\}_{n\geq 0}$ be a d-orthogonal MPS with respect to the functional $w = (w_0, \ldots, w_{d-1})$. The following statements are equivalent:*

a) *The functional Γ is d-symmetric.*

b) *$\{W_n\}_{n\geq 0}$ is d-symmetric.*

c) *$\{W_n\}_{n\geq 0}$ fulfils the following recurrence relation:*

$$W_{n+d+1}(x) = xW_{n+d}(x) - \gamma_{n+1}^0 W_n(x), \;\; n \geq 0,$$

$$W_n(x) = x^n, \;\; 0 \leq n \leq d.$$

1.4 Finite-type relations between polynomial sequences

The definitions and results of this section can be read in reference [33], expect the last one (lemma 1.35).

Let Φ be a monic polynomial and let t denote its degree, $deg(\Phi)$.

Definition 1.28. *We say that a PS* $\{W_n\}_{n\geq 0}$ *is compatible with* Φ *if* $\Phi w_n \neq 0,\ \ n \geq 0$.

Remark 1.29. *Any OPS is compatible with any monic polynomial.*

Definition 1.30. *Given two MPSs* $\{Q_n\}_{n\geq 0}$ *and* $\{R_n\}_{n\geq 0}$*, if there is an integer* $s \geq 0$ *such that*

$$\Phi(x)Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} R_\nu(x),\ \ n \geq s, \tag{1.16}$$

$$\exists r \geq s:\ \ \lambda_{r,r-s} \neq 0, \tag{1.17}$$

we say that (1.16)-(1.17) is a finite-type relation between sequences $\{R_n\}_{n\geq 0}$ *and* $\{Q_n\}_{n\geq 0}$*, with respect to* $\Phi(x)$.

When, instead of (1.17), we take

$$\lambda_{n,n-s} \neq 0,\ \ n \geq s, \tag{1.18}$$

we shall say that (1.16)-(1.18) is a strictly finite-type relation.

Remark 1.31. *A finite-type relation is not a commutative relation.*

Theorem 1.32. *Let* $\{Q_n\}_{n\geq 0}$ *and* $\{R_n\}_{n\geq 0}$ *be two MPS and* $\{v_n\}_{n\geq 0}$ *and* $\{r_n\}_{n\geq 0}$ *their dual sequences, respectively.*

Let $\{R_n\}_{n\geq 0}$ *be compatible with a polynomial* Φ*. The following properties are equivalent:*

i) *There is an integer* $s \geq 0$ *such that*

$$\Phi(x)Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} R_\nu(x),\ \ n \geq s,$$

$$\exists r \geq s:\ \ \lambda_{r,r-s} \neq 0,$$

ii) *There are an integer $s \geq 0$ and an application from $\mathbb{N}$ into $\mathbb{N} : m \mapsto \mu_m$ satisfying*

$$\max(0, m-t) \leq \mu_m \leq m+s, \quad m \geq 0,$$

$$\exists m_0 \geq 0 : \mu_{m_0} = m_0 + s,$$

and such that

$$\Phi r_m = \sum_{\nu=m-t}^{\mu_m} \lambda_{\nu,m} v_\nu, \;\; m \geq t,$$

$$\lambda_{\mu_m,m} \neq 0, \;\; m \geq 0.$$

Remark 1.33. *When the relation between $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ is strictly of finite-type, we have $\mu_m = m+s, \;\; m \geq 0$.*

Theorem 1.34. *For any MOPSs $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ and any monic polynomial Φ, the following assertions are equivalent:*

a) *There is an integer $s \geq 0$ such that*

$$\Phi(x)Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} R_\nu(x), \;\; n \geq s,$$

$$\exists r \geq s : \;\; \lambda_{r,r-s} \neq 0.$$

b) *There is an integer $s \geq 0$ such that*

$$\Phi(x)Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} R_\nu(x), \;\; n \geq s,$$

$$\lambda_{n,n-s} \neq 0, \;\; n \geq s.$$

c) *There exist a number $k_0 \neq 0$ and a monic polynomial Λ_s, $\deg \Lambda_s = s$, such that*

$$\Phi r_0 = k_0 \Lambda_s v_0.$$

d) *There exist an integer* $t \geq 0$ *and a monic polynomial* Λ_s, $\deg \Lambda_s = s$, *such that*

$$\Lambda_s(x)R_m(x) = \sum_{\nu=m-t}^{m+s} \tilde{\lambda}_{m,\nu} Q_\nu(x), \ \ m \geq t,$$

$$\tilde{\lambda}_{m,m-t} \neq 0, \ \ m \geq t.$$

We may write

$$\tilde{\lambda}_{m,\nu} = \frac{< v_0, Q_s^2 >}{\lambda_{s,0}} \frac{< r_0, R_m^2 >}{< v_0, Q_\nu^2 >} \lambda_{\nu,m}, \ \ 0 \leq \nu \leq m+s,$$

$$\lambda_{n,\nu} = \frac{< r_0, R_t^2 >}{\tilde{\lambda}_{t,0}} \frac{< v_0, Q_n^2 >}{< r_0, R_\nu^2 >} \tilde{\lambda}_{\nu,n}, \ \ 0 \leq \nu \leq n+t,$$

$$\Lambda_s(x) = \sum_{\nu=0}^{s} \frac{< v_0, Q_s^2 >}{\lambda_{s,0}} \frac{\lambda_{\nu,0}}{< v_0, Q_\nu^2 >} Q_\nu(x),$$

$$k_0 = \frac{\lambda_{s,0}}{< v_0, Q_s^2 >} = \frac{< r_0, R_t^2 >}{\tilde{\lambda}_{t,0}}.$$

We finalize by proving a lemma which will be helpful later on.

Lemma 1.35. *Let us suppose a MOPS* $\{R_n\}_{n \geq 0}$ *such that*

$$R_{n+1}(x) = (x-\beta)R_n(x) - \gamma R_{n-1}(x), \ n \geq 0, \ R_{-1}(x) = 0, \ R_0(x) = 1,$$

where $\beta \in \mathbb{C}$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$. *Taking* $s \geq 1$, *let us define* $B_n(x) = \sum_{\nu=0}^{s} \lambda_\nu R_{n-\nu}(x), \ n \geq s-1$, *where* $\lambda_\nu \in \mathbb{C}, \ 0 \leq \nu \leq s$. *Then,*

$$B_{n+2}(x) = (x-\beta)B_{n+1}(x) - \gamma B_n(x), \ n \geq s-1.$$

Proof.

$$\begin{aligned}
B_{n+2}(x) - xB_{n+1}(x) &= \sum_{\nu=0}^{s} \lambda_\nu R_{n+2-\nu}(x) - x \sum_{\nu=0}^{s} \lambda_\nu R_{n+1-\nu}(x) \\
&= \sum_{\nu=0}^{s} \lambda_\nu \Big(R_{n+2-\nu}(x) - xR_{n+1-\nu}(x) \Big) \\
&= \sum_{\nu=0}^{s} \lambda_\nu \Big(-\beta R_{n+1-\nu}(x) - \gamma R_{n-\nu}(x) \Big) \\
&= -\beta \sum_{\nu=0}^{s} \lambda_\nu R_{n+1-\nu}(x) - \gamma \sum_{\nu=0}^{s} \lambda_\nu R_{n-\nu}(x) \\
&= -\beta B_{n+1}(x) - \gamma B_n(x),
\end{aligned}$$

thus, $B_{n+2}(x) = (x-\beta)B_{n+1}(x) - \gamma B_n(x), \ n \geq s-1.$ □

1.5 Quasi-orthogonality

Definition 1.36. *[29] Let s be a non-negative integer. A sequence $\{B_n\}_{n\geq 0}$ is quasi-orthogonal of order s with respect to $u \in \mathcal{P}$ if it satisfies*

$$< u, B_m B_n >= 0, \;\; 0 \leq m \leq n-(s+1), \; n \geq s+1;$$
$$\exists r \geq s: \; < u, B_{r-s} B_r > \neq 0.$$

Definition 1.37. *[29] Let s be a non-negative integer. A sequence $\{B_n\}_{n\geq 0}$ is strictly quasi-orthogonal of order s with respect to $u \in \mathcal{P}$ if it satisfies*

$$< u, B_m B_n >= 0, \;\; 0 \leq m \leq n-(s+1), \; n \geq s+1;$$

$$\forall n \geq s: \; < u, B_{n-s} B_n > \neq 0.$$

From its definition, we know that every strictly quasi-orthogonal sequence of order s, $s \geq 0$, with respect to a form u, is free [26].

Chapter 2

Cubic decomposition of a monic polynomial sequence

In this chapter, we present the cubic decomposition of any MPS $\{W_n\}_{n\geq 0}$, establishing its existence and uniqueness (for each set of parameters involved). We also prove a constructive theorem which calculates every component sequence of the CD, from the structure coefficients of the given sequence $\{W_n\}_{n\geq 0}$. The diagonal CD, which is the most simple CD, is characterized in several manners, namely through the forms of the dual sequence of $\{W_n\}_{n\geq 0}$, and we, also, describe the component sequences of the CD of the canonical sequence.

2.1 The cubic decomposition existence and uniqueness

Let us consider the polynomial $\varpi(x) = x^3 + px^2 + qx + r$; $p, q, r \in \mathbb{C}$. Given $u \in \mathcal{P}'$, let us consider the form $\sigma_\varpi u$ defined in (1.5).

Every monic polynomial $W(x)$ can be rearranged in terms of polynomials evaluated in $\varpi(x)$. This rearrangement demands a monic polynomial of degree one, say $x - a$, and another of degree two, say $(x - b)(x - c)$, where a, b and c are complex constants, both necessary for adjusting the degrees. We will call these two polynomials as auxiliary polynomials and the constants a, b and c as their zeros. The result will be called a cubic decomposition (CD) of $W(x)$ and it is precisely described in the following lemma.

Lemma 2.1 (Cubic decomposition existence)**.** *Let $W(x)$ be a monic polynomial.*

If $\deg W = 3n$*, then there are three polynomials $P_n(x)$, $a^1_{n-1}(x)$ and $a^2_{n-1}(x)$ such that $P_n(x)$ is monic,* $\deg P_n = n$, $\deg a^1_{n-1} \leq n - 1$, $\deg a^2_{n-1} \leq n - 1$ *and*

$$W(x) = P_n(\varpi(x)) + (x - a)a^1_{n-1}(\varpi(x)) + (x - b)(x - c)a^2_{n-1}(\varpi(x)),$$

considering $a^1_{-1} = a^2_{-1} = 0$.

If $\deg W = 3n+1$, *then there are three polynomials* $Q_n(x)$, $b^1_n(x)$ *and* $b^2_{n-1}(x)$ *such that* $Q_n(x)$ *is monic,* $\deg Q_n = n$, $\deg b^1_n \leq n$, $\deg b^2_{n-1} \leq n-1$ *and*

$$W(x) = b^1_n(\varpi(x)) + (x-a)Q_n(\varpi(x)) + (x-b)(x-c)b^2_{n-1}(\varpi(x)),$$

considering $b^2_{-1} = 0$.

If $\deg W = 3n+2$, *then there are three polynomials* $R_n(x)$, $c^1_n(x)$ *and* $c^2_n(x)$ *so that* $R_n(x)$ *is monic,* $\deg R_n = n$, $\deg c^1_n \leq n$, $\deg c^2_n \leq n$ *and*

$$W(x) = c^1_n(\varpi(x)) + (x-a)c^2_n(\varpi(x)) + (x-b)(x-c)R_n(\varpi(x)).$$

Proof. Proceeding by induction over the degree of $W(x)$, let us calculate the initial statements related with $n=0,\ n=1$ and $n=2$.

If $\deg W = 0$, then $W(x) = 1 = P_0(\varpi(x)) + (x-a)a^1_{-1}(\varpi(x)) + (x-b)(x-c)a^2_{-1}(\varpi(x))$, since $P_0(x) = 1$ and $a^1_{-1} = a^2_{-1} = 0$.

If $\deg W = 1$, then

$$W(x) = x - \beta_0 = a - \beta_0 + (x-a)Q_0(\varpi(x)) + (x-b)(x-c)b^2_{-1}(\varpi(x)),$$

where $b^1_0(x) = a - \beta_0$, $Q_0(x) = 1$ and $b^2_{-1}(x) = 0$.

If $\deg W = 2$, then

$$\begin{aligned} W(x) &= (x-\beta_1)(x-\beta_0) - \chi_{0,0} \\ &= (x-b)(x-c) + (c-\beta_0)(x-b) + (b-\beta_1)(x-\beta_0) - \chi_{0,0} \\ &= (a-b)(c-\beta_0) + (a-\beta_0)(b-\beta_1) - \chi_{0,0} + (b+c-\beta_0-\beta_1)(x-a) + (x-b)(x-c) \\ &= c^1_0(\varpi(x)) + (x-a)c^2_0(\varpi(x)) + (x-b)(x-c)R_0(\varpi(x)); \end{aligned}$$

where $c^1_0(x) = (a-b)(c-\beta_0) + (a-\beta_0)(b-\beta_1) - \chi_{0,0},\ c^2_0(x) = b+c-\beta_0-\beta_1$ and $R_0(x) = 1$.

Let us suppose that any polynomial $W(x)$ so that $\deg W \leq 3n+2$ admits the enunciated decomposition.

Let $W(x)$ be a monic polynomial so that $\deg W = 3n+3$. Then, by Euclidean division, of $W(x)$ by $\varpi(x)$, we obtain

$$W(x) = \varpi(x)p_{3n}(x) + \alpha_n x^2 + \theta_n x + \xi_n,$$

where $p_{3n}(x)$ is a monic polynomial with degree $3n$ and $\alpha_n,\ \theta_n$ and ξ_n are constants.

Let us note that $\alpha_n x^2 + \theta_n x + \xi_n$ can be written as *span*$\{1, (x-a), (x-b)(x-c)\}$:

$$\alpha_n x^2 + \theta_n x + \xi_n = \big(\theta_n + \alpha_n(b+c)\big)a + \xi_n - \alpha_n bc + \big(\theta_n + \alpha_n(b+c)\big)(x-a) + \alpha_n(x-b)(x-c).$$

Regarding the induction hypothesis,

$$p_{3n}(x) = P_n(\varpi(x)) + (x-a)a^1_{n-1}(\varpi(x)) + (x-b)(x-c)a^2_{n-1}(\varpi(x)),$$

and therefore,

$$\begin{aligned}
W(x) =&\varpi(x)P_n(\varpi(x)) + (x-a)\varpi(x)a^1_{n-1}(\varpi(x)) + (x-b)(x-c)\varpi(x)a^2_{n-1}(\varpi(x))\\
&+ \alpha_n x^2 + \theta_n x + \xi_n\\
=&(xP_n)(\varpi(x)) + (x-a)(xa^1_{n-1})(\varpi(x)) + (x-b)(x-c)(xa^2_{n-1})(\varpi(x))\\
&+ \big(\theta_n + \alpha_n(b+c)\big)a + \xi_n - \alpha_n bc + \big(\theta_n + \alpha_n(b+c)\big)(x-a) + \alpha_n(x-b)(x-c)\\
=&\Big((xP_n)(\varpi(x)) + \big(\theta_n + \alpha_n(b+c)\big)a + \xi_n - \alpha_n bc\Big)\\
&+ (x-a)\Big(\theta_n + \alpha_n(b+c) + (xa^1_{n-1})(\varpi(x))\Big)\\
&+ (x-b)(x-c)\Big(\alpha_n + (xa^2_{n-1})(\varpi(x))\Big).
\end{aligned}$$

Thus, we have obtained the claimed decomposition for $W(x)$, where

$$P_{n+1}(x) = xP_n(x) + \big(\theta_n + \alpha_n(b+c)\big)a + \xi_n - \alpha_n bc,$$

$$a^1_n(x) = \theta_n + \alpha_n(b+c) + xa^1_{n-1}(x) \text{ and } a^2_n(x) = \alpha_n + xa^2_{n-1}(x).$$

Let $W(x)$ be a monic polynomial so that $\deg W = 3n+4$. Then, by Euclidean division, of $W(x)$ by $\varpi(x)$, we obtain

$$W(x) = \varpi(x)p_{3n+1}(x) + \alpha_n x^2 + \theta_n x + \xi_n,$$

where p_{3n+1} is a monic polynomial with degree $3n+1$ and $\alpha_n,\ \theta_n,\ \xi_n$ are constants.

By induction hypothesis,

$$p_{3n+1}(x) = b^1_n(\varpi(x)) + (x-a)Q_n(\varpi(x)) + (x-b)(x-c)b^2_{n-1}(\varpi(x)),$$

and consequently,

$$\begin{aligned}
W(x) =&\varpi(x)b^1_n(\varpi(x)) + (x-a)\varpi(x)Q_n(\varpi(x)) + (x-b)(x-c)\varpi(x)b^2_{n-1}(\varpi(x))\\
&+ \alpha_n x^2 + \theta_n x + \xi_n\\
=&(xb^1_n)(\varpi(x)) + (x-a)(xQ_n)(\varpi(x)) + (x-b)(x-c)(xb^2_{n-1})(\varpi(x))\\
&+ \big(\theta_n + \alpha_n(b+c)\big)a + \xi_n - \alpha_n bc + \big(\theta_n + \alpha_n(b+c)\big)(x-a) + \alpha_n(x-b)(x-c)\\
=&\Big((xb^1_n)(\varpi(x)) + \big(\theta_n + \alpha_n(b+c)\big)a + \xi_n - \alpha_n bc\Big)\\
&+ (x-a)\Big(\theta_n + \alpha_n(b+c) + (xQ_n)(\varpi(x))\Big)\\
&+ (x-b)(x-c)\Big(\alpha_n + (xb^2_{n-1})(\varpi(x))\Big).
\end{aligned}$$

In this manner, we have obtained the claimed decomposition for $W(x)$, where

$$Q_{n+1}(x) = \theta_n + \alpha_n(b+c) + xQ_n(x),$$

$$b^1_{n+1}(x)=xb^1_n(x)+\big(\theta_n+\alpha_n(b+c)\big)a+\xi_n-\alpha_n bc \text{ and } b^2_n(x)=\alpha_n+xb^2_{n-1}(x).$$

Let $W(x)$ be a monic polynomial so that $\deg W=3n+5$. Then, by Euclidean division, of $W(x)$ by $\varpi(x)$, we obtain

$$W(x)=\varpi(x)p_{3n+2}(x)+\alpha_n x^2+\theta_n x+\xi_n,$$

where p_{3n+2} is a monic polynomial with degree $3n+2$ and $\alpha_n,\ \theta_n,\ \xi_n$ are constants.

By induction hypothesis,

$$p_{3n+2}(x)=c^1_n(\varpi(x))+(x-a)c^2_n(\varpi(x))+(x-b)(x-c)R_n(\varpi(x)),$$

and therefore,

$$\begin{aligned}
W(x)=&\varpi(x)c^1_n(\varpi(x))+(x-a)\varpi(x)c^2_n(\varpi(x))+(x-b)(x-c)\varpi(x)R_n(\varpi(x))\\
&+\alpha_n x^2+\theta_n x+\xi_n\\
=&(xc^1_n)(\varpi(x))+(x-a)(xc^2_n)(\varpi(x))+(x-b)(x-c)(xR_n)(\varpi(x))\\
&+\big(\theta_n+\alpha_n(b+c)\big)a+\xi_n-\alpha_n bc+\big(\theta_n+\alpha_n(b+c)\big)(x-a)+\alpha_n(x-b)(x-c)\\
=&\Big((xc^1_n)(\varpi(x))+\big(\theta_n+\alpha_n(b+c)\big)a+\xi_n-\alpha_n bc\Big)\\
&+(x-a)\Big(\theta_n+\alpha_n(b+c)+(xc^2_n)(\varpi(x))\Big)\\
&+(x-b)(x-c)\Big(\alpha_n+(xR_n)(\varpi(x))\Big).
\end{aligned}$$

Thus, we have obtained the desired decomposition for $W(x)$, where

$$R_{n+1}(x)=\alpha_n+xR_n(x),$$

$$c^1_{n+1}(x)=xc^1_n(x)+\big(\theta_n+\alpha_n(b+c)\big)a+\xi_n-\alpha_n bc \text{ and } c^2_{n+1}(x)=\theta_n+\alpha_n(b+c)+xc^2_n(x).$$

□

Consequently, for any MPS $\{W_n\}_{n\geq0}$, there are three MPSs $\{P_n\}_{n\geq0}$, $\{Q_n\}_{n\geq0}$ and $\{R_n\}_{n\geq0}$, so that

$$\begin{aligned}
W_{3n}(x) &= P_n(\varpi(x))+(x-a)a^1_{n-1}(\varpi(x))+(x-b)(x-c)a^2_{n-1}(\varpi(x)), &(2.1)\\
W_{3n+1}(x) &= b^1_n(\varpi(x))+(x-a)Q_n(\varpi(x))+(x-b)(x-c)b^2_{n-1}(\varpi(x)), &(2.2)\\
W_{3n+2}(x) &= c^1_n(\varpi(x))+(x-a)c^2_n(\varpi(x))+(x-b)(x-c)R_n(\varpi(x)). &(2.3)
\end{aligned}$$

Let us remark that considering the parameters a,b and c, and the coefficients p,q and r, we obtain the most general CD, allowing the study of a wide number of particular choices.

We will call the decomposition (2.1)-(2.3) of $\{W_n\}_{n\geq0}$ a cubic decomposition (CD) of $\{W_n\}_{n\geq0}$ where the sequences:

- $\{P_n\}_{n\geq 0}, \{Q_n\}_{n\geq 0}, \{R_n\}_{n\geq 0}$ are called the principal components;
- $\{a^1_{n-1}\}_{n\geq 0}$, $\{a^2_{n-1}\}_{n\geq 0}$, $\{b^1_n\}_{n\geq 0}$, $\{b^2_{n-1}\}_{n\geq 0}$, $\{c^1_n\}_{n\geq 0}$, $\{c^2_n\}_{n\geq 0}$ are called the secondary components.

We will denote by $\{w_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$ and $\{r_n\}_{n\geq 0}$ the dual sequences of $\{W_n\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, respectively.

Lemma 2.2. *Let $P(x)$, $Q(x)$ and $R(x)$ be three polynomials.*
$P(\varpi(x))+(x-a)Q(\varpi(x))+(x-b)(x-c)R(\varpi(x)) = 0$ *implies* $P(x) = Q(x) = R(x) = 0$.

Proof. Let us suppose that the polynomials $P(x)$, $Q(x)$, $R(x)$ are non trivial and $\deg(P(x)) = n$, $\deg(Q(x)) = k$, $\deg(R(x)) = m$, for some non negative integers n, k and m.

If $n < k$, then $3n < 3k < 3k + 1$ and $\deg\big(P(\varpi(x)) + (x-a)Q(\varpi(x))\big) = 3k + 1$.

If $n > k$, then $3n > 3k + 1$ and $\deg\big(P(\varpi(x)) + (x-a)Q(\varpi(x))\big) = 3n$.

If $n = k$, then $\deg\big(P(\varpi(x)) + (x-a)Q(\varpi(x))\big) = 3k + 1$.

So, the identity

$$P(\varpi(x)) + (x-a)Q(\varpi(x)) = -(x-b)(x-c)R(\varpi(x))$$

yields $\max\{3n, 3k+1\} = 3m + 2$, which is impossible. □

Corollary 2.3 (Uniqueness of a CD)**.** *For each choice of coefficients p, q and r of the monic polynomial $\varpi(x)$, and constants a, b and c (zeros of the auxiliary polynomials $x - a$ and $(x-b)(x-c)$), the corresponding CD of a MPS is unique.*

Proof. It is an obvious corollary of lemma 2.2. □

Given a MPS $\{W_n\}_{n\geq 0}$ CD with parameters a, b, c and a particular cubic transformation $\varpi(x)$ we may obtain another CD for an enlarged $\varpi(x)$, through a shift. Next, we analyse three cases: $p = 0$, $q = 0$ and $r = 0$.

Let us consider that a CD of $\{W_n\}_{n\geq 0}$ is easily calculated for every parameters a, b, c and $\varpi(x) = x^3 + qx + r$ (i.e., with $p = 0$).

For any constants p_0, q_0 and r_0, we may get a CD for $\varpi(x) = x^3 + p_0x^2 + q_0x + r_0$, performing the following steps.

- Determine the constants α, q and r such that

$$\begin{cases} 3\alpha = p_0 \\ 3\alpha^2 + q = q_0 \\ \alpha^3 + q\alpha + r = r_0 \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{p_0}{3} \\ q = q_0 - \frac{p_0^2}{3} \\ r = r_0 - \left(\frac{p_0}{3}\right)^3 - q\frac{p_0}{3} \end{cases}$$

- Calculate the $\{W_n\}_{n\geq 0}$ CD for a, b, c and $\varpi(x) = x^3 + qx + r$.
- Apply the affine transformation $x \to x + \alpha$.

After these steps, we obtain the CD of the MPS $\{\tilde{W}_n\}_{n\geq 0}$, defined by $\tilde{W}_n(x) = W_n(x+\alpha)$, with parameters $a-\alpha, b-\alpha, c-\alpha$ and $\varpi(x) = x^3 + p_0x^2 + q_0x + r_0$, because if $\varpi(x) = x^3 + qx + r$, then $\varpi(x+\alpha) = x^3 + 3\alpha x^2 + (3\alpha^2 + q)x + \alpha^3 + q\alpha + r$.

Let us consider that a CD of $\{W_n\}_{n\geq 0}$ is easily calculated for every parameters a, b, c and $\varpi(x) = x^3 + px^2 + r$ (i.e., with $q = 0$).

For any constants p_0, q_0 and r_0, we may get a CD for $\varpi(x) = x^3 + p_0x^2 + q_0x + r_0$, performing the following steps.

- Determine the constants α, p and r such that

$$\left\{\begin{array}{c} 3\alpha + p = p_0 \\ 3\alpha^2 + 2\alpha p = q_0 \\ \alpha^3 + p\alpha^2 + r = r_0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} 3\alpha^2 - 2\alpha p_0 + q_0 = 0 \\ p = p_0 - 3\alpha \\ r = r_0 - \alpha^3 - p\alpha^2 \end{array}\right.$$

- Calculate the $\{W_n\}_{n\geq 0}$ CD for a, b, c and $\varpi(x) = x^3 + px^2 + r$.

- Apply the affine transformation $x \to x + \alpha$.

After these steps, we obtain the CD of the MPS $\{\tilde{W}_n\}_{n\geq 0}$, defined by $\tilde{W}_n(x) = W_n(x+\alpha)$, with parameters $a-\alpha, b-\alpha, c-\alpha$ and $\varpi(x) = x^3 + p_0x^2 + q_0x + r_0$, because if $\varpi(x) = x^3 + px^2 + r$, then $\varpi(x+\alpha) = x^3 + (3\alpha + p)x^2 + (3\alpha^2 + 2\alpha p)x + \alpha^3 + p\alpha^2 + r$.

Let us consider that a CD of $\{W_n\}_{n\geq 0}$ is easily calculated for every parameters a, b, c and $\varpi(x) = x^3 + px^2 + qx$ (i.e., with $r = 0$).

For any constants p_0, q_0 and r_0, we may get a CD for $\varpi(x) = x^3 + p_0x^2 + q_0x + r_0$, performing the following steps.

- Determine the constants α, p and q such that

$$\left\{\begin{array}{c} 3\alpha + p = p_0 \\ 3\alpha^2 + 2\alpha p + q = q_0 \\ \alpha^3 + p\alpha^2 + q\alpha = r_0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} \alpha^3 - p_0\alpha^2 + q_0\alpha - r_0 = 0 \\ p = p_0 - 3\alpha \\ q = q_0 + 3\alpha^2 - 2\alpha p_0 \end{array}\right.$$

- Calculate the $\{W_n\}_{n\geq 0}$ CD for a, b, c and $\varpi(x) = x^3 + px^2 + qx$.

- Apply the affine transformation $x \to x + \alpha$.

After these steps, we obtain the CD of the MPS $\{\tilde{W}_n\}_{n\geq 0}$, defined by $\tilde{W}_n(x) = W_n(x+\alpha)$, with parameters $a-\alpha, b-\alpha, c-\alpha$ and $\varpi(x) = x^3 + p_0x^2 + q_0x + r_0$, because if $\varpi(x) = x^3 + px^2 + qx$, then $\varpi(x+\alpha) = x^3 + (3\alpha + p)x^2 + (3\alpha^2 + 2\alpha p + q)x + \alpha^3 + p\alpha^2 + q\alpha$.

In brief, when we take one coefficient of $\varpi(x)$ null, it implies a translation of the variable x. Remembering that some properties of a MPS, as for example, regular orthogonality, d-orthogonality and classical character are invariant under shifts, these options may signify, for some analyses, no loss of generality.

We may, also, see how changes in the parameters a, b, c, alter each component of a CD. These considerations are better written, using a matrix notation of the CD that will be introduced in theorem 2.5.

2.2 Constructive characterization

A MPS, $\{W_n\}_{n\geq 0}$, is defined by its structure coefficients (1.1-1.2). Thus, we can characterize the component sequences of a CD of $\{W_n\}_{n\geq 0}$ in terms of those coefficients.

In this characterization, we will use the lemma 2.2, some matrix formalism, in particular, the natural matrix representation of the nine component sequences of the CD, and the following identities.

Lemma 2.4.

$$I1.\;\; x(x-a) = -(a-b)(a-c) + (b+c-a)(x-a) + (x-b)(x-c)$$

$$I2.\;\; x(x-b)(x-c) = \Theta(\varpi(x)) + L(x-a) - (b+c+p)(x-b)(x-c)$$

where

$$\Theta(x) = x - r + aL + bc(b+c+p), \tag{2.4}$$

$$L = bc - q - (b+c+p)(b+c). \tag{2.5}$$

Proof. By simple calculations, we can assure each equality. □

Theorem 2.5 (Constructive characterization of the CD component sequences of a MPS in terms of its structure coefficients)**.** *A MPS $\{W_n\}_{n\geq 0}$, with structure coefficients (1.1-1.2), admits the CD (2.1)-(2.3) if and only if the following relations are fulfilled for $n \geq 0$,*

(Z_0) $b_0^1 = a - \beta_0$,

$$(Z_1)\;\; c_n^1(x) = -\sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}\, b_\nu^1(x) - (\beta_{3n+1} - a) b_n^1(x) + \Theta(x) b_{n-1}^2(x) - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}\, c_\nu^1(x)$$
$$- \sum_{\nu=0}^{n} \chi_{3n,3\nu}\, P_\nu(x) - (a-b)(a-c) Q_n(x),$$

$$(Z_2)\;\; c_n^2(x) = -\sum_{\nu=0}^{n} \chi_{3n,3\nu}\, a_{\nu-1}^1(x) + b_n^1(x) + L b_{n-1}^2(x) - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}\, c_\nu^2(x)$$
$$- \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}\, Q_\nu(x) - (\beta_{3n+1} + a - b - c) Q_n(x),$$

$$(Z_3)\;\; R_n(x) = -\sum_{\nu=0}^{n} \chi_{3n,3\nu}\, a^2_{\nu-1}(x) - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}\, b^2_{\nu-1}(x) - (\beta_{3n+1} + b + c + p) b^2_{n-1}(x)$$

$$+Q_n(x) - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}\, R_\nu(x),$$

$$(Z_4)\;\; P_{n+1}(x) = -\sum_{\nu=0}^{n} \chi_{3n+1,3\nu}\, P_\nu(x) - (\beta_{3n+2} - a) c^1_n(x) - \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2}\, c^1_\nu(x)$$

$$-\sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1}\, b^1_\nu(x) - (a-b)(a-c) c^2_n(x) + \Theta(x) R_n(x),$$

$$(Z_5)\;\; a^1_n(x) = -\sum_{\nu=0}^{n} \chi_{3n+1,3\nu}\, a^1_{\nu-1}(x) + c^1_n(x) - \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2}\, c^2_\nu(x)$$

$$-(\beta_{3n+2} + a - b - c) c^2_n(x) - \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1}\, Q_\nu(x) + L R_n(x),$$

$$(Z_6)\;\; a^2_n(x) = -\sum_{\nu=0}^{n} \chi_{3n+1,3\nu}\, a^2_{\nu-1}(x) - \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1}\, b^2_{\nu-1}(x) + c^2_n(x)$$

$$-\sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2}\, R_\nu(x) - (\beta_{3n+2} + b + c + p) R_n(x),$$

$$(Z_7)\;\; b^1_{n+1}(x) = -(a-b)(a-c) a^1_n(x) + \Theta(x) a^2_n(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1}\, b^1_\nu(x)$$

$$-\sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2}\, c^1_\nu(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu}\, P_\nu(x) - (\beta_{3n+3} - a) P_{n+1}(x),$$

$$(Z_8)\;\; Q_{n+1}(x) = -\sum_{\nu=0}^{n} \chi_{3n+2,3\nu}\, a^1_{\nu-1}(x) - (\beta_{3n+3} + a - b - c) a^1_n(x) + L a^2_n(x)$$

$$-\sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2}\, c^2_\nu(x) + P_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1}\, Q_\nu(x),$$

$$(Z_9)\;\; b^2_n(x) = a^1_n(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu}\, a^2_{\nu-1}(x) - (\beta_{3n+3} + b + c + p) a^2_n(x)$$

$$-\sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1}\, b^2_{\nu-1}(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2}\, R_\nu(x),$$

where by convention $\sum_{\nu=0}^{-1} . = 0.$

Proof. Let us write the CD (2.1)-(2.3) in the following matrix notation

$$\begin{pmatrix} W_{3n}(x) \\ W_{3n+1}(x) \\ W_{3n+2}(x) \end{pmatrix} = \begin{pmatrix} P_n(\varpi(x)) & a^1_{n-1}(\varpi(x)) & a^2_{n-1}(\varpi(x)) \\ b^1_n(\varpi(x)) & Q_n(\varpi(x)) & b^2_{n-1}(\varpi(x)) \\ c^1_n(\varpi(x)) & c^2_n(\varpi(x)) & R_n(\varpi(x)) \end{pmatrix} \begin{pmatrix} 1 \\ x-a \\ (x-b)(x-c) \end{pmatrix}.$$

Introducing, also, the following notation

$$M_n(x) = \begin{pmatrix} L_{1,n}(x) \\ L_{2,n}(x) \\ L_{3,n}(x) \end{pmatrix} = \begin{pmatrix} P_n(x) & a^1_{n-1}(x) & a^2_{n-1}(x) \\ b^1_n(x) & Q_n(x) & b^2_{n-1}(x) \\ c^1_n(x) & c^2_n(x) & R_n(x) \end{pmatrix},$$

$G(x) = \begin{pmatrix} 1 \\ x-a \\ (x-b)(x-c) \end{pmatrix}$, the CD (2.1)-(2.3) is equivalent to the equalities:

$$W_{3n}(x) = L_{1,n}(\varpi(x))G(x);\ W_{3n+1}(x) = L_{2,n}(\varpi(x))G(x); \\ W_{3n+2}(x) = L_{3,n}(\varpi(x))G(x). \tag{2.6}$$

The following identity, obtained by (1.1-1.2),

$$W_{3n+2}(x) + \beta_{3n+1}W_{3n+1}(x) + \sum_{\nu=0}^{3n} \chi_{3n,\nu}W_\nu(x) = xW_{3n+1}(x)$$

$$\Leftrightarrow W_{3n+2}(x) + \beta_{3n+1}W_{3n+1}(x) + \sum_{\nu=0}^{n} \chi_{3n,3\nu}W_{3\nu}(x) + \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}W_{3\nu+1}(x) \\ + \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}W_{3\nu+2}(x) = xW_{3n+1}(x), \tag{2.7}$$

is equivalent to the next one, using equalities (2.6)

$$\Big\{L_{3,n}(\varpi(x)) + \beta_{3n+1}L_{2,n}(\varpi(x)) + \sum_{\nu=0}^{n} \chi_{3n,3\nu}L_{1,\nu}(\varpi(x)) + \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}L_{2,\nu}(\varpi(x)) \\ + \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}L_{3,\nu}(\varpi(x))\Big\}G(x) = xL_{2,n}(\varpi(x))G(x).$$

Regarding *I1* and *I2* (lema 2.4) we obtain

$$
\begin{aligned}
xG(x) &= \begin{pmatrix} x \\ x(x-a) \\ x(x-b)(x-c) \end{pmatrix} \\
&= \begin{pmatrix} a \\ -(a-b)(a-c) \\ \Theta(\varpi(x)) \end{pmatrix} + (x-a) \begin{pmatrix} 1 \\ b+c-a \\ L \end{pmatrix} + (x-b)(x-c) \begin{pmatrix} 0 \\ 1 \\ -(b+c+p) \end{pmatrix}.
\end{aligned}
$$

Now, by lemma 2.2, we know that $\big(P(\varpi(x)) \;\; Q(\varpi(x)) \;\; R(\varpi(x))\big)G(x) = 0$ implies $P = Q = R = 0$, thus, we obtain $(Z_1), (Z_2)$ and (Z_3) for $n \geq 0$:

$$
\begin{aligned}
&c_n^1(x) + \beta_{3n+1} b_n^1(x) + \sum_{\nu=0}^{n} \chi_{3n,3\nu} P_\nu(x) + \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1} b_\nu^1(x) + \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2} c_\nu^1(x) \\
&\quad = L_{2,n}(x) \begin{pmatrix} a \\ -(a-b)(a-c) \\ \Theta(x) \end{pmatrix} = a b_n^1(x) - (a-b)(a-c) Q_n(x) + \Theta(x) b_{n-1}^2(x),
\end{aligned}
$$

$$
\begin{aligned}
&c_n^2(x) + \beta_{3n+1} Q_n(x) + \sum_{\nu=0}^{n} \chi_{3n,3\nu} a_{\nu-1}^1(x) + \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1} Q_\nu(x) + \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2} c_\nu^2(x) \\
&\qquad = L_{2,n}(x) \begin{pmatrix} 1 \\ b+c-a \\ L \end{pmatrix} = b_n^1(x) + (b+c-a) Q_n(x) + L b_{n-1}^2(x),
\end{aligned}
$$

$$
\begin{aligned}
&R_n(x) + \beta_{3n+1} b_{n-1}^2(x) + \sum_{\nu=0}^{n} \chi_{3n,3\nu} a_{\nu-1}^2(x) + \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1} b_{\nu-1}^2(x) + \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2} R_\nu(x) \\
&\qquad = L_{2,n}(x) \begin{pmatrix} 0 \\ 1 \\ -(b+c+p) \end{pmatrix} = Q_n(x) - (b+c+p) b_{n-1}^2(x).
\end{aligned}
$$

Similarly, we obtain the other six. Let us consider, now, the identity:

$$
W_{3n+3}(x) + \beta_{3n+2} W_{3n+2}(x) + \sum_{\nu=0}^{3n+1} \chi_{3n+1,\nu} W_\nu(x) = x W_{3n+2}(x)
$$

$$
\begin{aligned}
\Leftrightarrow \Big\{ &L_{1,n+1}(\varpi(x)) + \beta_{3n+2} L_{3,n}(\varpi(x)) + \sum_{\nu=0}^{n} \chi_{3n+1,3\nu} L_{1,\nu}(\varpi(x)) + \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1} L_{2,\nu}(\varpi(x)) \\
&+ \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2} L_{3,\nu}(\varpi(x)) \Big\} G(x) = x L_{3,n}(\varpi(x)) G(x).
\end{aligned}
$$

By lemma 2.2, this last one and the following three are equivalent (where $n \geq 0$):

$$P_{n+1}(x) + \beta_{3n+2} c_n^1(x) + \sum_{\nu=0}^{n} \chi_{3n+1,3\nu} P_\nu(x) + \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1} b_\nu^1(x) + \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2} c_\nu^1(x)$$

$$= L_{3,n}(x) \begin{pmatrix} a \\ -(a-b)(a-c) \\ \Theta(x) \end{pmatrix} = a c_n^1(x) - (a-b)(a-c) c_n^2(x) + \Theta(x) R_n(x),$$

$$a_n^1(x) + \beta_{3n+2} c_n^2(x) + \sum_{\nu=0}^{n} \chi_{3n+1,3\nu} a_{\nu-1}^1(x) + \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1} Q_\nu(x) + \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2} c_\nu^2(x)$$

$$= L_{3,n}(x) \begin{pmatrix} 1 \\ b+c-a \\ L \end{pmatrix} = c_n^1(x) + (b+c-a) c_n^2(x) + L R_n(x),$$

$$a_n^2(x) + \beta_{3n+2} R_n(x) + \sum_{\nu=0}^{n} \chi_{3n+1,3\nu} a_{\nu-1}^2(x) + \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1} b_{\nu-1}^2(x) + \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2} R_\nu(x)$$

$$= L_{3,n}(x) \begin{pmatrix} 0 \\ 1 \\ -(b+c+p) \end{pmatrix} = c_n^2(x) - (b+c+p) R_n(x).$$

Finally,

$$W_{3n+4}(x) + \beta_{3n+3} W_{3n+3}(x) + \sum_{\nu=0}^{3n+2} \chi_{3n+2,\nu} W_\nu(x) = x W_{3n+3}(x)$$

$$\Leftrightarrow \Big\{ L_{2,n+1}(\varpi(x)) + \beta_{3n+3} L_{1,n+1}(\varpi(x)) + \sum_{\nu=0}^{n} \chi_{3n+2,3\nu} L_{1,\nu}(\varpi(x)) + \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1} L_{2,\nu}(\varpi(x))$$

$$+ \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2} L_{3,\nu}(\varpi(x)) \Big\} G(x) = x L_{1,n+1}(\varpi(x)) G(x).$$

By lemma 2.2, this last identity and the following three are equivalent (where $n \geq 0$):

$$b_{n+1}^1(x) + \beta_{3n+3} P_{n+1}(x) + \sum_{\nu=0}^{n} \chi_{3n+2,3\nu} P_\nu(x) + \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1} b_\nu^1(x) + \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2} c_\nu^1(x)$$

$$= L_{1,n+1}(x) \begin{pmatrix} a \\ -(a-b)(a-c) \\ \Theta(x) \end{pmatrix} = a P_{n+1}(x) - (a-b)(a-c) a_n^1(x) + \Theta(x) a_n^2(x),$$

$$Q_{n+1}(x)+\beta_{3n+3}a_n^1(x)+\sum_{\nu=0}^{n}\chi_{3n+2,3\nu}a_{\nu-1}^1(x)+\sum_{\nu=0}^{n}\chi_{3n+2,3\nu+1}Q_\nu(x)+\sum_{\nu=0}^{n}\chi_{3n+2,3\nu+2}c_\nu^2(x)$$

$$=L_{1,n+1}(x)\begin{pmatrix}1\\ b+c-a\\ L\end{pmatrix}=P_{n+1}(x)+(b+c-a)a_n^1(x)+La_n^2(x),$$

$$b_n^2(x)+\beta_{3n+3}a_n^2(x)+\sum_{\nu=0}^{n}\chi_{3n+2,3\nu}a_{\nu-1}^2(x)+\sum_{\nu=0}^{n}\chi_{3n+2,3\nu+1}b_{\nu-1}^2(x)+\sum_{\nu=0}^{n}\chi_{3n+2,3\nu+2}R_\nu(x)$$

$$=L_{1,n+1}(x)\begin{pmatrix}0\\ 1\\ -(b+c+p)\end{pmatrix}=a_n^1(x)-(b+c+p)a_n^2(x).$$

Note that, in this last step, we used $W_{3n+4}(x)$ and the corresponding CD given by (2.2) with $n\leftarrow n+1$. Considering (2.2) with $n=0$, we get: $x-\beta_0=W_1(x)=b_0^1(\varpi(x))+x-a$, which yields (Z_0). □

2.3 Diagonal cubic decomposition

With respect to the quadratic decomposition of a MPS, related to the quadratic transformation x^2, studied, for example, in [28], we know that if in a such quadratic decomposition the secondary components are null, then the decomposed MPS is symmetric. In order to discuss possible generalizations of this property, we introduce a designation for a MPS whose CD has null secondary components, and we give several characterizations for such MPS, using the dual sequence elements and the structure coefficients.

Definition 2.6. *A MPS for which the CD (2.1)-(2.3) has null secondary components will be called* $\begin{pmatrix}a&b&c\\ p&q&r\end{pmatrix}$*- symmetric.*

Theorem 2.7. *Let* $\{W_n\}_{n\geq0}$ *be a MPS. The following assertions are equivalent.*

(a) *(Conditions in terms of the component sequences)*

$$a_m^1=a_m^2=b_m^1=b_m^2=c_m^1=c_m^2=0,\ m\geq0.$$

(b) *(Conditions in terms of the dual sequence of* $\{W_n\}_{n\geq0}$*)*

$$\begin{array}{lll}\sigma_\varpi(w_{3n+1})=0, & \sigma_\varpi((x-a)w_{3n})=0, & \sigma_\varpi((x-b)(x-c)w_{3n})=0,\\ \sigma_\varpi(w_{3n+2})=0, & \sigma_\varpi((x-a)w_{3n+2})=0, & \sigma_\varpi((x-b)(x-c)w_{3n+1})=0.\end{array}$$

(c) *(Conditions in terms of the dual sequence of $\{W_n\}_{n\geq 0}$ and the relation between this sequence and dual sequences of principal components)*

The content of (b) and

$$u_n = \sigma_\varpi(w_{3n}), \quad v_n = \sigma_\varpi((x-a)w_{3n+1}), \quad r_n = \sigma_\varpi((x-b)(x-c)w_{3n+2}).$$

(d) *(Conditions in terms of the structure coefficients of $\{W_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\chi_{n,\nu}\}_{0\leq\nu\leq n,\, n\geq 0}$)*

1. $\beta_{3n} = a$,
2. $\beta_{3n+1} = b + c - a$,
3. $\beta_{3n+2} = -(b + c + p)$,
4. $\chi_{3n,3\nu} = -(a-b)(a-c) < u_\nu, Q_n >$, $0 \leq \nu \leq n$,
5. $\chi_{3n,3\nu+1} = 0$, $0 \leq \nu < n$,
6. $\chi_{3n,3\nu+2} =< r_\nu, Q_n >$, $0 \leq \nu < n$,
7. $\chi_{3n+1,3\nu} =< u_\nu, xR_n(x) > +[aL + bc(b+c+p) - r] < u_\nu, R_n(x) >$, $0 \leq \nu \leq n$,
8. $\chi_{3n+1,3\nu+1} = L < v_\nu, R_n(x) >$, $0 \leq \nu \leq n$,
9. $\chi_{3n+1,3\nu+2} = 0$, $0 \leq \nu < n$,
10. $\chi_{3n+2,3\nu} = 0$, $0 \leq \nu \leq n$,
11. $\chi_{3n+2,3\nu+1} =< v_\nu, P_{n+1}(x) >$, $0 \leq \nu \leq n$,
12. $\chi_{3n+2,3\nu+2} = 0$, $0 \leq \nu \leq n$,

where L is defined by (2.5) and $n \geq 0$.

Proof. (a) $\Rightarrow$ (b)

$< \sigma_\varpi(w_{3n+1}), P_m(x) >=< w_{3n+1}, P_m(\varpi(x)) > \overset{(a)}{=} < w_{3n+1}, W_{3m}(x) >= \delta_{3n+1,3m} = 0$. By lemma 1.3, $\sigma_\varpi(w_{3n+1}) = 0$.

Similarly, $< \sigma_\varpi(w_{3n+2}), P_m(x) >=< w_{3n+2}, P_m(\varpi(x)) >=< w_{3n+2}, W_{3m}(x) >= 0$,

$< \sigma_\varpi((x-a)w_{3n}), Q_m(x) >=< w_{3n}, (x-a)Q_m(\varpi(x)) >=< w_{3n}, W_{3m+1}(x) >= 0$,

$< \sigma_\varpi((x-a)w_{3n+2}), Q_m(x) >=< w_{3n+2}, (x-a)Q_m(\varpi(x)) >=< w_{3n+2}, W_{3m+1}(x) >= 0$,

$< \sigma_\varpi((x-b)(x-c)w_{3n}), R_m(x) >=< w_{3n}, (x-b)(x-c)R_m(\varpi(x)) >$
$=< w_{3n}, W_{3m+2}(x) >= 0$, and

$< \sigma_\varpi((x-b)(x-c)w_{3n+1}), R_m(x) >=< w_{3n+1}, W_{3m+2}(x) >= 0, \quad n, m \geq 0.$

(b) $\Rightarrow$ (c)

$< \sigma_\varpi(w_{3n}), P_m(x) >=< w_{3n}, P_m(\varpi(x)) >$
$\overset{(2.1)}{=} < w_{3n}, W_{3m}(x) - (x-a)a^1_{m-1}(\varpi(x)) - (x-b)(x-c)a^2_{m-1}(\varpi(x)) >$
$=< w_{3n}, W_{3m}(x) > - < \sigma_\varpi((x-a)w_{3n}), a^1_{m-1}(x) > - < \sigma_\varpi((x-b)(x-c)w_{3n}), a^2_{m-1}(x) >$
$\overset{(b)}{=} \delta_{n,m}$

and therefore, $\sigma_\varpi(w_{3n}) = u_n$, regarding the uniqueness of the dual sequence.

$$
\begin{aligned}
&< \sigma_\varpi((x-a)w_{3n+1}), Q_m(x) > = < w_{3n+1}, (x-a)Q_m(\varpi(x)) > \\
&\overset{(2.2)}{=} < w_{3n+1}, W_{3m+1}(x) - b_m^1(\varpi(x)) - (x-b)(x-c)b_{m-1}^2(\varpi(x)) > \\
&= < w_{3n+1}, W_{3m+1}(x) > - < \sigma_\varpi(w_{3n+1}), b_m^1(x) > - < \sigma_\varpi((x-b)(x-c)w_{3n+1}), b_{m-1}^2(x) > \\
&\overset{(b)}{=} \delta_{n,m}
\end{aligned}
$$

and therefore, $\sigma_\varpi((x-a)w_{3n+1}) = v_n$.

$$
\begin{aligned}
&< \sigma_\varpi((x-b)(x-c)w_{3n+2}), R_m(x) > = < w_{3n+2}, (x-b)(x-c)R_m(\varpi(x)) > \\
&\overset{(2.3)}{=} < w_{3n+2}, W_{3m+2}(x) - c_m^1(\varpi(x)) - (x-a)c_m^2(\varpi(x)) > \\
&= < w_{3n+2}, W_{3m+2}(x) > - < \sigma_\varpi(w_{3n+2}), c_m^1(x) > - < \sigma_\varpi((x-a)w_{3n+2}), c_m^2(x) > \\
&\overset{(b)}{=} \delta_{n,m}
\end{aligned}
$$

and therefore, $\sigma_\varpi((x-b)(x-c)w_{3n+2}) = r_n$.

$(c) \Rightarrow (a)$

$$
\begin{aligned}
&< v_n, a_{m-1}^1(x) > \overset{(c)}{=} < \sigma_\varpi((x-a)w_{3n+1}), a_{m-1}^1(x) > \\
&= < w_{3n+1}, (x-a)a_{m-1}^1(\varpi(x)) > \\
&\overset{(2.1)}{=} < w_{3n+1}, W_{3m}(x) - P_m(\varpi(x)) - (x-b)(x-c)a_{m-1}^2(\varpi(x)) > \\
&= < w_{3n+1}, W_{3m}(x) > - < \sigma_\varpi(w_{3n+1}), P_m(x) > - < \sigma_\varpi((x-b)(x-c)w_{3n+1}), a_{m-1}^2(x) > \\
&\overset{(c)}{=} \delta_{3n+1,3m} = 0
\end{aligned}
$$

so, by lemma 1.4, $a_{m-1}^1(x) = 0, \;\; m \geq 1$.

$$
\begin{aligned}
&< r_n, a_{m-1}^2(x) > = < \sigma_\varpi((x-b)(x-c)w_{3n+2}), a_{m-1}^2(x) > \\
&= < w_{3n+2}, (x-b)(x-c)a_{m-1}^2(\varpi(x)) > \\
&= < w_{3n+2}, W_{3m}(x) - P_m(\varpi(x)) - (x-a)a_{m-1}^1(\varpi(x)) > \\
&= < w_{3n+2}, W_{3m}(x) > - < \sigma_\varpi(w_{3n+2}), P_m(x) > - < \sigma_\varpi((x-a)w_{3n+2}), a_{m-1}^1(x) > \\
&= \delta_{3n+2,3m} = 0
\end{aligned}
$$

so, $a_{m-1}^2(x) = 0, \;\; m \geq 1$.

$$
\begin{aligned}
&< u_n, b_m^1(x) > = < \sigma_\varpi(w_{3n}), b_m^1(x) > \\
&= < w_{3n}, b_m^1(\varpi(x)) > \\
&= < w_{3n}, W_{3m+1}(x) - (x-a)Q_m(\varpi(x)) - (x-b)(x-c)b_{m-1}^2(\varpi(x)) > \\
&= < w_{3n}, W_{3m+1}(x) > - < \sigma_\varpi((x-a)w_{3n}), Q_m(x) > - < \sigma_\varpi((x-b)(x-c)w_{3n}), b_{m-1}^2(x) > \\
&= 0
\end{aligned}
$$

so, $b^1_m(x)=0,\ \ m\geq 0.$

$$\begin{aligned}
&< r_n, b^2_{m-1}(x) >=< \sigma_\varpi((x-b)(x-c)w_{3n+2}), b^2_{m-1}(x) >\\
&=< w_{3n+2}, (x-b)(x-c)b^2_{m-1}(\varpi(x)) >\\
&=< w_{3n+2}, W_{3m+1}(x) - b^1_m(\varpi(x)) - (x-a)Q_m(\varpi(x)) >\\
&=< w_{3n+2}, W_{3m+1}(x) > - < \sigma_\varpi(w_{3n+2}), b^1_m(x) > - < \sigma_\varpi((x-a)w_{3n+2}), Q_m(x) >\\
&= 0
\end{aligned}$$

so, $b^2_{m-1}(x)=0,\ \ m\geq 1.$

$$\begin{aligned}
&< u_n, c^1_m(x) >=< \sigma_\varpi(w_{3n}), c^1_m(x) >\\
&=< w_{3n}, c^1_m(\varpi(x)) >\\
&=< w_{3n}, W_{3m+2}(x) - (x-a)c^2_m(\varpi(x)) - (x-b)(x-c)R_m(\varpi(x)) >\\
&=< w_{3n}, W_{3m+2}(x) > - < \sigma_\varpi((x-a)w_{3n}), c^2_m(x) > - < \sigma_\varpi((x-b)(x-c)w_{3n}), R_m(x) >\\
&= 0
\end{aligned}$$

so, $c^1_m(x)=0,\ \ m\geq 0.$

$$\begin{aligned}
&< v_n, c^2_m(x) >=< \sigma_\varpi((x-a)w_{3n+1}), c^2_m(x) >\\
&=< w_{3n+1}, (x-a)c^2_m(\varpi(x)) >\\
&=< w_{3n+1}, W_{3m+2}(x) - c^1_m(\varpi(x)) - (x-b)(x-c)R_m(\varpi(x)) >\\
&=< w_{3n+1}, W_{3m+2}(x) > - < \sigma_\varpi(w_{3n+1}), c^1_m(x) > - < \sigma_\varpi((x-b)(x-c)w_{3n+1}), R_m(x) >\\
&= 0
\end{aligned}$$

so, $c^2_m(x)=0,\ \ m\geq 0.$

$(c)\Rightarrow(d)$ Let us first recall the identities (1.3) and (1.4), and the fact that (c) is equivalent to (a). Thus,

$$\begin{aligned}
&\beta_{3n} \overset{(1.3)}{=} < w_{3n}, xW_{3n}(x) > \overset{(2.1),(a)}{=} < w_{3n}, xP_n(\varpi(x)) >\\
&=< w_{3n}, (x-a)P_n(\varpi(x)) > +a < w_{3n}, P_n(\varpi(x)) >\\
&=< \sigma_\varpi((x-a)w_{3n}), P_n(x) > +a < \sigma_\varpi(w_{3n}), P_n(x) > \overset{(c)}{=} a < u_n, P_n(x) >= a.
\end{aligned}$$

$$\begin{aligned}
&\beta_{3n+1} \overset{(1.3)}{=} < w_{3n+1}, xW_{3n+1}(x) > \overset{(2.2),(a)}{=} < w_{3n+1}, x(x-a)Q_n(\varpi(x)) >\\
&\overset{I1}{=} -(a-b)(a-c) < w_{3n+1}, Q_n(\varpi(x)) > +(b+c-a) < w_{3n+1}, (x-a)Q_n(\varpi(x)) >\\
&+ < w_{3n+1}, (x-b)(x-c)Q_n(\varpi(x)) >\\
&= -(a-b)(a-c) < \sigma_\varpi(w_{3n+1}), Q_n(x) > +(b+c-a) < \sigma_\varpi((x-a)w_{3n+1}), Q_n(x) >\\
&+ < \sigma_\varpi((x-b)(x-c)w_{3n+1}), Q_n(x) > \overset{(c)}{=} (b+c-a) < v_n, Q_n(x) >= b+c-a.
\end{aligned}$$

$$\beta_{3n+2} \overset{(1.3)}{=} < w_{3n+2}, xW_{3n+2}(x) > \overset{(2.3),(a)}{=} < w_{3n+2}, x(x-b)(x-c)R_n(\varpi(x)) >$$
$$\overset{I2}{=} < w_{3n+2}, \Theta(\varpi(x))R_n(\varpi(x)) > + L < w_{3n+2}, (x-a)R_n(\varpi(x)) >$$
$$-(b+c+p) < w_{3n+2}, (x-b)(x-c)R_n(\varpi(x)) >$$
$$\overset{(c)}{=} -(b+c+p) < \sigma_\varpi((x-b)(x-c)w_{3n+2}), R_n(x) > \overset{(c)}{=} -(b+c+p).$$

$$\chi_{3n,3\nu} \overset{(1.4)}{=} < w_{3\nu}, xW_{3n+1}(x) > \overset{(2.2),(a)}{=} < w_{3\nu}, x(x-a)Q_n(\varpi(x)) >$$
$$\overset{I1}{=} -(a-b)(a-c) < w_{3\nu}, Q_n(\varpi(x)) > + (b+c-a) < w_{3\nu}, (x-a)Q_n(\varpi(x)) >$$
$$+ < w_{3\nu}, (x-b)(x-c)Q_n(\varpi(x)) >$$
$$= -(a-b)(a-c) < \sigma_\varpi(w_{3\nu}), Q_n(x) > = -(a-b)(a-c) < u_\nu, Q_n(x) >, \; 0 \le \nu \le n.$$

$$\chi_{3n,3\nu+1} \overset{(1.4)}{=} < w_{3\nu+1}, xW_{3n+1}(x) > \overset{(2.2),(a)}{=} < w_{3\nu+1}, x(x-a)Q_n(\varpi(x)) >$$
$$\overset{I1}{=} -(a-b)(a-c) < w_{3\nu+1}, Q_n(\varpi(x)) > + (b+c-a) < w_{3\nu+1}, (x-a)Q_n(\varpi(x)) >$$
$$+ < w_{3\nu+1}, (x-b)(x-c)Q_n(\varpi(x)) >$$
$$\overset{(c)}{=} (b+c-a) < \sigma_\varpi((x-a)w_{3\nu+1}), Q_n(x) >$$
$$\overset{(c)}{=} (b+c-a) < v_\nu, Q_n(x) > = (b+c-a)\delta_{\nu,n} = 0, \quad \text{for } 0 \le \nu < n.$$

$$\chi_{3n,3\nu+2} = < w_{3\nu+2}, xW_{3n+1}(x) > = < w_{3\nu+2}, x(x-a)Q_n(\varpi(x)) >$$
$$\overset{I1}{=} -(a-b)(a-c) < w_{3\nu+2}, Q_n(\varpi(x)) > + (b+c-a) < w_{3\nu+2}, (x-a)Q_n(\varpi(x)) >$$
$$+ < w_{3\nu+2}, (x-b)(x-c)Q_n(\varpi(x)) >$$
$$= < \sigma_\varpi((x-b)(x-c)w_{3\nu+2}), Q_n(x) >$$
$$= < r_\nu, Q_n(x) >, \quad \text{for } 0 \le \nu < n.$$

$$\chi_{3n+1,3\nu} = < w_{3\nu}, xW_{3n+2}(x) > = < w_{3\nu}, x(x-b)(x-c)R_n(\varpi(x)) >$$
$$\overset{I2}{=} < w_{3\nu}, \varpi(x)R_n(\varpi(x)) > + [aL + bc(b+c+p) - r] < w_{3\nu}, R_n(\varpi(x)) >$$
$$+ L < w_{3\nu}, (x-a)R_n(\varpi(x)) > - (b+c+p) < w_{3\nu}, (x-b)(x-c)R_n(\varpi(x)) >$$
$$= < u_\nu, xR_n(x) > + [aL + bc(b+c+p) - r] < u_\nu, R_n(x) >, \quad \text{for } 0 \le \nu \le n.$$

$$\chi_{3n+1,3\nu+1} = < w_{3\nu+1}, xW_{3n+2}(x) > = < w_{3\nu+1}, x(x-b)(x-c)R_n(\varpi(x)) >$$
$$\overset{I2}{=} < w_{3\nu+1}, \varpi(x)R_n(\varpi(x)) > + [aL + bc(b+c+p) - r] < w_{3\nu+1}, R_n(\varpi(x)) >$$
$$+ L < w_{3\nu+1}, (x-a)R_n(\varpi(x)) > - (b+c+p) < w_{3\nu+1}, (x-b)(x-c)R_n(\varpi(x)) >$$
$$= L < \sigma_\varpi((x-a)w_{3\nu+1}), R_n(x) >$$
$$= L < v_\nu, R_n(x) >, \quad \text{for } 0 \le \nu \le n.$$

$$
\begin{aligned}
\chi_{3n+1,3\nu+2} &=< w_{3\nu+2}, xW_{3n+2}(x) > \\
&= -(b+c+p) < w_{3\nu+2}, (x-b)(x-c)R_n(\varpi(x)) > \\
&\stackrel{I2}{=} -(b+c+p) < \sigma_{\varpi}((x-b)(x-c)w_{3\nu+2}), R_n(x) > \\
&= -(b+c+p) < r_\nu, R_n(x) >= -(b+c+p)\delta_{\nu,n} = 0, \quad \text{for } 0 \le \nu < n.
\end{aligned}
$$

$$
\begin{aligned}
\chi_{3n+2,3\nu} &=< w_{3\nu}, xW_{3n+3}(x) >=< w_{3\nu}, xP_{n+1}(\varpi(x)) > \\
&=< w_{3\nu}, (x-a)P_{n+1}(\varpi(x)) > +a < w_{3\nu}, P_{n+1}(\varpi(x)) > \\
&=< \sigma_{\varpi}((x-a)w_{3\nu}), P_{n+1}(x) > +a < \sigma_{\varpi}(w_{3\nu}), P_{n+1}(x) > \\
&= a < u_\nu, P_{n+1}(x) >= a\delta_{\nu,n+1} = 0, \quad \text{for } 0 \le \nu \le n.
\end{aligned}
$$

$$
\begin{aligned}
\chi_{3n+2,3\nu+1} &=< w_{3\nu+1}, xW_{3n+3}(x) >=< w_{3\nu+1}, xP_{n+1}(\varpi(x)) > \\
&=< w_{3\nu+1}, (x-a)P_{n+1}(\varpi(x)) > +a < w_{3\nu+1}, P_{n+1}(\varpi(x)) > \\
&=< \sigma_{\varpi}((x-a)w_{3\nu+1}), P_{n+1}(x) > +a < \sigma_{\varpi}(w_{3\nu+1}), P_{n+1}(x) > \\
&=< v_\nu, P_{n+1}(x) >, \quad \text{for } 0 \le \nu \le n.
\end{aligned}
$$

$$
\begin{aligned}
\chi_{3n+2,3\nu+2} &=< w_{3\nu+2}, xW_{3n+3}(x) > \\
&=< w_{3\nu+2}, (x-a)P_{n+1}(\varpi(x)) > +a < w_{3\nu+2}, P_{n+1}(\varpi(x)) > \\
&=< \sigma_{\varpi}((x-a)w_{3\nu+2}), P_{n+1}(x) > +a < \sigma_{\varpi}(w_{3\nu+2}), P_{n+1}(x) > \\
&= 0, \quad \text{for } 0 \le \nu \le n.
\end{aligned}
$$

$(d) \Rightarrow (a)$

We will proceed by induction over the degree of W_n. For $n = 0$, we have:

$$W_0(x) = 1 = P_0(\varpi(x)), \text{ because } P_0(x) = 1;$$

$$W_1(x) = x - \beta_0 = (x-a)Q_0(\varpi(x)), \text{ because } Q_0(x) = 1 \text{ and } \beta_{3n} = a, \ n \ge 0;$$

$$
\begin{aligned}
W_2(x) &\stackrel{(1.2)}{=} (x-\beta_1)W_1(x) - \chi_{0,0}W_0(x) \\
&= (x-\beta_1)(x-\beta_0) - \chi_{0,0} \\
&\stackrel{(d)}{=} (x-b-c+a)(x-a) + (a-b)(a-c) < u_0, Q_0(x) > \\
&= (x-b)(x-a) + (a-c)(x-a) + (a-b)(a-c) \\
&= (x-b)(x-c) \\
&= (x-b)(x-c)R_0(\varpi(x)), \text{ because } R_0(x) = 1.
\end{aligned}
$$

Let us suppose that the decomposition

$$
\begin{aligned}
W_{3k}(x) &= P_k(\varpi(x)) \\
W_{3k+1}(x) &= (x-a)Q_k(\varpi(x)) \\
W_{3k+2}(x) &= (x-b)(x-c)R_k(\varpi(x))
\end{aligned}
$$

is fulfilled for $k = 0, 1, \ldots, n$. This means that $a^1_{k-1}(x) = a^2_{k-1}(x) = b^1_k(x) = b^2_{k-1}(x) = c^1_k(x) = c^2_k(x) = 0$, for $k = 0, 1, \ldots, n$. Let us consider the equalities of theorem 2.5.

Recall that from lemma 1.6, $R_n(x) = \sum_{\nu=0}^{n} < v_\nu, R_n(x) > \ Q_\nu(x)$. Thus, by (Z_5),

$$a^1_n(x) + \sum_{\nu=0}^{n} L < v_\nu, R_n(x) > \ Q_\nu(x) = LR_n(x),$$

i.e., $a^1_n(x) = 0$.

The relations (Z_6) and (Z_7) assert that $a^2_n(x) = 0$ and $b^1_{n+1}(x) = 0$, respectively, and by (Z_9), we have $b^2_n(x) = 0$. Changing n by $n+1$ in (Z_1) and (Z_2), we conclude, also, that $c^1_{n+1}(x) = -(a-b)(a-c)\Big(Q_{n+1} - \sum_{\nu=0}^{n+1} < u_\nu, Q_{n+1}(x) > \ P_\nu(x)\Big) = 0$ and $c^2_{n+1}(x) = 0$. □

Corollary 2.8. *Let $\{W_n\}_{n\geq 0}$ be a MPS. If $\{W_n\}_{n\geq 0}$ is a MOPS, then $\{W_n\}_{n\geq 0}$ is not $\begin{pmatrix} a & b & c \\ p & q & r \end{pmatrix}$- symmetric, for every constants a, b, c, p, q, r.*

Proof. Theorem 2.7 gives us several characterizations of a $\begin{pmatrix} a & b & c \\ p & q & r \end{pmatrix}$-symmetric MPS. We can find there, for instance, the statement $\gamma_{3n+3} = \chi_{3n+2,3n+2} = 0$, which contradicts regular orthogonality. □

2.4 Canonical sequence cubic decomposition

Let us consider the canonical sequence $W_n(x) = x^n, \ n \geq 0$.

For some simple choices of constants a, b, c and p, q, r, we can easily obtain the corresponding CD of $\{W_n\}_{n\geq 0}$, as we may see in the following list.

a) For $a = b = c = p = q = r = 0$, we have

$$\begin{aligned} W_{3n}(x) &= (x^3)^n \\ W_{3n+1}(x) &= x(x^3)^n \\ W_{3n+2}(x) &= x^2(x^3)^n \end{aligned}$$

that is, $P_n(x) = x^n, \ a^1_{n-1}(x) = 0, \ a^2_{n-1}(x) = 0, b^1_n(x) = 0, \ Q_n(x) = x^n, \ b^2_{n-1}(x) = 0$ and $c^1_n(x) = 0, \ c^2_n(x) = 0, \ R_n(x) = x^n, n \geq 0$.

Note that the canonical sequence is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$- symmetric.

b) For $p=q=r=0$, we have

$$\begin{aligned} W_{3n}(x) &= (x^3)^n \\ W_{3n+1}(x) &= a(x^3)^n + (x-a)(x^3)^n \\ W_{3n+2}(x) &= \big(-bc+a(b+c)\big)(x^3)^n + (x-a)(b+c)(x^3)^n + (x-b)(x-c)(x^3)^n \end{aligned}$$

that is, $P_n(x)=x^n$, $a^1_{n-1}(x)=0$, $a^2_{n-1}(x)=0$, $b^1_n(x)=ax^n$, $Q_n(x)=x^n$, $b^2_{n-1}(x)=0$ and $c^1_n(x)=\big(-bc+a(b+c)\big)x^n$, $c^2_n(x)=(b+c)x^n$, $R_n(x)=x^n$, $n\geq 0$.

Thus, for the cubic transformation $\varpi(x)=x^3$, the corresponding CD is clear. Hence, we expect that the general transformation $\varpi(x)=x^3+px^2+qx+r$ will bring us more interesting component sequences.

The structure coefficients of the canonical sequence are the following:

$$\beta_n = 0,\ n\geq 0 \text{ and } \chi_{n,\nu}=0,\ \ n\geq 0,\ \ 0\leq \nu\leq n.$$

Therefore, we can characterize its CD by the next proposition, which lists the relations presented in theorem 2.5 for the specific case of the canonical sequence.

Proposition 2.9 (Characterization of Canonical Sequence Cubic Decomposition)**.** *The component sequences of the CD of the canonical sequence satisfy the following relations, where $\Theta(x)$ and L are defined by (2.4) and (2.5), and $n\geq 0$.*

(0) $b^1_0(x)=a,$

(1) $c^1_n(x)=ab^1_n(x)-(a-b)(a-c)Q_n(x)+\Theta(x)b^2_{n-1}(x),$

(2) $c^2_n(x)=b^1_n(x)+(b+c-a)Q_n(x)+Lb^2_{n-1}(x),$

(3) $R_n(x)=Q_n(x)-(b+c+p)b^2_{n-1}(x),$

(4) $P_{n+1}(x)=ac^1_n(x)-(a-b)(a-c)c^2_n(x)+\Theta(x)R_n(x),$

(5) $a^1_n(x)=c^1_n(x)+(b+c-a)c^2_n(x)+LR_n(x),$

(6) $a^2_n(x)=c^2_n(x)-(b+c+p)R_n(x),$

(7) $b^1_{n+1}(x)=aP_{n+1}(x)-(a-b)(a-c)a^1_n(x)+\Theta(x)a^2_n(x),$

(8) $Q_{n+1}(x)=P_{n+1}(x)+(b+c-a)a^1_n(x)+La^2_n(x),$

(9) $b^2_n(x)=a^1_n(x)-(b+c+p)a^2_n(x).$

We may rewrite the above identities (0)-(9) using matrix notation. From identities (1), (2) and (3), we have

$$\begin{pmatrix} c_n^1(x) \\ c_n^2(x) \\ R_n(x) \end{pmatrix} = M \begin{pmatrix} b_n^1(x) \\ Q_n(x) \\ b_{n-1}^2(x) \end{pmatrix},$$

where

$$M(x) = \begin{pmatrix} a & -(a-b)(a-c) & \Theta(x) \\ 1 & b+c-a & L \\ 0 & 1 & -(b+c+p) \end{pmatrix}. \qquad (2.8)$$

Note that $det(M) = x - r$.

Using the notation introduced in the proof of theorem 2.5, we obtain $L_{3,n}^T = ML_{2,n}^T$, where A^T denotes the transpose of the matrix A.

From the remaining identities we obtain $L_{1,n+1}^T = ML_{3,n}^T$ and $L_{2,n+1}^T = ML_{1,n+1}^T$.

Consequently,

$$L_{1,n+2}^T = M^3 L_{1,n+1}^T; \quad L_{2,n+1}^T = M^3 L_{2,n}^T; \quad L_{3,n+1}^T = M^3 L_{3,n}^T.$$

Let A denote the matrix M^3 and $y_n^i := L_{i,n}^T$. We conclude that

$$y_{n+1}^2 = A^{n+1} y_0^2, \qquad y_{n+1}^3 = A^{n+1} y_0^3, \qquad y_{n+2}^1 = A^{n+1} y_1^1, \qquad \text{that is,}$$

$$\begin{pmatrix} b_{n+1}^1(x) \\ Q_{n+1}(x) \\ b_n^2(x) \end{pmatrix} = A^{n+1} \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}; \qquad (2.9)$$

$$\begin{pmatrix} c_{n+1}^1(x) \\ c_{n+1}^2(x) \\ R_{n+1}(x) \end{pmatrix} = A^{n+1} \begin{pmatrix} c_0^1(x) \\ c_0^2(x) \\ 1 \end{pmatrix}; \qquad (2.10)$$

$$\begin{pmatrix} P_{n+2}(x) \\ a_{n+1}^1(x) \\ a_{n+1}^2(x) \end{pmatrix} = A^{n+1} \begin{pmatrix} P_1(x) \\ a_0^1(x) \\ a_0^2(x) \end{pmatrix}; \qquad (2.11)$$

where $c_0^1(x) = -bc + a(b+c)$, $c_0^2(x) = b + c$, $P_1(x) = bcp - a((b+c)p + q) - r + x$, $a_0^1(x) = -bp - cp - q$ and $a_0^2(x) = -p$.

In this manner, we have rewritten the constructive algorithm for the component sequences of a CD, presented in theorem 2.5, for the specific case of the canonical sequence. Notice that the component sequences of the CD of the canonical sequence are determined by the matrix A^{n+1}, $n \geq 0$.

Theorem 2.10. *Let $A = M^3$, where M is defined by (2.8), be the matrix involved in the calculation of the CD of the canonical sequence, as the identities (2.9-2.11) indicate. There is a PS $\{B_n\}_{n\geq 0}$ defined by*

$$B_0(x) = 1,\ B_1(x) = p_1(x),\ B_2(x) = p_2(x) + \big(p_1(x)\big)^2,$$
$$B_{n+3}(x) = p_3(x)B_n(x) + p_2(x)B_{n+1}(x) + p_1(x)B_{n+2}(x),\ \ n \geq 0,$$

and such that

$$A^{n+3} = p_3(x)B_n(x)I + \Big(p_3(x)B_{n-1}(x) + p_2(x)B_n(x)\Big)A + B_{n+1}(x)A^2,\ \ n \geq 0,$$

where $B_{-1}(x) = 0$, $p_1(x) = 3(x-r) + 3pq - p^3$, $p_2(x) = -3(x-r)^2 - 3pq(x-r) - q^3$, and $p_3(x) = (x-r)^3$.

Proof. Let $f(z)$ denote the characteristic polynomial of A. Then,

$$f(z) = -z^3 + \big(3x - p^3 + 3pq - 3r\big)z^2 - \big(3x^2 - 6rx + 3pqx + 3r^2 - 3pqr + q^3\big)z + (x-r)^3.$$

Suppose z^n is divided by $f(z)$. We obtain a quocient polynomial $q(z)$ and a remainder polynomial $r(z)$, i.e.,

$$z^n = f(z)q(z) + r(z),\ \ n \geq 3.$$

Let us replace z by A,

$$A^n = f(A)q(A) + r(A),\ \ n \geq 3.$$

By Cayley-Hamilton theorem (every square matrix satisfies its characteristic equation), we obtain $f(A) = 0$, thus, $A^n = r(A)$ or

$$A^n = \alpha_{1,n}I + \alpha_{2,n}A + \alpha_{3,n}A^2,\ \ n \geq 3,$$

where $\alpha_{1,n}, \alpha_{2,n}$ and $\alpha_{3,n}$ are polynomials (with variable x).

Defining $\alpha_{1,0} = 1,\ \alpha_{2,0} = 0,\ \alpha_{3,0} = 0,\ \alpha_{1,1} = 0,\ \alpha_{2,1} = 1,\ \alpha_{3,1} = 0,\ \alpha_{1,2} = 0,\ \alpha_{2,2} = 0$ and $\alpha_{3,2} = 1$, we obtain

$$A^n = \alpha_{1,n}I + \alpha_{2,n}A + \alpha_{3,n}A^2,\ \ n \geq 0.$$

For $n \geq 0$, we get:

$$\begin{aligned} A^{n+1} &= A^n A \\ &= \Big(\alpha_{1,n}I + \alpha_{2,n}A + \alpha_{3,n}A^2\Big)A \\ &= \alpha_{1,n}A + \alpha_{2,n}A^2 + \alpha_{3,n}A^3 \\ &= \alpha_{1,n}A + \alpha_{2,n}A^2 + \alpha_{3,n}\Big(\alpha_{1,3}I + \alpha_{2,3}A + \alpha_{3,3}A^2\Big) \\ &= \alpha_{1,3}\alpha_{3,n}I + \Big(\alpha_{1,n} + \alpha_{2,3}\alpha_{3,n}\Big)A + \Big(\alpha_{2,n} + \alpha_{3,3}\alpha_{3,n}\Big)A^2. \end{aligned} \tag{2.12}$$

Therefore, for $n \geq 0$, we have:

$$\alpha_{1,n+1} = \alpha_{1,3}\alpha_{3,n};$$

$$\alpha_{2,n+1} = \alpha_{1,n} + \alpha_{2,3}\alpha_{3,n};$$

$$\alpha_{3,n+1} = \alpha_{2,n} + \alpha_{3,3}\alpha_{3,n};$$

and consequently,

$$\begin{aligned}\alpha_{3,n+3} &= \alpha_{2,n+2} + \alpha_{3,3}\alpha_{3,n+2}\\ &= \alpha_{1,n+1} + \alpha_{2,3}\alpha_{3,n+1} + \alpha_{3,3}\alpha_{3,n+2}\\ &= \alpha_{1,3}\alpha_{3,n} + \alpha_{2,3}\alpha_{3,n+1} + \alpha_{3,3}\alpha_{3,n+2};\end{aligned}$$

and $\alpha_{2,n+2} = \alpha_{1,3}\alpha_{3,n} + \alpha_{2,3}\alpha_{3,n+1}$, or $\alpha_{2,n+2} = \alpha_{3,n+3} - \alpha_{3,3}\alpha_{3,n+2}$, $n \geq 0$. In other words, A^{n+1} may be expressed only in terms of polynomials $\alpha_{3,n}$.

Basic calculations about the equality $A^3 = \alpha_{1,3}I + \alpha_{2,3}A + \alpha_{3,3}A^2$ gives us

$$p_3(x) := \alpha_{1,3} = (x-r)^3;$$

$$p_2(x) := \alpha_{2,3} = -3(x-r)^2 - 3pq(x-r) - q^3$$
$$\text{and } \; p_1(x) := \alpha_{3,3} = 3(x-r) + 3pq - p^3.$$

From the above recurrence relation obtained for the polynomials $\alpha_{3,n}$, $n \geq 0$, and defining the polynomials $B_n(x) := \alpha_{3,n+2}$, $n \geq 0$, we conclude that they satisfy the following identity, for $n \geq 0$,

$$B_{n+1}(x) = p_3(x)B_{n-2}(x) + p_2(x)B_{n-1}(x) + p_1(x)B_n(x), \tag{2.13}$$

where $B_{-2}(x) = B_{-1}(x) = 0$, or equivalently

$$B_{n+3}(x) = p_3(x)B_n(x) + p_2(x)B_{n+1}(x) + p_1(x)B_{n+2}(x), \tag{2.14}$$

where $B_0(x) = 1$, $B_1(x) = p_1(x)$ and $B_2(x) = p_2(x) + \big(p_1(x)\big)^2$.

Considering the relation (2.12) and since $\alpha_{1,n} = \alpha_{1,3}\alpha_{3,n-1}$, $n \geq 1$, and

$$\alpha_{2,n} = \alpha_{1,3}\alpha_{3,n-2} + \alpha_{2,3}\alpha_{3,n-1},\ n \geq 2, \quad \text{we have}$$

$$A^{n+1} = \alpha_{1,3}\alpha_{3,n}I + \Big(\alpha_{1,3}\alpha_{3,n-1} + \alpha_{2,3}\alpha_{3,n}\Big)A + \Big(\alpha_{1,3}\alpha_{3,n-2} + \alpha_{2,3}\alpha_{3,n-1} + \alpha_{3,3}\alpha_{3,n}\Big)A^2,\ n \geq 2.$$

Hence, for $n \geq 2$,

$$\begin{aligned}A^{n+1} \quad &= p_3(x)B_{n-2}(x)I + \Big(p_3(x)B_{n-3}(x) + p_2(x)B_{n-2}(x)\Big)A\\ &+\Big(p_3(x)B_{n-4}(x) + p_2(x)B_{n-3}(x) + p_1(x)B_{n-2}(x)\Big)A^2,\end{aligned}$$

where $B_{-2}(x)=B_{-1}(x)=0$. Applying the relation (2.13), we obtain

$$A^{n+1}=p_3(x)B_{n-2}(x)I+\Big(p_3(x)B_{n-3}(x)+p_2(x)B_{n-2}(x)\Big)A+B_{n-1}(x)A^2,\ n\geq 2,$$

where $B_{-1}(x)=0$, or equivalently

$$A^{n+3}=p_3(x)B_n(x)I+\Big(p_3(x)B_{n-1}(x)+p_2(x)B_n(x)\Big)A+B_{n+1}(x)A^2,\ n\geq 0,$$

where $B_{-1}(x)=0$.

Let us now prove that $\{B_n\}_{n\geq 0}$ is a PS. In virtue of $B_0(x)=1,\ B_1(x)=p_1(x)$ and

$$B_2(x)=p^6-6qp^4+6rp^3+9q^2p^2-15qrp-q^3+6r^2+\big(-6p^3+15qp-12r\big)\,x+6x^2,$$

we have $\deg(B_i)\leq i,\ i=0,1,2$, and inductively, we conclude that $\deg(B_n)\leq n,\ \ n\geq 0$, regarding (2.14).

Let us insert $B_n(x)=\sum_{\nu=0}^{n}a_{n,\nu}(x-r)^{\nu}$ in (2.14). Then, we obtain the following set of difference equations:

- $\nu=0$

$$-q^3a_{n+1,0}+(3pq-p^3)a_{n+2,0}=a_{n+3,0}$$

- $\nu=1$

$$-3pqa_{n+1,0}+3a_{n+2,0}+\Big(-q^3a_{n+1,1}+(3pq-p^3)a_{n+2,1}\Big)=a_{n+3,1}$$

- $2\leq\nu\leq n+1$

$$a_{n,\nu-3}-3a_{n+1,\nu-2}+\Big(-3pqa_{n+1,\nu-1}+3a_{n+2,\nu-1}\Big)+\Big(-q^3a_{n+1,\nu}+(3pq-p^3)a_{n+2,\nu}\Big)=a_{n+3,\nu}$$

- $\nu=n+2$

$$a_{n,n-1}-3a_{n+1,n}+\Big(-3pqa_{n+1,n+1}+3a_{n+2,n+1}\Big)+(3pq-p^3)a_{n+2,n+2}=a_{n+3,n+2}$$

- $\nu=n+3$

$$a_{n,n}-3a_{n+1,n+1}+3a_{n+2,n+2}=a_{n+3,n+3}$$

considering $a_{n,-1} = 0,\ n \geq 0$.

The last equation gives us the expression of the coefficient of $(x-r)^n$ of $B_n(x)$, denoted by $a_{n,n}$.

In order to solve that equation, let us consider $a_{n,n} := a_n$. Thus, the equation becomes:

$$a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0.$$

The general solution is $a_n = A + Bn + Cn^2$. Attending to the initial polynomials $B_0(x)$, $B_1(x)$ and $B_2(x)$, we know that $a_0 = 1$, $a_1 = 3$ and $a_2 = 6$, which yields the following system

$$\begin{cases} A = 1 \\ A + B + C = 3 \\ A + 2B + 4C = 6 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = \frac{3}{2} \\ C = \frac{1}{2} \end{cases}.$$

We conclude that $a_n = 1 + \frac{3}{2}n + \frac{1}{2}n^2,\ n \geq 0$. Since $a_n = 0 \Leftrightarrow n = -2 \vee n = -1$, we just proved that $\deg B_n = n,\ n \geq 0$, and $\{B_n\}_{n\geq 0}$ is a PS (non monic). □

Considering $q = 0$, we can establish a connection between the PS $\{B_n\}_{n\geq 0}$ and the 2-Tchebyshev MPS (2-classical) of first kind $\{T_n\}_{n\geq 0}$ (see definition 1.24) studied in [13], and defined by

$$T_{n+3}(x) = xT_{n+2}(x) - \alpha T_{n+1}(x) - \gamma T_n(x),\ n \geq 0,\ \gamma \neq 0,$$
$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = x^2 - \alpha,$$

with generating function given by $(1 - xt + \alpha t^2 + \gamma t^3)^{-1} = \sum_{n\geq 0} T_n(x)t^n$.

Proposition 2.11. *Let us suppose $q = 0$ and consider the sequence $\{B_n\}_{n\geq 0}$ defined in theorem 2.10. Let us, also, consider the 2-Tchebyshev MPS of first kind, $\{T_n\}_{n\geq 0}$, defined by*

$$T_{n+3}(x) = xT_{n+2}(x) - 3T_{n+1}(x) + T_n(x),\ n \geq 0,\ T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = x^2 - 3.$$

Then, $(J\bar{B}_n)(x) = T_n(x)$ and $B_n(x) = (x-r)^n (GT_n)\left(\frac{1}{x-r}\right),\ n \geq 0$,

where $J = \tau_3 \circ h_{-\frac{1}{p^3}},\ G = h_{-p^3} \circ \tau_{-3}$ (isomorphisms of $\mathcal{P}$)

and $\bar{B}_n(x) = x^n B_n\Big(\frac{1}{x} + r\Big),\ n \geq 0$.

Proof. Let us recall that, for $q = 0$, the sequence $\{B_n\}_{n\geq 0}$ fulfils the recurrence relation

$$B_{n+3}(x) = (x-r)^3 B_n(x) - 3(x-r)^2 B_{n+1}(x) + \Big(3(x-r) - p^3\Big) B_{n+2}(x),\ n \geq 0, \quad (2.15)$$

with initial conditions

$$B_0(x) = 1,\ B_1(x) = 3(x-r) - p^3,\ B_2(x) = 6(x-r)^2 - 6p^3(x-r) + p^6.$$

We will now insert the following transformations in (2.15):

1. $x \to x + r$

2. $x \to \frac{1}{x}$

3. multiply both members by x^{n+3}

yielding,
from 1.: $B_{n+3}(x+r) = x^3 B_n(x+r) - 3x^2 B_{n+1}(x+r) + \Big(3x - p^3\Big) B_{n+2}(x+r),\ \ n \geq 0,$
from 2.: $B_{n+3}\Big(\frac{1}{x}+r\Big) = \frac{1}{x^3} B_n\Big(\frac{1}{x}+r\Big) - \frac{3}{x^2} B_{n+1}\Big(\frac{1}{x}+r\Big) + \Big(\frac{3}{x} - p^3\Big) B_{n+2}\Big(\frac{1}{x}+r\Big),\ n \geq 0,$
from 3.:
$x^{n+3} B_{n+3}\Big(\frac{1}{x}+r\Big) =$
$x^n B_n\Big(\frac{1}{x}+r\Big) - 3x^{n+1} B_{n+1}\Big(\frac{1}{x}+r\Big) + 3x^{n+2} B_{n+2}\Big(\frac{1}{x}+r\Big) - p^3 x^{n+3} B_{n+2}\Big(\frac{1}{x}+r\Big),\ \ n \geq 0.$

Consequently, the sequence $\bar{B}_n(x) = x^n B_n\Big(\frac{1}{x}+r\Big)$, $n \geq 0$, fulfils the following recurrence relation

$$\bar{B}_{n+3}(x) = (3 - p^3 x)\bar{B}_{n+2}(x) - 3\bar{B}_{n+1}(x) + \bar{B}_n(x), n \geq 0.$$

Let us consider the affine transformation $J = \tau_3 \circ h_{-\frac{1}{p^3}}$ (isomorphism of $\mathcal{P}$) defined by

$$f(x) \to f\Big(-\frac{1}{p^3}(x-3)\Big),\ \ \forall f \in \mathcal{P}.$$

In particular, $J(3 - p^3 x) = x$ and the sequence

$$(J\bar{B}_n)(x) = \Big(-\frac{1}{p^3}(x-3)\Big)^n B_n\Big(\frac{1}{-\frac{1}{p^3}(x-3)} + r\Big),\ \ n \geq 0,$$

fulfils

$$(J\bar{B}_{n+3})(x) = x(J\bar{B}_{n+2})(x) - 3(J\bar{B}_{n+1})(x) + (J\bar{B}_n)(x), n \geq 0,$$

with initial conditions
$(J\bar{B}_0)(x) = B_0(x) = 1,$
$(J\bar{B}_1)(x) = \Big(-\frac{1}{p^3}(x-3)\Big) B_1\Big(\frac{p^3}{3-x} + r\Big) = \Big(-\frac{1}{p^3}(x-3)\Big)\Big(3\frac{p^3}{3-x} - p^3\Big) = x,$
$(J\bar{B}_2)(x) = \Big(-\frac{1}{p^3}(x-3)\Big)^2 B_2\Big(\frac{p^3}{3-x} + r\Big) = \Big(-\frac{1}{p^3}(x-3)\Big)^2 \Big(6\left(\frac{p^3}{3-x}\right)^2 - 6p^3\left(\frac{p^3}{3-x}\right) + p^6\Big)$
$= x^2 - 3.$

We conclude that the sequence $\{J\bar{B}_n\}_{n\geq 0}$ is the 2-Tchebyshev MPS of first kind, where $\alpha = 3$ and $\gamma = -1$.

Conversely, let us consider the 2-Tchebyshev MPS of first kind, defined by

$$T_{n+3}(x) = xT_{n+2}(x) - 3T_{n+1}(x) + T_n(x),\ \ n \geq 0,$$

$$T_0(x) = 1,\ \ T_1(x) = x,\ \ T_2(x) = x^2 - 3,$$

and also the affine transformation $G = h_{-p^3} \circ \tau_{-3}$, isomorphism of $\mathcal{P}$, defined by $f(x) \to f(-p^3x+3)$, $\forall f \in \mathcal{P}$. In particular, $G(x) = -p^3x+3$.

Therefore, the sequence $\{GT_n\}_{n\geq 0}$ fulfils

$$GT_{n+3}(x) = (-p^3x+3)GT_{n+2}(x) - 3GT_{n+1}(x) + GT_n(x), \ \ n \geq 0,$$

with initial conditions $GT_0(x) = 1$, $GT_1(x) = -p^3x+3$ and $GT_2(x) = (-p^3x+3)^2 - 3 = p^6x^2 - 6p^3x + 6$.

Applying the transformation $x \to \frac{1}{x}$ and multiplying both members by x^{n+3}, we obtain:

$$x^{n+3}(GT_{n+3})\left(\frac{1}{x}\right) = x^{n+3}\left(3-\frac{p^3}{x}\right)(GT_{n+2})\left(\frac{1}{x}\right) - 3x^{n+3}(GT_{n+1})\left(\frac{1}{x}\right) + x^{n+3}(GT_n)\left(\frac{1}{x}\right),$$

or

$$x^{n+3}(GT_{n+3})\left(\frac{1}{x}\right) = (3x-p^3)x^{n+2}(GT_{n+2})\left(\frac{1}{x}\right) - 3x^2x^{n+1}(GT_{n+1})\left(\frac{1}{x}\right) + x^3x^n(GT_n)\left(\frac{1}{x}\right).$$

Inserting the transformation $x \to x-r$, we define the sequence

$$W_n(x) = (x-r)^n(GT_n)\left(\frac{1}{x-r}\right), \ \ n \geq 0.$$

Then,

$$W_{n+3}(x) = \big(3(x-r)-p^3\big)W_{n+2}(x) - 3(x-r)^2W_{n+1}(x) + (x-r)^3W_n(x), \ \ n \geq 0,$$

with initial conditions, $W_0(x) = 1$,

$W_1(x) = (x-r)(GT_1)\left(\frac{1}{x-r}\right) = (x-r)\left(3 - p^3\frac{1}{x-r}\right) = 3(x-r) - p^3$ and

$W_2(x) = (x-r)^2\left(p^6\frac{1}{(x-r)^2} - 6p^3\frac{1}{x-r} + 6\right) = p^6 - 6p^3(x-r) + 6(x-r)^2.$

In other words, $B_n(x) = (x-r)^n(GT_n)\left(\dfrac{1}{x-r}\right), \ \ n \geq 0.$ □

Remark 2.12. *Let us consider three constants p_0, q_0 and r_0 and determine α, p and r such that*

$$\begin{cases} 3\alpha^2 - 2\alpha p_0 + q_0 = 0 \\ p = p_0 - 3\alpha \\ r = r_0 - \alpha^3 - p\alpha^2 \end{cases}.$$

The CD of the canonical sequence with respect to the cubic transformation $\varpi(x) = x^3 + px^2 + r$ and to the parameters a, b and c, is determined by the matrix A^{n+1}, $n \geq 0$, where

$$A^{n+3} = p_3(x)B_n(x)I + \Big(p_3(x)B_{n-1}(x) + p_2(x)B_n(x)\Big)A + B_{n+1}(x)A^2, \ \ n \geq 0,$$

considering the definitions of theorem 2.10, and, from proposition 2.15, since $q = 0$*, we have*

$$B_n(x) = (x-r)^n (GT_n)\left(\frac{1}{x-r}\right), \;\; n \geq 0.$$

Taking into consideration the arguments presented in page 19, applying the shift $x \to x+\alpha$*, we obtain the CD of the MPS* $\{(x+\alpha)^n\}_{n\geq 0}$*, with respect to the cubic transformation* $\varpi(x) = x^3 + p_0 x^2 + q_0 x + r_0$ *and parameters* $a-\alpha, b-\alpha, c-\alpha$*.*

Chapter 3

Cubic decomposition of an orthogonal sequence

The present chapter is dedicated to the study of the CD of an orthogonal MPS $\{W_n\}_{n\geq 0}$. We know that $\{W_n\}_{n\geq 0}$ fulfils the recurrence relation (1.9-1.10), that is, the structure coefficients are reduced to the recurrence coefficients.

We begin to give two equivalent characterizations of the CD of a MOPS, from which we can establish sufficient conditions for the orthogonality of the principal components. The CDs where specific secondary components vanish are described, in particular, we approach the case where the two secondary components $\{a_n^1\}_{n\geq 0}$ and $\{a_n^2\}_{n\geq 0}$ vanish and we exemplify it with some symmetric MOPS.

In the previous chapter, we discussed the possibility of taking one of the three parameters p, q or r equal to zero, using an affine transformation (see page 19). Since the orthogonality is preserved by affine transformations, we will consider in this chapter $r = 0$, without lost of generality.

3.1 Characterizations of the cubic decomposition of an orthogonal sequence

Taking into consideration theorem (2.5) , we begin to give necessary and sufficient relations concerning the decomposed MPS orthogonality.

Theorem 3.1 (First Characterization of a MOPS CD). *A MPS $\{W_n\}_{n\geq 0}$ with CD given by (2.1)-(2.3) is orthogonal if and only if the following relations are satisfied, for $n \geq 0$, where $\Theta(x)$ and L are defined by (2.4) and (2.5).*

(A_0) $b_0^1 = a - \beta_0,$

(A_1) $(\beta_{3n+1} - a)b_n^1(x) - \Theta(x)b_{n-1}^2(x) + c_n^1(x) = -\gamma_{3n+1}P_n(x) - (a-b)(a-c)Q_n(x),$

(A_2) $\gamma_{3n+1}a_{n-1}^1(x) - b_n^1(x) - Lb_{n-1}^2(x) + c_n^2(x) = -(\beta_{3n+1} + a - b - c)Q_n(x),$

(A_3) $\gamma_{3n+1}a^2_{n-1}(x) + (\beta_{3n+1} + b + c + p)b^2_{n-1}(x) = Q_n(x) - R_n(x)$,

(A_4) $\gamma_{3n+2}b^1_n(x) + (\beta_{3n+2} - a)c^1_n(x) + (a - b)(a - c)c^2_n(x) = -P_{n+1}(x) + \Theta(x)R_n(x)$,

(A_5) $a^1_n(x) - c^1_n(x) + (\beta_{3n+2} + a - b - c)c^2_n(x) = -\gamma_{3n+2}Q_n(x) + LR_n(x)$,

(A_6) $a^2_n(x) + \gamma_{3n+2}b^2_{n-1}(x) - c^2_n(x) = -(\beta_{3n+2} + b + c + p)R_n(x)$,

(A_7) $(a - b)(a - c)a^1_n(x) - \Theta(x)a^2_n(x) + b^1_{n+1}(x) + \gamma_{3n+3}c^1_n(x) = -(\beta_{3n+3} - a)P_{n+1}(x)$,

(A_8) $(\beta_{3n+3} + a - b - c)a^1_n(x) - La^2_n(x) + \gamma_{3n+3}c^2_n(x) = P_{n+1}(x) - Q_{n+1}(x)$,

(A_9) $a^1_n(x) - (\beta_{3n+3} + b + c + p)a^2_n(x) - b^2_n(x) = \gamma_{3n+3}R_n(x)$.

Proof. The sequence $\{W_n\}_{n\geq 0}$ is orthogonal if and only if its structure coefficients satisfy the following:

$$\chi_{n,n} = \gamma_{n+1} \neq 0, \text{ and } \chi_{n,\nu} = 0,\ 0 \leq \nu < n.$$

Thus, theorem 2.5 concludes the proof. □

In order to obtain, from theorem 3.1, conditions relating $\{W_n\}_{n\geq 0}$ orthogonality and its principal components orthogonality, we investigated the existence of second order recurrence relations for these latest. The result is another list of ten identities, that characterizes a MOPS CD, sharing the majority of the identities with theorem 3.1.

Theorem 3.2 (Second Characterization of a MOPS Cubic Decomposition)**.** *A MPS with CD given by (2.1)-(2.3) is orthogonal if and only if the following relations are fulfilled, for* $n \geq 0$*:*

identities (A_0), (A_1), (A_2), (A_5), (A_6), (A_7), *and* (A_9) *from theorem 3.1 and the following three, concerning the principal components.*

$$\begin{aligned} P_{n+2}(x) = &\big\{\Theta(x) - A_{3n}\big\}P_{n+1}(x) - B_{3n}P_n(x) \\ &- K_{3n}b^1_n(x) - H_{3n}b^1_{n+1}(x) - V_{3n}c^1_n(x) - S_{3n}c^1_{n+1}(x), \end{aligned} \tag{3.1}$$

with initial conditions $P_0(x) = 1$ *and*

$$\begin{aligned} P_1(x) =&\Theta(x) - \gamma_1(a - \beta_2) - \gamma_2(a - \beta_0) + (a - \beta_0)(a - \beta_1)(a - \beta_2) \\ &- (a - b)(a - c)(a + b + c - \beta_0 - \beta_1 - \beta_2)\,; \end{aligned}$$

$$\begin{aligned} Q_{n+2}(x) = &\big\{\Theta(x) - A_{3n+1}\big\}Q_{n+1}(x) - B_{3n+1}Q_n(x) \\ &- K_{3n+1}c^2_n(x) - H_{3n+1}c^2_{n+1}(x) - V_{3n+1}a^1_n(x) - S_{3n+1}a^1_{n+1}(x), \end{aligned} \tag{3.2}$$

with initial conditions $Q_0(x) = 1$ *and*

$$Q_1(x) = \Theta(x) - \gamma_1\,(b+c-\beta_2-\beta_3) - \gamma_2\,(b+c-\beta_0-\beta_3) - \gamma_3\,(b+c-\beta_0-\beta_1) \\ + (a-\beta_0)\,(a-\beta_1)\,(b+c-\beta_2-\beta_3) + (b+c-\beta_0-\beta_1)\,(a-b-c+\beta_2)\,(a-b-c+\beta_3) \\ - (a-b)(a-c)\,(2b+2c-\beta_0-\beta_1-\beta_2-\beta_3) - L\,(a-b-c+p+\beta_0+\beta_1+\beta_2+\beta_3)\,;$$

$$\begin{aligned} R_{n+2}(x) = {}& \big\{\Theta(x) - A_{3n+2}\big\}R_{n+1}(x) - B_{3n+2}R_n(x) \\ & - K_{3n+2}a_n^2(x) - H_{3n+2}a_{n+1}^2(x) - V_{3n+2}b_n^2(x) - S_{3n+2}b_{n+1}^2(x), \end{aligned} \tag{3.3}$$

with initial conditions $R_0(x) = 1$ *and*

$$R_1(x) = \Theta(x) + \gamma_1\,(p+\beta_2+\beta_3+\beta_4) + \gamma_2\,(p+\beta_0+\beta_3+\beta_4) + \gamma_3\,(p+\beta_0+\beta_1+\beta_4) \\ + \gamma_4\,(p+\beta_0+\beta_1+\beta_2) - (p+\beta_0+\beta_1+\beta_2)\,(b+c+p+\beta_3)\,(b+c+p+\beta_4) \\ + (b+c-\beta_0-\beta_1)\,(a-b-c+\beta_2)\,(a+p+\beta_3+\beta_4) - (a-\beta_0)\,(a-\beta_1)\,(p+\beta_2+\beta_3+\beta_4) \\ - (a-b)(a-c)\,(b+c-p-\beta_0-\beta_1-\beta_2-\beta_3-\beta_4) - L\,(a+2p+\beta_0+\beta_1+\beta_2+\beta_3+\beta_4)\,;$$

where $\Theta(x)$ *and* L *are defined by (2.4) and (2.5), and*

$$\begin{aligned} A_n \quad & = \gamma_{n+3}(\beta_{n+2}+2\beta_{n+3}+p) + \gamma_{n+4}(2\beta_{n+3}+\beta_{n+4}+p) \\ & +(\beta_{n+3}-a)(\beta_{n+3}+a-b-c)(\beta_{n+3}+b+c+p) - (\beta_{n+3}-a)L \\ & +(a-b)(a-c)(\beta_{n+3}+b+c+p); \end{aligned} \tag{3.4}$$

$$B_n = \gamma_{n+1}\gamma_{n+2}\gamma_{n+3}; \tag{3.5}$$

$$K_n = \gamma_{n+2}\gamma_{n+3}\,(\beta_{n+1}+\beta_{n+2}+\beta_{n+3}+p)\,; \tag{3.6}$$

$$\begin{aligned} H_n = {}& \gamma_{n+3}+\gamma_{n+4}+\gamma_{n+5}+(a-b)(a-c)-L \\ & + (\beta_{n+3}+a-b-c)(\beta_{n+3}+b+c+p) + (\beta_{n+4}-a)(\beta_{n+3}+\beta_{n+4}+a+p); \end{aligned} \tag{3.7}$$

$$\begin{aligned} V_n = {}& \gamma_{n+3}\Big\{\gamma_{n+2}+\gamma_{n+3}+\gamma_{n+4}+(a-b)(a-c)-L \\ & + (\beta_{n+3}+a-b-c)(\beta_{n+3}+b+c+p) + (\beta_{n+2}-a)(\beta_{n+2}+\beta_{n+3}+a+p)\Big\}; \end{aligned} \tag{3.8}$$

$$S_n = \beta_{n+3}+\beta_{n+4}+\beta_{n+5}+p. \tag{3.9}$$

Proof. Let us begin to remark the structure of identities (3.1), (3.2) and (3.3): they begin as a recurrence relation of second order for each principal component and are completed with elements of only two secondary component sequences. These two secondary components are exactly the sequences that appear in the respective column, when we use the matrix notation to write all component sequences of a CD (see the theorem 2.5 proof). For example, in (3.1), polynomial $P_{n+1}(x)$ is written in terms of elements of the tree sequences $\{P_n\}_{n\geq 0}$, $\{b_n^1\}_{n\geq 0}$ and $\{c_n^1\}_{n\geq 0}$.

This result is accomplished, firstly, by deducting identities (3.1), (3.2) and (3.3) from theorem 3.1 relations (Part II), and secondly, obtaining the three absent relations of theorem 3.1, (A_3), (A_4) and (A_9), from the ones enunciated (Part III).

In both procedures, it will be useful to have some components of the CD (or algebraic expressions of some polynomials) written in terms of the elements of one of the columns $(P_n(x), b_n^1(x), c_n^1(x))^T$, $(a_{n-1}^1(x), Q_n(x), c_n^2(x))^T$ and $(a_{n-1}^2(x), b_{n-1}^2(x), R_n(x))^T$. This will be done by manipulating theorem 3.1 relations. Therefore, we will do, previously, these extensive calculations (Part I).

Part I

Let us begin to write every component in terms of elements of the sequences $\{a_{n-1}^2(x)\}_{n\geq 0}$, $\{b_{n-1}^2(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$.

Let us note (A_9), (A_6) and (A_3) (respectively):

$$a_n^1(x) = (\beta_{3n+3} + b + c + p)a_n^2(x) + b_n^2(x) + \gamma_{3n+3}R_n(x), \quad n \geq 0,$$

$$c_n^2(x) = a_n^2(x) + \gamma_{3n+2}b_{n-1}^2(x) + (\beta_{3n+2} + b + c + p)R_n(x), \quad n \geq 0,$$

$$Q_n(x) = \gamma_{3n+1}a_{n-1}^2(x) + (\beta_{3n+1} + b + c + p)b_{n-1}^2(x) + R_n(x), \quad n \geq 0.$$

By identities (A_5), (A_8) and (A_2) (this last one with $n \leftarrow n+1$), we can also write $c_n^1(x)$, $P_{n+1}(x)$ and $b_{n+1}^1(x)$ (respectively) as a linear combination of elements of the sequences $\{a_{n-1}^2(x)\}_{n\geq 0}$, $\{b_{n-1}^2(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$, as follows.

$$\begin{aligned} c_n^1(x) &= a_n^1(x) + (\beta_{3n+2} + a - b - c)c_n^2(x) + \gamma_{3n+2}Q_n(x) - LR_n(x) \\ &= (\beta_{3n+3} + b + c + p)a_n^2(x) + b_n^2(x) + \gamma_{3n+3}R_n(x) \\ &+ (\beta_{3n+2} + a - b - c)\Big(a_n^2(x) + \gamma_{3n+2}b_{n-1}^2(x) + (\beta_{3n+2} + b + c + p)R_n(x)\Big) \\ &+ \gamma_{3n+2}\Big(\gamma_{3n+1}a_{n-1}^2(x) + (\beta_{3n+1} + b + c + p)b_{n-1}^2(x) + R_n(x)\Big) - LR_n(x); \end{aligned}$$

$$\begin{aligned} c_n^1(x) &= \gamma_{3n+1}\gamma_{3n+2}a_{n-1}^2(x) + (\beta_{3n+2} + \beta_{3n+3} + a + p)a_n^2(x) \\ &+ \gamma_{3n+2}(\beta_{3n+1} + \beta_{3n+2} + a + p)b_{n-1}^2(x) + b_n^2(x) \\ &+ \Big(\gamma_{3n+2} + \gamma_{3n+3} + (\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p) - L\Big)R_n(x), \quad n \geq 0. \end{aligned} \tag{3.10}$$

$$\begin{aligned} P_{n+1}(x) &= (\beta_{3n+3} + a - b - c)a_n^1(x) - La_n^2(x) + \gamma_{3n+3}c_n^2(x) + Q_{n+1}(x) \\ &= (\beta_{3n+3} + a - b - c)\Big((\beta_{3n+3} + b + c + p)a_n^2(x) + b_n^2(x) + \gamma_{3n+3}R_n(x)\Big) \\ &- La_n^2(x) + \gamma_{3n+3}\Big(a_n^2(x) + \gamma_{3n+2}b_{n-1}^2(x) + (\beta_{3n+2} + b + c + p)R_n(x)\Big) \\ &+ \gamma_{3n+4}a_n^2(x) + (\beta_{3n+4} + b + c + p)b_n^2(x) + R_{n+1}(x); \end{aligned}$$

$$
\begin{aligned}
P_{n+1}(x) = &\Big(\gamma_{3n+3}+\gamma_{3n+4}+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)-L\Big)a_n^2(x)\\
&+\gamma_{3n+2}\gamma_{3n+3}b_{n-1}^2(x)+(\beta_{3n+3}+\beta_{3n+4}+a+p)b_n^2(x)\\
&+\gamma_{3n+3}(\beta_{3n+2}+\beta_{3n+3}+a+p)R_n(x)+R_{n+1}(x),\quad n\geq 0.
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
b_{n+1}^1(x) &= \gamma_{3n+4}a_n^1(x)-Lb_n^2(x)+c_{n+1}^2(x)+(\beta_{3n+4}+a-b-c)Q_{n+1}(x)\\
&=\gamma_{3n+4}\Big((\beta_{3n+3}+b+c+p)a_n^2(x)+b_n^2(x)+\gamma_{3n+3}R_n(x)\Big)-Lb_n^2(x)\\
&+a_{n+1}^2(x)+(\beta_{3n+5}+b+c+p)R_{n+1}(x)+\gamma_{3n+5}b_n^2(x)\\
&+(\beta_{3n+4}+a-b-c)\Big(\gamma_{3n+4}a_n^2(x)+(\beta_{3n+4}+b+c+p)b_n^2(x)+R_{n+1}(x)\Big);
\end{aligned}
$$

$$
\begin{aligned}
b_{n+1}^1(x) = &\gamma_{3n+4}(\beta_{3n+3}+\beta_{3n+4}+a+p)a_n^2(x)+a_{n+1}^2(x)\\
&+\Big(\gamma_{3n+4}+\gamma_{3n+5}+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)-L\Big)b_n^2(x)\\
&+\gamma_{3n+3}\gamma_{3n+4}R_n(x)+(\beta_{3n+4}+\beta_{3n+5}+a+p)R_{n+1}(x),\quad n\geq 0.
\end{aligned}
\tag{3.12}
$$

The next step consists in writing polynomial $\Theta(x)R_{n+1}(x)-(a-b)(a-c)c_{n+1}^2(x)-\Theta(x)P_{n+1}(x)$ in terms of elements of the sequences $\{P_n(x)\}_{n\geq 0}$, $\{b_n^1(x)\}_{n\geq 0}$ and $\{c_n^1(x)\}_{n\geq 0}$.

Let us note identities (A_7), (A_1) and (A_4) (respectively):

$$\Theta(x)a_n^2(x)=(a-b)(a-c)a_n^1(x)+b_{n+1}^1(x)+\gamma_{3n+3}c_n^1(x)+(\beta_{3n+3}-a)P_{n+1}(x),\ n\geq 0,$$

$$\Theta(x)b_{n-1}^2(x)=(\beta_{3n+1}-a)b_n^1(x)+c_n^1(x)+\gamma_{3n+1}P_n(x)+(a-b)(a-c)Q_n(x),\ n\geq 0,$$

$$\Theta(x)R_n(x)=\gamma_{3n+2}b_n^1(x)+(\beta_{3n+2}-a)c_n^1(x)+(a-b)(a-c)c_n^2(x)+P_{n+1}(x),\ n\geq 0.$$

These identities permit to write, through (A_9) multiplied by $\Theta(x)$, the following:

$$
\begin{aligned}
&\Theta(x)a_n^1(x)=(\beta_{3n+3}+b+c+p)\Theta(x)a_n^2(x)+\Theta(x)b_n^2(x)+\gamma_{3n+3}\Theta(x)R_n(x)\\
&\Rightarrow \Theta(x)a_n^1(x)=(\beta_{3n+3}+b+c+p)\Big((a-b)(a-c)a_n^1(x)+b_{n+1}^1(x)+\gamma_{3n+3}c_n^1(x)\\
&+(\beta_{3n+3}-a)P_{n+1}(x)\Big)\\
&+(\beta_{3n+4}-a)b_{n+1}^1(x)+c_{n+1}^1(x)+\gamma_{3n+4}P_{n+1}(x)+(a-b)(a-c)Q_{n+1}(x)\\
&+\gamma_{3n+3}\gamma_{3n+2}b_n^1(x)+\gamma_{3n+3}(\beta_{3n+2}-a)c_n^1(x)+(a-b)(a-c)\gamma_{3n+3}c_n^2(x)+\gamma_{3n+3}P_{n+1}(x)
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \Theta(x)a_n^1(x)=\gamma_{3n+2}\gamma_{3n+3}b_n^1(x)+(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)b_{n+1}^1(x)\\
&+\gamma_{3n+3}(\beta_{3n+2}+\beta_{3n+3}-a+b+c+p)c_n^1(x)+c_{n+1}^1(x)\\
&+\Big(\gamma_{3n+3}+\gamma_{3n+4}+(\beta_{3n+3}-a)(\beta_{3n+3}+b+c+p)\Big)P_{n+1}(x)\\
&+(a-b)(a-c)\Big((\beta_{3n+3}+b+c+p)a_n^1(x)+\gamma_{3n+3}c_n^2(x)+Q_{n+1}(x)\Big),
\end{aligned}
\tag{3.13}
$$

and through (A_6) multiplied by $\Theta(x)$, the following:

$$\Theta(x)c_n^2(x) = \Theta(x)a_n^2(x) + \gamma_{3n+2}\Theta(x)b_{n-1}^2(x) + (\beta_{3n+2} + b + c + p)\Theta(x)R_n(x)$$
$$\Rightarrow \Theta(x)c_n^2(x) = (a-b)(a-c)a_n^1(x) + b_{n+1}^1(x) + \gamma_{3n+3}c_n^1(x) + (\beta_{3n+3} - a)P_{n+1}(x)$$
$$+ \gamma_{3n+2}\Big((\beta_{3n+1} - a)b_n^1(x) + c_n^1(x) + \gamma_{3n+1}P_n(x) + (a-b)(a-c)Q_n(x)\Big)$$
$$+ (\beta_{3n+2} + b + c + p)\Big(\gamma_{3n+2}b_n^1(x) + (\beta_{3n+2} - a)c_n^1(x) + (a-b)(a-c)c_n^2(x) + P_{n+1}(x)\Big)$$

$$\begin{aligned}
&\Rightarrow \Theta(x)c_n^2(x) = \gamma_{3n+2}(\beta_{3n+1} - a + \beta_{3n+2} + b + c + p)b_n^1(x) + b_{n+1}^1(x) \\
&+ \big\{\gamma_{3n+2} + \gamma_{3n+3} + (\beta_{3n+2} - a)(\beta_{3n+2} + b + c + p)\big\}c_n^1(x) \\
&+ \gamma_{3n+1}\gamma_{3n+2}P_n(x) + (\beta_{3n+3} - a + \beta_{3n+2} + b + c + p)P_{n+1}(x) \\
&+ (a-b)(a-c)\Big(a_n^1(x) + (\beta_{3n+2} + b + c + p)c_n^2(x) + \gamma_{3n+2}Q_n(x)\Big).
\end{aligned} \tag{3.14}$$

Let us also remark the following expression for $\Theta(x)R_{n+1}$, that comes from (A_3) with $n \leftarrow n+1$ and multiplied by $\Theta(x)$, (where $Q_{n+1}(x)$ is replaced by the expression given by (A_8) and from identities (3.14), (3.13), (A_7) and (A_1), with $n \leftarrow n+1$.

$$\Theta(x)R_{n+1}(x) = \Theta(x)P_{n+1}(x) - (\beta_{3n+3} + a - b - c)\Theta(x)a_n^1(x) + (L - \gamma_{3n+4})\Theta(x)a_n^2(x)$$
$$- (\beta_{3n+4} + b + c + p)\Theta(x)b_n^2(x) - \gamma_{3n+3}\Theta(x)c_n^2(x)$$

$$\begin{aligned}
&\Rightarrow \Theta(x)R_{n+1}(x) = \Theta(x)P_{n+1}(x) \\
&- (\beta_{3n+3} + a - b - c)\Big\{\gamma_{3n+2}\gamma_{3n+3}b_n^1(x) + (\beta_{3n+3} + \beta_{3n+4} - a + b + c + p)b_{n+1}^1(x) \\
&+ \gamma_{3n+3}(\beta_{3n+2} + \beta_{3n+3} - a + b + c + p)c_n^1(x) + c_{n+1}^1(x) \\
&+ \Big(\gamma_{3n+3} + \gamma_{3n+4} + (\beta_{3n+3} - a)(\beta_{3n+3} + b + c + p)\Big)P_{n+1}(x) \\
&+ (a-b)(a-c)\Big((\beta_{3n+3} + b + c + p)a_n^1(x) + \gamma_{3n+3}c_n^2(x) + Q_{n+1}(x)\Big)\Big\} \\
&+ (L - \gamma_{3n+4})\Big\{b_{n+1}^1(x) + \gamma_{3n+3}c_n^1(x) + (\beta_{3n+3} - a)P_{n+1}(x) + (a-b)(a-c)a_n^1(x))\Big\} \\
&- (\beta_{3n+4} + b + c + p)\Big\{(\beta_{3n+4} - a)b_{n+1}^1(x) + c_{n+1}^1(x) + \gamma_{3n+4}P_{n+1}(x) + (a-b)(a-c)Q_{n+1}(x)\Big\} \\
&- \gamma_{3n+2}\gamma_{3n+3}(\beta_{3n+1} + \beta_{3n+2} - a + b + c + p)b_n^1(x) - \gamma_{3n+3}b_{n+1}^1(x) \\
&- \gamma_{3n+3}\Big(\gamma_{3n+2} + \gamma_{3n+3} + (\beta_{3n+2} - a)(\beta_{3n+2} + b + c + p)\Big)c_n^1(x) \\
&- \gamma_{3n+3}\Big(\gamma_{3n+1}\gamma_{3n+2}P_n(x) + (\beta_{3n+2} + \beta_{3n+3} - a + b + c + p)P_{n+1}(x)\Big) \\
&- \gamma_{3n+3}(a-b)(a-c)\Big(a_n^1(x) + (\beta_{3n+2} + b + c + p)c_n^2(x) + \gamma_{3n+2}Q_n(x)\Big)
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \Theta(x)R_{n+1}(x) = \Big\{\Theta(x) - \gamma_{3n+3}(\beta_{3n+2} + \beta_{3n+3} - a + b + c + p) \\
&- (\beta_{3n+3} + a - b - c)\big(\gamma_{3n+3} + \gamma_{3n+4} + (\beta_{3n+3} - a)(\beta_{3n+3} + b + c + p)\big) \\
&+ (L - \gamma_{3n+4})(\beta_{3n+3} - a) - \gamma_{3n+4}(\beta_{3n+4} + b + c + p)\Big\}P_{n+1}(x) - \gamma_{3n+1}\gamma_{3n+2}\gamma_{3n+3}P_n(x) \\
&- \gamma_{3n+2}\gamma_{3n+3}(\beta_{3n+1} + \beta_{3n+2} - a + b + c + p + \beta_{3n+3} + a - b - c)b_n^1(x) \\
&- \Big\{\gamma_{3n+3} + \gamma_{3n+4} - L + (\beta_{3n+3} + a - b - c)(\beta_{3n+3} + \beta_{3n+4} - a + b + c + p) \\
&+ (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p)\Big\}b_{n+1}^1(x) \\
&- \gamma_{3n+3}\Big\{\gamma_{3n+2} + \gamma_{3n+3} + \gamma_{3n+4} - L + (\beta_{3n+2} - a)(\beta_{3n+2} + b + c + p) \\
&+ (\beta_{3n+3} + a - b - c)(\beta_{3n+2} + \beta_{3n+3} - a + b + c + p)\Big\}c_n^1(x) \\
&- (\beta_{3n+3} + a - b - c + \beta_{3n+4} + b + c + p)c_{n+1}^1(x) \\
&- (a - b)(a - c)\Big\{\gamma_{3n+3}\Big(a_n^1(x) + (\beta_{3n+2} + b + c + p)c_n^2(x) + \gamma_{3n+2}Q_n(x)\Big) \\
&+ (\beta_{3n+3} + a - b - c)\Big((\beta_{3n+3} + b + c + p)a_n^1(x) + \gamma_{3n+3}c_n^2(x) + Q_{n+1}(x)\Big) \\
&+ (\gamma_{3n+4} - L)a_n^1(x) + (\beta_{3n+4} + b + c + p)Q_{n+1}(x)\Big\}.
\end{aligned}
$$

Adding the term $-(a-b)(a-c)c_{n+1}^2(x)$, and simplifying, we have:

$$
\begin{aligned}
&\Theta(x)R_{n+1}(x) - (a - b)(a - c)c_{n+1}^2(x) = \Big\{\Theta(x) - \gamma_{3n+3}(\beta_{3n+2} + 2\beta_{3n+3} + p) \\
&- \gamma_{3n+4}(2\beta_{3n+3} + \beta_{3n+4} + p) - (\beta_{3n+3} - a)(\beta_{3n+3} + a - b - c)(\beta_{3n+3} + b + c + p) \\
&+ (\beta_{3n+3} - a)L\Big\}P_{n+1}(x) - \gamma_{3n+1}\gamma_{3n+2}\gamma_{3n+3}P_n(x) \\
&- \gamma_{3n+2}\gamma_{3n+3}(\beta_{3n+1} + \beta_{3n+2} + \beta_{3n+3} + p)b_n^1(x) \\
&- \Big\{\gamma_{3n+3} + \gamma_{3n+4} - L + (\beta_{3n+3} + a - b - c)(\beta_{3n+3} + \beta_{3n+4} - a + b + c + p) \\
&+ (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p)\Big\}b_{n+1}^1(x) \\
&- \gamma_{3n+3}\Big\{\gamma_{3n+2} + \gamma_{3n+3} + \gamma_{3n+4} - L + (\beta_{3n+2} - a)(\beta_{3n+2} + b + c + p) \\
&+ (\beta_{3n+3} + a - b - c)(\beta_{3n+2} + \beta_{3n+3} - a + b + c + p)\Big\}c_n^1(x) \\
&- (\beta_{3n+3} + \beta_{3n+4} + a + p)c_{n+1}^1(x) \\
&- (a - b)(a - c)\Big\{\big(\gamma_{3n+3} + \gamma_{3n+4} - L + (\beta_{3n+3} + a - b - c)(\beta_{3n+3} + b + c + p)\big)a_n^1(x) \\
&+ \gamma_{3n+3}(\beta_{3n+2} + \beta_{3n+3} + a + p)c_n^2(x) + c_{n+1}^2(x) \\
&+ \gamma_{3n+3}\gamma_{3n+2}Q_n(x) + (\beta_{3n+3} + \beta_{3n+4} + a + p)Q_{n+1}(x)\Big\}.
\end{aligned}
$$

The following calculations will simplify the last term of the above identity, with coefficient $-(a-b)(a-c)$.

$$\begin{aligned}
&\big(\gamma_{3n+3}+\gamma_{3n+4}-L+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)\big)a_n^1(x)\\
&+\gamma_{3n+3}(\beta_{3n+2}+\beta_{3n+3}+a+p)c_n^2(x)+c_{n+1}^2(x)\\
&+\gamma_{3n+3}\gamma_{3n+2}Q_n(x)+(\beta_{3n+3}+\beta_{3n+4}+a+p)Q_{n+1}(x)\\
&=\gamma_{3n+3}\Big(a_n^1(x)+(\beta_{3n+2}+a-b-c)c_n^2(x)+\gamma_{3n+2}Q_n(x)\Big)\\
&+(\beta_{3n+3}+b+c+p)\Big((\beta_{3n+3}+a-b-c)a_n^1(x)+\gamma_{3n+3}c_n^2(x)+Q_{n+1}(x)\Big)\\
&+\Big(\gamma_{3n+4}a_n^1(x)+c_{n+1}^2(x)+(\beta_{3n+4}+a-b-c)Q_{n+1}(x)\Big)-La_n^1(x)\\
&=\gamma_{3n+3}\big(c_n^1(x)+LR_n(x)\big)+(\beta_{3n+3}+b+c+p)\big(P_{n+1}(x)+La_n^2(x)\big)\\
&+b_{n+1}^1(x)+Lb_n^2(x)-La_n^1(x),\ \text{using } (A_5),(A_8) \text{ and } (A_2) \text{ with } n\leftarrow n+1;\\
&=b_{n+1}^1(x)+\gamma_{3n+3}c_n^1(x)+(\beta_{3n+3}+b+c+p)P_{n+1}(x)\\
&-L\Big(a_n^1(x)-(\beta_{3n+3}+b+c+p)a_n^2(x)-b_n^2(x)-\gamma_{3n+3}R_n(x)\Big)\\
&\overset{(A_9)}{=}b_{n+1}^1(x)+\gamma_{3n+3}c_n^1(x)+(\beta_{3n+3}+b+c+p)P_{n+1}(x).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\Theta(x)R_{n+1}(x)-(a-b)(a-c)c_{n+1}^2(x)=\Big\{\Theta(x)-\gamma_{3n+3}(\beta_{3n+2}+2\beta_{3n+3}+p)\\
&-\gamma_{3n+4}(2\beta_{3n+3}+\beta_{3n+4}+p)-(\beta_{3n+3}-a)(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)\\
&+(\beta_{3n+3}-a)L-(a-b)(a-c)(\beta_{3n+3}+b+c+p)\Big\}P_{n+1}(x)-\gamma_{3n+1}\gamma_{3n+2}\gamma_{3n+3}P_n(x)\\
&-\gamma_{3n+2}\gamma_{3n+3}(\beta_{3n+1}+\beta_{3n+2}+\beta_{3n+3}+p)b_n^1(x)\\
&-\Big\{\gamma_{3n+3}+\gamma_{3n+4}+(a-b)(a-c)-L+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)\\
&+(\beta_{3n+4}-a)(\beta_{3n+3}+\beta_{3n+4}+a+p)\Big\}b_{n+1}^1(x)\\
&-\gamma_{3n+3}\Big\{\gamma_{3n+2}+\gamma_{3n+3}+\gamma_{3n+4}+(a-b)(a-c)-L\\
&+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)+(\beta_{3n+2}-a)(\beta_{3n+2}+\beta_{3n+3}+a+p)\Big\}c_n^1(x)\\
&-(\beta_{3n+3}+\beta_{3n+4}+a+p)c_{n+1}^1(x),
\end{aligned}$$

that is,

$$\begin{aligned}
&\Theta(x)R_{n+1}(x)-(a-b)(a-c)c_{n+1}^2(x)=\big(\Theta(x)-A_{3n}\big)P_{n+1}(x)-B_{3n}P_n(x)\\
&-K_{3n}b_n^1(x)+\big(-H_{3n}+\gamma_{3n+5}\big)b_{n+1}^1(x)-V_{3n}c_n^1(x)+\big(-S_{3n}+\beta_{3n+5}-a\big)c_{n+1}^1(x).
\end{aligned}\tag{3.15}$$

Let us remark that identity (3.15) was deduced by the use of all the identities of theorem 3.1, where only (A_1), (A_3) and (A_2) were involved after the transformation $n\leftarrow n+1$.

The last step of Part I consists in writing polynomial $P_{n+2}(x)+La^2_{n+1}(x)-\Theta(x)Q_{n+1}(x)$ in terms of elements of the component sequences $\{a^1_{n-1}\}_{n\geq 0}$, $\{c^2_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$.

Let us begin to note identities (A_2), (A_5) and (A_8) (respectively):

$$b^1_n(x) = \gamma_{3n+1}a^1_{n-1}(x) - Lb^2_{n-1}(x) + c^2_n(x) + (\beta_{3n+1} + a - b - c)Q_n(x),\ \geq 0,$$

$$c^1_n(x) = a^1_n(x) + (\beta_{3n+2} + a - b - c)c^2_n(x) + \gamma_{3n+2}Q_n(x) - LR_n(x),\ \geq 0,$$

$$P_{n+1}(x) = (\beta_{3n+3} + a - b - c)a^1_n(x) - La^2_n(x) + \gamma_{3n+3}c^2_n(x) + Q_{n+1}(x),\ \geq 0.$$

From (A_3) $(n \leftarrow n+1)$, multiplied by $\Theta(x)$, we have
$\Theta(x)R_{n+1}(x) = \Theta(x)Q_{n+1}(x) - \gamma_{3n+4}\Theta(x)a^2_n(x) - (\beta_{3n+4} + b + c + p)\Theta(x)b^2_n(x)$.
Inserting (A_1) (with $n \leftarrow n+1$) and (A_7):

$$\begin{aligned}
&\Theta(x)R_{n+1}(x) = \Theta(x)Q_{n+1}(x)\\
&-\gamma_{3n+4}\Big((a-b)(a-c)a^1_n(x) + b^1_{n+1}(x) + \gamma_{3n+3}c^1_n(x) + (\beta_{3n+3} - a)P_{n+1}(x)\Big)\\
&-(\beta_{3n+4} + b + c + p)\Big((\beta_{3n+4} - a)b^1_{n+1}(x) + c^1_{n+1}(x) + \gamma_{3n+4}P_{n+1}(x)\\
&+(a-b)(a-c)Q_{n+1}(x)\Big)\\
&\Rightarrow \Theta(x)R_{n+1}(x) = \Theta(x)Q_{n+1}(x) - (\beta_{3n+4} + b + c + p)(a-b)(a-c)Q_{n+1}(x)\\
&-\gamma_{3n+4}(a-b)(a-c)a^1_n(x) - (\beta_{3n+4} + b + c + p)(\beta_{3n+4} - a)b^1_{n+1}(x) - \gamma_{3n+4}b^1_{n+1}(x)\\
&-\gamma_{3n+4}\gamma_{3n+3}c^1_n(x) - (\beta_{3n+4} + b + c + p)c^1_{n+1}(x)\\
&-\gamma_{3n+4}(\beta_{3n+3} + \beta_{3n+4} - a + b + c + p)P_{n+1}(x).
\end{aligned}$$

Introducing (A_5), (A_8) and (A_2):

$$\begin{aligned}
&\Theta(x)R_{n+1}(x) = \Theta(x)Q_{n+1}(x) - (\beta_{3n+4} + b + c + p)(a-b)(a-c)Q_{n+1}(x)\\
&-\gamma_{3n+4}(a-b)(a-c)a^1_n(x)\\
&-\Big(\gamma_{3n+4} + (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p)\Big)\Big(\gamma_{3n+4}a^1_n(x) - Lb^2_n(x)\\
&+c^2_{n+1}(x) + (\beta_{3n+4} + a - b - c)Q_{n+1}(x)\Big)\\
&-\gamma_{3n+3}\gamma_{3n+4}\Big(a^1_n(x) + (\beta_{3n+2} + a - b - c)c^2_n(x) + \gamma_{3n+2}Q_n(x) - LR_n(x)\Big)\\
&-(\beta_{3n+4} + b + c + p)\Big(a^1_{n+1}(x) + (\beta_{3n+5} + a - b - c)c^2_{n+1}(x) + \gamma_{3n+5}Q_{n+1}(x) - LR_{n+1}(x)\Big)\\
&-\gamma_{3n+4}(\beta_{3n+3} + \beta_{3n+4} - a + b + c + p)\Big((\beta_{3n+3} + a - b - c)a^1_n(x) - La^2_n(x)\\
&+\gamma_{3n+3}c^2_n(x) + Q_{n+1}(x)\Big),
\end{aligned}$$

that is,

$$
\begin{aligned}
&\Theta(x)R_{n+1}(x) = \Big\{\Theta(x) - (\beta_{3n+4}+b+c+p)(a-b)(a-c) - \gamma_{3n+5}(\beta_{3n+4}+b+c+p) \\
&- \gamma_{3n+4}(\beta_{3n+4}+a-b-c) - (\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p)(\beta_{3n+4}+a-b-c) \\
&- \gamma_{3n+4}(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)\Big\}Q_{n+1}(x) - \gamma_{3n+2}\gamma_{3n+3}\gamma_{3n+4}Q_n(x) \\
&- \Big\{\gamma_{3n+4}(a-b)(a-c) + \gamma_{3n+3}\gamma_{3n+4} + \gamma_{3n+4}\big(\gamma_{3n+4} + (\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p)\big) \\
&+ \gamma_{3n+4}(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)(\beta_{3n+3}+a-b-c)\Big\}a_n^1(x) \\
&- (\beta_{3n+4}+b+c+p)a_{n+1}^1(x) \\
&- \Big\{\gamma_{3n+3}\gamma_{3n+4}(\beta_{3n+2}+a-b-c) + \gamma_{3n+3}\gamma_{3n+4}(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)\Big\}c_n^2(x) \\
&- \Big\{(\beta_{3n+4}+b+c+p)(\beta_{3n+5}+a-b-c) + \gamma_{3n+4} + (\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p)\Big\}c_{n+1}^2(x) \\
&+ L\Big\{\gamma_{3n+4}(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)a_n^2(x) \\
&+ \Big(\gamma_{3n+4} + (\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p)\Big)b_n^2(x) \\
&+ \gamma_{3n+3}\gamma_{3n+4}R_n(x) + (\beta_{3n+4}+b+c+p)R_{n+1}(x)\Big\}.
\end{aligned}
\tag{3.16}
$$

So, in identity (A_4) $(n \leftarrow n+1)$ we may replace $c_{n+1}^1(x)$, $b_{n+1}^1(x)$ and $\Theta(x)R_{n+1}(x)$ by (A_5), (A_2) (both with $n \leftarrow n+1$) and (3.16), respectively, yielding the following.

$$
P_{n+2}(x) + \gamma_{3n+5}b_{n+1}^1(x) + (\beta_{3n+5}-a)c_{n+1}^1(x) + (a-b)(a-c)c_{n+1}^2(x) = \Theta(x)R_{n+1}(x)
$$

$$
\begin{aligned}
&\Rightarrow P_{n+2}(x) + \gamma_{3n+5}\Big(\gamma_{3n+4}a_n^1(x) - Lb_n^2(x) + c_{n+1}^2(x) + (\beta_{3n+4}+a-b-c)Q_{n+1}(x)\Big) \\
&+ (\beta_{3n+5}-a)\Big(a_{n+1}^1(x) + (\beta_{3n+5}+a-b-c)c_{n+1}^2(x) + \gamma_{3n+5}Q_{n+1}(x) - LR_{n+1}(x)\Big) \\
&+ (a-b)(a-c)c_{n+1}^2(x) \\
&= \Big\{\Theta(x) - \gamma_{3n+4}(\beta_{3n+3}+2\beta_{3n+4}+p) - \gamma_{3n+5}(\beta_{3n+4}+b+c+p) \\
&- (\beta_{3n+4}-a)(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p) - (a-b)(a-c)(\beta_{3n+4}+b+c+p)\Big\}Q_{n+1}(x) \\
&- \gamma_{3n+2}\gamma_{3n+3}\gamma_{3n+4}Q_n(x) \\
&- \gamma_{3n+4}\Big\{\gamma_{3n+3} + \gamma_{3n+4} + (a-b)(a-c) \\
&+ (\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p) + (\beta_{3n+3}+a-b-c)(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)\Big\}a_n^1(x) \\
&- (\beta_{3n+4}+b+c+p)a_{n+1}^1(x) - \gamma_{3n+3}\gamma_{3n+4}(\beta_{3n+2}+\beta_{3n+3}+\beta_{3n+4}+p)c_n^2(x) \\
&- \Big\{\gamma_{3n+4} + (\beta_{3n+4}+b+c+p)(\beta_{3n+4}+\beta_{3n+5}-b-c)\Big\}c_{n+1}^2(x)
\end{aligned}
$$

$$+L\Big\{\gamma_{3n+4}(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)a_n^2(x)$$
$$+\Big(\gamma_{3n+4}+(\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p)\Big)b_n^2(x)$$
$$+\gamma_{3n+3}\gamma_{3n+4}R_n(x)+(\beta_{3n+4}+b+c+p)R_{n+1}(x)\Big\}.$$

Adding the term $La_{n+1}^2(x)$, and simplifying, we have:

$$P_{n+2}(x)+La_{n+1}^2(x)=\Big\{\Theta(x)-\gamma_{3n+4}(\beta_{3n+3}+2\beta_{3n+4}+p)-\gamma_{3n+5}(2\beta_{3n+4}+\beta_{3n+5}+p)$$
$$-(\beta_{3n+4}-a)(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)$$
$$-(a-b)(a-c)(\beta_{3n+4}+b+c+p)\Big\}Q_{n+1}(x)-\gamma_{3n+2}\gamma_{3n+3}\gamma_{3n+4}Q_n(x)$$
$$-\gamma_{3n+4}\Big\{\gamma_{3n+3}+\gamma_{3n+4}+\gamma_{3n+5}+(a-b)(a-c)$$
$$+(\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p)+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)\Big\}a_n^1(x)$$
$$-(\beta_{3n+4}+\beta_{3n+5}-a+b+c+p)a_{n+1}^1(x)-\gamma_{3n+3}\gamma_{3n+4}(\beta_{3n+2}+\beta_{3n+3}+\beta_{3n+4}+p)c_n^2(x)$$
$$-\Big\{\gamma_{3n+4}+\gamma_{3n+5}+(\beta_{3n+4}+b+c+p)(\beta_{3n+4}+\beta_{3n+5}-b-c)$$
$$+(\beta_{3n+5}-a)(\beta_{3n+5}+a-b-c)+(a-b)(a-c)\Big\}c_{n+1}^2(x)$$
$$+L\Big\{\gamma_{3n+4}(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)a_n^2(x)+a_{n+1}^2(x)$$
$$+\Big(\gamma_{3n+4}+\gamma_{3n+5}+(\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p)\Big)b_n^2(x)$$
$$+\gamma_{3n+3}\gamma_{3n+4}R_n(x)+(\beta_{3n+4}+\beta_{3n+5}-a+b+c+p)R_{n+1}(x)\Big\}.$$

The following calculations will simplify the last term of the above identity, with coefficient L.

$$\gamma_{3n+4}(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)a_n^2(x)+a_{n+1}^2(x)$$
$$+\Big(\gamma_{3n+4}+\gamma_{3n+5}+(\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p)\Big)b_n^2(x)$$
$$+\gamma_{3n+3}\gamma_{3n+4}R_n(x)+(\beta_{3n+4}+\beta_{3n+5}-a+b+c+p)R_{n+1}(x)$$
$$=a_{n+1}^2(x)+\gamma_{3n+5}b_n^2(x)+(\beta_{3n+5}+b+c+p)R_{n+1}(x)$$
$$+(\beta_{3n+4}-a)\Big(\gamma_{3n+4}a_n^2(x)+(\beta_{3n+4}+b+c+p)b_n^2(x)+R_{n+1}(x)\Big)$$
$$+\gamma_{3n+4}\Big((\beta_{3n+3}+b+c+p)a_n^2(x)+b_n^2(x)+\gamma_{3n+3}R_n(x)\Big)$$
$$=c_{n+1}^2(x)+(\beta_{3n+4}-a)Q_{n+1}(x)+\gamma_{3n+4}a_n^1(x),$$

using (A_6) and (A_3), with $n\leftarrow n+1$, and (A_9).

Thus, we obtain:

$$P_{n+2}(x)+La^2_{n+1}(x)=\Big\{\Theta(x)-\gamma_{3n+4}(\beta_{3n+3}+2\beta_{3n+4}+p)-\gamma_{3n+5}(2\beta_{3n+4}+\beta_{3n+5}+p)$$
$$-(\beta_{3n+4}-a)(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)+L(\beta_{3n+4}-a)$$
$$-(a-b)(a-c)(\beta_{3n+4}+b+c+p)\Big\}Q_{n+1}(x)-\gamma_{3n+2}\gamma_{3n+3}\gamma_{3n+4}Q_n(x)$$
$$-\gamma_{3n+4}\Big\{\gamma_{3n+3}+\gamma_{3n+4}+\gamma_{3n+5}+(a-b)(a-c)-L$$
$$+(\beta_{3n+4}-a)(\beta_{3n+3}+\beta_{3n+4}+a+p)+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)\Big\}a^1_n(x)$$
$$-(\beta_{3n+4}+\beta_{3n+5}-a+b+c+p)a^1_{n+1}(x)-\gamma_{3n+3}\gamma_{3n+4}(\beta_{3n+2}+\beta_{3n+3}+\beta_{3n+4}+p)c^2_n(x)$$
$$-\Big\{\gamma_{3n+4}+\gamma_{3n+5}+(a-b)(a-c)-L$$
$$+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)+(\beta_{3n+5}-a)(\beta_{3n+4}+\beta_{3n+5}+a+p)\Big\}c^2_{n+1}(x).$$

For all constants N and M, we have

$$(M-a)(N+M+a+p)+(N+a-b-c)(N+b+c+p)$$
$$=(N-a)(N+M+a+p)+(M+a-b-c)(M+b+c+p),$$

then, $(\beta_{3n+4}-a)(\beta_{3n+3}+\beta_{3n+4}+a+p)+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)$

$=(\beta_{3n+3}-a)(\beta_{3n+3}+\beta_{3n+4}+a+p)+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)$, and consequently

$$\begin{aligned}P_{n+2}(x)+La^2_{n+1}(x)&=\big(\Theta(x)-A_{3n+1}\big)Q_{n+1}(x)-B_{3n+1}Q_n(x)\\&-K_{3n+1}c^2_n(x)+\big(-H_{3n+1}+\gamma_{3n+6}\big)c^2_{n+1}(x)-V_{3n+1}a^1_n(x)\\&+\big(-S_{3n+1}+\beta_{3n+6}-a-b-c\big)a^1_{n+1}(x).\end{aligned}\tag{3.17}$$

Let us remark that identity (3.17) was deduced by the use of all the identities of theorem 3.1, where only (A_2), (A_3), (A_4), (A_5) and (A_6) were involved after the transformation $n\leftarrow n+1$.

Part II ($\Rightarrow$) To deduce (3.3), we consider (A_4), with $n\leftarrow n+1$, and we replace $P_{n+2}(x)$, $c^1_{n+1}(x)$, $b^1_{n+1}(x)$ by (3.11), (3.10) and (3.12), (where in (3.11) and (3.10) we take

$n \leftarrow n+1)$ and $c^2_{n+1}(x)$ by (A_6) (also with $n \leftarrow n+1$).

$$
\begin{aligned}
&P_{n+2}(x) + \gamma_{3n+5} b^1_{n+1}(x) + (\beta_{3n+5} - a) c^1_{n+1}(x) + (a-b)(a-c) c^2_{n+1}(x) = \Theta(x) R_{n+1}(x) \\
&\Rightarrow \Big(\gamma_{3n+6} + \gamma_{3n+7} + (\beta_{3n+6} + a - b - c)(\beta_{3n+6} + b + c + p) - L\Big) a^2_{n+1}(x) \\
&+ \gamma_{3n+5}\gamma_{3n+6} b^2_n(x) + (\beta_{3n+6} + \beta_{3n+7} + a + p) b^2_{n+1}(x) \\
&+ \gamma_{3n+6}(\beta_{3n+5} + \beta_{3n+6} + a + p) R_{n+1}(x) + R_{n+2}(x) \\
&+ \gamma_{3n+5}\Big\{\gamma_{3n+4}(\beta_{3n+3} + \beta_{3n+4} + a + p) a^2_n(x) + a^2_{n+1}(x) \\
&+ \Big(\gamma_{3n+4} + \gamma_{3n+5} + (\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p) - L\Big) b^2_n(x) \\
&+ \gamma_{3n+3}\gamma_{3n+4} R_n(x) + (\beta_{3n+4} + \beta_{3n+5} + a + p) R_{n+1}(x)\Big\} \\
&+ (\beta_{3n+5} - a)\Big\{\gamma_{3n+4}\gamma_{3n+5} a^2_n(x) + (\beta_{3n+5} + \beta_{3n+6} + a + p) a^2_{n+1}(x) \\
&+ \gamma_{3n+5}(\beta_{3n+4} + \beta_{3n+5} + a + p) b^2_n(x) + b^2_{n+1}(x) \\
&+ \Big(\gamma_{3n+5} + \gamma_{3n+6} + (\beta_{3n+5} + a - b - c)(\beta_{3n+5} + b + c + p) - L\Big) R_{n+1}(x)\Big\} \\
&+ (a-b)(a-c)\Big(a^2_{n+1}(x) + \gamma_{3n+5} b^2_n(x) + (\beta_{3n+5} + b + c + p) R_{n+1}(x)\Big) = \Theta(x) R_{n+1}(x),
\end{aligned}
$$

and rearranging, we get:

$$
\begin{aligned}
&R_{n+2}(x) = \Theta(x) R_{n+1}(x) - \gamma_{3n+6}(\beta_{3n+5} + \beta_{3n+6} + a + p) R_{n+1}(x) \\
&- (\beta_{3n+5} - a)\Big(\gamma_{3n+5} + \gamma_{3n+6} + (\beta_{3n+5} + a - b - c)(\beta_{3n+5} + b + c + p) - L\Big) R_{n+1}(x) \\
&- \gamma_{3n+5}(\beta_{3n+4} + \beta_{3n+5} + a + p) R_{n+1}(x) - (a-b)(a-c)(\beta_{3n+5} + b + c + p) R_{n+1}(x) \\
&- \gamma_{3n+3}\gamma_{3n+4}\gamma_{3n+5} R_n(x) \\
&- (\beta_{3n+5} - a)\gamma_{3n+4}\gamma_{3n+5} a^2_n(x) - \gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+3} + \beta_{3n+4} + a + p) a^2_n(x) \\
&- \Big(\gamma_{3n+6} + \gamma_{3n+7} + (\beta_{3n+6} + a - b - c)(\beta_{3n+6} + b + c + p) - L\Big) a^2_{n+1}(x) \\
&- (\beta_{3n+5} - a)(\beta_{3n+5} + \beta_{3n+6} + a + p) a^2_{n+1}(x) - \gamma_{3n+5} a^2_{n+1}(x) - (a-b)(a-c) a^2_{n+1}(x) \\
&- \gamma_{3n+5}\gamma_{3n+6} b^2_n(x) - \gamma_{3n+5}(\beta_{3n+5} - a)(\beta_{3n+4} + \beta_{3n+5} + a + p) b^2_n(x) \\
&- \gamma_{3n+5}\Big(\gamma_{3n+4} + \gamma_{3n+5} + (\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p) - L\Big) b^2_n(x) \\
&- (a-b)(a-c)\gamma_{3n+5} b^2_n(x) \\
&- (\beta_{3n+6} + \beta_{3n+7} + a + p) b^2_{n+1}(x) - (\beta_{3n+5} - a) b^2_{n+1}(x)
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow R_{n+2}(x) = \Big\{\Theta(x) - \gamma_{3n+5}(\beta_{3n+4} + 2\beta_{3n+5} + p) - \gamma_{3n+6}(2\beta_{3n+5} + \beta_{3n+6} + p) \\
&- (\beta_{3n+5} - a)(\beta_{3n+5} + a - b - c)(\beta_{3n+5} + b + c + p) \\
&+ (\beta_{3n+5} - a)L - (a-b)(a-c)(\beta_{3n+5} + b + c + p)\Big\} R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+4}\gamma_{3n+5} R_n(x) \\
&- \gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+3} + \beta_{3n+4} + \beta_{3n+5} + p) a_n^2(x) \\
&- \Big\{\gamma_{3n+5} + \gamma_{3n+6} + \gamma_{3n+7} + (a-b)(a-c) - L \\
&+ (\beta_{3n+6} + a - b - c)(\beta_{3n+6} + b + c + p) + (\beta_{3n+5} - a)(\beta_{3n+5} + \beta_{3n+6} + a + p)\Big\} a_{n+1}^2(x) \\
&- \gamma_{3n+5}\Big\{\gamma_{3n+4} + \gamma_{3n+5} + \gamma_{3n+6} + (a-b)(a-c) - L \\
&+ (\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p) + (\beta_{3n+5} - a)(\beta_{3n+4} + \beta_{3n+5} + a + p)\Big\} b_n^2(x) \\
&- (\beta_{3n+5} + \beta_{3n+6} + \beta_{3n+7} + p) b_{n+1}^2(x).
\end{aligned}
\tag{3.18}
$$

As we remarked before, for all constants N and M, we have:

$$
\begin{aligned}
(M-a)(N+M+a+p) &+ (N+a-b-c)(N+b+c+p) = \\
&(N-a)(N+M+a+p) + (M+a-b-c)(M+b+c+p),
\end{aligned}
$$

which justifies that in identity (3.18) the coefficient of $a_{n+1}^2(x)$ is $-H_{3n+2}$, the coefficient of $b_n^2(x)$ is $-V_{3n+2}$ and that identity (3.18) is in fact identity (3.3).

In order to obtain identity (3.1), we replace in (A_4), with $n \leftarrow n+1$, the polynomial

$$
\Theta(x) R_{n+1}(x) - (a-b)(a-c) c_{n+1}^2(x)
$$

by the linear relation between $\Theta(x)P_{n+1}(x)$ and elements of the sequences $\{P_n\}_{n\geq 0}$, $\{b_n^1\}_{n\geq 0}$ and $\{c_n^1\}_{n\geq 0}$, given by (3.15), as follows.

$$
\begin{aligned}
&\gamma_{3n+5} b_{n+1}^1(x) + (\beta_{3n+5} - a) c_{n+1}^1(x) + P_{n+2}(x) = \Theta(x) R_{n+1}(x) - (a-b)(a-c) c_{n+1}^2(x) \\
&\Rightarrow \gamma_{3n+5} b_{n+1}^1(x) + (\beta_{3n+5} - a) c_{n+1}^1(x) + P_{n+2}(x) = \big(\Theta(x) - A_{3n}\big) P_{n+1}(x) - B_{3n} P_n(x) \\
&- K_{3n} b_n^1(x) + \big(-H_{3n} + \gamma_{3n+5}\big) b_{n+1}^1(x) - V_{3n} c_n^1(x) + \big(-S_{3n} + \beta_{3n+5} - a\big) c_{n+1}^1(x).
\end{aligned}
$$

Identity (3.2) comes from (A_8), with $n \leftarrow n+1$,

$$
P_{n+2}(x) + L a_{n+1}^2(x) = (\beta_{3n+6} + a - b - c) a_{n+1}^1(x) + \gamma_{3n+6} c_{n+1}^2(x) + Q_{n+2}(x),
$$

where $P_{n+2}(x) + L a_{n+1}^2(x)$ is replaced by (3.17).

Let us conclude part II with the initial conditions enunciated. Polynomials $P_1(x)$, $Q_1(x)$ and $R_1(x)$ are determined trough theorem 3.1 relations with $n = 0$ and $n = 1$, as follows.

Considering $n = 0$, the relations (A_1) and (A_2) determine, respectively, $c_0^1(x)$ and $c_0^2(x)$,

$$
c_0^1(x) = -(\beta_1 - a) b_0^1(x) - \gamma_1 P_0(x) - (a-b)(a-c) Q_0(x),
$$

$$c_0^2(x) = b_0^1(x) - (\beta_1 + a - b - c)Q_0(x),$$

and from (A_4),

$$P_1(x) = -\gamma_2 b_0^1(x) - (\beta_2 - a)c_0^1(x) - (a-b)(a-c)c_0^2(x) + \Theta(x),$$

which is polynomial $P_1(x)$ presented in (3.1).

Identities (A_5) and (A_6) determine, respectively, $a_0^1(x)$ and $a_0^2(x)$, and from (A_8) we have:

$$Q_1(x) = -(\beta_3 + a - b - c)a_0^1(x) + La_0^2(x) - \gamma_3 c_0^2(x) + P_1(x),$$

which is polynomial $Q_1(x)$ presented in (3.2). Relation (A_9) calculates $b_0^2(x)$ and from (A_3), with $n = 1$, we get

$$R_1(x) = -\gamma_4 a_0^2(x) - (\beta_4 + b + c + p)b_0^2(x) + Q_1(x),$$

which is polynomial $R_1(x)$ presented in (3.3).

The polynomials $P_1(x)$, $Q_1(x)$ and $R_1(x)$ presented were calculated using *Mathematica 6* software. These polynomials are easily computed using the procedure described in chapter 5.

Part III ($\Leftarrow$) Let us suppose now the enunciated relations. We will demonstrate relations (A_3), (A_4) and (A_9) of theorem 3.1 by induction over n.

Firstly, we recall that (A_4) and (A_9) are fulfilled for $n = 0$ and (A_3) is fulfilled for $n = 0$ and $n = 1$, because the initial conditions enunciated were determined in such a way they fulfill theorem 3.1 and particularly, using the identities that both theorem 3.1 and this one, 3.2, have in common.

Let us take as induction hypotheses that the relations (A_4) and (A_9) are fulfilled for $n \leq k$ and (A_3) is fulfilled for $n \leq k+1$, for some $k \geq 0$. Next, we will show that relations (A_4) and (A_9) are fulfilled for $n = k+1$ and (A_3) is fulfilled for $n = k+2$, which concludes the proof.

Let us begin with (A_3) for $n = k+2$:

$$\gamma_{3k+7} a_{k+1}^2(x) + (\beta_{3k+7} + b + c + p)b_{k+1}^2(x) - Q_{k+2}(x) + R_{k+2}(x) = 0.$$

The left member will be transformed regarding the hypotheses considered and we will show that it is zero. Inserting (3.3) yields:

$$\begin{aligned}
&\gamma_{3k+7} a_{k+1}^2(x) + (\beta_{3k+7} + b + c + p)b_{k+1}^2(x) - Q_{k+2}(x) + R_{k+2}(x) \\
&= \big\{\Theta(x) - A_{3k+2}\big\} R_{k+1}(x) - B_{3k+2} R_k(x) - K_{3k+2} a_k^2(x) - H_{3k+2} a_{k+1}^2(x) \\
&- V_{3k+2} b_k^2(x) - S_{3k+2} b_{k+1}^2(x) + \gamma_{3k+7} a_{k+1}^2(x) + (\beta_{3k+7} + b + c + p)b_{k+1}^2(x) - Q_{k+2}(x) \\
&= \Big(\Theta(x) R_{k+1}(x) - Q_{k+2}(x)\Big) - A_{3k+2} R_{k+1}(x) - B_{3k+2} R_k(x) - K_{3k+2} a_k^2(x) \\
&- (H_{3k+2} - \gamma_{3k+7}) a_{k+1}^2(x) - V_{3k+2} b_k^2(x) - \big(S_{3k+2} - (\beta_{3k+7} + b + c + p)\big) b_{k+1}^2(x).
\end{aligned} \tag{3.19}$$

We will now write the difference $\Theta(x)R_{k+1}(x) - Q_{k+2}(x)$ in terms of elements of the sequences $\{R_n(x)\}_{n\geq 0}$, $\{a^2_{n-1}(x)\}_{n\geq 0}$ and $\{b^2_{n-1}(x)\}_{n\geq 0}$.

By hypothesis, (A_3) is fulfilled for $n = k+1$, hence, we begin to consider it, multiplied by $\Theta(x)$:

$$\Theta(x)R_{k+1}(x) = \Theta(x)Q_{k+1}(x) - \gamma_{3k+4}\Theta(x)a^2_k(x) - (\beta_{3k+4} + b + c + p)\Theta(x)b^2_k(x).$$

Introducing (3.2), (A_7) (with $n = k$) and (A_1) (with $n = k+1$) we get:

$$\begin{aligned}
&\Theta(x)R_{k+1}(x) = Q_{k+2}(x) + A_{3k+1}Q_{k+1}(x) + B_{3k+1}Q_k(x) + V_{3k+1}a^1_k(x)\\
&+ S_{3k+1}a^1_{k+1}(x) + K_{3k+1}c^2_k(x) + H_{3k+1}c^2_{k+1}(x)\\
&- (\beta_{3k+4} + b + c + p)\Big((\beta_{3k+4} - a)b^1_{k+1}(x) + c^1_{k+1}(x) + \gamma_{3k+4}P_{k+1}(x) + (a-b)(a-c)Q_{k+1}(x)\Big)\\
&- \gamma_{3k+4}\Big((a-b)(a-c)a^1_k(x) + b^1_{k+1}(x) + \gamma_{3k+3}c^1_k(x) + (\beta_{3k+3} - a)P_{k+1}(x)\Big)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \Theta(x)R_{k+1}(x) - Q_{k+2}(x) = \Big(A_{3k+1} - (\beta_{3k+4} + b + c + p)(a-b)(a-c)\Big)Q_{k+1}(x)\\
&+ B_{3k+1}Q_k(x) + \Big(V_{3k+1} - \gamma_{3k+4}(a-b)(a-c)\Big)a^1_k(x) + S_{3k+1}a^1_{k+1}(x)\\
&- \Big((\beta_{3k+4} + b + c + p)(\beta_{3k+4} - a) + \gamma_{3k+4}\Big)b^1_{k+1}(x)\\
&- \gamma_{3k+3}\gamma_{3k+4}c^1_k(x) - (\beta_{3k+4} + b + c + p)c^1_{k+1}(x)\\
&+ K_{3k+1}c^2_k(x) + H_{3k+1}c^2_{k+1}(x) - \gamma_{3k+4}(\beta_{3k+3} + \beta_{3k+4} - a + b + c + p)P_{k+1}(x).
\end{aligned}$$

We will now proceed with the following replacements:

-$a^1_k(x)$ and $a^1_{k+1}(x)$ by (A_9) (with $n = k$ and $n = k+1$, respectively);

-$b^1_{k+1}(x)$ by (3.12) (with $n = k$);

-$c^1_k(x)$ and $c^1_{k+1}(x)$ by (3.10) (with $n = k$ and $n = k+1$, respectively);

-$c^2_k(x)$ and $c^2_{k+1}(x)$ by the expression given by (A_6) (with $n = k$ and $n = k+1$, respectively);

-$P_{k+1}(x)$ by (3.11) with $n = k$; and

-$Q_k(x)$ and $Q_{k+1}(x)$ by the expression given by (A_3) (with $n = k$ and $n = k+1$, respectively).

Let us note that the expression of $c^1_k(x)$ (3.10) was obtained trough identities (A_5), (A_6), (A_9) and (A_3) with $n = k$; the expression of $b^1_{k+1}(x)$ (3.12) was obtained trough (A_2), (A_6), (A_9) and (A_3) with $n \leq k+1$, and the expression of $P_{k+1}(x)$ (3.11) was deduced by (A_8), with $n \leq k$, and (A_6), (A_9) and (A_3) with $n \leq k+1$. In other words, the use that we will give to these three identities is coherent with the list of hypotheses taken.

The indicated replacements yield the following identity:

$$
\begin{aligned}
&\Theta(x)R_{k+1}(x) - Q_{k+2}(x)\\
&= \Big(A_{3k+1} - (\beta_{3k+4}+b+c+p)(a-b)(a-c)\Big)\Big(\gamma_{3k+4}a_k^2(x) + (\beta_{3k+4}+b+c+p)b_k^2(x) + R_{k+1}\Big)\\
&+ B_{3k+1}\Big(\gamma_{3k+1}a_{k-1}^2(x) + (\beta_{3k+1}+b+c+p)b_{k-1}^2(x) + R_k\Big)\\
&+ \Big(V_{3k+1} - \gamma_{3k+4}(a-b)(a-c)\Big)\Big((\beta_{3k+3}+b+c+p)a_k^2(x) + b_k^2(x) + \gamma_{3k+3}R_k(x)\Big)\\
&+ S_{3k+1}\Big((\beta_{3k+6}+b+c+p)a_{k+1}^2(x) + b_{k+1}^2(x) + \gamma_{3k+6}R_{k+1}(x)\Big)\\
&- \Big((\beta_{3k+4}+b+c+p)(\beta_{3k+4}-a) + \gamma_{3k+4}\Big)\Big\{\gamma_{3k+4}(\beta_{3k+3}+\beta_{3k+4}+a+p)a_k^2(x)\\
&+ a_{k+1}^2(x) + \Big(\gamma_{3k+4}+\gamma_{3k+5}+(\beta_{3k+4}+a-b-c)(\beta_{3k+4}+b+c+p) - L\Big)b_k^2(x)\\
&+ \gamma_{3k+3}\gamma_{3k+4}R_k(x) + (\beta_{3k+4}+\beta_{3k+5}+a+p)R_{k+1}(x)\Big\}\\
&- \gamma_{3k+3}\gamma_{3k+4}\Big\{\gamma_{3k+1}\gamma_{3k+2}a_{k-1}^2(x) + (\beta_{3k+2}+\beta_{3k+3}+a+p)a_k^2(x)\\
&+ \gamma_{3k+2}(\beta_{3k+1}+\beta_{3k+2}+a+p)b_{k-1}^2(x) + b_k^2(x)\\
&+ \Big(\gamma_{3k+2}+\gamma_{3k+3}+(\beta_{3k+2}+a-b-c)(\beta_{3k+2}+b+c+p) - L\Big)R_k(x)\Big\}\\
&- (\beta_{3k+4}+b+c+p)\Big\{\gamma_{3k+4}\gamma_{3k+5}a_k^2(x) + (\beta_{3k+5}+\beta_{3k+6}+a+p)a_{k+1}^2(x)\\
&+ \gamma_{3k+5}(\beta_{3k+4}+\beta_{3k+5}+a+p)b_k^2(x) + b_{k+1}^2(x)\\
&+ \Big(\gamma_{3k+5}+\gamma_{3k+6}+(\beta_{3k+5}+a-b-c)(\beta_{3k+5}+b+c+p) - L\Big)R_{k+1}(x)\Big\}\\
&+ K_{3k+1}\Big(a_k^2(x) + \gamma_{3k+2}b_{k-1}^2(x) + (\beta_{3k+2}+b+c+p)R_k(x)\Big)\\
&+ H_{3k+1}\Big(a_{k+1}^2(x) + \gamma_{3k+5}b_k^2(x) + (\beta_{3k+5}+b+c+p)R_{k+1}(x)\Big)\\
&- \gamma_{3k+4}(\beta_{3k+3}+\beta_{3k+4}-a+b+c+p)\Big\{\Big(\gamma_{3k+3}+\gamma_{3k+4}\\
&+ (\beta_{3k+3}+a-b-c)(\beta_{3k+3}+b+c+p) - L\Big)a_k^2(x) + \gamma_{3k+2}\gamma_{3k+3}b_{k-1}^2(x)\\
&+ (\beta_{3k+3}+\beta_{3k+4}+a+p)b_k^2(x) + \gamma_{3k+3}(\beta_{3k+2}+\beta_{3k+3}+a+p)R_k(x) + R_{k+1}(x)\Big\}.
\end{aligned}
$$

Therefore, in (3.19), we see that:

1. $a_{k-1}^2(x)$ coefficient is:

$$\gamma_{3k+1}B_{3k+1} - \gamma_{3k+1}\gamma_{3k+2}\gamma_{3k+3}\gamma_{3k+4};$$

2. $a_k^2(x)$ coefficient is:

$$
\begin{aligned}
&- K_{3k+2} + \gamma_{3k+4}A_{3k+1} - (a-b)(a-c)(\beta_{3k+4}+b+c+p)\gamma_{3k+4}\\
&+ (\beta_{3k+3}+b+c+p)\Big(V_{3k+1} - (a-b)(a-c)\gamma_{3k+4}\Big)\\
&- \gamma_{3k+4}(\beta_{3k+3}+\beta_{3k+4}+a+p)\Big(\gamma_{3k+4} + (\beta_{3k+4}-a)(\beta_{3k+4}+b+c+p)\Big)\\
&- \gamma_{3k+3}\gamma_{3k+4}(\beta_{3k+2}+\beta_{3k+3}+a+p) - \gamma_{3k+4}\gamma_{3k+5}(\beta_{3k+4}+b+c+p)\\
&+ K_{3k+1} - \gamma_{3k+4}(\beta_{3k+3}+\beta_{3k+4}-a+b+c+p)\Big(\gamma_{3k+3}+\gamma_{3k+4}\\
&+ (\beta_{3k+3}+a-b-c)(\beta_{3k+3}+b+c+p) - L\Big);
\end{aligned}
$$

3. $a_{k+1}^2(x)$ coefficient is:

$$
\begin{aligned}
&- (H_{3k+2} - \gamma_{3k+7}) + S_{3k+1}(\beta_{3k+6}+b+c+p) - (\beta_{3k+4}+b+c+p)(\beta_{3k+4}-a)\\
&- \gamma_{3k+4} - (\beta_{3k+4}+b+c+p)(\beta_{3k+5}+\beta_{3k+6}+a+p) + H_{3k+1};
\end{aligned}
$$

4. $b_{k-1}^2(x)$ coefficient is:

$$
\begin{aligned}
&B_{3k+1}(\beta_{3k+1}+b+c+p) - \gamma_{3k+2}\gamma_{3k+3}\gamma_{3k+4}(\beta_{3k+1}+\beta_{3k+2}+a+p)\\
&+ \gamma_{3k+2}K_{3k+1} - \gamma_{3k+2}\gamma_{3k+3}\gamma_{3k+4}(\beta_{3k+3}+\beta_{3k+4}-a+b+c+p);
\end{aligned}
$$

5. $b_k^2(x)$ coefficient is:

$$
\begin{aligned}
&- V_{3k+2} + (\beta_{3k+4}+b+c+p)A_{3k+1} - (a-b)(a-c)(\beta_{3k+4}+b+c+p)^2\\
&+ V_{3k+1} - (a-b)(a-c)\gamma_{3k+4}\\
&- \Big((\beta_{3k+4}+b+c+p)(\beta_{3k+4}-a) + \gamma_{3k+4}\Big)\Big(\gamma_{3k+4}+\gamma_{3k+5}\\
&+ (\beta_{3k+4}+a-b-c)(\beta_{3k+4}+b+c+p) - L\Big) - \gamma_{3k+3}\gamma_{3k+4}\\
&- \gamma_{3k+5}(\beta_{3k+4}+b+c+p)(\beta_{3k+4}+\beta_{3k+5}+a+p) + \gamma_{3k+5}H_{3k+1}\\
&- \gamma_{3k+4}(\beta_{3k+3}+\beta_{3k+4}-a+b+c+p)(\beta_{3k+3}+\beta_{3k+4}+a+p);
\end{aligned}
$$

6. $b_{k+1}^2(x)$ coefficient is:

$$
-S_{3k+2} + (\beta_{3k+7}+b+c+p) + S_{3k+1} - (\beta_{3k+4}+b+c+p);
$$

7. $R_k(x)$ coefficient is:

$$
\begin{aligned}
&- B_{3k+2} + B_{3k+1} + \gamma_{3k+3}\Big(V_{3k+1} - \gamma_{3k+4}(a-b)(a-c)\Big)\\
&- \gamma_{3k+3}\gamma_{3k+4}\Big((\beta_{3k+4}+b+c+p)(\beta_{3k+4}-a) + \gamma_{3k+4}\Big)\\
&- \gamma_{3k+3}\gamma_{3k+4}\Big(\gamma_{3k+2}+\gamma_{3k+3} + (\beta_{3k+2}+a-b-c)(\beta_{3k+2}+b+c+p) - L\Big)\\
&+ K_{3k+1}(\beta_{3k+2}+b+c+p)\\
&- \gamma_{3k+3}\gamma_{3k+4}(\beta_{3k+3}+\beta_{3k+4}-a+b+c+p)(\beta_{3k+2}+\beta_{3k+3}+a+p);
\end{aligned}
$$

8. $R_{k+1}(x)$ coefficient is:

$$
\begin{aligned}
&- A_{3k+2} + A_{3k+1} - (\beta_{3k+4} + b + c + p)(a-b)(a-c) + S_{3k+1}\gamma_{3k+6} \\
&- \Big((\beta_{3k+4} + b + c + p)(\beta_{3k+4} - a) + \gamma_{3k+4}\Big)(\beta_{3k+4} + \beta_{3k+5} + a + p) \\
&- (\beta_{3k+4} + b + c + p)\Big(\gamma_{3k+5} + \gamma_{3k+6} + (\beta_{3k+5} + a - b - c)(\beta_{3k+5} + b + c + p) - L\Big) \\
&+ H_{3k+1}(\beta_{3k+5} + b + c + p) \\
&- \gamma_{3k+4}(\beta_{3k+3} + \beta_{3k+4} - a + b + c + p).
\end{aligned}
$$

Every one of these coefficients was simplified in *Mathematica 6* (just defining the coefficients A_k, B_k, H_k, K_k, S_k and V_k and applying the command FullSimplify [39]), and for every one we have obtained the null result, for $k \geq 0$.

Let us follow with identity (A_4), with $n = k+1$:

$$\gamma_{3k+5}b^1_{k+1}(x) + (\beta_{3k+5} - a)c^1_{k+1}(x) + (a-b)(a-c)c^2_{k+1}(x) + P_{k+2}(x) - \Theta(x)R_{k+1}(x) = 0.$$

Inserting (3.1), we have:

$$
\begin{aligned}
&\gamma_{3k+5}b^1_{k+1}(x) + (\beta_{3k+5} - a)c^1_{k+1}(x) + (a-b)(a-c)c^2_{k+1}(x) + P_{k+2}(x) - \Theta(x)R_{k+1}(x) \\
&= \big\{\Theta(x) - A_{3k}\big\}P_{k+1}(x) - B_{3k}P_k(x) - K_{3k}b^1_k(x) - H_{3k}b^1_{k+1}(x) - V_{3k}c^1_k(x) - S_{3k}c^1_{k+1}(x) \\
&+ \gamma_{3k+5}b^1_{k+1}(x) + (\beta_{3k+5} - a)c^1_{k+1}(x) + (a-b)(a-c)c^2_{k+1}(x) - \Theta(x)R_{k+1}(x) \\
&= -A_{3k}P_{k+1}(x) - B_{3k}P_k(x) - K_{3k}b^1_k(x) - (H_{3k} - \gamma_{3k+5})b^1_{k+1}(x) \\
&- V_{3k}c^1_k(x) - (S_{3k} - \beta_{3k+5} + a)c^1_{k+1}(x) \\
&- \Big(\Theta(x)R_{k+1}(x) - (a-b)(a-c)c^2_{k+1}(x) - \Theta(x)P_{k+1}(x)\Big).
\end{aligned}
\tag{3.20}
$$

We will now replace $\Theta(x)R_{k+1}(x) - (a-b)(a-c)c^2_{k+1}(x) - \Theta(x)P_{k+1}(x)$ by the expression given by (3.15). Let us remark that the relation (3.15) was deduced by identities in the list of hypotheses considered. Therefore, in (3.20), we see that, for $k \geq 0$,

$$\gamma_{3k+5}b^1_{k+1}(x) + (\beta_{3k+5} - a)c^1_{k+1}(x) + (a-b)(a-c)c^2_{k+1}(x) + P_{k+2}(x) - \Theta(x)R_{k+1}(x) = 0.$$

Finally, let us consider identity (A_8), with $n = k+1$:

$$(\beta_{3k+6} + a - b - c)a^1_{k+1}(x) - La^2_{k+1}(x) + \gamma_{3k+6}c^2_{k+1}(x) - P_{k+2}(x) + Q_{k+2}(x) = 0.$$

Inserting (3.2), we obtain:

$$
\begin{aligned}
&(\beta_{3k+6}+a-b-c)a^1_{k+1}(x) - La^2_{k+1}(x) + \gamma_{3k+6}c^2_{k+1}(x) - P_{k+2}(x) + Q_{k+2}(x)\\
&= \big\{\Theta(x) - A_{3k+1}\big\}Q_{k+1}(x) - B_{3k+1}Q_k(x) - K_{3k+1}c^2_k(x) - H_{3k+1}c^2_{k+1}(x)\\
&- V_{3k+1}a^1_k(x) - S_{3k+1}a^1_{k+1}(x) + (\beta_{3k+6}+a-b-c)a^1_{k+1}(x)\\
&- La^2_{k+1}(x) + \gamma_{3k+6}c^2_{k+1}(x) - P_{k+2}(x)\\
&= -A_{3k+1}Q_{k+1}(x) - B_{3k+1}Q_k(x) - K_{3k+1}c^2_k(x) - (H_{3k+1}-\gamma_{3k+6})c^2_{k+1}(x)\\
&- V_{3k+1}a^1_k(x) - \big(S_{3k+1} - (\beta_{3k+6}+a-b-c)\big)a^1_{k+1}(x)\\
&- \Big(P_{k+2}(x) + La^2_{k+1}(x) - \Theta(x)Q_{k+1}(x)\Big).
\end{aligned}
\tag{3.21}
$$

Replacing, in (3.21), $P_{k+2}(x) + La^2_{k+1}(x) - \Theta(x)Q_{k+1}(x)$ by expression (3.17), we see that, for $k \geq 0$:

$$(\beta_{3k+6}+a-b-c)a^1_{k+1}(x) - La^2_{k+1}(x) + \gamma_{3k+6}c^2_{k+1}(x) - P_{k+2}(x) + Q_{k+2}(x) = 0.$$

Let us note that the relation (3.17) was deduced by identities considered as hypotheses.

□

Remark 3.3.

- *Theorem 3.1 and theorem 3.2 generalize the results presented in [17], where* $\varpi(x) = x^3$ *and* $a = b = c = 0$.
- $B_n \neq 0, \forall n \geq 0$ *(regularity conditions).*
- *If* $K_n = H_n = V_n = S_n = 0,\ n \geq 0$*, then the principal components are orthogonal.*
- *In the symmetric case* ($\beta_n = 0,\ n \geq 0$)*, when* $p = 0$, $K_n = S_n = 0,\ n \geq 0$.

3.2 Principal components orthogonality conditions

The following result gives us sufficient relations for the orthogonality of each principal component of a MOPS CD. Those relations consist in having the secondary components of the corresponding column null.

Corollary 3.4. *Let* $\{W_n\}_{n\geq 0}$ *be a MOPS given by (2.1)-(2.3). Then*

a) $b^1_n = c^1_n = 0,\ n \geq 0 \ \Rightarrow\ \{P_n\}_{n\geq 0}$ *is a MOPS;*

b) $c^2_n = a^1_n = 0,\ n \geq 0 \ \Rightarrow\ \{Q_n\}_{n\geq 0}$ *is a MOPS;*

c) $a^2_n = b^2_n = 0,\ n \geq 0 \ \Rightarrow\ \{R_n\}_{n\geq 0}$ *is a MOPS.*

Proof. Considering identity (3.1) and $b_n^1 = c_n^1 = 0, \ n \geq 0$, we obtain a second order recurrence relation for $\{P_n\}_{n\geq 0}$. Similarly, from identities (3.2) and (3.3), hypotheses b) and c), imply a second order recurrence relation for $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, respectively. □

Let us consider the polynomials $b_n^1(x)$ and $c_n^1(x)$ written in terms of elements of the MPS $\{P_n\}_{n\geq 0}$, the polynomials $a_n^1(x)$ and $c_n^2(x)$ written in terms of elements of the MPS $\{Q_n\}_{n\geq 0}$, and the polynomials $a_n^2(x)$ and $b_n^2(x)$ written in terms of elements of the MPS $\{R_n\}_{n\geq 0}$:

$$b_n^1(x) = \sum_{\nu=0}^{n} \theta_{n,\nu}^{1,P} P_\nu(x), \tag{3.22}$$

$$c_n^1(x) = \sum_{\nu=0}^{n} \zeta_{n,\nu}^{1,P} P_\nu(x), \tag{3.23}$$

$$a_n^1(x) = \sum_{\nu=0}^{n} \lambda_{n,\nu}^{1,Q} Q_\nu(x), \tag{3.24}$$

$$c_n^2(x) = \sum_{\nu=0}^{n} \zeta_{n,\nu}^{2,Q} Q_\nu(x), \tag{3.25}$$

$$a_n^2(x) = \sum_{\nu=0}^{n} \lambda_{n,\nu}^{2,R} R_\nu(x), \tag{3.26}$$

$$b_n^2(x) = \sum_{\nu=0}^{n} \theta_{n,\nu}^{2,R} R_\nu(x). \tag{3.27}$$

The following result gives us necessary and sufficient conditions for the orthogonality of each principal component of a MOPS CD.

Corollary 3.5. *Let $\{W_n\}_{n\geq 0}$ be a MOPS given by (2.1)-(2.3). Then*

a) $\{P_n\}_{n\geq 0}$ *is a MOPS if and only if*

$$K_{3n}\theta_{n,\nu}^{1,P} + H_{3n}\theta_{n+1,\nu}^{1,P} + V_{3n}\zeta_{n,\nu}^{1,P} + S_{3n}\zeta_{n+1,\nu}^{1,P} = 0, \ \ 0 \leq \nu \leq n-1, \ \ n \geq 1,$$

and

$$B_{3n} + K_{3n}\theta_{n,n}^{1,P} + H_{3n}\theta_{n+1,n}^{1,P} + V_{3n}\zeta_{n,n}^{1,P} + S_{3n}\zeta_{n+1,n}^{1,P} \neq 0, \ \ n \geq 0.$$

b) $\{Q_n\}_{n\geq 0}$ *is a MOPS if and only if*

$$K_{3n+1}\zeta_{n,\nu}^{2,Q} + H_{3n+1}\zeta_{n+1,\nu}^{2,Q} + V_{3n+1}\lambda_{n,\nu}^{1,Q} + S_{3n+1}\lambda_{n+1,\nu}^{1,Q} = 0, \ \ 0 \leq \nu \leq n-1, \ \ n \geq 1,$$

and

$$B_{3n+1} + K_{3n+1}\zeta_{n,n}^{2,Q} + H_{3n+1}\zeta_{n+1,n}^{2,Q} + V_{3n+1}\lambda_{n,n}^{1,Q} + S_{3n+1}\lambda_{n+1,n}^{1,Q} \neq 0, \ \ n \geq 0.$$

c) $\{R_n\}_{n\geq 0}$ *is a MOPS if and only if*

$$K_{3n+2}\lambda^{2,R}_{n,\nu}+H_{3n+2}\lambda^{2,R}_{n+1,\nu}+V_{3n+2}\theta^{2,R}_{n,\nu}+S_{3n+2}\theta^{2,R}_{n+1,\nu}=0,\ \ 0\leq\nu\leq n-1,\ \ n\geq 1,$$

and

$$B_{3n+2}+K_{3n+2}\lambda^{2,R}_{n,n}+H_{3n+2}\lambda^{2,R}_{n+1,n}+V_{3n+2}\theta^{2,R}_{n,n}+S_{3n+2}\theta^{2,R}_{n+1,n}\neq 0,\ \ n\geq 0.$$

Proof. Inserting (3.22) and (3.23) in identity (3.1) we obtain the structure relation for $\{P_n\}_{n\geq 0}$, where
$\chi^P_{n,n}=B_{3n}+K_{3n}\theta^{1,P}_{n,n}+H_{3n}\theta^{1,P}_{n+1,n}+V_{3n}\zeta^{1,P}_{n,n}+S_{3n}\zeta^{1,P}_{n+1,n},\ \ n\geq 0$, and
$\chi^P_{n,\nu}=K_{3n}\theta^{1,P}_{n,\nu}+H_{3n}\theta^{1,P}_{n+1,\nu}+V_{3n}\zeta^{1,P}_{n,\nu}+S_{3n}\zeta^{1,P}_{n+1,\nu},\ \ 0\leq\nu\leq n-1,\ \ n\geq 1.$

Inserting (3.24) and (3.25) in identity (3.2) we obtain the structure relation for $\{Q_n\}_{n\geq 0}$, where
$\chi^Q_{n,n}=B_{3n+1}+K_{3n+1}\zeta^{2,Q}_{n,n}+H_{3n+1}\zeta^{2,Q}_{n+1,n}+V_{3n+1}\lambda^{1,Q}_{n,n}+S_{3n+1}\lambda^{1,Q}_{n+1,n},\ \ n\geq 0$, and
$\chi^Q_{n,\nu}=K_{3n+1}\zeta^{2,Q}_{n,\nu}+H_{3n+1}\zeta^{2,Q}_{n+1,\nu}+V_{3n+1}\lambda^{1,Q}_{n,\nu}+S_{3n+1}\lambda^{1,Q}_{n+1,\nu},\ \ 0\leq\nu\leq n-1,\ \ n\geq 1.$

Inserting (3.26) and (3.27) in identity (3.3) we obtain the structure relation for $\{R_n\}_{n\geq 0}$, where
$\chi^R_{n,n}=B_{3n+2}+K_{3n+2}\lambda^{2,R}_{n,n}+H_{3n+2}\lambda^{2,R}_{n+1,n}+V_{3n+2}\theta^{2,R}_{n,n}+S_{3n+2}\theta^{2,R}_{n+1,n},\ \ n\geq 0$, and
$\chi^R_{n,\nu}=K_{3n+2}\lambda^{2,R}_{n,\nu}+H_{3n+2}\lambda^{2,R}_{n+1,\nu}+V_{3n+2}\theta^{2,R}_{n,\nu}+S_{3n+2}\theta^{2,R}_{n+1,\nu},\ \ 0\leq\nu\leq n-1,\ \ n\geq 1.$ □

Returning to the coefficients A_n, B_n, K_n, H_n, V_n, S_n of theorem 3.2, given by the relations (3.4),(3.5),(3.6),(3.7),(3.8) and (3.9), the next result identifies the orthogonal sequences $\{W_n\}_{n\geq 0}$ such that $K_n=H_n=V_n=S_n=0,\ \ n\geq 0$. Recall that if $K_n=H_n=V_n=S_n=0,\ \ n\geq 0$, then the three principal components are orthogonal.

Proposition 3.6. *Given a MOPS $\{W_n\}_{n\geq 0}$, the coefficients K_n, H_n, V_n and S_n, of theorem 3.2, are all null if and only if the recurrence coefficients of $\{W_n\}_{n\geq 0}$ satisfy the following, where L is defined by (2.5).*

$$\beta_{3n+1}=\beta_1,\ \ \beta_{3n+2}=\beta_2,\ \ \beta_{3n+3}=\beta_3,\ \ \beta_1+\beta_2+\beta_3+p=0,\ \ n\geq 0,$$

$$\gamma_{3n+2}=\gamma_2,\ \ \gamma_{3n+3}=\gamma_3,\ \ \gamma_{3n+4}=\gamma_4,\ \ n\geq 0,$$

$$\gamma_2+\gamma_3+\gamma_4+(a-b)(a-c)-L+(\beta_3+a-b-c)(\beta_3+b+c+p)+(\beta_2-a)(a-\beta_1)=0.$$

Proof. Let us suppose that $K_n=H_n=V_n=S_n=0,\ \ n\geq 0$. Then,

$$\beta_{n+1}+\beta_{n+2}+\beta_{n+3}+p=0; \tag{3.28}$$

$$\beta_{n+3}+\beta_{n+4}+\beta_{n+5}+p=0; \tag{3.29}$$

$$\begin{aligned}&\gamma_{n+3}+\gamma_{n+4}+\gamma_{n+5}+(a-b)(a-c)-L\\&+(\beta_{n+3}+a-b-c)(\beta_{n+3}+b+c+p)+(\beta_{n+4}-a)(\beta_{n+3}+\beta_{n+4}+a+p)=0;\end{aligned} \tag{3.30}$$

$$\begin{aligned}&\gamma_{n+2}+\gamma_{n+3}+\gamma_{n+4}+(a-b)(a-c)-L\\&+(\beta_{n+3}+a-b-c)(\beta_{n+3}+b+c+p)+(\beta_{n+2}-a)(\beta_{n+2}+\beta_{n+3}+a+p)=0.\end{aligned} \tag{3.31}$$

Subtracting relation (3.29) to relation (3.28), with $n \leftarrow n+3$, we obtain

$$\beta_{n+3}=\beta_{n+6}, \quad n\geq 0.$$

Consequently:

$$\beta_{3n+3}=\beta_3, \quad \beta_{3n+4}=\beta_4, \quad \beta_{3n+5}=\beta_5, \quad n\geq 0.$$

From (3.28), with $n=0$, $n=1$ and $n=2$, we have:

$$\beta_1+\beta_2+\beta_3+p=0, \quad \beta_2+\beta_3+\beta_4+p=0, \quad \beta_3+\beta_4+\beta_5+p=0,$$

which implies that $\beta_1=\beta_4$ and $\beta_2=\beta_5$, therefore:

$$\beta_{3n+1}=\beta_1, \quad \beta_{3n+2}=\beta_2, \quad \beta_{3n+3}=\beta_3, \quad n\geq 0,$$

where

$$\beta_1+\beta_2+\beta_3+p=0. \tag{3.32}$$

Subtracting relation (3.31) to relation (3.30), we obtain:

$$\gamma_{n+5}=\gamma_{n+2}+(\beta_{n+2}-a)(\beta_{n+2}+\beta_{n+3}+a+p)-(\beta_{n+4}-a)(\beta_{n+3}+\beta_{n+4}+a+p).$$

Replacing n by $3n$, $3n+1$ and $3n+2$, we obtain:

$$\begin{aligned}\gamma_{3n+5}&=\gamma_{3n+2}+(\beta_2-a)(\beta_2+\beta_3+a+p)-(\beta_1-a)(\beta_3+\beta_1+a+p)\\&\overset{(3.32)}{=}\gamma_{3n+2}+(\beta_2-a)(a-\beta_1)-(\beta_1-a)(a-\beta_2)\\&=\gamma_{3n+2};\end{aligned}$$

$$\begin{aligned}\gamma_{3n+6}&=\gamma_{3n+3}+(\beta_3-a)(\beta_3+\beta_1+a+p)-(\beta_2-a)(\beta_1+\beta_2+a+p)\\&\overset{(3.32)}{=}\gamma_{3n+3}+(\beta_3-a)(a-\beta_2)-(\beta_2-a)(a-\beta_3)\\&=\gamma_{3n+3};\end{aligned}$$

$$\begin{aligned}\gamma_{3n+7}&=\gamma_{3n+4}+(\beta_1-a)(\beta_1+\beta_2+a+p)-(\beta_3-a)(\beta_2+\beta_3+a+p)\\&\overset{(3.32)}{=}\gamma_{3n+4}+(\beta_1-a)(a-\beta_3)-(\beta_3-a)(a-\beta_1)\\&=\gamma_{3n+4}.\end{aligned}$$

Then,

$$\gamma_{3n+2}=\gamma_2, \quad \gamma_{3n+3}=\gamma_3, \quad \gamma_{3n+4}=\gamma_4, \quad n\geq 0,$$

where, from (3.31) with $n=0$, we have also:

$$\gamma_2+\gamma_3+\gamma_4+(a-b)(a-c)-L+(\beta_3+a-b-c)(\beta_3+b+c+p)+(\beta_2-a)(a-\beta_1)=0. \tag{3.33}$$

Reciprocally, let us suppose that the recurrence coefficients of $\{W_n\}_{n\geq 0}$ satisfy the enunciated conditions.

Then, we can easily confirm that $K_{3n} = K_{3n+1} = K_{3n+2} = 0, \quad n \geq 0$, $S_{3n} = S_{3n+1} = S_{3n+2} = 0, \quad n \geq 0$, and

$$\begin{aligned}
H_{3n} &= \gamma_3 + \gamma_4 + \gamma_2 + (a-b)(a-c) - L \\
&\quad + (\beta_3 + a - b - c)(\beta_3 + b + c + p) + (\beta_1 - a)(\beta_3 + \beta_1 + a + p) \\
&\overset{(3.32)}{=} \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p) + (\beta_1 - a)(a - \beta_2) \\
&\overset{(3.33)}{=} 0.
\end{aligned}$$

In the next calculations it will be useful to attend to the following identity, fulfilled for all constants M and N.

$$(M+a-b-c)(M+b+c+p)-(N+a-b-c)(N+b+c+p) = (M-N)(M+N+a+p) \quad (3.34)$$

$$\begin{aligned}
&H_{3n+1} = \gamma_4 + \gamma_5 + \gamma_6 + (a-b)(a-c) - L \\
&+ (\beta_4 + a - b - c)(\beta_4 + b + c + p) + (\beta_5 - a)(\beta_4 + \beta_5 + a + p) \\
&\overset{(3.34)}{=} \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p) \\
&+ (\beta_4 - \beta_3)(\beta_3 + \beta_4 + a + p) + (\beta_2 - a)(\beta_1 + \beta_2 + a + p) \\
&\overset{(3.32)}{=} \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p) \\
&+ (\beta_1 - \beta_3)(a - \beta_2) + (\beta_2 - a)(a - \beta_3) \\
&= \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p) + (\beta_2 - a)(a - \beta_1) \\
&\overset{(3.33)}{=} 0.
\end{aligned}$$

$$\begin{aligned}
H_{3n+2} &= \gamma_5 + \gamma_6 + \gamma_7 + (a-b)(a-c) - L \\
&+ (\beta_5 + a - b - c)(\beta_5 + b + c + p) + (\beta_6 - a)(\beta_5 + \beta_6 + a + p) \\
&= \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L \\
&+ (\beta_2 + a - b - c)(\beta_2 + b + c + p) + (\beta_3 - a)(\beta_2 + \beta_3 + a + p) \\
&\overset{(3.34)}{=} \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p) \\
&+ (\beta_2 - \beta_3)(\beta_2 + \beta_3 + a + p) + (\beta_3 - a)(\beta_2 + \beta_3 + a + p) \\
&= \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p) \\
&+ (\beta_2 - a)(\beta_2 + \beta_3 + a + p) \overset{(3.33)}{=} 0.
\end{aligned}$$

$$\begin{aligned}
V_{3n} &= \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p) \\
&+ (\beta_2 - a)(\beta_2 + \beta_3 + a + p) \overset{(3.33)}{=} 0.
\end{aligned}$$

$$V_{3n+1} = \gamma_3 + \gamma_4 + \gamma_2 + (a-b)(a-c) - L + (\beta_1 + a - b - c)(\beta_1 + b + c + p)$$
$$+ (\beta_3 - a)(\beta_3 + \beta_1 + a + p)$$
$$\overset{(3.34)}{=} \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p)$$
$$+ (\beta_1 - \beta_3)(\beta_1 + \beta_3 + a + p) + (\beta_3 - a)(\beta_3 + \beta_1 + a + p)$$
$$= \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p)$$
$$+ (a - \beta_2)(\beta_1 - a) \overset{(3.33)}{=} 0.$$

$$V_{3n+2} = \gamma_4 + \gamma_5 + \gamma_6 + (a-b)(a-c) - L + (\beta_5 + a - b - c)(\beta_5 + b + c + p)$$
$$+ (\beta_4 - a)(\beta_4 + \beta_5 + a + p)$$
$$\overset{(3.34)}{=} \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p)$$
$$+ (\beta_2 - \beta_3)(\beta_2 + \beta_3 + a + p) + (\beta_1 - a)(\beta_1 + \beta_2 + a + p)$$
$$= \gamma_2 + \gamma_3 + \gamma_4 + (a-b)(a-c) - L + (\beta_3 + a - b - c)(\beta_3 + b + c + p)$$
$$+ (\beta_2 - a)(a - \beta_1) + (a - \beta_3)(a - \beta_1) + (\beta_1 - a)(\beta_1 + \beta_2 + a + p) \overset{(3.33)}{=} 0.$$

□

Remark 3.7. *The coefficients K_n, H_n, V_n and S_n of theorem 3.2 do not depend of the recurrence coefficients β_0 and γ_1. That is the reason why, in proposition 3.6, these coefficients are free.*

This aspect also tell us that, given a MOPS $\{W_n\}_{n\geq 0}$, the coefficients K_n, H_n, V_n and S_n are invariable in the family of the co-recursive sequences of $\{W_n\}_{n\geq 0}$ or in the family of the 1-perturbed sequence $\left\{W_n\Big(0; \begin{matrix} 0 \\ \lambda \end{matrix}; 1; x\Big)\right\}_{n\geq 0}$, where $\lambda \in \mathbb{C}\backslash\{0\}$ (see definitions 1.16 and 1.17).

Using the coefficients K_n, H_n, V_n and S_n, we can easily look for conditions which guarantee principal component sequences orthogonality. We next present an example, namely, those coefficients for a symmetric MOPS, in particular, for the Tchebyshev sequence of first kind and for the Tchebyshev sequence of second kind.

Example 3.8. *(Symmetric MOPS)*

If $\{W_n\}_{n\geq 0}$ is a symmetric MOPS, we obtain: $\Theta(x) - A_n = x - p\gamma_{n+3} - p\gamma_{n+4}$, $B_n = \gamma_{n+1}\gamma_{n+2}\gamma_{n+3}$, $K_n = p\gamma_{n+2}\gamma_{n+3}$, $H_n = q + \gamma_{n+3} + \gamma_{n+4} + \gamma_{n+5}$, $V_n = \gamma_{n+3}(q + \gamma_{n+2} + \gamma_{n+3} + \gamma_{n+4})$ and $S_n = p$.

Therefore, if we choose $p = 0$ and the recurrence coefficients of $\{W_n\}_{n\geq 0}$ satisfy: $q + \gamma_{n+3} + \gamma_{n+4} + \gamma_{n+5} = 0$ and $q + \gamma_{n+2} + \gamma_{n+3} + \gamma_{n+4} = 0$, we conclude that the principal components are all orthogonal.

Notice that such symmetric sequence fulfils:

$$\gamma_{3n+2} = \gamma_2,\ \gamma_{3n+3} = \gamma_3,\ \gamma_{3n+4} = \gamma_4,\ n \geq 0,\ and\ q + \gamma_2 + \gamma_3 + \gamma_4 = 0.$$

If $\{W_n\}_{n\geq 0}$ is the first kind Tchebyshev sequence, then $\gamma_1 = \frac{1}{2},\ \gamma_{n+1} = \frac{1}{4},\ n \geq 1$, and we need to choose $p = 0$ and $q = -\frac{3}{4}$, so that the principal components are orthogonal. The same choice is adequate for the second kind Tchebyshev sequence, since for this symmetric sequence, we have $\gamma_{n+1} = \frac{1}{4},\ n \geq 0$.

3.3 Some secondary components null

Let us now analyse, trough the conditions of theorem 3.1, the CD of a MOPS for which some secondary components vanish.

The following result excludes the possibility of getting cubic decompositions for a MOPS for which some secondary components are absent.

Theorem 3.9. *Let $\{W_n\}_{n\geq 0}$ be a MOPS defined by (2.1)-(2.3).*

a) *The following situations concerning the secondary components in (2.1)-(2.3) are impossible:*

1. $a_n^1 = a_n^2 = b_n^2 = 0,\ \ n \geq 0$;
2. *Only one of the six secondary component non null.*

b) *If only two of the six secondary components are non null, then the pair of such sequences is one of the following:*

1. $\{a_n^1\}_{n\geq 0}$ *and* $\{b_n^1\}_{n\geq 0}$;
2. $\{a_n^2\}_{n\geq 0}$ *and* $\{c_n^1\}_{n\geq 0}$;
3. $\{b_n^2\}_{n\geq 0}$ *and* $\{c_n^2\}_{n\geq 0}$.

Remark 3.10. *As an obvious consequence of the sentence a) 1. of theorem 3.9, we can conclude that if $\{W_n\}_{n\geq 0}$ is orthogonal, then $\{W_n\}_{n\geq 0}$ can not have a diagonal CD, because $\{W_n\}_{n\geq 0}$ can not have a lower triangular CD. This is the content of corollary 2.8, where the same fact was deduced from theorem 2.7.*

Proof. **a)** If $a_n^1 = a_n^2 = b_n^2 = 0,\ \ n \geq 0$, then (A_9)

$$a_n^1(x) - (\beta_{3n+3} + b + c + p)a_n^2(x) - b_n^2(x) = \gamma_{3n+3}R_n(x)$$

becomes $\gamma_{3n+3}R_n(x) = 0,\ n \geq 0$, which contradicts sequence $\{W_n\}_{n\geq 0}$ regular orthogonality.

This conclusion excludes the possibility of a CD with b_n^1 as the only secondary component non null, or with only c_n^1 or c_n^2, as the secondary non null component. Therefore, we only have three more situations to examine, which are, the CD where the only non null secondary component is a_n^1, a_n^2 and b_n^2, respectively.

Let us first suppose that $a_n^2 = b_n^1 = b_n^2 = c_n^1 = c_n^2 = 0, \quad n \geq 0$, where a_n^1 is the only secondary component nontrivial.

By (A_9), we know that $a_n^1(x) = \gamma_{3n+3} R_n(x), \ n \geq 0$.

From (A_1), $(\beta_{3n+1} - a) b_n^1(x) - \Theta(x) b_{n-1}^2(x) + c_n^1(x) = -\gamma_{3n+1} P_n(x) - (a-b)(a-c) Q_n(x)$, we get

$$\gamma_{3n+1} P_n(x) = -(a-b)(a-c) Q_n(x),$$

hence, $\gamma_{3n+1} = -(a-b)(a-c)$, because the polynomials involved are monic.

Moreover, identity (A_7), $(a-b)(a-c) a_n^1(x) - \Theta(x) a_n^2(x) + b_{n+1}^1(x) + \gamma_{3n+3} c_n^1(x) = -(\beta_{3n+3} - a) P_{n+1}(x)$, can be written as follows:

$$(\beta_{3n+3} - a) P_{n+1}(x) = \gamma_{3n+1} \gamma_{3n+3} R_n(x).$$

Then, we obtain $\beta_{3n+3} - a = 0, \quad n \geq 0$, and thus $\gamma_{3n+1} \gamma_{3n+3} R_n(x) = 0$, which contradicts $\{W_n\}_{n \geq 0}$ regular orthogonality.

Let us now suppose that $a_n^1 = b_n^1 = b_n^2 = c_n^1 = c_n^2 = 0, \quad n \geq 0$, remaining the non null secondary component a_n^2.

By (A_9), we get $\gamma_{3n+3} R_n(x) = -(\beta_{3n+3} + b + c + p) a_n^2(x), n \geq 0$. Hence, $\beta_{3n+3} + b + c + p \neq 0, \quad n \geq 0$, otherwise, if for some $k \geq 0$, $\beta_{3k+3} + b + c + p = 0$, then $\gamma_{3k+3} R_k(x) = 0$, which contradicts the orthogonality of the sequence $\{W_n\}_{n \geq 0}$. Therefore, we conclude that $\deg a_n^2 = n, \quad n \geq 0$.

By identity (A_6), $a_n^2(x) + \gamma_{3n+2} b_{n-1}^2(x) - c_n^2(x) = -(\beta_{3n+2} + b + c + p) R_n(x)$, we get

$$a_n^2(x) = -(\beta_{3n+2} + b + c + p) R_n(x), \quad n \geq 0,$$

therefore, $\beta_{3n+2} + b + c + p \neq 0, \quad n \geq 0$.

From identity (A_5) we have $\gamma_{3n+2} Q_n(x) = L R_n(x)$ and (A_3) $(n \leftarrow n+1)$, multiplied by γ_{3n+5}, gives us

$$\gamma_{3n+5} R_{n+1}(x) + \gamma_{3n+5} \gamma_{3n+4} a_n^2(x) = \gamma_{3n+5} Q_{n+1}(x).$$

Hence, $\gamma_{3n+5} R_{n+1}(x) - \gamma_{3n+5} \gamma_{3n+4} (\beta_{3n+2} + b + c + p) R_n(x) = L R_{n+1}(x)$, that is,

$$(\gamma_{3n+5} - L) R_{n+1}(x) - \gamma_{3n+5} \gamma_{3n+4} (\beta_{3n+2} + b + c + p) R_n(x) = 0, \ n \geq 0,$$

which implies $\beta_{3n+2} + b + c + p = 0$, contradicting previous conclusions.

Finally, let us suppose that $a_n^1 = a_n^2 = b_n^1 = c_n^1 = c_n^2 = 0, \quad n \geq 0$, remaining the non null secondary component b_n^2. By (A_9), $b_n^2(x) = -\gamma_{3n+3} R_n(x)$, in particular, $\deg b_n^2 = n$, $n \geq 0$.

By identity (A_6) $(n \leftarrow n+1)$, $\gamma_{3n+5} b_n^2(x) = -(\beta_{3n+5} + b + c + p) R_{n+1}(x)$, or

$$-\gamma_{3n+5} \gamma_{3n+3} R_n(x) = -(\beta_{3n+5} + b + c + p) R_{n+1}(x)$$

implying, in particular, $\gamma_{3n+5} \gamma_{3n+3} = 0$ which contradicts the orthogonality of the sequence $\{W_n\}_{n \geq 0}$.

b) To demonstrate item b), we will analyse the twelve possible pairs of non null secondary components. In fact, there are 15 pairs, but three of them are excluded by the first statement of item a). In nine of those cases we will obtain assertions contradicting sequence $\{W_n\}_{n\geq 0}$ regular orthogonality, remaining only the three announced cases.

Let us consider that the pair of non null secondary components is constituted by $\{a_n^1\}_{n\geq 0}$ and $\{a_n^2\}_{n\geq 0}$. Identity (A_2), $\gamma_{3n+1}a_{n-1}^1(x) - b_n^1(x) - Lb_{n-1}^2(x) + c_n^2(x) = -(\beta_{3n+1} + a - b - c)Q_n(x)$, becomes $\gamma_{3n+1}a_{n-1}^1(x) = -(\beta_{3n+1} + a - b - c)Q_n(x)$. This last implies $\beta_{3n+1} + a - b - c = 0$, and then, $\gamma_{3n+1}a_{n-1}^1(x) = 0$, which implies $\gamma_{3n+1} = 0$, at least for some n, since the sequence $\{a_n^1\}_{n\geq 0}$ is nontrivial. This last conclusion is not possible, due to the orthogonality of $\{W_n\}_{n\geq 0}$.

Let us consider that the pair of non null secondary components is constituted by $\{a_n^2\}_{n\geq 0}$ and $\{b_n^2\}_{n\geq 0}$. Identity (A_5), $a_n^1(x) - c_n^1(x) + (\beta_{3n+2} + a - b - c)c_n^2(x) = -\gamma_{3n+2}Q_n(x) + LR_n(x)$, becomes $\gamma_{3n+2}Q_n(x) = LR_n(x)$; thus, $\gamma_{3n+2} = L$. Identity (A_2), $\gamma_{3n+1}a_{n-1}^1(x) - b_n^1(x) - Lb_{n-1}^2(x) + c_n^2(x) = -(\beta_{3n+1} + a - b - c)Q_n(x)$, becomes $(\beta_{3n+1} + a - b - c)Q_n(x) = Lb_{n-1}^2(x)$, which implies $\beta_{3n+1} + a - b - c = 0$ and $Lb_{n-1}^2(x) = 0$. Hence, $L = 0$, which is absurd.

Let us consider that the pair of non null secondary components is constituted by $\{a_n^1\}_{n\geq 0}$ and $\{b_n^2\}_{n\geq 0}$.

Identity (A_7), $(a-b)(a-c)a_n^1(x) - \Theta(x)a_n^2(x) + b_{n+1}^1(x) + \gamma_{3n+3}c_n^1(x) = -(\beta_{3n+3} - a)P_{n+1}(x)$, becomes $(\beta_{3n+3} - a)P_{n+1}(x) = -(a-b)(a-c)a_n^1(x)$, which yields $\beta_{3n+3} - a = 0$ and $-(a-b)(a-c) = 0$.

Identity (A_1), $(\beta_{3n+1} - a)b_n^1(x) - \Theta(x)b_{n-1}^2(x) + c_n^1(x) = -\gamma_{3n+1}P_n(x) - (a-b)(a-c)Q_n(x)$, becomes $\gamma_{3n+1}P_n(x) = \Theta(x)b_{n-1}^2(x)$.

Also, identity (A_6), multiplied by $\Theta(x)$, says that $(\beta_{3n+2} + b + c + p)\Theta(x)R_n(x) + \gamma_{3n+2}\Theta(x)b_{n-1}^2(x) = 0$. Inserting the result of (A_1) and the information of (A_4), $P_{n+1}(x) = \Theta(x)R_n(x)$, we get,

$$(\beta_{3n+2} + b + c + p)P_{n+1}(x) + \gamma_{3n+2}\gamma_{3n+1}P_n(x) = 0,$$

implying $\gamma_{3n+2}\gamma_{3n+1} = 0$, which is absurd.

Let us consider that the pair of non null secondary components is constituted by $\{a_n^1\}_{n\geq 0}$ and $\{c_n^1\}_{n\geq 0}$.

From (A_9), we get $\gamma_{3n+3}R_n(x) = a_n^1(x)$ and from (A_3), $R_n(x) = Q_n(x)$, $n \geq 0$; consequently, by (A_2) $(n \leftarrow n+1)$ we obtain

$$(\beta_{3n+4} + a - b - c)R_{n+1}(x) + \gamma_{3n+4}\gamma_{3n+3}R_n(x) = 0$$

which implies $\gamma_{3n+4}\gamma_{3n+3} = 0$.

Let us consider that the pair of non null secondary components is constituted by $\{a_n^1\}_{n\geq 0}$ and $\{c_n^2\}_{n\geq 0}$.

By (A_9), we get $\gamma_{3n+3}R_n(x) = a_n^1(x)$, $n \geq 0$, from (A_3) we get $R_n(x) = Q_n(x)$, $n \geq 0$, and from (A_1) $(n \leftarrow n+1)$

$$\gamma_{3n+4}P_{n+1}(x) = -(a-b)(a-c)R_{n+1}(x).$$

Considering (A_7), multiplied by γ_{3n+4}, we obtain,

$$-(\beta_{3n+3}-a)(a-b)(a-c)R_{n+1}(x)=-(a-b)(a-c)\gamma_{3n+3}\gamma_{3n+4}R_n(x),$$

yielding $(a-b)(a-c)=0$, since $(\beta_{3n+3}-a)R_{n+1}(x)-\gamma_{3n+3}\gamma_{3n+4}R_n(x)=0$ is impossible.
However, if $(a-b)(a-c)=0$, then $\gamma_{3n+4}P_{n+1}(x)=0$, which is absurd.

Let us consider that the pair of non null secondary components is constituted by $\{b_n^1\}_{n\geq 0}$ and $\{b_n^2\}_{n\geq 0}$.
Identity (A_9) states that $b_n^2(x)=-\gamma_{3n+3}R_n(x)$ and thus, the relation (A_6) $(n\leftarrow n+1)$ is rewritten as follows

$$(\beta_{3n+5}+b+c+p)R_{n+1}(x)-\gamma_{3n+5}\gamma_{3n+3}R_n(x)=0.$$

Thus, $\gamma_{3n+5}\gamma_{3n+3}=0$.

Let us consider that the pair of non null secondary components is constituted by $\{b_n^1\}_{n\geq 0}$ and $\{a_n^2\}_{n\geq 0}$.
From (A_5), we get $\gamma_{3n+2}Q_n(x)=LR_n(x)$ and from (A_6),

$$a_n^2(x)=-(\beta_{3n+2}+b+c+p)R_n(x),\ n\geq 0.$$

Therefore, by (A_9), we obtain

$$-(\beta_{3n+3}+b+c+p)(\beta_{3n+2}+b+c+p)R_n(x)+\gamma_{3n+3}R_n(x)=0,$$

thus, $\gamma_{3n+3}=(\beta_{3n+3}+b+c+p)(\beta_{3n+2}+b+c+p)\neq 0,\ n\geq 0$.
On the other hand, (A_3) $(n\leftarrow n+1)$ multiplied by γ_{3n+5}, can be rewritten as follows:

$$\gamma_{3n+5}R_{n+1}(x)-\gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+2}+b+c+p)R_n(x)=LR_{n+1}(x).$$

Therefore, $\gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+2}+b+c+p)=0$, that is, $\beta_{3n+2}+b+c+p=0$, contradicting the above inequality.

Let us consider that the pair of non null secondary components is constituted by $\{c_n^1\}_{n\geq 0}$ and $\{b_n^2\}_{n\geq 0}$.
From (A_7), $\gamma_{3n+3}c_n^1(x)+(\beta_{3n+3}-a)P_{n+1}(x)=0, n\geq 0$, we conclude $\beta_{3n+3}-a=0, n\geq 0$; however, this implies $\gamma_{3n+3}c_n^1(x)=0,\ n\geq 0$, i.e., $c_n^1(x)=0,\ n\geq 0$, which reduces this situation to the one where the only secondary component nontrivial is $\{b_n^2\}_{n\geq 0}$, already treated in item a).

Let us consider that the pair of non null secondary components is constituted by $\{c_n^2\}_{n\geq 0}$ and $\{a_n^2\}_{n\geq 0}$.
From (A_9), $(\beta_{3n+3}+b+c+p)a_n^2(x)=-\gamma_{3n+3}R_n(x)$, we conclude that $\beta_{3n+3}+b+c+p\neq 0,\ n\geq 0$, and $\deg(a_n^2)=n,\ n\geq 0$. In this manner,

$$a_n^2(x)=-\gamma_{3n+3}(\beta_{3n+3}+b+c+p)^{-1}R_n(x),\ n\geq 0.$$

From (A_6) we get $c_n^2(x) = a_n^2(x) + (\beta_{3n+2} + b + c + p)R_n(x)$, $n \geq 0$, that is,
$c_n^2(x) = -\gamma_{3n+3}(\beta_{3n+3} + b + c + p)^{-1}R_n(x) + (\beta_{3n+2} + b + c + p)R_n(x)$, $n \geq 0$.
From (A_3) $(n \leftarrow n+1)$, we obtain $Q_{n+1}(x) = R_{n+1}(x) + \gamma_{3n+4}a_n^2(x)$, $n \geq 0$,
or more precisely, $Q_{n+1}(x) = R_{n+1}(x) - \gamma_{3n+4}\gamma_{3n+3}(\beta_{3n+3} + b + c + p)^{-1}R_n(x)$, $n \geq 0$.
Then, we can read in (A_5) $(n \leftarrow n+1)$ the following

$$(\beta_{3n+5} + a - b - c)\Big(- \gamma_{3n+6}(\beta_{3n+6} + b + c + p)^{-1} + (\beta_{3n+5} + b + c + p)\Big)R_{n+1}(x)$$
$$+ \gamma_{3n+5}\Big(R_{n+1}(x) - \gamma_{3n+4}\gamma_{3n+3}(\beta_{3n+3} + b + c + p)^{-1}R_n(x)\Big) = LR_{n+1}(x),\ n \geq 0.$$

Nevertheless, this last identity implies $\gamma_{3n+5}\gamma_{3n+4}\gamma_{3n+3}(\beta_{3n+3} + b + c + p)^{-1} = 0$, $n \geq 0$, which is absurd. □

We will now focus on the three cases listed on item b) of theorem 3.9. In fact, those are the most simple CDs of an orthogonal MPS. Identities of theorem 3.1 will bring us necessary and sufficient conditions for the sequence $\{W_n\}_{n\geq 0}$ orthogonality, when this sequence has one of the three emphasized CDs. We will also notice the central role played by the sequence $\{R_n\}_{n\geq 0}$ in every one of these cases.

Theorem 3.11. *Let $\{W_n\}_{n\geq 0}$ be a MPS defined by (2.1)-(2.3), such that*

$$a_n^2 = b_n^2 = c_n^1 = c_n^2 = 0, \quad n \geq 0.$$

Then, $\{W_n\}_{n\geq 0}$ is a MOPS if and only if the following relations are fulfilled, where L is defined by (2.5).

(a_1) $\beta_{3n+2} = -(b + c + p), \quad n \geq 0,$

(a_2) $\beta_{3n} + \beta_{3n+1} = b + c, \quad n \geq 0,$

(a_3) $\gamma_{3n+1} = (a - \beta_{3n})(\beta_{3n} + a - b - c) - (a - b)(a - c) = \beta_{3n}(b + c - \beta_{3n}) - bc, \quad n \geq 0,$

(a_4) $\gamma_{3n+2} + \gamma_{3n+3} = L, \quad n \geq 0,$

(a_5) $Q_n(x) = R_n(x), \quad n \geq 0,$

(a_6) $P_{n+1}(x) = R_{n+1}(x) + (a - \beta_{3n+4})\gamma_{3n+3}R_n(x), \quad n \geq 0,$

(a_7) $a_n^1(x) = \gamma_{3n+3}R_n(x), \quad n \geq 0,$

(a_8) $b_{n+1}^1(x) = (a - \beta_{3n+3})R_{n+1}(x) + \gamma_{3n+3}\gamma_{3n+4}R_n(x), \quad n \geq 0,\ b_0^1(x) = a - \beta_0,$

(a_9)

$$R_{n+2}(x) = \big(x - \beta_{n+1}^R\big) R_{n+1}(x) - \gamma_{n+1}^R R_n(x), \;\; n \geq 0,$$

$$R_0(x) = 1, \quad R_1(x) = x + aL + bc(b + c + p) - \gamma_3(\beta_3 + a - b - c) - \gamma_2(a - \beta_0),$$

with $\beta_{n+1}^R = -(aL + bc(b + c + p)) + \gamma_{3n+6}(\beta_{3n+6} + a - b - c) + \gamma_{3n+5}(a - \beta_{3n+3})$
and $\gamma_{n+1}^R = \gamma_{3n+3}\gamma_{3n+4}\gamma_{3n+5}, \;\; n \geq 0.$

Proof. Let us suppose the relations of theorem 3.1 with $a_n^2 = b_n^2 = c_n^1 = c_n^2 = 0, \;\; n \geq 0.$

Identities (A_3), (A_9) and (A_6) become very simple, saying (a_5), (a_7) and (a_1), respectively.

Therefore, from (A_5) we obtain

$$a_n^1(x) = -\gamma_{3n+2} Q_n(x) + L R_n(x) \Rightarrow \gamma_{3n+3} R_n(x) = (L - \gamma_{3n+2}) R_n(x),$$

implying (a_4).

From (A_8) we get $P_{n+1}(x) = R_{n+1}(x) + (\beta_{3n+3} + a - b - c)\gamma_{3n+3} R_n(x)$, $n \geq 0$, which will give place to (a_6), after proving (a_2).

Considering (A_2), with $n \leftarrow n + 1$,

$$b_{n+1}^1(x) = (\beta_{3n+4} + a - b - c) R_{n+1}(x) + \gamma_{3n+4}\gamma_{3n+3} R_n(x), \; n \geq 0,$$

we have justified (a_8), after proving (a_2). On the other hand, in (A_7) we find also that

$$(a - b)(a - c) a_n^1(x) + b_{n+1}^1(x) = -(\beta_{3n+3} - a) P_{n+1}(x)$$

$$\Rightarrow b_{n+1}^1(x) = -(\beta_{3n+3} - a)\Big(R_{n+1}(x) + (\beta_{3n+3} + a - b - c)\gamma_{3n+3} R_n(x)\Big) - (a - b)(a - c)\gamma_{3n+3} R_n(x)$$

$$\Rightarrow b_{n+1}^1(x) = -\gamma_{3n+3}\Big((a - b)(a - c) + (\beta_{3n+3} - a)(\beta_{3n+3} + a - b - c)\Big) R_n(x) - (\beta_{3n+3} - a) R_{n+1}(x).$$

Putting together the two informations about b_{n+1}^1, we get

$$\begin{aligned}
&(\beta_{3n+4} + a - b - c) R_{n+1}(x) + \gamma_{3n+4}\gamma_{3n+3} R_n(x) \\
&= -\gamma_{3n+3}\Big((a - b)(a - c) + (\beta_{3n+3} - a)(\beta_{3n+3} + a - b - c)\Big) R_n(x) - (\beta_{3n+3} - a) R_{n+1}(x) \\
&\Rightarrow \beta_{3n+4} + a - b - c = -\beta_{3n+3} + a \;\wedge\; \gamma_{3n+4} = -(a - b)(a - c) - (\beta_{3n+3} - a)(\beta_{3n+3} + a - b - c) \\
&\Rightarrow \beta_{3n+3} + \beta_{3n+4} = b + c \;\wedge\; \gamma_{3n+4} = \beta_{3n+3}(b + c - \beta_{3n+3}) - bc, \; n \geq 0.
\end{aligned}$$

Let us note that (A_2) with $n = 0$, tell us that $\beta_0 + \beta_1 = b + c$ (since $b_0^1(x) = a - \beta_0$), and from (A_1) with $n = 0$ we obtain

$$\gamma_1 = -(a - b)(a - c) - (\beta_1 - a)(a - \beta_0) = \beta_0(b + c - \beta_0) - bc.$$

We then obtain (a_2) and (a_3):

$\beta_{3n}+\beta_{3n+1} = b+c \wedge \gamma_{3n+1} = (a-\beta_{3n})(\beta_{3n}+a-b-c)-(a-b)(a-c) = \beta_{3n}(b+c-\beta_{3n})-bc, n \geq 0.$

Finally, from (A_4) $(n \leftarrow n+1)$ and introducing the expressions above obtained for $P_{n+1}(x)$ and $b^1_{n+1}(x)$, we get
$R_{n+2}(x) + (\beta_{3n+6}+a-b-c)\gamma_{3n+6}R_{n+1}(x)$
$+\gamma_{3n+5}\Big((\beta_{3n+4}+a-b-c)R_{n+1}(x)+\gamma_{3n+4}\gamma_{3n+3}R_n(x)\Big) = \Theta(x)R_{n+1}(x)$, which gives us (a_9), since $\Theta(x) = x + aL + bc(b+c+p)$.

Taking (A_4), with $n=0$, we get the polynomial $R_1(x)$:

$$\gamma_2 b^1_0(x) = -P_1(x) + \Theta(x) \Rightarrow R_1(x) + (\beta_3+a-b-c)\gamma_3 + \gamma_2(a-\beta_0) = \Theta(x).$$

Reciprocally, let us suppose the enunciated list and let us prove all the relations of theorem 3.1 with $a^2_n = b^2_n = c^1_n = c^2_n = 0, \ \ n \geq 0$.

With respect to (A_1), it is obviously fulfilled for $n=0$, because $(\beta_1 - a)b^1_0(x) + \gamma_1 + (a-b)(a-c) = 0$. Furthermore,

$$\begin{aligned}
&(\beta_{3n+4}-a)b^1_{n+1}(x) + \gamma_{3n+4}P_{n+1}(x) + (a-b)(a-c)Q_{n+1}(x)\\
&= (\beta_{3n+4}-a)\Big((a-\beta_{3n+3})R_{n+1}(x) + \gamma_{3n+3}\gamma_{3n+4}R_n(x)\Big)\\
&+ \gamma_{3n+4}\Big(R_{n+1}(x) + (a-\beta_{3n+4})\gamma_{3n+3}R_n(x)\Big) + (a-b)(a-c)R_{n+1}(x)\\
&= \Big((\beta_{3n+4}-a)(a-\beta_{3n+3}) + \gamma_{3n+4} + (a-b)(a-c)\Big)R_{n+1}(x)\\
&+ \gamma_{3n+3}\Big((\beta_{3n+4}-a)\gamma_{3n+4} + \gamma_{3n+4}(a-\beta_{3n+4})\Big)R_n(x) = 0, \ n \geq 0.
\end{aligned}$$

With respect to (A_2), it is also fulfilled for $n=0$, because $\beta_1 + a - b - c = b^1_0(x)$, and

$\gamma_{3n+4}a^1_n(x) - b^1_{n+1}(x) + (\beta_{3n+4}+a-b-c)Q_{n+1}(x)$
$= \gamma_{3n+4}\gamma_{3n+3}R_n(x) - (a-\beta_{3n+3})R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+4}R_n(x) + (\beta_{3n+4}+a-b-c)R_{n+1}(x)$
$= (\beta_{3n+4}+a-b-c)R_{n+1}(x) - (a-\beta_{3n+3})R_{n+1}(x) = 0, \ n \geq 0.$

Identities (A_3), (A_6) and (A_9) are clearly satisfied.

With respect to (A_4), we calculate previously for $n=0$:
$P_1(x) + \gamma_2 b^1_0(x) - \Theta(x) = R_1(x) + (a-\beta_4)\gamma_3 + \gamma_2(a-\beta_0) - \Theta(x) = 0$, since

$$R_1(x) = \Theta(x) - \gamma_3(\beta_3+a-b-c) - \gamma_2(a-\beta_0)$$

and $a - \beta_4 = \beta_3 + a - b - c$. On the other hand,

$\gamma_{3n+5}b^1_{n+1}(x) - \Theta(x)R_{n+1}(x) + P_{n+2}(x)$
$= \gamma_{3n+5}\Big((a-\beta_{3n+3})R_{n+1}(x) + \gamma_{3n+3}\gamma_{3n+4}R_n(x)\Big) - \Theta(x)R_{n+1}(x)$
$+ R_{n+2}(x) + (a-\beta_{3n+7})\gamma_{3n+6}R_{n+1}(x)$
$= R_{n+2}(x) - \Theta(x)R_{n+1}(x) + (\beta_{3n+6}+a-b-c)\gamma_{3n+6}R_{n+1}(x) + \gamma_{3n+5}(a-\beta_{3n+3})R_{n+1}(x)$
$+ \gamma_{3n+3}\gamma_{3n+4}\gamma_{3n+5}R_n(x) = 0, \ n \geq 0$, by (a_9).

Relations (a_7), (a_5) and (a_4), guarantee that

$$a_n^1(x) + \gamma_{3n+2}Q_n(x) - LR_n(x) = \gamma_{3n+3}R_n(x) + \gamma_{3n+2}R_n(x) - LR_n(x) = 0,\ n \geq 0,$$

which fulfils (A_5).

With respect to (A_7),

$$\begin{aligned}
&(a-b)(a-c)a_n^1(x) + b_{n+1}^1(x) + (\beta_{3n+3} - a)P_{n+1}(x) \\
&= (a-b)(a-c)\gamma_{3n+3}R_n(x) + (a - \beta_{3n+3})R_{n+1}(x) + \gamma_{3n+3}\gamma_{3n+4}R_n(x) \\
&+ (\beta_{3n+3} - a)\Big(R_{n+1}(x) + (a - \beta_{3n+4})\gamma_{3n+3}R_n(x)\Big) \\
&= \gamma_{3n+3}\Big(\gamma_{3n+4} + (a - \beta_{3n+4})(\beta_{3n+3} - a) + (a-b)(a-c)\Big)R_n(x) \\
&= 0,\ n \geq 0,\ \text{by } (a_3) \text{ and } (a_2).
\end{aligned}$$

Finally, with respect to (A_8),

$$\begin{aligned}
&(\beta_{3n+3} + a - b - c)a_n^1(x) - P_{n+1}(x) + Q_{n+1}(x) \\
&= (\beta_{3n+3} + a - b - c)\gamma_{3n+3}R_n(x) - R_{n+1}(x) - (a - \beta_{3n+4})\gamma_{3n+3}R_n(x) + R_{n+1}(x) = 0,\ n \geq 0.
\end{aligned}$$

□

Using the relations (a_1)-(a_4) of theorem 3.11, we can present simpler expressions to the coefficients A_n, K_n, H_n, V_n and S_n of theorem 3.2 (given by the relations (3.4),(3.6),(3.7),(3.8) and (3.9)). If $\{W_n\}_{n\geq 0}$ is a MOPS satisfying theorem 3.11 hypotheses, we have:

$A_{3n} = \gamma_{3n+3}(2\beta_{3n+3} - b - c) - (\beta_{3n+3} - a)L$;

$A_{3n+1} = \gamma_{3n+5}(2\beta_{3n+4} - b - c) - (\beta_{3n+4} - a)L$;

$A_{3n+2} = \gamma_{3n+5}(a - \beta_{3n+3}) + \gamma_{3n+6}(\beta_{3n+6} + a - b - c)$;

$K_{3n} = \gamma_{3n+2}\gamma_{3n+3}(\beta_{3n+3} - \beta_{3n})$;

$H_{3n} = \gamma_{3n+3} - \gamma_{3n+6}$;

$V_{3n} = 0$;

$S_{3n} = 0$;

$K_{3n+1} = 0$;

$H_{3n+1} = 0$;

$V_{3n+1} = \gamma_{3n+4}(\gamma_{3n+5} - \gamma_{3n+2})$;

$S_{3n+1} = \beta_{3n+6} - \beta_{3n+3}$;

$K_{3n+2} = 0$;

$H_{3n+2} = 0$;

$V_{3n+2} = 0$;

$S_{3n+2} = 0$.

Consequently, by theorem 3.2, we have also:

$$P_{n+2}(x) = \Big(\Theta(x) - \gamma_{3n+3}(2\beta_{3n+3} - b - c) + (\beta_{3n+3} - a)L\Big)P_{n+1}(x) - \gamma_{3n+1}\gamma_{3n+2}\gamma_{3n+3}P_n(x)$$
$$-\gamma_{3n+2}\gamma_{3n+3}(\beta_{3n+3} - \beta_{3n})b^1_n(x) - (\gamma_{3n+3} - \gamma_{3n+6})b^1_{n+1}(x), \ \ n \geq 0;$$

$$Q_{n+2}(x) = \Big(\Theta(x) - \gamma_{3n+5}(2\beta_{3n+4} - b - c) + (\beta_{3n+4} - a)L\Big)Q_{n+1}(x) - \gamma_{3n+2}\gamma_{3n+3}\gamma_{3n+4}Q_n(x)$$
$$-\gamma_{3n+4}(\gamma_{3n+5} - \gamma_{3n+2})a^1_n(x) - (\beta_{3n+6} - \beta_{3n+3})a^1_{n+1}(x), \ \ n \geq 0; \text{ and}$$

$$R_{n+2}(x) = \Big(\Theta(x) - \gamma_{3n+5}(a - \beta_{3n+3}) - \gamma_{3n+6}(\beta_{3n+6} + a - b - c)\Big)R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+4}\gamma_{3n+5}R_n(x),$$

$n \geq 0$, which is relation (a_9).

Theorem 3.11 statements put in evidence sequence $\{R_n\}_{n\geq 0}$ orthogonality (that we could also conclude from corollary 3.4, since $a^2_n = b^2_n = 0$) and the fact that all components of $\{W_n\}_{n\geq 0}$ CD are expressed in terms of elements of sequence $\{R_n\}_{n\geq 0}$. Sequences $\{a^1_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are clearly determined. Next, we analyse what further information we can obtain from the expressions of the other two sequences $\{P_n\}_{n\geq 0}$ and $\{b^1_n\}_{n\geq 0}$.

With respect to the sequence $\{P_n\}_{n\geq 0}$, under the relations of theorem 3.11, we may conclude the following aspects.

1. If $\beta_{3n+1} = a, \ n \geq 1$, then, by (a_6), $P_n(x) = R_n(x), \ n \geq 0$ and by (a_5), the three principal components are identical and orthogonal.

2. If there is $r \geq 1$ such that $\beta_{3r+1} \neq a$, then we obtain

$$P_n(x) = \sum_{\nu=n-1}^{n} \lambda_{n,\nu} R_\nu(x), \ \ n \geq 1,$$
$$\exists r \geq 1 : \ \lambda_{r,r-1} \neq 0,$$

   where $\lambda_{n,n} = 1$ and $\lambda_{n,n-1} = (a - \beta_{3n+1})\gamma_{3n}$. In other words, there is a finite-type relation between $\{R_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$, with respect to $\Phi(x) = 1$, where $s = 1$ and $t = 0$ (see definition 1.30).

   Sequence $\{R_n\}_{n\geq 0}$ is compatible with any monic polynomial Φ, because it is orthogonal. Thus, by theorem 1.32, the remarked finite-type relation is equivalent to the existence of an application from $\mathbb{N}$ into $\mathbb{N} : m \mapsto \mu_m$ fulfilling

$$m \leq \mu_m \leq m + 1, \ \ m \geq 0,$$
$$\exists m_0 \geq 0 : \mu_{m_0} = m_0 + 1,$$

   and such that

$$r_m = \sum_{\nu=m}^{\mu_m} \lambda_{\nu,m} u_\nu, \ \ m \geq 0,$$

$$\lambda_{\mu_m,m} \neq 0, \ \ m \geq 0.$$

Attending to sequence $\{R_n\}_{n\geq 0}$ orthogonality, we may also write

$$R_m r_0 =< r_0, R_m^2 > \sum_{\nu=m}^{\mu_m} \lambda_{\nu,m} u_\nu, \ \ m \geq 0,$$

$$\lambda_{\mu_m,m} \neq 0, \ \ m \geq 0.$$

3. If $\beta_{3n+1} \neq a, \ \ n \geq 1$, then the relation between $\{R_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ is strictly of finite-type, thus

$$< r_0, R_m^2 >^{-1} R_m r_0 = u_m + (a - \beta_{3m+4})\gamma_{3m+3} u_{m+1}, \ \ m \geq 0.$$

With respect to the sequence $\{b_n^1\}_{n\geq 0}$, under the relations of theorem 3.11, we may conclude the following aspects.

1. If $\beta_{3n+3} = a, \ \ n \geq 0$, then $b_{n+1}^1(x) = \gamma_{3n+3}\gamma_{3n+4} R_n(x), \ \ \ n \geq 0$, i.e., $\deg b_{n+1}^1(x) = n, \ \ n \geq 0$.
2. If there is $r \geq 1$ such that $\beta_{3r-3} \neq a$, then we may write the following

$$< r_0, b_m^1 b_n^1 >= 0, \ \ 0 \leq m \leq n-2, \ \ n \geq 2,$$
$$\exists r \geq 1 : \ < r_0, b_{r-1}^1 b_r^1 > \neq 0,$$

which means that $\{b_n^1\}_{n\geq 0}$ is quasi-orthogonal of order $s = 1$ with respect to r_0 (see definitions 1.36 and 1.37).

Proof. If $\exists k \geq 2 : \ \ \beta_{3k-3} \neq a$, then

$< r_0, b_{k-1}^1 b_k^1 >$
$=< r_0, \Big((a - \beta_{3k-3})R_{k-1}(x) + \gamma_{3k-3}\gamma_{3k-2}R_{k-2}(x)\Big)\Big((a - \beta_{3k})R_k(x) + \gamma_{3k}\gamma_{3k+1}R_{k-1}(x)\Big) >$
$= (a - \beta_{3k-3})\gamma_{3k}\gamma_{3k+1} < r_0, R_{k-1}^2(x) >$, by orthogonality conditions
$\neq 0$.

If $\beta_0 \neq a$, then

$$\begin{aligned} &< r_0, b_0^1 b_1^1 > \\ &=< r_0, (a - \beta_0)\Big((a - \beta_6)R_1(x) + \gamma_3\gamma_4 R_0(x)\Big) > \\ &= (a - \beta_0)\gamma_3\gamma_4 < r_0, R_0^2(x) >, \text{ by orthogonality conditions} \\ &\neq 0. \end{aligned}$$

If $1 \leq m \leq n-2$ and $n \geq 2$, then

$$< r_0, b_m^1 b_n^1 >$$
$$=< r_0, \Big((a-\beta_{3m})R_m(x) + \gamma_{3m}\gamma_{3m+1}R_{m-1}(x)\Big)\Big((a-\beta_{3n})R_n(x) + \gamma_{3n}\gamma_{3n+1}R_{n-1}(x)\Big) >$$
$= 0$, by orthogonality conditions and regarding the inequalities $m \leq n-2 < n-1 < n$.

If $0 = m \leq n-2$ and $n \geq 2$, then

$$< r_0, b_0^1 b_n^1 >$$
$$=< r_0, (a-\beta_0)\Big((a-\beta_{3n})R_n(x) + \gamma_{3n}\gamma_{3n+1}R_{n-1}(x)\Big) >$$
$= 0$, by orthogonality conditions.

□

3. If $\beta_{3n-3} \neq a$, $\forall n \geq 1$, then $\{b_n^1\}_{n\geq 0}$ is strictly quasi-orthogonal of order $s = 1$ with respect to r_0, and consequently, $\{b_n^1\}_{n\geq 0}$ is a free sequence.

Let us proceed with the description of the CD of a MOPS where the only two secondary components nontrivial are $\{a_n^2(x)\}_{n\geq 0}$ and $\{c_n^1(x)\}_{n\geq 0}$.

Theorem 3.12. *Let $\{W_n\}_{n\geq 0}$ be a MPS defined by (2.1)-(2.3), such that*

$$a_n^1 = b_n^1 = b_n^2 = c_n^2 = 0, \quad n \geq 0.$$

Then $\{W_n\}_{n\geq 0}$ is a MOPS if and only if the following assertions are fulfilled, where L is defined by (2.5).

(a_1) $\beta_{3n+1} = b + c - a, \quad n \geq 0,$

(a_2) $\beta_0 = a, \;\; \beta_{3n+2} + \beta_{3n+3} = a - b - c - p, \quad n \geq 0,$

(a_3) $\gamma_{3n+3} = (a - \beta_{3n+2})(a - \beta_{3n+3}), \quad n \geq 0,$

(a_4) $\gamma_{3n+1} + \gamma_{3n+2} = L - (a-b)(a-c), \quad n \geq 0,$

(a_5) $P_{n+1}(x) = R_{n+1}(x) + (a - \beta_{3n+3})(L - \gamma_{3n+4})R_n(x), \quad n \geq 0,$

(a_6) $Q_{n+1}(x) = R_{n+1}(x) - \gamma_{3n+4}(a - \beta_{3n+3})R_n(x), \quad n \geq 0,$

(a_7) $a_n^2(x) = -(a - \beta_{3n+3})R_n(x), \quad n \geq 0,$

(a_8)

$$c_{n+1}^1(x) = (\gamma_{3n+5} - L)R_{n+1}(x) - \gamma_{3n+5}\gamma_{3n+4}(a - \beta_{3n+3})R_n(x), \quad n \geq 0, \; c_0^1 = \gamma_2 - L,$$

(a_9)

$$R_{n+2}(x) = \big(x - \beta_{n+1}^R\big) R_{n+1}(x) - \gamma_{n+1}^R R_n(x), \quad n \geq 0,$$

$$R_0(x) = 1, \quad R_1(x) = x + aL + bc(b+c+p) + (a - \beta_3)(\gamma_4 - L) + (\beta_2 - a)(L - \gamma_2),$$

with $\beta_{n+1}^R = -(aL + bc(b+c+p)) - (a - \beta_{3n+6})(\gamma_{3n+7} - L) - (\beta_{3n+5} - a)(L - \gamma_{3n+5})$
and $\gamma_{n+1}^R = \gamma_{3n+4}\gamma_{3n+5}(a - \beta_{3n+5})(a - \beta_{3n+3}), \quad n \geq 0.$

Proof. Let us suppose the relations listed on theorem 3.1 with $a_n^1 = b_n^1 = b_n^2 = c_n^2 = 0, \ \ n \geq 0$. From (A_2) and (A_6) we obtain (a_1) and

$$a_n^2(x) = -(\beta_{3n+2} + b + c + p) R_n(x), \ n \geq 0,$$

which will justify (a_7), after proving (a_2). From (A_9) we get $-(\beta_{3n+3} + b + c + p)(\beta_{3n+2} + b + c + p) R_n(x) + \gamma_{3n+3} R_n(x) = 0$, that is,

$$\gamma_{3n+3} = (\beta_{3n+3} + b + c + p)(\beta_{3n+2} + b + c + p), \ n \geq 0,$$

which will justify (a_3), after proving (a_2).

Considering (A_3), with $n \leftarrow n+1$, we get

$$Q_{n+1}(x) = R_{n+1}(x) - \gamma_{3n+4}(\beta_{3n+2} + b + c + p) R_n(x), \ n \geq 0,$$

which will justify (a_6), after proving (a_2).

Therefore, from (A_8), we may write

$$P_{n+1}(x) = R_{n+1}(x) - \gamma_{3n+4}(\beta_{3n+2} + b + c + p) R_n(x) + L(\beta_{3n+2} + b + c + p) R_n(x), \ n \geq 0$$

$$\Leftrightarrow P_{n+1}(x) = R_{n+1}(x) - (\beta_{3n+2} + b + c + p)(\gamma_{3n+4} - L) R_n(x), \ n \geq 0,$$

which will justify (a_5), after proving (a_2).

From (A_5) $(n \leftarrow n+1)$, we will get (a_8):

$$c_{n+1}^1(x) = \gamma_{3n+5}\Big(R_{n+1}(x) - \gamma_{3n+4}(\beta_{3n+2} + b + c + p) R_n(x)\Big) - L R_{n+1}(x), \ n \geq 0$$
$$\Rightarrow c_{n+1}^1(x) = (\gamma_{3n+5} - L) R_{n+1}(x) - \gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+2} + b + c + p) R_n(x), \ n \geq 0.$$

Moreover, from (A_5), with $n = 0$, we may determine $c_0^1(x)$:

$$\gamma_2 = c_0^1(x) + L \Leftrightarrow c_0^1(x) = \gamma_2 - L.$$

Let us take identity (A_1), firstly with $n = 0$ and secondly with $n \leftarrow n+1$:

$$c_0^1(x) + \gamma_1 = -(a - b)(a - c) \Rightarrow \gamma_1 + \gamma_2 = L - (a - b)(a - c);$$

$$
\begin{aligned}
&c^1_{n+1}(x)+\gamma_{3n+4}P_{n+1}(x)=-(a-b)(a-c)Q_{n+1}(x)\\
&\Rightarrow (\gamma_{3n+5}-L)R_{n+1}(x)-\gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+2}+b+c+p)R_n(x)\\
&+\gamma_{3n+4}\Big(R_{n+1}(x)-(\gamma_{3n+4}-L)(\beta_{3n+2}+b+c+p)R_n(x)\Big)\\
&=-(a-b)(a-c)\Big(R_{n+1}(x)-\gamma_{3n+4}(\beta_{3n+2}+b+c+p)R_n(x)\Big)\\
&\Rightarrow \Big(\gamma_{3n+4}+\gamma_{3n+5}+(a-b)(a-c)-L\Big)R_{n+1}(x)\\
&-\gamma_{3n+4}(\beta_{3n+2}+b+c+p)\Big(\gamma_{3n+4}+\gamma_{3n+5}+(a-b)(a-c)-L\Big)R_n(x)=0\\
&\Leftrightarrow \gamma_{3n+4}+\gamma_{3n+5}=L-(a-b)(a-c),\ n\geq 0.
\end{aligned}
$$

In this manner, $\gamma_{3n+1}+\gamma_{3n+2}=L-(a-b)(a-c),\ n\geq 0$.

Identity (A_4), with $n=0$, tell us that

$$P_1(x)+(\beta_2-a)c^1_0(x)=\Theta(x)\Rightarrow P_1(x)=\Theta(x)-(\beta_2-a)(\gamma_2-L),$$

where $\Theta(x)=x+aL+bc(b+c+p)$. However, from (A_8), with $n=0$, we get $P_1(x)=R_1(x)-(\gamma_4-L)(\beta_2+b+c+p)$; consequently

$$R_1(x)=\Theta(x)-(\beta_2-a)(\gamma_2-L)+(\gamma_4-L)(\beta_2+b+c+p).$$

Attending to (A_4), with $n\leftarrow n+1$, and also to identities above, we will obtain (a_9):

$$
\begin{aligned}
&P_{n+2}(x)+(\beta_{3n+5}-a)c^1_{n+1}(x)=\Theta(x)R_{n+1}(x)\\
&\Rightarrow R_{n+2}(x)-(\beta_{3n+5}+b+c+p)(\gamma_{3n+7}-L)R_{n+1}(x)\\
&+(\beta_{3n+5}-a)\Big((\gamma_{3n+5}-L)R_{n+1}(x)-\gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+2}+b+c+p)R_n(x)\Big)=\Theta(x)R_{n+1}(x)\\
&\Rightarrow R_{n+2}(x)=\Big\{\Theta(x)+(\beta_{3n+5}+b+c+p)(\gamma_{3n+7}-L)+(\beta_{3n+5}-a)(L-\gamma_{3n+5})\Big\}R_{n+1}(x)\\
&-\gamma_{3n+4}\gamma_{3n+5}(a-\beta_{3n+5})(\beta_{3n+2}+b+c+p)R_n(x),\ n\geq 0.
\end{aligned}
$$

From (A_7), with $n=0$,

$$
\begin{aligned}
&(\beta_3-a)P_1(x)+\gamma_3c^1_0(x)=\Theta(x)a^2_0(x)\\
&\Rightarrow (\beta_3-a)\big(\Theta(x)-(\beta_2-a)(\gamma_2-L)\big)+\gamma_3(\gamma_2-L)=-\Theta(x)(\beta_2+b+c+p)\\
&\Rightarrow \beta_3-a=-(\beta_2+b+c+p)\ \wedge\ -(\beta_3-a)(\beta_2-a)(\gamma_2-L)+\gamma_3(\gamma_2-L)=0\\
&\Rightarrow \beta_2+\beta_3=a-b-c-p\ \wedge\ -(\gamma_2-L)\big((\beta_3-a)(\beta_2-a)-\gamma_3\big)=0\\
&\Rightarrow \beta_2+\beta_3=a-b-c-p\ \wedge\ -(\gamma_2-L)\big((\beta_2+b+c+p)(\beta_3+b+c+p)-\gamma_3\big)=0.\\
&\Rightarrow \beta_2+\beta_3=a-b-c-p,\ \text{since}\ \gamma_3=(\beta_2+b+c+p)(\beta_3+b+c+p).
\end{aligned}
$$

Regarding (A_7), with $n \leftarrow n+1$, and identities above, we obtain

$$
\begin{aligned}
&(\beta_{3n+6}-a)P_{n+2}(x)+\gamma_{3n+6}c^1_{n+1}(x)=\Theta(x)a^2_{n+1}(x)\\
&\Rightarrow (\beta_{3n+6}-a)\Big(R_{n+2}(x)-(\beta_{3n+5}+b+c+p)(\gamma_{3n+7}-L)R_{n+1}(x)\Big)\\
&+\gamma_{3n+6}\Big((\gamma_{3n+5}-L)R_{n+1}(x)-\gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+2}+b+c+p)R_n(x)\Big)\\
&=-\Theta(x)(\beta_{3n+5}+b+c+p)R_{n+1}(x)\\
&\Rightarrow (\beta_{3n+6}-a)\Big(\Theta(x)+(\beta_{3n+5}+b+c+p)(\gamma_{3n+7}-L)+(\beta_{3n+5}-a)(L-\gamma_{3n+5})\Big)R_{n+1}(x)\\
&-\gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+6}-a)(a-\beta_{3n+5})(\beta_{3n+2}+b+c+p)R_n(x)\\
&-(\beta_{3n+6}-a)(\beta_{3n+5}+b+c+p)(\gamma_{3n+7}-L)R_{n+1}(x)\\
&+\gamma_{3n+6}\Big((\gamma_{3n+5}-L)R_{n+1}(x)-\gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+2}+b+c+p)R_n(x)\Big)\\
&=-\Theta(x)(\beta_{3n+5}+b+c+p)R_{n+1}(x)\\
&\Rightarrow (\beta_{3n+6}-a+\beta_{3n+5}+b+c+p)\Theta(x)R_{n+1}(x)\\
&+(L-\gamma_{3n+5})\Big((\beta_{3n+6}-a)(\beta_{3n+5}-a)-\gamma_{3n+6}\Big)R_{n+1}(x)\\
&-\gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+2}+b+c+p)\Big((\beta_{3n+6}-a)(a-\beta_{3n+5})+\gamma_{3n+6}\Big)R_n(x)=0\\
&\Rightarrow \beta_{3n+6}-a=-(\beta_{3n+5}+b+c+p),\ n\geq 0,\\
&\text{since } \gamma_{3n+6}=(\beta_{3n+5}+b+c+p)(\beta_{3n+6}+b+c+p),\\
&\text{or } \gamma_{3n+6}=(\beta_{3n+6}-a)(\beta_{3n+5}-a).
\end{aligned}
$$

Then, $\beta_{3n+2}+\beta_{3n+3}=a-b-c-p, n\geq 0$, and thus, we can replace factors $\beta_{3n+2}+b+c+p$ and $\beta_{3n+3}+b+c+p$ by $a-\beta_{3n+3}$ and $a-\beta_{3n+2}$, respectively, in the above identities.

Reciprocally, let us suppose the enunciated list of relations, along with $a^1_n=b^1_n=b^2_n=c^2_n=0,\ \ n\geq 0$, and let us guarentee relations of theorem 3.1. Calculations made to obtain relations enunciated are also useful to fulfill this reciprocity, as follows: (A_2), (A_6) and (A_9) are obviously fulfilled and identities (a_6), (a_5), (a_8), (a_4), (a_9) and (a_2) correspond to identities (A_3), (A_8), (A_5), (A_1), (A_4) and (A_7), respectively, under the list of hypotheses considered. □

Using the relations (a_1)-(a_4) of theorem 3.12, we can present simpler expressions to the coefficients A_n, K_n, H_n, V_n and S_n of theorem 3.2, defined by the relations (3.4),(3.6),(3.7),(3.8) and (3.9). If $\{W_n\}_{n\geq 0}$ is a MOPS satisfying theorem 3.12 hypotheses, we have:

$$A_{3n}=\gamma_{3n+4}(2\beta_{3n+3}-a+b+c+p)-(\beta_{3n+3}-a)L+(a-b)(a-c)(\beta_{3n+3}+b+c+p);$$

$$
\begin{aligned}
A_{3n+1}\ &=\gamma_{3n+4}(\beta_{3n+3}-2a+2b+2c+p)+\gamma_{3n+5}(\beta_{3n+5}-2a+2b+2c+p)\\
&\quad -(b+c-2a)L+(a-b)(a-c)(2b+2c-a+p);
\end{aligned}
$$

$$A_{3n+2}=\gamma_{3n+5}(2\beta_{3n+5}-a+b+c+p)-(\beta_{3n+5}-a)L+(a-b)(a-c)(\beta_{3n+5}+b+c+p);$$

$K_{3n} = 0;$
$H_{3n} = 0;$
$V_{3n} = \gamma_{3n+3}(\gamma_{3n+2} - \gamma_{3n+5});$
$S_{3n} = \beta_{3n+3} - \beta_{3n+6};$
$K_{3n+1} = 0;$
$H_{3n+1} = 0;$
$V_{3n+1} = 0;$
$S_{3n+1} = 0;$
$K_{3n+2} = \gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+3} - \beta_{3n+6});$
$H_{3n+2} = \gamma_{3n+5} - \gamma_{3n+8};$
$V_{3n+2} = 0;$
$S_{3n+2} = 0.$
Consequently, by theorem 3.2, we have also:
$P_{n+2}(x) = \Big(\Theta(x) - A_{3n}\Big)P_{n+1}(x) - \gamma_{3n+1}\gamma_{3n+2}\gamma_{3n+3}P_n(x)$
$-\gamma_{3n+3}(\gamma_{3n+2} - \gamma_{3n+5})c_n^1(x) - (\gamma_{3n+3} - \gamma_{3n+6})c_{n+1}^1(x), \quad n \geq 0;$

$Q_{n+2}(x) = \Big(\Theta(x) - A_{3n+1}\Big)Q_{n+1}(x) - \gamma_{3n+2}\gamma_{3n+3}\gamma_{3n+4}Q_n(x), \quad n \geq 0$; that is, $\{Q_n\}_{n\geq 0}$ is orthogonal, fulfilling the following recurrence relation of second order:

$$\begin{aligned} &Q_{n+2}(x) = \big(x - \beta_{n+1}^Q\big)Q_{n+1}(x) - \gamma_{n+1}^Q Q_n(x), \quad n \geq 0,\\ &Q_0(x) = 1,\\ &Q_1(x) = x + aL + bc(b+c+p) - \gamma_1(2b+2c-a+p) - \gamma_2(b+c-a-\beta_3)\\ &\ -(a-b)(a-c)(2b+2c-a+p) - L(2a-b-c), \end{aligned} \tag{3.35}$$

with $\beta_{n+1}^Q = -(aL + bc(b+c+p)) + \gamma_{3n+4}(\beta_{3n+3} - 2a + 2b + 2c + p) + \gamma_{3n+5}(\beta_{3n+5} - 2a + 2b + 2c + p) - (b+c-2a)L + (a-b)(a-c)(2b+2c-a+p)$
and $\gamma_{n+1}^Q = \gamma_{3n+2}\gamma_{3n+3}\gamma_{3n+4}, \quad n \geq 0$; and

$R_{n+2}(x) = \Big(\Theta(x) - A_{3n+2}\Big)R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+4}\gamma_{3n+5}R_n(x)$
$-\gamma_{3n+4}\gamma_{3n+5}(\beta_{3n+3} - \beta_{3n+6})a_n^2(x) - (\gamma_{3n+5} - \gamma_{3n+8})a_{n+1}^2(x), \quad n \geq 0;$
which is relation (a_9).

As in theorem 3.11, the sequence $\{R_n\}_{n\geq 0}$ is orthogonal at theorem 3.12, and all components are expressed in terms of elements of $\{R_n\}_{n\geq 0}$. Sequence $\{a_n^2\}_{n\geq 0}$ is clearly determined. Next, we point out further information about the other three sequences $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{c_n^1\}_{n\geq 0}$.

With respect to the sequence $\{Q_n\}_{n\geq 0}$, under the relations of theorem 3.12, we may conclude the following aspects.

1. The sequence $\{Q_n\}_{n\geq 0}$ is orthogonal, either by corollary 3.4, since we consider $a_n^1 = c_n^2 = 0, \quad n \geq 0$, or by theorem 3.2, as we seen above, establishing the recurrence relation (3.35).

2. Identity $Q_n(x) = R_n(x) - \gamma_{3n+1}(a-\beta_{3n})R_{n-1}(x), \quad n \geq 1$, reflects a strictly finite-type relation between $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, with respect to $\Phi(x) = 1$, given by

$$Q_n(x) = \sum_{\nu=n-1}^{n} \lambda_{n,\nu} R_\nu(x), \;\; n \geq 1,$$
$$\forall n \geq 1 : \;\; \lambda_{n,n-1} \neq 0,$$

   where $\lambda_{n,n} = 1$ and $\lambda_{n,n-1} = -\gamma_{3n+1}(a - \beta_{3n})$.

   In fact, $a - \beta_{3n} \neq 0, \;\; n \geq 1$, because $\gamma_{3n+3} = (a - \beta_{3n+2})(a - \beta_{3n+3}) \neq 0, \quad n \geq 0$.

   Taking into account that both sequences are orthogonal, by theorem 1.34, we can conclude that there exist a constant $k_0 \neq 0$ and a monic polynomial Λ_1, with degree one, such that

$$r_0 = k_0 \Lambda_1 v_0,$$

   where

$$k_0 = \frac{\lambda_{1,0}}{< v_0, Q_1^2 >} = -\frac{\gamma_4(a - \beta_3)}{< v_0, Q_1^2 >},$$
$$\Lambda_1(x) = -\frac{1}{\gamma_4(a - \beta_3)} \frac{< v_0, Q_1^2 >}{< v_0, Q_0^2 >} + Q_1(x).$$

With respect to the sequence $\{P_n\}_{n\geq 0}$, under the relations of theorem 3.12, we may conclude the following aspects.

1. If $\gamma_{3n+1} = L, \;\; n \geq 1$, then $P_n = R_n, \;\; n \geq 0$, thus the three principal components are orthogonal.

2. If there is $r \geq 1$ such that $\gamma_{3r+1} \neq L$, then

$$P_n(x) = \sum_{\nu=n-1}^{n} \lambda_{n,\nu} R_\nu(x), \;\; n \geq 1,$$
$$\exists r \geq 1 : \;\; \lambda_{r,r-1} \neq 0,$$

   where $\lambda_{n,n} = 1$ and $\lambda_{n,n-1} = (a - \beta_{3n})(L - \gamma_{3n+1})$. In other words, there is a finite-type relation between $\{R_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$, with respect to $\Phi(x) = 1$ (where $s = 1$ and $t = 0$).

So, by theorem 1.32, this finite-type relation is equivalent to the existence of an application from $\mathbb{N}$ into $\mathbb{N} : m \mapsto \mu_m$ satisfying

$$m \leq \mu_m \leq m+1, \quad m \geq 0,$$
$$\exists m_0 \geq 0 : \mu_{m_0} = m_0 + 1,$$

and also

$$r_m = \sum_{\nu=m}^{\mu_m} \lambda_{\nu,m} u_\nu, \;\; m \geq 0,$$

$$\lambda_{\mu_m,m} \neq 0, \;\; m \geq 0.$$

Since sequence $\{R_n\}_{n\geq 0}$ is orthogonal, we may write

$$R_m r_0 =< r_0, R_m^2 > \sum_{\nu=m}^{\mu_m} \lambda_{\nu,m} u_\nu, \;\; m \geq 0,$$

$$\lambda_{\mu_m,m} \neq 0, \;\; m \geq 0.$$

3. If $\gamma_{3n+1} \neq L, \;\; n \geq 1$, then we have a strictly finite-type relation between $\{R_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$, therefore,

$$< r_0, R_m^2 >^{-1} R_m r_0 = u_m + (a - \beta_{3m+3})(L - \gamma_{3m+4}) u_{m+1}, \;\; m \geq 0.$$

At last, concerning the sequence $\{c_n^1\}_{n\geq 0}$, under the relations of theorem 3.12, we may conclude the following aspects.

1. If $\gamma_{3n+5} = L, \;\; n \geq 0$, then $c_{n+1}^1(x) = -\gamma_{3n+5}\gamma_{3n+4}(a - \beta_{3n+3}) R_n(x), \quad n \geq 0$, and $\deg(c_{n+1}^1) = n, \;\; n \geq 0$.

2. If there is $r \geq 1$ such that $\gamma_{3r-1} \neq L$, then

$$< r_0, c_m^1 c_n^1 >= 0, \;\; 0 \leq m \leq n-2, \;\; n \geq 2,$$
$$\exists r \geq 1 : < r_0, c_{r-1}^1 c_r^1 > \neq 0,$$

which means that $\{c_n^1\}_{n\geq 0}$ is quasi-orthogonal of order $s = 1$ with respect to r_0, (see definition 1.36).

Proof. If $\exists k \geq 2: \quad \gamma_{3k-1} \neq L$, then

$$
\begin{aligned}
&< r_0, c^1_{k-1} c^1_k > \\
&=< r_0, \Big((\gamma_{3k-1} - L) R_{k-1}(x) - \gamma_{3k-1}\gamma_{3k-2}(a - \beta_{3k-3}) R_{k-2}(x) \Big) \Big((\gamma_{3k+2} - L) R_k(x) \\
&- \gamma_{3k+2}\gamma_{3k+1}(a - \beta_{3k}) R_{k-1}(x) \Big) > \\
&= -\gamma_{3k+2}\gamma_{3k+1}(a - \beta_{3k})(\gamma_{3k-1} - L) < r_0, R^2_{k-1}(x) >, \text{ by orthogonality conditions} \\
&\neq 0.
\end{aligned}
$$

If $\gamma_2 \neq L$, then

$$
\begin{aligned}
&< r_0, c^1_0 c^1_1 > \\
&=< r_0, (\gamma_2 - L)\Big((\gamma_5 - L) R_1(x) - \gamma_5\gamma_4(a - \beta_3) R_0(x) \Big) > \\
&= -(\gamma_2 - L)\gamma_5\gamma_4(a - \beta_3) < r_0, R^2_0(x) >, \text{ by orthogonality conditions} \\
&\neq 0.
\end{aligned}
$$

If $1 \leq m \leq n-2$ and $n \geq 2$, then

$$
\begin{aligned}
&< r_0, c^1_m c^1_n > \\
&=< r_0, \Big((\gamma_{3m+2} - L) R_m(x) - \gamma_{3m+2}\gamma_{3m+1}(a - \beta_{3m}) R_{m-1}(x) \Big) \Big((\gamma_{3n+2} - L) R_n(x) \\
&- \gamma_{3n+2}\gamma_{3n+1}(a - \beta_{3n}) R_{n-1}(x) \Big) \Big) > \\
&= 0,
\end{aligned}
$$

by orthogonality conditions and regarding the inequalities $m \leq n-2 < n-1 < n$.
If $0 = m \leq n-2$ and $n \geq 2$, then

$$
\begin{aligned}
&< r_0, c^1_0 c^1_n > \\
&=< r_0, (\gamma_2 - L)\Big((\gamma_{3n+2} - L) R_n(x) - \gamma_{3n+2}\gamma_{3n+1}(a - \beta_{3n}) R_{n-1}(x) \Big) > \\
&= 0, \text{ by orthogonality conditions.}
\end{aligned}
$$

□

3. If $\gamma_{3n-1} \neq L, \ \forall n \geq 1$, then$\{c^1_n\}_{n \geq 0}$ is strictly quasi-orthogonal of order $s = 1$ with respect to r_0 (see definition 1.37); consequently $\{c^1_n\}_{n \geq 0}$ is a free sequence.

Let us continue with the description of the CD of a MOPS where the only two secondary components nontrivial are $\{b^2_n(x)\}_{n \geq 0}$ and $\{c^2_n(x)\}_{n \geq 0}$.

Theorem 3.13. *Let* $\{W_n\}_{n\geq 0}$ *be a MPS defined by (2.1)-(2.3), such that*

$$a_n^1 = a_n^2 = b_n^1 = c_n^1 = 0, \quad n \geq 0.$$

Then $\{W_n\}_{n\geq 0}$ *is a MOPS if and only if the following relations are fulfilled, where* L *is defined by (2.5).*

(a_1) $\beta_{3n} = a, \quad n \geq 0,$

(a_2) $\beta_{3n+1} + \beta_{3n+2} = -(a+p), \quad n \geq 0,$

(a_3) $\gamma_{3n+2} = L - (\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p), \quad n \geq 0,$

(a_4) $\gamma_1 = -(a-b)(a-c), \quad \gamma_{3n+3} + \gamma_{3n+4} = -(a-b)(a-c), \quad n \geq 0,$

(a_5) $P_{n+2}(x) = R_{n+2}(x) - \gamma_{3n+6}(\beta_{3n+7} - \beta_{3n+5})R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+5}\gamma_{3n+6}R_n(x), \quad n \geq 0,$

$$P_1(x) = x + aL + bc(b+c+p) - (a-b)(a-c)(\beta_2 + b + c + p),$$

(a_6) $Q_{n+1}(x) = R_{n+1}(x) - \gamma_{3n+3}(\beta_{3n+4} + b + c + p)R_n(x), \quad n \geq 0,$

(a_7) $b_n^2(x) = -\gamma_{3n+3}R_n(x), \quad n \geq 0,$

(a_8) $c_{n+1}^2(x) = (\beta_{3n+5}+b+c+p)R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+5}R_n(x), \quad n \geq 0, \; c_0^2(x) = \beta_2+b+c+p,$

(a_9)

$$R_{n+2}(x) = \big(x - \beta_{n+1}^R\big)R_{n+1}(x) - \gamma_{n+1}^R R_n(x), \quad n \geq 0, \quad R_0(x) = 1,$$

$$R_1(x) = x + aL + bc(b+c+p) - \gamma_4(\beta_4 - \beta_2) - (a-b)(a-c)(\beta_4 + b + c + p),$$

with $\beta_{n+1}^R = -(aL+bc(b+c+p)) + \gamma_{3n+7}(\beta_{3n+7} - \beta_{3n+5}) + (a-b)(a-c)(\beta_{3n+7}+b+c+p)$ *and* $\gamma_{n+1}^R = \gamma_{3n+3}\gamma_{3n+5}\gamma_{3n+7}, \quad n \geq 0.$

Proof. Let us suppose the list of identities of theorem 3.1 with $a_n^1 = a_n^2 = b_n^1 = c_n^1 = 0, \; n \geq 0$. Identity (A_7) and the initial condition $b_0^1 = a - \beta_0$ are equivalent to (a_1). Relation (A_9) corresponds to (a_7) where $b_n^2(x) = -\gamma_{3n+3}R_n(x)$, $n \geq 0$, therefore, in (A_3) $(n \leftarrow n+1)$ we may read (a_6):

$$Q_{n+1}(x) = R_{n+1}(x) - (\beta_{3n+4} + b + c + p)\gamma_{3n+3}R_n(x), \; n \geq 0.$$

From (A_6), with $n = 0$ and with $n \leftarrow n+1$, we obtain (a_8):

$$c_0^2(x) = \beta_2 + b + c + p,$$

$$c_{n+1}^2(x) = (\beta_{3n+5} + b + c + p)R_{n+1}(x) - \gamma_{3n+5}\gamma_{3n+3}R_n(x), \; n \geq 0.$$

On the other hand, (A_2), with $n=0$, tell us that $c_0^2(x) = -(\beta_1 + a - b - c)$, conducting us to the identity $-(\beta_1 + a - b - c) = \beta_2 + b + c + p$, or,

$$\beta_1 + \beta_2 = -(a+p).$$

Relation (A_2), with $n \leftarrow n+1$, corresponds to the identity

$$\begin{aligned}
&- Lb_n^2(x) + c_{n+1}^2(x) + (\beta_{3n+4} + a - b - c)Q_{n+1}(x) = 0\\
&\Rightarrow L\gamma_{3n+3}R_n(x) + (\beta_{3n+5} + b + c + p)R_{n+1}(x) - \gamma_{3n+5}\gamma_{3n+3}R_n(x)\\
&+ (\beta_{3n+4} + a - b - c)\Big(R_{n+1}(x) - (\beta_{3n+4} + b + c + p)\gamma_{3n+3}R_n(x)\Big) = 0\\
&\Rightarrow \beta_{3n+4} + \beta_{3n+5} = -(a+p) \ \wedge \ \gamma_{3n+5} = L - (\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p)\\
&\Rightarrow \beta_{3n+4} + \beta_{3n+5} = -(a+p) \ \wedge \ \gamma_{3n+5} = L - (\beta_{3n+5} + a - b - c)(\beta_{3n+5} + b + c + p),\ n \geq 0.
\end{aligned}$$

Equality (A_5) reinforces (A_2). In fact, with $n=0$, we have $(\beta_2 + a - b - c)c_0^2(x) + \gamma_2 = L$, i.e.,

$$\gamma_2 = L - (\beta_2 + a - b - c)(\beta_2 + b + c + p),$$

and with $n \leftarrow n+1$,

$$\begin{aligned}
&(\beta_{3n+5} + a - b - c)c_{n+1}^2(x) + \gamma_{3n+5}Q_{n+1}(x) = LR_{n+1}(x)\\
&\Rightarrow (\beta_{3n+5} + a - b - c)\Big((\beta_{3n+5} + b + c + p)R_{n+1}(x) - \gamma_{3n+5}\gamma_{3n+3}R_n(x)\Big)\\
&+ \gamma_{3n+5}\Big(R_{n+1}(x) - (\beta_{3n+4} + b + c + p)\gamma_{3n+3}R_n(x)\Big) = LR_{n+1}(x)\\
&\Rightarrow \beta_{3n+4} + \beta_{3n+5} = -(a+p) \ \wedge \ \gamma_{3n+5} = L - (\beta_{3n+5} + a - b - c)(\beta_{3n+5} + b + c + p),\ n \geq 0.
\end{aligned}$$

In brief, (A_2) and (A_5) correspond to (a_2) and (a_3).

Identities (A_8) and (A_4) will give us information about $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$.

From (A_4), with $n=0$,

$$P_1(x) = -(a-b)(a-c)(\beta_2 + b + c + p) + \Theta(x)$$

and consequently, from (A_8), with $n=0$,

$$\begin{aligned}
&P_1(x) = Q_1(x) + \gamma_3 c_0^2(x)\\
&\Rightarrow P_1(x) = R_1(x) - (\beta_4 + b + c + p)\gamma_3 + \gamma_3(\beta_2 + b + c + p)\\
&\Rightarrow R_1(x) = P_1(x) + \gamma_3(\beta_4 - \beta_2)\\
&\Rightarrow R_1(x) = \Theta(x) + \gamma_3(\beta_4 - \beta_2) - (a-b)(a-c)(\beta_2 + b + c + p).
\end{aligned}$$

From (A_8) with $n \leftarrow n+1$,

$$\begin{aligned}
&P_{n+2}(x) = Q_{n+2}(x) + \gamma_{3n+6}c_{n+1}^2(x)\\
&\Rightarrow P_{n+2}(x) = R_{n+2}(x) - (\beta_{3n+7} + b + c + p)\gamma_{3n+6}R_{n+1}(x)\\
&+ \gamma_{3n+6}\Big((\beta_{3n+5} + b + c + p)R_{n+1}(x) - \gamma_{3n+5}\gamma_{3n+3}R_n(x)\Big)\\
&\Rightarrow P_{n+2}(x) = R_{n+2}(x) - \gamma_{3n+6}(\beta_{3n+7} - \beta_{3n+5})R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+5}\gamma_{3n+6}R_n(x),\ n \geq 0;
\end{aligned}$$

and from (A_4), with $n \leftarrow n+1$, we obtain:

$$P_{n+2}(x) = -(a-b)(a-c)c_{n+1}^2(x) + \Theta(x)R_{n+1}(x)$$
$$\Rightarrow R_{n+2}(x) - \gamma_{3n+6}(\beta_{3n+7} - \beta_{3n+5})R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+5}\gamma_{3n+6}R_n(x)$$
$$= -(a-b)(a-c)\Big((\beta_{3n+5} + b + c + p)R_{n+1}(x) - \gamma_{3n+5}\gamma_{3n+3}R_n(x)\Big) + \Theta(x)R_{n+1}(x)$$
$$\Rightarrow R_{n+2}(x) = \Big\{\Theta(x) + \gamma_{3n+6}(\beta_{3n+7} - \beta_{3n+5}) - (a-b)(a-c)(\beta_{3n+5} + b + c + p)\Big\}R_{n+1}(x)$$
$$+ \gamma_{3n+3}\gamma_{3n+5}\Big(\gamma_{3n+6} + (a-b)(a-c)\Big)R_n(x),\ n \geq 0.$$

At last, let us consider (A_1), with $n = 0$ and $n = 1$:

$$\gamma_1 = -(a-b)(a-c),$$
$$\gamma_4 P_1(x) = -(a-b)(a-c)Q_1(x) + \Theta(x)b_0^2(x)$$
$$\Rightarrow \gamma_4 P_1(x) = -(a-b)(a-c)\Big(P_1(x) - \gamma_3(\beta_2 + b + c + p)\Big) - \gamma_3\Theta(x)$$
$$\Rightarrow \Big(\gamma_4 + (a-b)(a-c)\Big)P_1(x) = -\gamma_3\Big(-(a-b)(a-c)(\beta_2 + b + c + p) + \Theta(x)\Big)$$
$$\Rightarrow \Big(\gamma_4 + (a-b)(a-c)\Big)P_1(x) = -\gamma_3 P_1(x)$$
$$\Leftrightarrow \gamma_4 + \gamma_3 = -(a-b)(a-c),$$

and with $n \leftarrow n+2$:

$$\gamma_{3n+7}P_{n+2}(x) = -(a-b)(a-c)Q_{n+2}(x) + \Theta(x)b_{n+1}^2(x)$$
$$\Rightarrow \gamma_{3n+7}\Big(R_{n+2}(x) - \gamma_{3n+6}(\beta_{3n+7} - \beta_{3n+5})R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+5}\gamma_{3n+6}R_n(x)\Big)$$
$$= -(a-b)(a-c)\Big(R_{n+2}(x) - (\beta_{3n+7} + b + c + p)\gamma_{3n+6}R_{n+1}(x)\Big) - \Theta(x)\gamma_{3n+6}R_{n+1}(x)$$
$$\Rightarrow \Big(\gamma_{3n+7} + (a-b)(a-c)\Big)R_{n+2}(x) - \gamma_{3n+6}\gamma_{3n+7}(\beta_{3n+7} - \beta_{3n+5})R_{n+1}(x)$$
$$- (a-b)(a-c)(\beta_{3n+7} + b + c + p)\gamma_{3n+6}R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+5}\gamma_{3n+6}\gamma_{3n+7}R_n(x)$$
$$+ \gamma_{3n+6}\Big\{R_{n+2}(x) - \gamma_{3n+6}(\beta_{3n+7} - \beta_{3n+5})R_{n+1}(x) + (a-b)(a-c)(\beta_{3n+5} + b + c + p)R_{n+1}(x)$$
$$- \gamma_{3n+3}\gamma_{3n+5}\Big(\gamma_{3n+6} + (a-b)(a-c)\Big)R_n(x)\Big\} = 0$$
$$\Leftrightarrow \Big(\gamma_{3n+6} + \gamma_{3n+7} + (a-b)(a-c)\Big)R_{n+2}(x)$$
$$- \gamma_{3n+6}(\beta_{3n+7} - \beta_{3n+5})\Big(\gamma_{3n+6} + \gamma_{3n+7} + (a-b)(a-c)\Big)R_{n+1}(x)$$
$$- \gamma_{3n+3}\gamma_{3n+5}\gamma_{3n+6}\Big(\gamma_{3n+6} + \gamma_{3n+7} + (a-b)(a-c)\Big)R_n(x) = 0$$
$$\Leftrightarrow \gamma_{3n+6} + \gamma_{3n+7} + (a-b)(a-c) = 0,\ n \geq 0.$$

So, (A_1) gives us (a_4) and since $\gamma_3 = -(a-b)(a-c) - \gamma_4$, we may rewrite $R_1(x)$ as follows:

$$R_1(x) = \Theta(x) + (-(a-b)(a-c) - \gamma_4)(\beta_4 - \beta_2) - (a-b)(a-c)(\beta_2 + b + c + p)$$
$$\Rightarrow R_1(x) = \Theta(x) - \gamma_4(\beta_4 - \beta_2) - (a-b)(a-c)(\beta_4 + b + c + p).$$

Reciprocally, let us suppose the enunciated list of identities and let us guarentee the relations of theorem 3.1, with $a_n^1 = a_n^2 = b_n^1 = c_n^1 = 0, \ \ n \geq 0$. Calculations made to obtain the enunciated identities are also useful to fulfill this reciprocity, as follows: (A_7) and (A_9) are obviously satisfied and identities (a_6), (a_8), (a_2), (a_3), (a_5), (a_9) and (a_4) correspond to identities (A_3), (A_6), (A_2), (A_5), (A_8), (A_4) and (A_1), respectively, under the list of hypotheses considered. □

Using the relations (a_1)-(a_4) of theorem 3.13, we can present simpler expressions to the coefficients A_n, K_n, H_n, V_n and S_n of theorem 3.2 (given by the relations (3.4),(3.6),(3.7),(3.8) and (3.9)). If $\{W_n\}_{n\geq 0}$ is a MOPS satisfying theorem 3.13 hypotheses, we have:

$A_{3n} = \gamma_{3n+3}(\beta_{3n+2} + 2a + p) + \gamma_{3n+4}(2a + \beta_{3n+4} + p) + (a - b)(a - c)(a + b + c + p);$

$A_{3n+1} = \gamma_{3n+4}(2\beta_{3n+4} + a + p) + (a - b)(a - c)(\beta_{3n+4} + b + c + p);$

$A_{3n+2} = \gamma_{3n+6}(2\beta_{3n+5} + a + p) + (a - b)(a - c)(\beta_{3n+5} + b + c + p);$

$K_{3n} = 0;$

$H_{3n} = 0;$

$V_{3n} = 0;$

$S_{3n} = 0;$

$K_{3n+1} = \gamma_{3n+3}\gamma_{3n+4}(\beta_{3n+2} + \beta_{3n+4} + a + p);$

$H_{3n+1} = \gamma_{3n+6} - \gamma_{3n+3};$

$V_{3n+1} = 0;$

$S_{3n+1} = 0;$

$K_{3n+2} = 0;$

$H_{3n+2} = 0;$

$V_{3n+2} = \gamma_{3n+5}(\gamma_{3n+4} - \gamma_{3n+7});$

$S_{3n+2} = \beta_{3n+5} - \beta_{3n+8}.$

Consequently, by theorem 3.2, we have also:

$P_{n+2}(x) = \Big(\Theta(x) - A_{3n}\Big)P_{n+1}(x) - \gamma_{3n+1}\gamma_{3n+2}\gamma_{3n+3}P_n(x), \ \ n \geq 0$, that is, $\{P_n\}_{n\geq 0}$ is orthogonal, fulfilling the following recurrence relation of second order:

$$\begin{aligned} &P_{n+2}(x) = \big(x - \beta^P_{n+1}\big)P_{n+1}(x) - \gamma^P_{n+1}P_n(x), \ \ n \geq 0, \\ &P_0(x) = 1, \\ &P_1(x) = x + aL + bc(b + c + p) - (a - b)(a - c)(\beta_2 + b + c + p), \end{aligned} \tag{3.36}$$

with $\beta^P_{n+1} = -(aL + bc(b + c + p)) + \gamma_{3n+3}(\beta_{3n+2} + 2a + p) + \gamma_{3n+4}(2a + \beta_{3n+4} + p) + (a - b)(a - c)(a + b + c + p)$

and $\gamma^P_{n+1} = \gamma_{3n+1}\gamma_{3n+2}\gamma_{3n+3}, \ \ n \geq 0$; and

$Q_{n+2}(x) = \Big(\Theta(x) - A_{3n+1}\Big)Q_{n+1}(x) - \gamma_{3n+2}\gamma_{3n+3}\gamma_{3n+4}Q_n(x)$

$-\gamma_{3n+3}\gamma_{3n+4}(\beta_{3n+2}+\beta_{3n+4}+a+p)c_n^2(x)-(\gamma_{3n+6}-\gamma_{3n+3})c_{n+1}^2(x),\ \ n\geq 0;$

$R_{n+2}(x)=\Big(\Theta(x)-A_{3n+2}\Big)R_{n+1}(x)-\gamma_{3n+3}\gamma_{3n+4}\gamma_{3n+5}R_n(x)$
$-\gamma_{3n+5}(\gamma_{3n+4}-\gamma_{3n+7})b_n^2(x)-(\beta_{3n+5}-\beta_{3n+8})b_{n+1}^2(x),\ \ n\geq 0;$
which is relation (a_9).

As in theorem 3.11 and theorem 3.12, the sequence $\{R_n\}_{n\geq 0}$ is orthogonal at theorem 3.13, and all components are expressed in terms of elements of $\{R_n\}_{n\geq 0}$. Sequence $\{b_n^2\}_{n\geq 0}$ is clearly determined. Next, we point out further information about the other three sequences $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{c_n^2\}_{n\geq 0}$.

With respect to the sequence $\{P_n\}_{n\geq 0}$, under the relations of theorem 3.13, we may conclude the following aspects.

1. The sequence $\{P_n\}_{n\geq 0}$ is orthogonal, either by corollary 3.4, since $b_n^1=c_n^1=0,\ \ n\geq 0$, or by theorem 3.2, as we seen above, establishing the recurrence relation (3.36).

2. The identity $P_n(x)=R_n(x)-\gamma_{3n}(\beta_{3n+1}-\beta_{3n-1})R_{n-1}(x)-\gamma_{3n-3}\gamma_{3n-1}\gamma_{3n}R_{n-2}(x),\ \ n\geq 2$, reflects a strictly finite-type relation between $\{R_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$, with respect to $\Phi(x)=1$ ($s=2$ and $t=0$), given by

$$P_n(x)=\sum_{\nu=n-2}^{n}\lambda_{n,\nu}R_\nu(x),\ \ n\geq 2,$$
$$\forall n\geq 2:\ \ \lambda_{n,n-2}\neq 0,$$

   where $\lambda_{n,n}=1$, $\lambda_{n,n-1}=-\gamma_{3n}(\beta_{3n+1}-\beta_{3n-1})$ and $\lambda_{n,n-2}=-\gamma_{3n-3}\gamma_{3n-1}\gamma_{3n}$.

   Since both sequences are orthogonal, by theorem 1.34, there exist a constant $k_0\neq 0$ and a monic polynomial Λ_2, with degree two, such that

$$r_0=k_0\Lambda_2u_0,$$

   where
$$k_0=\frac{\lambda_{2,0}}{<u_0,P_2^2>}=-\frac{\gamma_3\gamma_5\gamma_6}{<u_0,P_2^2>},$$
$$\Lambda_2(x)=\sum_{\nu=0}^{2}\frac{<u_0,P_2^2>}{\lambda_{2,0}}\frac{\lambda_{\nu,0}}{<u_0,P_\nu^2>}P_\nu(x)$$
$$=-\frac{<u_0,P_2^2>}{\gamma_3\gamma_5\gamma_6}+\frac{\gamma_3(\beta_4-\beta_2)<u_0,P_2^2>}{\gamma_3\gamma_5\gamma_6<u_0,P_1^2>}P_1(x)+P_2(x).$$

With respect to the sequence $\{Q_n\}_{n\geq 0}$, under the relations of theorem 3.13, we may conclude the following aspects.

1. If $\beta_{3n+1} + b + c + p = 0$, $n \geq 1$, then $Q_n = R_n$, $n \geq 0$, and the three principal components are orthogonal.

2. If there is $r \geq 1$ such that $\beta_{3r+1} + b + c + p \neq 0$, then

$$Q_n(x) = \sum_{\nu=n-1}^{n} \lambda_{n,\nu} R_\nu(x), \;\; n \geq 1,$$
$$\exists r \geq 1 : \;\; \lambda_{r,r-1} \neq 0,$$

   where $\lambda_{n,n} = 1$ and $\lambda_{n,n-1} = -\gamma_{3n}(\beta_{3n+1} + b + c + p)$; in other words, there is a finite-type relation between $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, with respect to $\Phi(x) = 1$ (where $s = 1$ and $t = 0$).

   Regarding theorem 1.32, this finite-type relation is equivalent to the existence of an application from $\mathbb{N}$ into $\mathbb{N} : m \mapsto \mu_m$ satisfying

$$m \leq \mu_m \leq m + 1, \;\;\; m \geq 0,$$
$$\exists m_0 \geq 0 : \mu_{m_0} = m_0 + 1,$$

   and also

$$r_m = \sum_{\nu=m}^{\mu_m} \lambda_{\nu,m} v_\nu, \;\; m \geq 0,$$

$$\lambda_{\mu_m,m} \neq 0, \;\; m \geq 0.$$

   Attending to $\{R_n\}_{n\geq 0}$ orthogonality, we may also write

$$R_m r_0 =< r_0, R_m^2 > \sum_{\nu=m}^{\mu_m} \lambda_{\nu,m} v_\nu, \;\; m \geq 0,$$

$$\lambda_{\mu_m,m} \neq 0, \;\; m \geq 0.$$

3. If $\beta_{3n+1} + b + c + p \neq 0$, $n \geq 1$, then we have a strictly finite-type relation between $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, therefore,

$$< r_0, R_m^2 >^{-1} R_m r_0 = v_m - \gamma_{3m+3}(\beta_{3m+4} + b + c + p) v_{m+1}, \;\; m \geq 0.$$

With respect to the sequence $\{c_n^2\}_{n\geq 0}$, under the relations of theorem 3.13, we may conclude the following aspects.

1. If $\beta_{3n+5}+b+c+p=0,\ \ n\geq 0$, then $c_{n+1}^2(x)=-\gamma_{3n+3}\gamma_{3n+5}R_n(x),\quad n\geq 0$, and $\deg(c_{n+1}^2(x))=n,\ \ n\geq 0$.

2. If there is $r\geq 1$ such that $\beta_{3r-1}+b+c+p\neq 0$, then

$$< r_0, c_m^2 c_n^2 >= 0,\ \ 0\leq m\leq n-2,\ \ n\geq 2,$$
$$\exists r\geq 1: < r_0, c_{r-1}^2 c_r^2 >\neq 0,$$

   which means that $\{c_n^2\}_{n\geq 0}$ is quasi-orthogonal of order $s=1$ with respect to r_0.

   The proof of this assertion is analogous to the proof of the same characteristic for $\{b_n^1\}_{n\geq 0}$ in theorem 3.11, or $\{c_n^1\}_{n\geq 0}$ in theorem 3.12.

   Proof. If $\exists k\geq 2:\ \ \beta_{3k-1}+b+c+p\neq 0$, then

$$\begin{aligned}
&< r_0, c_{k-1}^2 c_k^2 >\\
&=< r_0, \Big((\beta_{3k-1}+b+c+p)R_{k-1}(x)-\gamma_{3k-3}\gamma_{3k-1}R_{k-2}(x)\Big)\Big((\beta_{3k+2}+b+c+p)R_k(x)\\
&-\gamma_{3k}\gamma_{3k+2}R_{k-1}(x)\Big) >\\
&=-\gamma_{3k}\gamma_{3k+2}(\beta_{3k-1}+b+c+p) < r_0, R_{k-1}^2(x) >,\ \text{ by orthogonality conditions}\\
&\neq 0.
\end{aligned}$$

   If $\beta_2+b+c+p\neq 0$, then

$$\begin{aligned}
&< r_0, c_0^2 c_1^2 >\\
&=< r_0, (\beta_2+b+c+p)\Big((\beta_5+b+c+p)R_1(x)-\gamma_3\gamma_5 R_0(x)\Big) >\\
&=-\gamma_3\gamma_5(\beta_2+b+c+p) < r_0, R_0^2(x) >,\ \text{ by orthogonality conditions}\\
&\neq 0.
\end{aligned}$$

   If $1\leq m\leq n-2$ and $n\geq 2$, then

$$\begin{aligned}
&< r_0, c_m^2 c_n^2 >\\
&=< r_0, \Big((\beta_{3m+2}+b+c+p)R_m(x)-\gamma_{3m}\gamma_{3m+2}R_{m-1}(x)\Big)\Big((\beta_{3n+2}+b+c+p)R_n(x)\\
&-\gamma_{3n}\gamma_{3n+2}R_{n-1}(x)\Big) >\\
&=0,
\end{aligned}$$

   by orthogonality conditions and regarding the inequalities $m\leq n-2<n-1<n$.

If $0 = m \leq n-2$ and $n \geq 2$, then

$$\begin{aligned}
&< r_0, c_0^2 c_n^2 > \\
&=< r_0, (\beta_2 + b + c + p)\Big((\beta_{3n+2} + b + c + p)R_n(x) - \gamma_{3n}\gamma_{3n+2}R_{n-1}(x)\Big) > \\
&= 0, \text{ by orthogonality conditions.}
\end{aligned}$$

□

3. If $\beta_{3n-1} + b + c + p \neq 0, \ \forall n \geq 1$, then $\{c_n^2\}_{n\geq 0}$ is strictly quasi-orthogonal of order $s = 1$ with respect to r_0, and consequently, $\{c_n^2\}_{n\geq 0}$ is a free sequence.

As a consequence of theorem 3.2, we remarked above that if $K_n = H_n = V_n = S_n = 0, \ n \geq 0$, then the principal components are orthogonal. Nevertheless, the reciprocal is not true. Let us see an example of a MOPS in which CD, the principal components are orthogonal and the coefficients K_n, H_n, V_n, S_n, defined by (3.6),(3.7),(3.8) and (3.9), are not all trivial. Let us consider a MOPS $\{W_n\}_{n\geq 0}$ such that $a_n^2 = b_n^2 = c_n^1 = c_n^2 = 0, \ n \geq 0$, that is, it fulfils the theorem 3.11. Supposing $\beta_{3n+4} = a, \ n \geq 0$, then the principal components are all orthogonal and identical.

The recurrence coefficients of $\{W_n\}_{n\geq 0}$ must also fulfill the following:
$\beta_{3n+2} = -(b + c + p), \ \ n \geq 0,$
$\beta_0 + \beta_1 = b + c$, and $\beta_{3n+3} = b + c - a, \ \ n \geq 0,$
$\gamma_1 = \beta_0(b + c - \beta_0) - bc$, and $\gamma_{3n+4} = (b + c - a)a - bc, \ \ n \geq 0,$
$\gamma_{3n+2} + \gamma_{3n+3} = L, \ \ n \geq 0.$

Let us choose $a = 0, b = 1, c = 1, p = 1$ and $q = 1$. Then, the sequence $\{W_n\}_{n\geq 0}$ defined by the following recurrence coefficients:

$$\beta_{3n} = 2, \ \ \beta_{3n+1} = 0, \ \ \beta_{3n+2} = -3,$$

$$\gamma_{3n+1} = -1, \ \ \gamma_{3n+2} = n + 1, \ \ \gamma_{3n+3} = -n - 7, \ \ n \geq 0,$$

is a MOPS for which the principal components are orthogonal and $V_{3n+1} = -1, \ \ n \geq 0$, and the component sequences are the following:

$$a_n^1(x) = -(n + 7)R_n(x), \ \ n \geq 0,$$

$$b_n^1(x) = -2R_n(x) + (n + 6)R_{n-1}(x), \ \ n \geq 0,$$

$$R_0(x) = 1, \ \ R_{n+1}(x) = \Big(\Theta(x) + 2(n + 1)\Big)R_n(x) - (n + 1)(n + 6)R_{n-1}(x),$$

where $R_{-1}(x) = 0$.

In theorems 3.11, 3.12 and 3.13 we can see further examples. Under theorem 3.11 hypotheses, we found that $\{Q_n\}_{n\geq 0}$ is orthogonal and, nevertheless, $S_{3n+1} = \beta_{3n+6} - \beta_{3n+3}$

and $V_{3n+1} = \gamma_{3n+4}(\gamma_{3n+5} - \gamma_{3n+2})$. Also under theorem 3.13 hypotheses, we found that $\{R_n\}_{n\geq 0}$ is orthogonal and $S_{3n+2} = \beta_{3n+5} - \beta_{3n+8}$ and $V_{3n+2} = \gamma_{3n+5}(\gamma_{3n+4} - \gamma_{3n+7})$.

In this section, using the relations of theorem 3.1, we have excluded the CDs having five secondary components null and, also, twelve CDs having exactly four secondary components null, because each one contradicts regular orthogonality of the given sequence.

These two results can be continued by the examination of the CD of a MOPS such that exactly:

- three secondary components are null (20 cases, although one of these cases is already rejected as possible by theorem 3.9);
- two secondary components are null (15 cases);
- one secondary component is null (6 cases).

In the next section, we will study the case for which the two secondary components $\{a_n^1\}_{n\geq 0}$ and $\{a_n^2\}_{n\geq 0}$ vanish.

3.4 The study of an historical particular case

The following results deal with the case where $W_{3n}(x) = P_n(\varpi(x))$. Using the general CD presented at (2.1)-(2.3), the CD of the other two polynomials $W_{3n+1}(x)$ and $W_{3n+2}(x)$ is taken on account in all the steps of our analysis.

The next result describes the case where $W_{3n}(x) = P_n(\varpi(x))$, for any given MPS $\{W_n\}_{n\geq 0}$ (not necessarily orthogonal), in terms of the dual sequences of $\{W_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$.

Proposition 3.14. *Let $\{W_n\}_{n\geq 0}$ be a MPS defined by (2.1)-(2.3). The following assertions are equivalent.*

(a) $a_n^1 = a_n^2 = 0, \;\; n \geq 0.$

(b) $\sigma_\varpi(w_{3n+1}) = 0$*;* $\sigma_\varpi(w_{3n+2}) = 0$ *and* $u_n = \sigma_\varpi(w_{3n}), \;\; n \geq 0.$

Proof. (a) $\Rightarrow$ (b) As in theorem 2.7, we can prove that $< \sigma_\varpi(w_{3n+1}), P_m(x) >= 0$, $< \sigma_\varpi(w_{3n+2}), P_m(x) >= 0$ and $< \sigma_\varpi(w_{3n}), P_m(x) >= \delta_{n,m}$.

(b) $\Rightarrow$ (a) Since $\sigma_\varpi(w_{3n+1}) = 0$ and

$0 =< w_{3n+1}, W_{3m}(x) >$

$=< w_{3n+1}, P_m(\varpi(x)) + (x-a)a_{m-1}^1(\varpi(x)) + (x-b)(x-c)a_{m-1}^2(\varpi(x)) >$,

we have that $< w_{3n+1}, (x-a)a_{m-1}^1(\varpi(x)) + (x-b)(x-c)a_{m-1}^2(\varpi(x)) >= 0$.

In the same manner, $\sigma_\varpi(w_{3n+2}) = 0$ and

$0 =< w_{3n+2}, W_{3m}(x) >$

$=< w_{3n+2}, P_m(\varpi(x)) + (x-a)a_{m-1}^1(\varpi(x)) + (x-b)(x-c)a_{m-1}^2(\varpi(x)) >$

implies that $< w_{3n+2}, (x-a)a_{m-1}^1(\varpi(x)) + (x-b)(x-c)a_{m-1}^2(\varpi(x)) >= 0$.

Regarding that $u_n = \sigma_\varpi(w_{3n})$ and since $\delta_{n,m} =< w_{3n}, W_{3m}(x) >$ we have:

$\delta_{n,m} =< u_n, P_m(x) > + < w_{3n}, (x-a)a_{m-1}^1(\varpi(x)) + (x-b)(x-c)a_{m-1}^2(\varpi(x)) >$.

Thus, $< w_{3n}, (x-a)a_{m-1}^1(\varpi(x)) + (x-b)(x-c)a_{m-1}^2(\varpi(x)) >= 0$.

We conclude that

$< w_n, (x-a)a^1_{m-1}(\varpi(x)) + (x-b)(x-c)a^2_{m-1}(\varpi(x)) >= 0, \quad n, m \geq 0.$

Then, $(x-a)a^1_{m-1}(\varpi(x)) + (x-b)(x-c)a^2_{m-1}(\varpi(x)) = 0$ and, by the lemma 2.2, we obtain $a^1_{m-1} = a^2_{m-1} = 0$, $m \geq 0$. □

The study of the statements of theorem 3.1 when $a^1_n = a^2_n = 0, \quad n \geq 0$, brings us the next theorem.

Theorem 3.15. *Let $\{W_n\}_{n\geq 0}$ be a MPS defined by (2.1)-(2.3), such that $a^1_n = a^2_n = 0, \quad n \geq 0$. Then $\{W_n\}_{n\geq 0}$ is orthogonal if and only if the following relations are fulfilled, for $n \geq 0$, considering $R_{-1}(x) = 0$.*

(a_1) $b^1_n(x) = (\beta_{3n+1} + \beta_{3n+2} + a + p)R_n(x)$
$-\gamma_{3n}\Big(\gamma_{3n+2} - L + (\beta_{3n+1} + a - b - c)(\beta_{3n+1} + b + c + p)\Big)R_{n-1}(x),$

(a_2) $b^2_n(x) = -\gamma_{3n+3}R_n(x),$

(a_3) $c^1_n(x) = \Big(\gamma_{3n+2} - L + (\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p)\Big)R_n(x)$
$-\gamma_{3n}\gamma_{3n+2}(\beta_{3n+1} + \beta_{3n+2} + a + p)R_{n-1}(x),$

(a_4) $c^2_n(x) = (\beta_{3n+2} + b + c + p)R_n(x) - \gamma_{3n}\gamma_{3n+2}R_{n-1}(x),$

(a_5) $P_{n+1}(x) = R_{n+1}(x) - \gamma_{3n+3}(\beta_{3n+4} - \beta_{3n+2})R_n(x) - \gamma_{3n}\gamma_{3n+2}\gamma_{3n+3}R_{n-1}(x),$

(a_6) $Q_n(x) = R_n(x) - (\beta_{3n+1} + b + c + p)\gamma_{3n}R_{n-1}(x),$

(a_7) $R_{n+1}(x) = \Big\{\Theta(x) + \gamma_{3n+3}(\beta_{3n+4} - a) + \gamma_{3n+4}(\beta_{3n+2} - a) + \gamma_{3n+5}(\beta_{3n+3} - a)$
$-(\beta_{3n+3} - a)(\beta_{3n+4} - a)(\beta_{3n+5} - a) - (a-b)(a-c)(a+b+c+p)\Big)\Big\}R_n(x)$
$-\gamma_{3n}\gamma_{3n+2}\gamma_{3n+4}R_{n-1}(x),$

(a_8) $\beta_{3n} = \beta_{3n+3},$

(a_9) $\beta_{3n} + \beta_{3n+1} + \beta_{3n+2} + p = 0,$

(a_{10}) $\gamma_{3n+5} = \gamma_{3n+2} + (\beta_{3n+2} - \beta_{3n+4})(\beta_{3n+2} - \beta_{3n+5}),$

(a_{11}) $\gamma_{3n}(1 - \delta_{n,0}) + \gamma_{3n+1} + \gamma_{3n+2} = (\beta_{3n} - a)(\beta_{3n+1} - a)$
$-(\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p) + L - (a-b)(a-c),$

(a_{12}) $\gamma_{3n+2} + \gamma_{3n+3} + \gamma_{3n+4} = (\beta_{3n} - a)(\beta_{3n+2} - a)$
$-(\beta_{3n+1} + a - b - c)(\beta_{3n+1} + b + c + p) + L - (a-b)(a-c),$

(a_{13}) $\Big(\gamma_{3n+3} + \gamma_{3n+4} + (a-b)(a-c)\Big)(\beta_{3n+4} - \beta_{3n+2}) - \gamma_{3n+5}(\beta_{3n+3} - a) + \gamma_{3n+2}(\beta_{3n} - a)$
$-(\beta_{3n+2} - a)\Big(\gamma_{3n+2} - L + (\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p)\Big)$
$+(\beta_{3n+4} - a)\Big(\gamma_{3n+5} - L + (\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p)\Big) = 0.$

Proof. (⇒) Let us take on account that $a^1_n(x) = a^2_n(x) = 0$, $n \geq 0$, in the relations (A_0)-(A_9) of theorem 3.1. Let us consider $R_{-1}(x) = 0$ and replace, in (A_1)-(A_8), $b^2_n(x)$ and $b^2_{n-1}(x)$ by their expressions given by (A_9): $b^2_n(x) = -\gamma_{3n+3}R_n(x)$. We obtain the following:

$(\tilde{A}_1)$ $(\beta_{3n+4} - a)b^1_{n+1}(x) + c^1_{n+1}(x) = -\gamma_{3n+4}P_{n+1}(x) - (a-b)(a-c)Q_{n+1}(x)$
$-\gamma_{3n+3}\Theta(x)R_n(x),$

and from (A_1), with $n=0$, we get
$(\tilde{A}_1')$ $c_0^1(x) = -\gamma_1 - (a-b)(a-c) + (\beta_0 - a)(\beta_1 - a)$.
$(\tilde{A}_2)$ $b_{n+1}^1(x) - c_{n+1}^2(x) = (\beta_{3n+4} + a - b - c)Q_{n+1}(x) + L\gamma_{3n+3}R_n(x)$,
and from (A_2) with $n=0$, we get
$(\tilde{A}_2')$ $c_0^2(x) = b + c - (\beta_0 + \beta_1)$.
$(\tilde{A}_3)$ $Q_n(x) = R_n(x) - (\beta_{3n+1} + b + c + p)\gamma_{3n}R_{n-1}(x)$.
$(\tilde{A}_4)$ $\gamma_{3n+2}b_n^1(x) + (\beta_{3n+2} - a)c_n^1(x) + (a-b)(a-c)c_n^2(x) = -P_{n+1}(x) + \Theta(x)R_n(x)$.
$(\tilde{A}_5)$ $c_n^1(x) - (\beta_{3n+2} + a - b - c)c_n^2(x) = \gamma_{3n+2}Q_n(x) - LR_n(x)$.
$(\tilde{A}_6)$ $c_{n+1}^2(x) = (\beta_{3n+5} + b + c + p)R_{n+1}(x) - \gamma_{3n+3}\gamma_{3n+5}R_n(x)$,
and from (A_6) with $n=0$, we get
$(\tilde{A}_6')$ $c_0^2(x) = \beta_2 + b + c + p$.
$(\tilde{A}_7)$ $b_{n+1}^1(x) + \gamma_{3n+3}c_n^1(x) = -(\beta_{3n+3} - a)P_{n+1}(x)$.
$(\tilde{A}_8)$ $\gamma_{3n+3}c_n^2(x) = P_{n+1}(x) - Q_{n+1}(x)$.
$(\tilde{A}_9)$ $b_n^2(x) = -\gamma_{3n+3}R_n(x)$.
Regarding $\tilde{A}_2'$ and $\tilde{A}_6'$, we conclude that

$$\beta_0 + \beta_1 + \beta_2 + p = 0. \tag{3.37}$$

From the above list, we will be able to achieve the three following purposes:
1) to express each secondary component in terms of the principal component sequence $\{R_n(x)\}_{n\geq 0}$;
2) to express the other components sequences $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ in terms of the sequence $\{R_n(x)\}_{n\geq 0}$;
3) to obtain information on $\{\gamma_{n+1}\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ coefficients.

Notice that sequences $\{b_n^2(x)\}_{n\geq 0}$, $\{c_n^2(x)\}_{n\geq 0}$ and, also, $\{Q_n(x)\}_{n\geq 0}$ have been already written as intended in $(\tilde{A}_9)$, $(\tilde{A}_6)$ and $(\tilde{A}_3)$, respectively, constituting the identities (a_2), (a_4) (attending also to $\tilde{A}_6'$) and (a_6). Inserting $c_{n+1}^2(x)$ and $Q_{n+1}(x)$ expressions, given by (a_4) and (a_6), in $(\tilde{A}_2)$, we obtain:

$$\begin{aligned} b_{n+1}^1(x) &= (\beta_{3n+4} + \beta_{3n+5} + a + p)R_{n+1}(x) \\ &- \gamma_{3n+3}\Big(\gamma_{3n+5} - L + (\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p)\Big)R_n(x), \ \ n \geq 0. \end{aligned}$$

By (3.37), $b_0^1(x) = a - \beta_0 = \beta_1 + \beta_2 + a + p$; then, we may also write (a_1).
Proceeding similarly with $\tilde{A}_5$, we obtain:

$$\begin{aligned} c_n^1(x) &= (\beta_{3n+2} + a - b - c)\Big((\beta_{3n+2} + b + c + p)R_n(x) - \gamma_{3n}\gamma_{3n+2}R_{n-1}(x)\Big) \\ &+ \gamma_{3n+2}\Big(R_n(x) - (\beta_{3n+1} + b + c + p)\gamma_{3n}R_{n-1}(x)\Big) - LR_n(x), \text{ yielding } (a_3). \end{aligned}$$

Considering $(\tilde{A}_8)$, $P_{n+1}(x) = Q_{n+1}(x) + \gamma_{3n+3}c_n^2(x)$, and proceeding as above, we get (a_5). The purposes 1) and 2) are achieved, leaving $(\tilde{A}_1)$, $(\tilde{A}_4)$ and $(\tilde{A}_7)$ untouched. Let us now replace $b^1_{n+1}(x)$, $c^1_n(x)$ and $P_{n+1}(x)$ in $(\tilde{A}_7)$ by their expressions given by (a_1), (a_3) and (a_5). We get:

$$
\begin{aligned}
&(\beta_{3n+4}+\beta_{3n+5}+a+p)R_{n+1}(x)\\
&-\gamma_{3n+3}\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)R_n(x)\\
&+\gamma_{3n+3}\Big(\gamma_{3n+2}-L+(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)\Big)R_n(x)\\
&-\gamma_{3n}\gamma_{3n+2}\gamma_{3n+3}(\beta_{3n+1}+\beta_{3n+2}+a+p)R_{n-1}(x)\\
&=-(\beta_{3n+3}-a)\Big(R_{n+1}(x)-\gamma_{3n+3}(\beta_{3n+4}-\beta_{3n+2})R_n(x)\\
&-\gamma_{3n}\gamma_{3n+2}\gamma_{3n+3}R_{n-1}(x)\Big).
\end{aligned}
$$

That is,

$$
\begin{aligned}
&(\beta_{3n+3}+\beta_{3n+4}+\beta_{3n+5}+p)R_{n+1}(x)\\
&-\gamma_{3n+3}\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)R_n(x)\\
&+\gamma_{3n+3}\Big(\gamma_{3n+2}-L+(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)\Big)R_n(x)\\
&-\gamma_{3n+3}(\beta_{3n+3}-a)(\beta_{3n+4}-\beta_{3n+2})R_n(x)\\
&-\gamma_{3n}\gamma_{3n+2}\gamma_{3n+3}(\beta_{3n+1}+\beta_{3n+2}+\beta_{3n+3}+p)R_{n-1}(x)=0.
\end{aligned}
$$

Then, we obtain: $\beta_{3n+3}+\beta_{3n+4}+\beta_{3n+5}+p=0$ and by (3.37), we have (a_9) for $n\geq 0$; $\beta_{3n+1}+\beta_{3n+2}+\beta_{3n+3}+p=0$, $n\geq 1$, and

$$
\begin{aligned}
\gamma_{3n+5}-\gamma_{3n+2}+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\\
-(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)+(\beta_{3n+3}-a)(\beta_{3n+4}-\beta_{3n+2})=0.
\end{aligned}
\tag{3.38}
$$

According to the identity (3.34):

$$(M+a-b-c)(M+b+c+p)-(N+a-b-c)(N+b+c+p)=(M-N)(M+N+a+p),$$

which will be also used in further calculations, and by (a_9), we obtain, from (3.38), the relation (a_{10}). Notice that (a_9) and $\beta_{3n+1}+\beta_{3n+2}+\beta_{3n+3}+p=0$, $n\geq 1$, imply (a_8) for $n\geq 1$.

Let us now replace in $(\tilde{A}_1)$, $b^1_{n+1}(x)$, $c^1_{n+1}(x)$ and $Q_{n+1}(x)$ by their expressions given by (a_1), (a_3) and (a_6).

$$
\begin{aligned}
&(\beta_{3n+4}-a)\Big\{(\beta_{3n+4}+\beta_{3n+5}+a+p)R_{n+1}(x)\\
&-\gamma_{3n+3}\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)R_n(x)\Big\}\\
&+\Big(\gamma_{3n+5}-L+(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)\Big)R_{n+1}(x)\\
&-\gamma_{3n+3}\gamma_{3n+5}(\beta_{3n+4}+\beta_{3n+5}+a+p)R_n(x)+\gamma_{3n+4}P_{n+1}(x)\\
&+(a-b)(a-c)\Big\{R_{n+1}(x)-(\beta_{3n+4}+b+c+p)\gamma_{3n+3}R_n(x)\Big\}+\gamma_{3n+3}\Theta(x)R_n=0.
\end{aligned} \tag{3.39}
$$

Examining the leading coefficient of each member and attending to (a_9), we obtain (a_{11}), for $n \geq 1$. Let us remark that from (a_3), with $n=0$, we have: $c_0^1(x) = \gamma_2 - L + (\beta_2+a-b-c)(\beta_2+b+c+p)$, and, on the other hand, from $(\tilde{A}_1')$, $c_0^1(x) = -\gamma_1 - (a-b)(a-c) + (\beta_0-a)(\beta_1-a)$. We conclude that

$$
\gamma_1+\gamma_2 = (\beta_0-a)(\beta_1-a)-(\beta_2+a-b-c)(\beta_2+b+c+p)+L-(a-b)(a-c),
$$

which corresponds to identity (a_{11}) for $n=0$.

Using (a_{11}) and (a_5), the identity (3.39)

$$
\begin{aligned}
&\Big\{(\beta_{3n+4}-a)(\beta_{3n+4}+\beta_{3n+5}+a+p)+\gamma_{3n+5}\\
&+(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)-L+(a-b)(a-c)\Big\}R_{n+1}(x)+\gamma_{3n+4}P_{n+1}(x)\\
&+\gamma_{3n+3}\Theta(x)R_n(x)-\gamma_{3n+3}\Big\{(\beta_{3n+4}-a)\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)\\
&+\gamma_{3n+5}(\beta_{3n+4}+\beta_{3n+5}+a+p)+(a-b)(a-c)(\beta_{3n+4}+b+c+p)\Big\}R_n(x)=0
\end{aligned}
$$

can be rewritten as follows:

$$
\begin{aligned}
&-(\gamma_{3n+3}+\gamma_{3n+4})R_{n+1}(x)+\gamma_{3n+4}P_{n+1}(x)+\gamma_{3n+3}\Theta(x)R_n(x)\\
&-\gamma_{3n+3}\Big\{(\beta_{3n+4}-a)\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)\\
&+\gamma_{3n+5}(\beta_{3n+4}+\beta_{3n+5}+a+p)+(a-b)(a-c)(\beta_{3n+4}+b+c+p)\Big\}R_n(x)=0,
\end{aligned}
$$

or, by (a_5), we have:

$$
\begin{aligned}
&-\gamma_{3n+3}R_{n+1}(x)-\gamma_{3n+3}\gamma_{3n+4}\Big((\beta_{3n+4}-\beta_{3n+2})R_n(x)+\gamma_{3n}\gamma_{3n+2}R_{n-1}(x)\Big)+\gamma_{3n+3}\Theta(x)R_n(x)\\
&-\gamma_{3n+3}\Big\{(\beta_{3n+4}-a)\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)
\end{aligned}
$$

$+\gamma_{3n+5}(\beta_{3n+4}+\beta_{3n+5}+a+p)+(a-b)(a-c)(\beta_{3n+4}+b+c+p)\Big\}R_n(x)=0$, yielding to

$$R_{n+1}(x)=\Big\{\Theta(x)-\gamma_{3n+4}(\beta_{3n+4}-\beta_{3n+2})$$
$$-(\beta_{3n+4}-a)\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)$$
$$-\gamma_{3n+5}(\beta_{3n+4}+\beta_{3n+5}+a+p)-(a-b)(a-c)(\beta_{3n+4}+b+c+p)\Big\}R_n(x) \tag{3.40}$$
$$-\gamma_{3n}\gamma_{3n+2}\gamma_{3n+4}R_{n-1}(x).$$

Notice that the $R_n(x)$ coefficient in (3.40) can be transformed as we next indicate, justifying (a_7).

$$\Theta(x)-\gamma_{3n+4}(\beta_{3n+4}-\beta_{3n+2})-\gamma_{3n+5}(\beta_{3n+4}+\beta_{3n+5}+a+p)$$
$$-(\beta_{3n+4}-a)\Big(\gamma_{3n+5}+(a-b)(a-c)-L+(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)$$
$$+(\beta_{3n+4}-\beta_{3n+5})(\beta_{3n+4}+\beta_{3n+5}+a+p)\Big)-(a-b)(a-c)(a+b+c+p)$$
$$\overset{(a_9),(a_{11})}{=}\Theta(x)-\gamma_{3n+4}(\beta_{3n+4}-\beta_{3n+2})-\gamma_{3n+5}(\beta_{3n+4}+\beta_{3n+5}+a+p)$$
$$-(\beta_{3n+4}-a)\Big(-\gamma_{3n+3}-\gamma_{3n+4}+(\beta_{3n+3}-a)(\beta_{3n+5}-a)\Big)$$
$$-(a-b)(a-c)(a+b+c+p)$$
$$\overset{(a_9)}{=}\Theta(x)+\gamma_{3n+3}(\beta_{3n+4}-a)+\gamma_{3n+4}(\beta_{3n+2}-a)+\gamma_{3n+5}(\beta_{3n+3}-a)$$
$$-(\beta_{3n+3}-a)(\beta_{3n+4}-a)(\beta_{3n+5}-a)-(a-b)(a-c)(a+b+c+p).$$

Finally, we will eliminate the term $\Theta(x)R_n(x)$ of $(\tilde{A}_4)$ adding $(\tilde{A}_1)$ to $\gamma_{3n+3}(\tilde{A}_4)$. The resulting relation is the following.

$$(\beta_{3n+4}-a)b^1_{n+1}(x)+\gamma_{3n+2}\gamma_{3n+3}b^1_n(x)+c^1_{n+1}(x)+(\beta_{3n+2}-a)\gamma_{3n+3}c^1_n(x)$$
$$(\gamma_{3n+3}+\gamma_{3n+4})P_{n+1}(x)+(a-b)(a-c)\Big(Q_{n+1}(x)+\gamma_{3n+3}c^2_n(x)\Big)=0,\ \ n\geq 0.$$

Notice that by $(\tilde{A}_8)$, $Q_{n+1}(x)+\gamma_{3n+3}c^2_n(x)=P_{n+1}(x)$. Therefore, we can replace $b^1_{n+1}(x)$,

$b_n^1(x)$, $c_{n+1}^1(x)$, $c_n^1(x)$ and $P_{n+1}(x)$ by their expressions given by (a_1), (a_3) and (a_5), yielding:

$$\begin{aligned}
&(\beta_{3n+4}-a)\Big\{(\beta_{3n+4}+\beta_{3n+5}+a+p)R_{n+1}(x)\\
&-\gamma_{3n+3}\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)R_n(x)\Big\}\\
&+\gamma_{3n+2}\gamma_{3n+3}\Big\{(\beta_{3n+1}+\beta_{3n+2}+a+p)R_n(x)\\
&-\gamma_{3n}\Big(\gamma_{3n+2}-L+(\beta_{3n+1}+a-b-c)(\beta_{3n+1}+b+c+p)\Big)R_{n-1}(x)\Big\}\\
&+\Big(\gamma_{3n+5}-L+(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)\Big)R_{n+1}(x)\\
&-\gamma_{3n+3}\gamma_{3n+5}(\beta_{3n+4}+\beta_{3n+5}+a+p)R_n(x)\\
&+(\beta_{3n+2}-a)\gamma_{3n+3}\Big(\gamma_{3n+2}-L+(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)\Big)R_n(x)\\
&-(\beta_{3n+2}-a)\gamma_{3n}\gamma_{3n+2}\gamma_{3n+3}(\beta_{3n+1}+\beta_{3n+2}+a+p)R_{n-1}(x)\\
&+(\gamma_{3n+3}+\gamma_{3n+4}+(a-b)(a-c))\Big\{R_{n+1}(x)-\gamma_{3n+3}(\beta_{3n+4}-\beta_{3n+2})R_n(x)\\
&-\gamma_{3n}\gamma_{3n+2}\gamma_{3n+3}R_{n-1}(x)\Big\}=0.\ \text{That is,}
\end{aligned}$$

$$\begin{aligned}
&\Big\{\gamma_{3n+3}+\gamma_{3n+4}+\gamma_{3n+5}-L+(a-b)(a-c)\\
&+(\beta_{3n+4}-a)(\beta_{3n+4}+\beta_{3n+5}+a+p)+(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)\Big\}R_{n+1}(x)\\
&-\gamma_{3n+3}\Big\{(\beta_{3n+4}-a)\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)\\
&-\gamma_{3n+2}(\beta_{3n+1}+\beta_{3n+2}+a+p)+\gamma_{3n+5}(\beta_{3n+4}+\beta_{3n+5}+a+p)\\
&-(\beta_{3n+2}-a)\Big(\gamma_{3n+2}-L+(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)\Big)\\
&+(\beta_{3n+4}-\beta_{3n+2})\Big(\gamma_{3n+3}+\gamma_{3n+4}+(a-b)(a-c)\Big)\Big\}R_n(x)\\
&-\gamma_{3n}\gamma_{3n+2}\gamma_{3n+3}\Big\{\gamma_{3n+2}-L+(\beta_{3n+1}+a-b-c)(\beta_{3n+1}+b+c+p)\\
&+(\beta_{3n+2}-a)(\beta_{3n+1}+\beta_{3n+2}+a+p)+\gamma_{3n+3}+\gamma_{3n+4}+(a-b)(a-c)\Big\}R_{n-1}(x)=0.
\end{aligned}$$

By (a_{11}) and using (a_9), the coefficient of $R_{n+1}(x)$ is trivial, hence we get the two following relations, where we apply again (a_9):

$$\begin{aligned}&\Big(\gamma_{3n+3}+\gamma_{3n+4}+(a-b)(a-c)\Big)(\beta_{3n+4}-\beta_{3n+2})\\&-\gamma_{3n+5}(\beta_{3n+3}-a)+\gamma_{3n+2}(\beta_{3n}-a)\\&-(\beta_{3n+2}-a)\Big(\gamma_{3n+2}-L+(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)\Big)\\&+(\beta_{3n+4}-a)\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)=0,\ \ n\geq 0,\end{aligned} \tag{3.41}$$

$$\begin{aligned}&\gamma_{3n+2}+\gamma_{3n+3}+\gamma_{3n+4}=(\beta_{3n+2}-a)(\beta_{3n}-a)\\&-(\beta_{3n+1}+a-b-c)(\beta_{3n+1}+b+c+p)+L-(a-b)(a-c),\ \ n\geq 1.\end{aligned} \tag{3.42}$$

Identity (3.41) corresponds to (a_{13}), and identity (3.42) is (a_{12}) for $n \geq 1$.

We finalize by proving (a_8) and (a_{12}) for $n = 0$.

In order to prove that $\beta_3 = \beta_0$, we insert $\gamma_{3n+3}+\gamma_{3n+4}+(a-b)(a-c)$ expression given by (a_{11}), with $n \leftarrow n+1$, in the relation (a_{13}). The resulting relation is the following:

$$\begin{aligned}&\Big(-\gamma_{3n+5}+(\beta_{3n+4}-a)(\beta_{3n+3}-a)-(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)+L\Big)(\beta_{3n+4}-\beta_{3n+2})\\&-\gamma_{3n+5}(\beta_{3n+3}-a)+\gamma_{3n+2}(\beta_{3n}-a)\\&-(\beta_{3n+2}-a)\Big(\gamma_{3n+2}-L+(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)\Big)\\&+(\beta_{3n+4}-a)\Big(\gamma_{3n+5}-L+(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)=0\end{aligned}$$

$$\begin{aligned}&\Rightarrow -\gamma_{3n+5}(\beta_{3n+3}-\beta_{3n+2})-\gamma_{3n+2}(\beta_{3n+2}-\beta_{3n})\\&+(\beta_{3n+4}-\beta_{3n+2})\Big((\beta_{3n+4}-a)(\beta_{3n+3}-a)-(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)\Big)\\&-(\beta_{3n+2}-a)(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)\\&+(\beta_{3n+4}-a)(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)=0\end{aligned}$$

$$\begin{aligned}&\Rightarrow \gamma_{3n+5}(\beta_{3n+3}-\beta_{3n+2})+\gamma_{3n+2}(\beta_{3n+2}-\beta_{3n})\\&-(\beta_{3n+4}-a)(\beta_{3n+4}-a)(\beta_{3n+3}-a)+(\beta_{3n+2}-a)(\beta_{3n+4}-a)(\beta_{3n+3}-a)\\&+(\beta_{3n+4}-a)(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)\\&-(\beta_{3n+2}-a)(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)\\&-(\beta_{3n+4}-a)(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\\&+(\beta_{3n+2}-a)(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)=0\end{aligned}$$

$$\begin{aligned}&\Rightarrow \gamma_{3n+5}(\beta_{3n+3}-\beta_{3n+2})+\gamma_{3n+2}(\beta_{3n+2}-\beta_{3n})\\&+(\beta_{3n+2}-a)\Big((\beta_{3n+4}-a)(\beta_{3n+3}-a)-(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)\\&+(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)\Big)\\&+(\beta_{3n+4}-a)\Big(-(\beta_{3n+4}-a)(\beta_{3n+3}-a)+(\beta_{3n+5}+a-b-c)(\beta_{3n+5}+b+c+p)\\&-(\beta_{3n+4}+a-b-c)(\beta_{3n+4}+b+c+p)\Big)=0\end{aligned}$$

$$\Rightarrow \gamma_{3n+5}(\beta_{3n+3}-\beta_{3n+2})+\gamma_{3n+2}(\beta_{3n+2}-\beta_{3n})$$
$$+(\beta_{3n+2}-a)\Big((\beta_{3n+4}-a)(\beta_{3n+3}-a)+(\beta_{3n+2}-\beta_{3n+5})(\beta_{3n+2}+\beta_{3n+5}+a+p)\Big)$$
$$+(\beta_{3n+4}-a)\Big(-(\beta_{3n+4}-a)(\beta_{3n+3}-a)+(\beta_{3n+5}-\beta_{3n+4})(\beta_{3n+4}+\beta_{3n+5}+a+p)\Big)=0$$

$$\Rightarrow \gamma_{3n+5}(\beta_{3n+3}-\beta_{3n+2})+\gamma_{3n+2}(\beta_{3n+2}-\beta_{3n})$$
$$+(\beta_{3n+3}-a)(\beta_{3n+4}-a)(\beta_{3n+2}-\beta_{3n+4})+(\beta_{3n+2}-a)(\beta_{3n+2}-\beta_{3n+5})(\beta_{3n+2}+\beta_{3n+5}+a+p)$$
$$+(\beta_{3n+4}-a)(\beta_{3n+5}-\beta_{3n+4})(a-\beta_{3n+3})=0$$

$$\Rightarrow \gamma_{3n+5}(\beta_{3n+3}-\beta_{3n+2})+\gamma_{3n+2}(\beta_{3n+2}-\beta_{3n})$$
$$+(\beta_{3n+3}-a)(\beta_{3n+4}-a)(\beta_{3n+2}-\beta_{3n+4}-\beta_{3n+5}+\beta_{3n+4})$$
$$+(\beta_{3n+2}-a)(\beta_{3n+2}-\beta_{3n+5})(\beta_{3n+2}+\beta_{3n+5}+a+p)=0$$

$$\Rightarrow \gamma_{3n+5}(\beta_{3n+3}-\beta_{3n+2})+\gamma_{3n+2}(\beta_{3n+2}-\beta_{3n})$$
$$+(\beta_{3n+2}-\beta_{3n+5})\Big((\beta_{3n+3}-a)(\beta_{3n+4}-a)+(\beta_{3n+2}-a)(\beta_{3n+2}+\beta_{3n+5}+a+p)\Big)=0,\ \ n\geq 0.$$

Replacing γ_{3n+5} by the expression given by (a_{10}) we obtain a trivial identity, except for the case $n=0$, which we present next. Replacing γ_5 by $\gamma_2+(\beta_2-\beta_4)(\beta_2-\beta_5)$, we get:

$$\gamma_2(\beta_3-\beta_2)+(\beta_3-\beta_2)(\beta_2-\beta_4)(\beta_2-\beta_5)+\gamma_2(\beta_2-\beta_0)$$
$$+(\beta_2-\beta_5)\Big((\beta_3-a)(\beta_4-a)+(\beta_2-a)(\beta_2+\beta_5+a+p)\Big)=0$$

$$\Leftrightarrow \gamma_2(\beta_3-\beta_0)+(\beta_2-\beta_5)\Big((\beta_3-\beta_2)(\beta_2-\beta_4)$$
$$+(\beta_3-a)(\beta_4-a)+(\beta_2-a)(\beta_2+\beta_5+a+p)\Big)=0.$$

Let us simplify the last term of the first member of the above identity:
$(\beta_2-\beta_5)\Big((\beta_3-\beta_2)(\beta_2-a)-(\beta_3-\beta_2)(\beta_4-a)$
$+(\beta_3-a)(\beta_4-a)+(\beta_2-a)(\beta_2+\beta_5+a+p)\Big)$
$=(\beta_2-\beta_5)\Big((\beta_2-a)(\beta_3-\beta_2+\beta_2+\beta_5+a+p)+(\beta_4-a)(\beta_3-a-\beta_3+\beta_2)\Big)$
$=(\beta_2-\beta_5)(\beta_2-a)(\beta_3+\beta_4+\beta_5+p)=0$. Then, $\gamma_2(\beta_3-\beta_0)=0$ and $\beta_3=\beta_0$.

Let us now deal with (a_{12}) for $n=0$. By (a_{11}), with $n=1$ and (a_{10}) with $n=0$, we can write the following identity:

$\gamma_2+\gamma_3+\gamma_4+(\beta_2-\beta_4)(\beta_2-\beta_5)=(\beta_4-a)(\beta_3-a)$
$\quad -(\beta_5+a-b-c)(\beta_5+b+c+p)+L-(a-b)(a-c)$,

that is, $\gamma_2+\gamma_3+\gamma_4=(\beta_0-a)(\beta_2-a)-(\beta_1+a-b-c)(\beta_1+b+c+p)+L-(a-b)(a-c)$
$-(\beta_2-\beta_4)(\beta_2-\beta_5)-(\beta_0-a)(\beta_2-a)+(\beta_1+a-b-c)(\beta_1+b+c+p)$
$+(\beta_4-a)(\beta_3-a)-(\beta_5+a-b-c)(\beta_5+b+c+p)$.

Using the relations (a_8) and (a_9), the last five terms are simplified as follows:
$-(\beta_2-a)(\beta_2-\beta_5)+(\beta_4-a)(\beta_2-\beta_5)-(\beta_0-a)(\beta_2-a)+(\beta_4-a)(\beta_3-a)$
$+(\beta_1-\beta_5)(\beta_1+\beta_5+a+p)$

$$= -(\beta_2 - a)(\beta_0 + \beta_2 - \beta_5 - a) + (\beta_4 - a)(\beta_2 + \beta_3 - \beta_5 - a) + (\beta_1 - \beta_5)(\beta_1 + \beta_5 + a + p)$$
$$\overset{(a_8),(a_9)}{=} -(\beta_2 - a)(-p - \beta_1 - \beta_5 - a) + (\beta_4 - a)(-p - \beta_1 - \beta_5 - a) + (\beta_1 - \beta_5)(\beta_1 + \beta_5 + a + p)$$
$$\overset{(a_8),(a_9)}{=} (\beta_1 + \beta_5 + a + p)(\beta_1 + \beta_2 - \beta_4 - \beta_5) \overset{(a_9)}{=} (\beta_1 + \beta_5 + a + p)(-p - \beta_0 + p + \beta_3) \overset{(a_8)}{=} 0.$$

($\Leftarrow$) Reciprocally, let us suppose the enunciated relations. It is easy to see that they guarantee the relations of theorem 3.1, when we take $a_n^1 = a_n^2 = 0,\ n \geq 0$. $\square$

The next result allow us to write in a simpler way the relations $(a_{10}) - (a_{13})$ of theorem 3.15. Note that these relations, along with (a_8) and (a_9), characterize the recurrence coefficients of the orthogonal sequence $\{W_n\}_{n\geq 0}$, which admits a CD such that $a_n^1 = a_n^2 = 0,\ \ n \geq 0$.

Theorem 3.16. *If a MPS $\{W_n\}_{n\geq 0}$ fulfils the relations of theorem 3.15, then, for $n \geq 0$, we have:*

- $y_n + (\beta_{3n} - a)(\beta_{3n+1} - a) = \mu$;
- $\gamma_{3n}(1 - \delta_{n,0}) + \gamma_{3n+1} = \lambda$,
  in particular, $\lambda = \gamma_1$ *and* $\gamma_1 = -\big(\mu + (a-b)(a-c) + (\beta_0 - a)(\beta_0 + p + 2a)\big)$;
- $\mu + \lambda = -(\beta_{3n} - a)(\beta_{3n} + p + 2a) - (a-b)(a-c)$;
- $(\beta_{3n} - a)\Big(\gamma_{3n+2} - (\beta_{3n+1} - a)(\beta_{3n+2} - a)\Big) = \theta$;
  where $y_n = \gamma_{3n+2} - L + (\beta_{3n+1} + a - b - c)(\beta_{3n+1} + b + c + p)$ *and* $\mu,\ \lambda,\ \theta \in \mathbb{C}$.

Proof. The relation (a_{10}) can be written as follows.

$$\begin{aligned}&\gamma_{3n+5} - L + (\beta_{3n+3} - a)(\beta_{3n+4} - \beta_{3n+2})\\ &= \gamma_{3n+2} - L + (\beta_{3n+2} - \beta_{3n+4})(\beta_{3n+2} + \beta_{3n+4} + a + p);\end{aligned}$$

which is equivalent to the following, using identity (3.34),

$$\begin{aligned}&\gamma_{3n+5} - L + (\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p) + (\beta_{3n+3} - a)(\beta_{3n+4} - \beta_{3n+2})\\ &= \gamma_{3n+2} - L + (\beta_{3n+1} + a - b - c)(\beta_{3n+1} + b + c + p)\\ &+ (\beta_{3n+2} - \beta_{3n+1})(\beta_{3n+1} + \beta_{3n+2} + a + p).\end{aligned}$$

Using (a_9) of theorem 3.15, we have:

$$y_{n+1} + (\beta_{3n+3} - a)(\beta_{3n+4} - \beta_{3n+2}) = y_n - (\beta_{3n} - a)(\beta_{3n+2} - \beta_{3n+1})$$

or, by (a_8), $y_{n+1} + \beta_{3n+4}(\beta_{3n+3} - a) = y_n + \beta_{3n+1}(\beta_{3n} - a)$. Consequently, $y_n + \beta_{3n+1}(\beta_{3n} - a)$ is constant and by (a_8) we can say that $y_n + (\beta_{3n} - a)(\beta_{3n+1} - a) = \mu$, for some $\mu \in \mathbb{C}$ and $n \geq 0$.

Also the relation (a_{11}) can be rewritten as follows

$$\gamma_{3n}(1-\delta_{n,0})+\gamma_{3n+1}+\gamma_{3n+2}=(\beta_{3n}-a)(\beta_{3n+1}-a)$$
$$-(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)+L-(a-b)(a-c)$$

$$\Rightarrow \gamma_{3n+2}-L+(\beta_{3n+1}+a-b-c)(\beta_{3n+1}+b+c+p)+(\beta_{3n+2}-\beta_{3n+1})(\beta_{3n+1}+\beta_{3n+2}+a+p)$$
$$+(a-b)(a-c)-(\beta_{3n}-a)(\beta_{3n+1}-a)=-\big(\gamma_{3n}(1-\delta_{n,0})+\gamma_{3n+1}\big)$$

$$\Rightarrow y_n-(\beta_{3n+2}-\beta_{3n+1})(\beta_{3n}-a)$$
$$+(a-b)(a-c)-(\beta_{3n}-a)(\beta_{3n+1}-a)=-\big(\gamma_{3n}(1-\delta_{n,0})+\gamma_{3n+1}\big),\ \text{ by } (a_9),$$

$$\Rightarrow y_n-(\beta_{3n}-a)(\beta_{3n+2}-a)+(a-b)(a-c)=-\big(\gamma_{3n}(1-\delta_{n,0})+\gamma_{3n+1}\big)$$

$$\Rightarrow y_n+(\beta_{3n}-a)(\beta_{3n+1}-a)+(\beta_{3n}-a)(2a-\beta_{3n+1}-\beta_{3n+2})$$
$$+(a-b)(a-c)=-\big(\gamma_{3n}(1-\delta_{n,0})+\gamma_{3n+1}\big),\text{ that is,}$$

$$\mu+(a-b)(a-c)+(\beta_{3n}-a)(\beta_{3n}+p+2a)=-\big(\gamma_{3n}(1-\delta_{n,0})+\gamma_{3n+1}\big),\text{ by } (a_9).$$

Therefore, $\gamma_{3n}(1-\delta_{n,0})+\gamma_{3n+1}=\lambda,\ \ n\geq 0$, for some $\lambda\in\mathbb{C}$. In fact, $\lambda=\gamma_1$ and $\gamma_1=-\big(\mu+(a-b)(a-c)+(\beta_0-a)(\beta_0+p+2a)\big)$.

Let us proceed with identity (a_{12}). Introducing y_n, we obtain:

$$\gamma_{3n+2}-L+(\beta_{3n+1}+a-b-c)(\beta_{3n+1}+b+c+p)=(\beta_{3n+2}-a)(\beta_{3n}-a)$$
$$-(a-b)(a-c)-(\gamma_{3n+3}+\gamma_{3n+4})$$

$$\Rightarrow y_n-(\beta_{3n+2}-a)(\beta_{3n}-a)=-(a-b)(a-c)-\lambda$$

$$\Rightarrow y_n+(\beta_{3n}-a)(\beta_{3n+1}-a)=(\beta_{3n}-a)(\beta_{3n+1}+\beta_{3n+2}-2a)-(a-b)(a-c)-\lambda$$

$$\text{that is, } \mu+\lambda=(\beta_{3n}-a)(\beta_{3n+1}+\beta_{3n+2}-2a)-(a-b)(a-c); \tag{3.43}$$

or, by (a_9), we get the relation $\mu+\lambda=-(\beta_{3n}-a)(\beta_{3n}+p+2a)-(a-b)(a-c),\ n\geq 0$.

Proceeding similarly with identity (a_{13}), we obtain:

$$\Big(\lambda+(a-b)(a-c)\Big)(\beta_{3n+4}-\beta_{3n+2})-\gamma_{3n+5}(\beta_{3n+3}-a)+\gamma_{3n+2}(\beta_{3n}-a)$$
$$-(\beta_{3n+2}-a)\Big(\gamma_{3n+2}-L+(\beta_{3n+1}+a-b-c)(\beta_{3n+1}+b+c+p)\Big)$$
$$-(\beta_{3n+2}-a)(\beta_{3n+2}-\beta_{3n+1})(\beta_{3n+1}+\beta_{3n+2}+a+p)+(\beta_{3n+4}-a)y_{n+1}=0$$

$$\Rightarrow \gamma_{3n+2}(\beta_{3n}-a)-(\beta_{3n+2}-a)y_n-\Big(\lambda+(a-b)(a-c)\Big)(\beta_{3n+2}-a)$$
$$+(\beta_{3n+2}-a)(\beta_{3n+2}-\beta_{3n+1})(\beta_{3n}-a)$$
$$=\gamma_{3n+5}(\beta_{3n+3}-a)-(\beta_{3n+4}-a)y_{n+1}-\Big(\lambda+(a-b)(a-c)\Big)(\beta_{3n+4}-a)$$

$$\Rightarrow \gamma_{3n+5}(\beta_{3n+3}-a)-(\beta_{3n+4}-a)y_{n+1}-\Big(\lambda+(a-b)(a-c)\Big)(\beta_{3n+4}-a)$$
$$=\gamma_{3n+2}(\beta_{3n}-a)-(\beta_{3n+1}-a)y_n-\Big(\lambda+(a-b)(a-c)\Big)(\beta_{3n+1}-a)$$
$$-(\beta_{3n+2}-\beta_{3n+1})\Big(y_n+\lambda+(a-b)(a-c)\Big)+(\beta_{3n+2}-a)(\beta_{3n+2}-\beta_{3n+1})(\beta_{3n}-a)$$

$$\Rightarrow \gamma_{3n+5}(\beta_{3n+3}-a)-(\beta_{3n+4}-a)y_{n+1}-\Big(\lambda+(a-b)(a-c)\Big)(\beta_{3n+4}-a)$$
$$=\gamma_{3n+2}(\beta_{3n}-a)-(\beta_{3n+1}-a)y_n-\Big(\lambda+(a-b)(a-c)\Big)(\beta_{3n+1}-a)$$
$$-(\beta_{3n+2}-\beta_{3n+1})\Big(y_n+\lambda+(a-b)(a-c)-(\beta_{3n+2}-a)(\beta_{3n}-a)\Big),\ \ n\geq 0.$$

According to (3.43), the last term of the right member of the above identity becomes:

$$(\beta_{3n+2}-\beta_{3n+1})\Big(y_n+\lambda+(a-b)(a-c)-(\beta_{3n+2}-a)(\beta_{3n}-a)\Big)$$
$$=(\beta_{3n+2}-\beta_{3n+1})\Big(y_n-\mu+(\beta_{3n}-a)(\beta_{3n+1}-a)+(\beta_{3n}-a)(\beta_{3n+2}-a)$$
$$-(\beta_{3n+2}-a)(\beta_{3n}-a)\Big)$$
$$=(\beta_{3n+2}-\beta_{3n+1})\Big(y_n-\mu+(\beta_{3n}-a)(\beta_{3n+1}-a)\Big)=0,\ \ n\geq 0.$$

Hence, $\gamma_{3n+2}(\beta_{3n}-a)-(\beta_{3n+1}-a)\Big(y_n+\lambda+(a-b)(a-c)\Big)=\theta$, for some $\theta\in\mathbb{C}$. Notice that

$$\begin{aligned} y_n+\lambda+(a-b)(a-c) &= y_n-\mu-(\beta_{3n}-a)(\beta_{3n}+p+2a)\\ &= -(\beta_{3n}-a)(\beta_{3n+1}-a)-(\beta_{3n}-a)(\beta_{3n}+p+2a)\\ &= -(\beta_{3n}-a)(\beta_{3n}+\beta_{3n+1}+a+p),\ \ n\geq 0.\end{aligned}$$

Then,

$$(\beta_{3n}-a)\Big(\gamma_{3n+2}+(\beta_{3n+1}-a)(\beta_{3n}+\beta_{3n+1}+a+p)\Big)=\theta$$

or $(\beta_{3n}-a)\Big(\gamma_{3n+2}-(\beta_{3n+1}-a)(\beta_{3n+2}-a)\Big)=\theta,\ \ n\geq 0.$ □

The following result summarizes the equivalent descriptions obtained for the particular CD of a MOPS where $a_n^1=a_n^2=0,\ \ n\geq 0.$

Theorem 3.17. *Let $\{W_n\}_{n\geq 0}$ be a MOPS defined by (2.1)-(2.3). The following assertions are equivalent, considering $R_{-1}(x)=R_{-2}(x)=0$.*

(a) $a_n^1=a_n^2=0,\ \ n\geq 0.$

(b)
$$\begin{aligned} W_{3n}(x) &= R_n(\varpi(x))-\gamma_{3n}(\beta_{3n+1}-\beta_{3n-1})R_{n-1}(\varpi(x))\\ &\quad -\gamma_{3n-3}\gamma_{3n-1}\gamma_{3n}R_{n-2}(\varpi(x)),\ \ n\geq 0.\\ W_{3n+1}(x) &= (\beta_{3n+1}+\beta_{3n+2}+a+p)R_n(\varpi(x))\\ &\quad -\gamma_{3n}\Big(\gamma_{3n+2}-L+(\beta_{3n+1}+a-b-c)(\beta_{3n+1}+b+c+p)\Big)R_{n-1}(\varpi(x))\\ &\quad +(x-a)\Big(R_n(\varpi(x))-(\beta_{3n+1}+b+c+p)\gamma_{3n}R_{n-1}(\varpi(x))\Big)\\ &\quad -(x-b)(x-c)\gamma_{3n}R_{n-1}(\varpi(x)),\ \ n\geq 0.\\ W_{3n+2}(x) &= \Big(\gamma_{3n+2}-L+(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)\Big)R_n(\varpi(x))\end{aligned}$$

$$-\gamma_{3n}\gamma_{3n+2}(\beta_{3n+1}+\beta_{3n+2}+a+p)R_{n-1}(\varpi(x))$$
$$+(x-a)\Big((\beta_{3n+2}+b+c+p)R_n(\varpi(x))-\gamma_{3n}\gamma_{3n+2}R_{n-1}(\varpi(x))\Big)$$
$$+(x-b)(x-c)R_n(\varpi(x)),\quad n\geq 0,$$

where $\{R_n\}_{n\geq 0}$ *is the MOPS with recurrence relation given by* (a_7) *of theorem 3.15.*

(c) *The recurrence coefficients of* $\{W_n\}_{n\geq 0}$ *satisfy the relations* (a_8) *-* (a_{12}) *of theorem 3.15.*

(d) *The recurrence coefficients of* $\{W_n\}_{n\geq 0}$ *satisfy the following relations, for some constant* τ*.*

- $\beta_{3n}=\tau$,
- $\beta_{3n+1}+\beta_{3n+2}=-(p+\tau)$,
- $\gamma_{3n+2}=-\gamma_1-(\tau-a)(\beta_{3n+1}+\tau+a+p)$
  $-(\beta_{3n+1}+a-b-c)(\beta_{3n+1}+b+c+p)+L-(a-b)(a-c)$,
- $\gamma_{3n}(1-\delta_{n,0})+\gamma_{3n+1}=\gamma_1,\quad n\geq 0.$

Proof. By theorem 3.15 (a) implies (b). To prove that (b) implies (a), we calculate a_n^1 and a_n^2 using relations (A_5) and (A_6) of theorem 3.1, substituting every component sequence by its expression given by (b). We then find that $a_n^1=a_n^2=0,\quad n\geq 0$.

The theorem 3.15 justifies that (a) implies (c).

We will prove, by induction on n, that, if the recurrence coefficients of $\{W_n\}_{n\geq 0}$ satisfy the relations (a_8) -(a_{12}) of theorem 3.15, then $a_n^1=a_n^2=0,\quad n\geq 0$.

Considering (A_0) and identities (A_1) and (A_2), with $n=0$, we obtain:

$$c_0^1(x)=-(\beta_1-a)(a-\beta_0)-\gamma_1-(a-b)(a-c);\quad c_0^2(x)=b+c-\beta_0-\beta_1.$$

Taking (A_5), with $n=0$, we have:

$$a_0^1(x)=(\beta_0-a)(\beta_1-a)-\gamma_1-(\beta_2+a-b-c)(b+c-\beta_0-\beta_1)+L-(a-b)(a-c)-\gamma_2.$$

Using relations (a_9) and (a_{11}), we conclude that $a_0^1(x)=0$.

Taking (A_6), with $n=0$, we have:

$$a_0^2(x)=b+c-\beta_0-\beta_1-(\beta_2+b+c+p)\overset{(a_9)}{=}0.$$

Let us suppose that $a_k^1(x)=a_k^2(x)=0,\quad k=0,\ldots,n$. Notice that $a_{n+1}^1(x)$ and $a_{n+1}^2(x)$ are given by (A_5) and (A_6), with $n\leftarrow n+1$:

$$a_{n+1}^1(x)=c_{n+1}^1(x)-(\beta_{3n+5}+a-b-c)c_{n+1}^2(x)-\gamma_{3n+5}Q_{n+1}(x)+LR_{n+1}(x),$$

$$a^2_{n+1}(x) = -\gamma_{3n+5} b^2_n(x) + c^2_{n+1}(x) - (\beta_{3n+5} + b + c + p) R_{n+1}(x).$$

In order to prove that $a^1_{n+1}(x)$ and $a^2_{n+1}(x)$ are null, we will use the relations (a_8) - (a_{12}) and the induction hypotheses, to write $b^2_n(x)$, $Q_{n+1}(x)$, $c^2_{n+1}(x)$ and $c^1_{n+1}(x)$ as linear combinations of elements of the sequence $\{R_n\}_{n\geq 0}$.

Firstly, by (A_9), we know that

$$b^2_k(x) = -\gamma_{3k+3} R_k(x), \quad k = 0, \ldots, n, \tag{3.44}$$

and consequently, by (A_3), we have that:

$$Q_{k+1}(x) = -(\beta_{3k+4} + b + c + p)\gamma_{3k+3} R_k(x) + R_{k+1}(x), \quad k = 0, \ldots, n. \tag{3.45}$$

To continue this process with $c^2_{n+1}(x)$, we will first write $b^1_{n+1}(x)$ in the same form.

Using the relation (A_6) and identity (3.44) we have:

$$c^2_k(x) = -\gamma_{3k+2}\gamma_{3k} R_{k-1}(x) + (\beta_{3k+2} + b + c + p) R_k(x), \quad k = 0, \ldots, n; \tag{3.46}$$

and using the relation (A_5) and the identity (3.46) we have:

$$\begin{aligned} c^1_k(x) &= -\gamma_{3k}\gamma_{3k+2}(\beta_{3k+1} + \beta_{3k+2} + a + p) R_{k-1}(x) \\ &\quad + \Big(\gamma_{3k+2} + (\beta_{3k+2} + a - b - c)(\beta_{3k+2} + b + c + p) - L\Big) R_k(x), \quad k = 0, \ldots, n, \end{aligned} \tag{3.47}$$

considering $R_{-1}(x) = 0$.

Also by (A_8) and inserting (3.46) and (3.45), we get the following:

$$P_{k+1}(x) = -\gamma_{3k}\gamma_{3k+2}\gamma_{3k+3} R_{k-1}(x) - \gamma_{3k+3}(\beta_{3k+4} - \beta_{3k+2}) R_k(x) + R_{k+1}(x), \quad k = 0, \ldots, n; \tag{3.48}$$

and from (A_7), we obtain:

$$\begin{aligned} b^1_{k+1}(x) &= \gamma_{3k}\gamma_{3k+2}\gamma_{3k+3}(\beta_{3k+1} + \beta_{3k+2} + \beta_{3k+3} + p) R_{k-1}(x) \\ &\quad - \gamma_{3k+3}\Big(\gamma_{3k+2} + (\beta_{3k+2} + a - b - c)(\beta_{3k+2} + b + c + p) - L \\ &\quad - (\beta_{3k+3} - a)(\beta_{3k+4} + b + c + p) + (\beta_{3k+3} - a)(\beta_{3k+2} + b + c + p)\Big) R_k(x) \\ &\quad - (\beta_{3k+3} - a) R_{k+1}(x), \quad k = 0, \ldots, n. \end{aligned}$$

Notice that, by (a_8) and (a_9), the coefficient of $R_{k-1}(x)$ vanishes, and thus

$$\begin{aligned} b^1_{k+1}(x) &= -\gamma_{3k+3}\Big(\gamma_{3k+2} + (\beta_{3k+2} + a - b - c)(\beta_{3k+2} + b + c + p) - L \\ &\quad - (\beta_{3k+3} - a)(\beta_{3k+4} - \beta_{3k+2})\Big) R_k(x) - (\beta_{3k+3} - a) R_{k+1}(x), \quad k = 0, \ldots, n. \end{aligned} \tag{3.49}$$

The relation (A_2), with $n \leftarrow n+1$, permits to determine $c^2_{n+1}(x)$:

$$c^2_{n+1}(x) = b^1_{n+1}(x) + L b^2_n(x) - (\beta_{3n+4} + a - b - c) Q_{n+1}(x).$$

Introducing the identities (3.49), (3.44) and (3.45), we obtain:

$$\begin{aligned}c^2_{n+1}(x) = &-\gamma_{3n+3}\Big(\gamma_{3n+2} + (\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p)\\ &- (\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p) - (\beta_{3n+3} - a)(\beta_{3n+4} - \beta_{3n+2})\Big)R_n(x)\\ &- (\beta_{3n+3} + \beta_{3n+4} - b - c)R_{n+1}(x).\end{aligned}$$

Applying the relation (3.34) and simplifying, we get:

$$\begin{aligned}c^2_{n+1}(x) = &-\gamma_{3n+3}\Big(\gamma_{3n+2} + (\beta_{3n+2} - \beta_{3n+4})(\beta_{3n+2} + \beta_{3n+3} + \beta_{3n+4} + p)\Big)R_n(x)\\ &- (\beta_{3n+3} + \beta_{3n+4} - b - c)R_{n+1}(x);\end{aligned}$$

and by (a_9) and (a_{10}), we have:

$$c^2_{n+1}(x) = -\gamma_{3n+3}\gamma_{3n+5}R_n(x) - (\beta_{3n+3} + \beta_{3n+4} - b - c)R_{n+1}(x). \tag{3.50}$$

Let us now determine a similar expression for $c^1_{n+1}(x)$. Let us consider (A_1), with $n \leftarrow n+1$, where $\Theta(x)R_n(x)$ is replaced by its expression given by (A_4):

$$\begin{aligned}c^1_{n+1}(x) = &-(\beta_{3n+4} - a)b^1_{n+1}(x) - \gamma_{3n+2}\gamma_{3n+3}b^1_n(x) - \gamma_{3n+3}(\beta_{3n+2} - a)c^1_n(x)\\ &- \gamma_{3n+3}(a-b)(a-c)c^2_n(x) - (\gamma_{3n+3} + \gamma_{3n+4})P_{n+1}(x) - (a-b)(a-c)Q_{n+1}(x).\end{aligned}$$

Introducing the identities (3.49), (3.47), (3.46), (3.48) and (3.45), we obtain:

$$\begin{aligned}c^1_{n+1}(x) = \ &\gamma_{3n}\gamma_{3n+2}\gamma_{3n+3}\Big\{\gamma_{3n-1} + (\beta_{3n-1} + a - b - c)(\beta_{3n-1} + b + c + p)\\ &- (\beta_{3n} - a)(\beta_{3n+1} - \beta_{3n-1}) + (\beta_{3n+2} - a)(\beta_{3n+1} + \beta_{3n+2} + a + p)\\ &+ (a-b)(a-c) - L + \gamma_{3n+3} + \gamma_{3n+4}\Big\}R_{n-1}(x)\\ &+ \gamma_{3n+3}\Big\{(\beta_{3n+4} - a)\Big(\gamma_{3n+2} + (\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p) - L\\ &- (\beta_{3n+3} - a)(\beta_{3n+4} - \beta_{3n+2})\Big)\\ &+ \gamma_{3n+2}(\beta_{3n} - a) - (\beta_{3n+2} - a)\Big(\gamma_{3n+2} + (\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p) - L\Big)\\ &- (a-b)(a-c)(\beta_{3n+2} + b + c + p) + (\gamma_{3n+3} + \gamma_{3n+4})(\beta_{3n+4} - \beta_{3n+2})\\ &+ (a-b)(a-c)(\beta_{3n+4} + b + c + p)\Big\}R_n(x)\\ &+ \Big((\beta_{3n+3} - a)(\beta_{3n+4} - a) - \gamma_{3n+3} - \gamma_{3n+4} - (a-b)(a-c)\Big)R_{n+1}(x).\end{aligned}$$

In order to simplify the coefficient of $R_{n-1}(x)$, let us consider the following calculations.

$$\begin{aligned}
&\gamma_{3n-1} + (\beta_{3n-1} + a - b - c)(\beta_{3n-1} + b + c + p) - (\beta_{3n} - a)(\beta_{3n+1} - \beta_{3n-1})\\
&+ (\beta_{3n+2} - a)(\beta_{3n+1} + \beta_{3n+2} + a + p) + (a - b)(a - c) - L + \gamma_{3n+3} + \gamma_{3n+4}\\
&\overset{(a_9)}{=} \gamma_{3n-1} + (\beta_{3n-1} + a - b - c)(\beta_{3n-1} + b + c + p) - (\beta_{3n} - a)(\beta_{3n+1} - \beta_{3n-1})\\
&+ (\beta_{3n+2} - a)(a - \beta_{3n}) + (a - b)(a - c) - L + \gamma_{3n+3} + \gamma_{3n+4}\\
&\overset{(a_{12})}{=} \gamma_{3n-1} + (\beta_{3n-1} + a - b - c)(\beta_{3n-1} + b + c + p) - (\beta_{3n} - a)(\beta_{3n+1} - \beta_{3n-1})\\
&- (\beta_{3n+1} + a - b - c)(\beta_{3n+1} + b + c + p) - \gamma_{3n+2}\\
&\overset{(3.34)}{=} \gamma_{3n-1} - \gamma_{3n+2} + (\beta_{3n-1} - \beta_{3n+1})(\beta_{3n-1} + \beta_{3n+1} + a + p)\\
&+ (\beta_{3n-1} - \beta_{3n+1})(\beta_{3n} - a)\\
&= \gamma_{3n-1} - \gamma_{3n+2} + (\beta_{3n-1} - \beta_{3n+1})(\beta_{3n-1} + \beta_{3n} + \beta_{3n+1} + p)\\
&\overset{(a_9)}{=} \gamma_{3n-1} - \gamma_{3n+2} + (\beta_{3n-1} - \beta_{3n+1})(\beta_{3n-1} - \beta_{3n+2}) \overset{(a_{10})}{=} 0, \quad n \geq 1.
\end{aligned}$$

In order to simplify the coefficient of $R_n(x)$, let us attend to the following calculations.

$$\begin{aligned}
&\gamma_{3n+2}(\beta_{3n+4} - a) - \gamma_{3n+2}(\beta_{3n+2} - a) - L(\beta_{3n+4} - \beta_{3n+2}) + (a - b)(a - c)(\beta_{3n+4} - \beta_{3n+2})\\
&+ (\gamma_{3n+3} + \gamma_{3n+4})(\beta_{3n+4} - \beta_{3n+2}) + \gamma_{3n+2}(\beta_{3n} - a)\\
&+ (\beta_{3n+4} - a)\Big((\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p) - (\beta_{3n+3} - a)(\beta_{3n+4} - \beta_{3n+2})\Big)\\
&- (\beta_{3n+2} - a)(\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p)\\
&= (\beta_{3n+4} - \beta_{3n+2})\Big(\gamma_{3n+2} + \gamma_{3n+3} + \gamma_{3n+4} - L + (a - b)(a - c)\\
&+ (\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p) - (\beta_{3n+4} - a)(\beta_{3n+3} - a)\Big) + \gamma_{3n+2}(\beta_{3n} - a)\\
&\overset{(a_8),(a_{12})}{=} (\beta_{3n+4} - \beta_{3n+2})\Big((\beta_{3n} - a)(\beta_{3n+2} - a) - (\beta_{3n+1} + a - b - c)(\beta_{3n+1} + b + c + p)\\
&+ (\beta_{3n+2} + a - b - c)(\beta_{3n+2} + b + c + p) - (\beta_{3n+4} - a)(\beta_{3n} - a)\Big) + \gamma_{3n+2}(\beta_{3n} - a)\\
&\overset{(3.34)}{=} (\beta_{3n+4} - \beta_{3n+2})\Big((\beta_{3n} - a)(\beta_{3n+2} - \beta_{3n+4})\\
&+ (\beta_{3n+2} - \beta_{3n+1})(\beta_{3n+1} + \beta_{3n+2} + a + p)\Big) + \gamma_{3n+2}(\beta_{3n} - a)\\
&\overset{(a_9)}{=} (\beta_{3n+4} - \beta_{3n+2})\Big((\beta_{3n} - a)(\beta_{3n+2} - \beta_{3n+4})\\
&+ (\beta_{3n+2} - \beta_{3n+1})(a - \beta_{3n})\Big) + \gamma_{3n+2}(\beta_{3n} - a)
\end{aligned}$$

$$= (\beta_{3n+4} - \beta_{3n+2})(\beta_{3n} - a)(\beta_{3n+1} - \beta_{3n+4}) + \gamma_{3n+2}(\beta_{3n} - a)$$
$$= (\beta_{3n} - a)\Big(\gamma_{3n+2} - (\beta_{3n+4} - \beta_{3n+2})(\beta_{3n+4} - \beta_{3n+1})\Big)$$
$$\overset{(a_8),(a_9)}{=} (\beta_{3n} - a)\Big(\gamma_{3n+2} + (\beta_{3n+2} - \beta_{3n+4})(\beta_{3n+4} + \beta_{3n+3} + \beta_{3n+2} + p)$$
$$\overset{(a_9)}{=} (\beta_{3n} - a)\Big(\gamma_{3n+2} + (\beta_{3n+2} - \beta_{3n+4})(\beta_{3n+2} - \beta_{3n+5})$$
$$\overset{(a_{10})}{=} \gamma_{3n+5}(\beta_{3n} - a), \quad n \geq 0.$$

Consequently, we have:

$$\begin{aligned} c^1_{n+1}(x) &= \gamma_{3n+3}\gamma_{3n+5}(\beta_{3n} - a)R_n(x) \\ &- \Big(\gamma_{3n+3} + \gamma_{3n+4} - (\beta_{3n+3} - a)(\beta_{3n+4} - a) + (a - b)(a - c)\Big)R_{n+1}(x). \end{aligned} \tag{3.51}$$

Let us consider the necessary conditions to calculate $a^1_{n+1}(x)$ and $a^2_{n+1}(x)$, which are given by (A_5) and (A_6), as we focused earlier.

Introducing, in (A_5), the identities (3.51), (3.50) and (3.45), and applying identity (a_9), we obtain:

$$\begin{aligned} a^1_{n+1}(x) &= \gamma_{3n+3}\gamma_{3n+5}(\beta_{3n+3} + \beta_{3n+4} + \beta_{3n+5} + p)R_n(x) \\ &- \Big(\gamma_{3n+3} + \gamma_{3n+4} + \gamma_{3n+5} + (a - b)(a - c) - L - (\beta_{3n+3} - a)(\beta_{3n+4} - a) \\ &+ (\beta_{3n+5} + a - b - c)(\beta_{3n+5} + +b + c + p)\Big)R_{n+1}(x) \overset{(a_9),(a_{11})}{=} 0. \end{aligned}$$

Introducing, in (A_6), the identities (3.44) and (3.50), we obtain immediately $a^2_{n+1}(x) = 0$.

Finally, let us suppose the relations (a_8) - (a_{12}) of theorem 3.15. Relation (a_8) implies that $\beta_{3n} = \tau,\ n \geq 0$, for some constant τ.

Hence, by (a_9), we have $\beta_{3n+1} + \beta_{3n+2} = -(p+\tau)$, and theorem 3.16 allows us to rewrite (a_{10}) - (a_{12}) as follows:

$\gamma_{3n+2} - L + (\beta_{3n+1} + a - b - c)(\beta_{3n+1} + b + c + p) + (\tau - a)(\beta_{3n+1} - a) = \mu$,

$\gamma_{3n}(1 - \delta_{n,0}) + \gamma_{3n+1} = \lambda$,

$\mu + \lambda = -(\tau - a)(\tau + p + 2a) - (a - b)(a - c)$, where $\lambda = \gamma_1$.

Then, we obtain

$$\begin{aligned} \gamma_{3n+2} &= -\gamma_1 - (\tau - a)(\beta_{3n+1} + \tau + a + p) \\ &\quad -(\beta_{3n+1} + a - b - c)(\beta_{3n+1} + b + c + p) + L - (a - b)(a - c), \end{aligned}$$

$\gamma_{3n}(1 - \delta_{n,0}) + \gamma_{3n+1} = \gamma_1, \quad n \geq 0.$

Conversely, if we suppose the four relations of item (d), and also using relation (3.34), the relations (a_8) - (a_{12}) of theorem 3.15 are satisfied, which concludes the proof. □

Next, we will analyze the coefficients A_n, B_n, K_n, H_n, V_n and S_n of the theorem 3.2, when the CD of $\{W_n\}_{n\geq 0}$ satisfies $a^1_n = a^2_n = 0, \quad n \geq 0$.

Identities (a_8), (a_9), (a_{11}) and (a_{12}) guarantee that $K_{3n} = H_{3n} = V_{3n} = S_{3n} = 0$, yielding the orthogonality of the sequence $\{P_n\}_{n\geq 0}$, which satisfies the following recurrence relation of second order

$$P_{n+2}(x) = \left(x - \beta^P_{n+1}\right) P_{n+1}(x) - \gamma^P_{n+1} P_n(x), \tag{3.52}$$

where $\beta^P_{n+1} = -(aL + bc(b+c+p)) + A_{3n}$ and $\gamma^P_{n+1} = B_{3n}$.

The sequence $\{Q_n\}_{n\geq 0}$ is not necessarily orthogonal. Let us express $\{Q_n\}_{n\geq 0}$ structure relation. In identity (3.2), the two last terms are null ($a^1_n(x) = 0$ and also $V_{3n+1} = S_{3n+1} = 0$, $n \geq 0$). By (a_9) and (a_{11}), we obtain $H_{3n+1} = \gamma_{3n+6} - \gamma_{3n+3}$, yielding

$$\begin{aligned} &Q_{n+2}(x) = \left\{\Theta(x) - A_{3n+1}\right\} Q_{n+1}(x) - B_{3n+1} Q_n(x) \\ &- \gamma_{3n+3}\gamma_{3n+4}(\beta_{3n+2} + \beta_{3n+3} + \beta_{3n+4} + p) c^2_n(x) - (\gamma_{3n+6} - \gamma_{3n+3}) c^2_{n+1}(x). \end{aligned} \tag{3.53}$$

Remark 3.18. *Let us note that if $\beta_{3n+2} + \beta_{3n+3} + \beta_{3n+4} + p = 0$, which implies $\beta_{3n+4} = \beta_{3n+1}$, and $\gamma_{3n+6} = \gamma_{3n+3}$, $n \geq 0$, then the sequence $\{Q_n\}_{n\geq 0}$ is orthogonal.*

Using (a_4), we can replace $c^2_n(x)$ and $c^2_{n+1}(x)$ by linear combinations of $R_{n+1}(x)$, $R_n(x)$ and $R_{n-1}(x)$. Also, the identity (a_6) allows us to write $R_{n+1}(x)$ in terms of $\{Q_n\}_{n\geq 0}$ elements. By induction, we can prove that

$$R_{n+1}(x) = Q_{n+1}(x) + \sum_{i=0}^{n} c_{i,n} Q_i,$$

where $c_{i,n} = \prod_{k=i+1}^{n+1} \gamma_{3k}(\beta_{3k+1} + b + c + p)$, $n \geq 0$. Introducing this identity in (3.53), we obtain:

$$\begin{aligned} &Q_{n+2}(x) = \left\{\Theta(x) - A_{3n+1} - (\gamma_{3n+6} - \gamma_{3n+3})(\beta_{3n+5} + b + c + p)\right\} Q_{n+1}(x) \\ &+ \gamma_{3n+3}\Bigg\{ (\gamma_{3n+6} - \gamma_{3n+3}) z_{n+1} \\ &- \gamma_{3n+4}\Big(\gamma_{3n+2} + (\beta_{3n+2} + \beta_{3n+3} + \beta_{3n+4} + p)(\beta_{3n+2} + b + c + p)\Big)\Bigg\} Q_n(x) \\ &+ \gamma_{3n}\gamma_{3n+3}\Bigg\{ (\beta_{3n+1} + b + c + p)(\gamma_{3n+6} - \gamma_{3n+3}) z_{n+1} \\ &+ \gamma_{3n+4}(\beta_{3n+2} + \beta_{3n+3} + \beta_{3n+4} + p) z_n \Bigg\} \left(Q_{n-1}(x) + \sum_{i=0}^{n-2} c_{i,n-2} Q_i \right), \end{aligned}$$

where $z_n = \gamma_{3n+2} - (\beta_{3n+1} + b + c + p)(\beta_{3n+2} + b + c + p)$.

A similar approach to the identity (3.3) brings us again the identity (a_7).

Remark 3.19. *The results presented in [1] and [23], for the particular cases taken in these references, concerning the recurrence coefficients of the two orthogonal sequences*

$\{W_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$, are generalized by the relations of item (d) of theorem 3.17 and by the expressions of β^P_{n+1} and γ^P_{n+1}, written in terms of the recurrence coefficients of $\{W_n\}_{n\geq 0}$ and the parameters of the CD.

In [24], in the problem P1 mentioned in the introduction, the recurrence coefficients of the orthogonal sequence $\{W_n\}_{n\geq 0}$ are written as follows: β_{3n} is constant, β_{3n+1}, β_{3n+2}, γ_{3n+1} and γ_{3n+2} are expressed in terms of the sequence $\{P_n\}_{n\geq 0}$ and the monic kernel polynomials of K-parameter c associated with $\{P_n\}_{n\geq 0}$ [8], and γ_{3n} is written in terms of these two last sequences and also the recurrence coefficients of $\{P_n\}_{n\geq 0}$, γ^P_n. We can see, in one proof of that reference, the relations indicated in item (d) of theorem 3.17, as the expressions of the coefficients β^P_{n+1} and γ^P_{n+1}, when $a = b = c = 0$. Also in [24], the authors present relations for the forms w_0 and u_0, which correspond to the ones indicated in item (b) of proposition 3.14, when $n = 0$ and $\{W_n\}_{n\geq 0}$ is orthogonal. Let us remark that the content of proposition 3.14, which characterizes the dual sequence $\{w_n\}_{n\geq 0}$ and relates the form w_{3n} with the form u_n, $n \geq 0$, does not depend of the orthogonality of $\{W_n\}_{n\geq 0}$.

In the two contributions [24] and [25], as in [1] and [23], the polynomials $W_{3n+1}(x)$ and $W_{3n+2}(x)$ are written as rational fractions involving elements of the sequence $\{P_n\}_{n\geq 0}$, that is, they do not participate in the cubic decomposition, making impossible further comparisons with the results, here presented, concerning these two subsequences of $\{W_n\}_{n\geq 0}$ and the correspondent six component sequences.

As in the theorems 3.11, 3.12 and 3.13, the theorem 3.15 states that the sequence $\{R_n\}_{n\geq 0}$ is orthogonal and all the components of $\{W_n\}_{n\geq 0}$ CD are expressed in terms of elements of the sequence $\{R_n\}_{n\geq 0}$. Moreover, in the CD of an orthogonal sequence $\{W_n\}_{n\geq 0}$ such that $a^1_n = a^2_n = 0$, $n \geq 0$, the principal component $\{P_n\}_{n\geq 0}$ is also orthogonal. By proposition 3.14, we have

$$u_0 = \sigma_{\varpi}(w_0).$$

The next result analyses the relations that we can establish between the elements of the dual sequences of the principal sequences, more precisely between u_0, $\{v_n\}_{n\geq 0}$ and r_0.

Theorem 3.20. *Given a MOPS $\{W_n\}_{n\geq 0}$ defined by (2.1)-(2.3), such that $a^1_n = a^2_n = 0$, $n \geq 0$, we have:*

- *If $\beta_{3n+1} + b + c + p = 0$, $n \geq 1$, then $v_0 = r_0$.*
- *If there is $\kappa \geq 1$ such that $\beta_{3\kappa+1} + b + c + p \neq 0$, then,*

$$R_m r_0 = < r_0, R^2_m > \sum_{\nu=m}^{\mu_m} \lambda_{\nu,m} v_\nu, \;\; m \geq 0,$$

where $m \leq \mu_m \leq m+1$ and $\lambda_{\mu_m,m} \neq 0$, $m \geq 0$.

In particular, if $\beta_{3n+1} + b + c + p \neq 0$, $n \geq 1$, then

$$< r_0, R^2_m >^{-1} R_m r_0 = v_m - (\beta_{3m+4} + b + c + p)\gamma_{3m+3} v_{m+1}, \;\; m \geq 0.$$

- $r_0 = k_0 \Lambda_2 u_0$, *where* $k_0 = -\frac{\gamma_3\gamma_5\gamma_6}{<u_0, P_2^2>}$ *and*
$$\Lambda_2(x) = -\frac{<u_0, P_2^2>}{\gamma_3\gamma_5\gamma_6} + \gamma_4(\beta_4 - \beta_2)P_1(x) + P_2(x).$$

Proof. The relations (a_5) and (a_6) of theorem 3.15 and the relations (a_5) and (a_6) of theorem 3.13 are the same, respectively. Also, in both cases, the sequence $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are orthogonal. Then, the relations enunciated can be obtained by the same arguments presented in the discussion of theorem 3.13. □

Remark 3.21. *The theorem 3.15 generalizes the situation described in theorem 3.13, thus, the coincidence of some results was expected.*

3.4.1 A cubic decomposition of a symmetric orthogonal sequence

The next result is a particular scenery of theorem 3.15. We specify all components for the symmetric orthogonal case, yielding a generalization of the main result of [1]. We will see that the parameter p (coefficient of x^2 in $\varpi(x)$) is necessarily null.

Corollary 3.22. *Let* $\{W_n\}_{n\geq 0}$ *be a symmetric MPS defined by (2.1)-(2.3), such that* $a_n^1 = a_n^2 = 0, \quad n \geq 0$. *Then* $\{W_n\}_{n\geq 0}$ *is orthogonal if and only if the following relations are satisfied, for* $n \geq 0$ *and considering* $R_{-1}(x) = 0$.

(b_1) $b_n^1(x) = aR_n(x) - \big(\gamma_{3n+2} - L + (a-b-c)(b+c)\big)\gamma_{3n}R_{n-1}(x)$,
(b_2) $b_n^2(x) = -\gamma_{3n+3}R_n(x)$,
(b_3) $c_n^1(x) = \big(\gamma_{3n+2} - L + (a-b-c)(b+c)\big)R_n(x) - a\gamma_{3n}\gamma_{3n+2}R_{n-1}(x)$,
(b_4) $c_n^2(x) = (b+c)R_n(x) - \gamma_{3n}\gamma_{3n+2}R_{n-1}(x)$,
(b_5) $P_{n+1}(x) = R_{n+1}(x) - \gamma_{3n}\gamma_{3n+2}\gamma_{3n+3}R_{n-1}(x)$,
(b_6) $Q_n(x) = R_n(x) - (b+c)\gamma_{3n}R_{n-1}(x)$,
(b_7) $R_{n+1}(x) = xR_n(x) - \gamma_{3n}\gamma_{3n+2}\gamma_{3n+4}R_{n-1}(x)$,
(b_8) $p = 0$,
(b_9) $\gamma_{3n+2} = -q - \gamma_1$,
(b_{10}) $\gamma_{3n}(1 - \delta_{n,0}) + \gamma_{3n+1} = \gamma_1$.

Proof. Let us take $\beta_n = 0$, $n \geq 0$, in theorem 3.15. Inserting the constant (2.5) in the relations (a_9), (a_{10}), (a_{11}), we get (b_8), (b_9) and (b_{10}). Using these identities, the relations (a_1)-(a_7) and inserting the polynomial (2.4), we find the relations (b_1)-(b_7). □

As we remarked earlier, the sequence $\{P_n\}_{n\geq 0}$ is orthogonal, and its recurrence relation is the following: $P_0(x) = 1$, $P_1(x) = x$,

$$P_{n+2}(x) = xP_{n+1}(x) - \gamma_{3n+1}\gamma_{3n+2}\gamma_{3n+3}P_n(x), \quad n \geq 0.$$

Remark 3.23. *Since we took* $r = 0$ *in this chapter, the sequences* $\{P_n\}_{n\geq 0}$ *and* $\{R_n\}_{n\geq 0}$ *are also symmetric. Otherwise, we would have* $\beta_n^P = r$ *and* $\beta_n^R = r$, $n \geq 0$.

Example 3.24. *Let us consider the symmetric MOPS* $\{W_n\}_{n\geq 0}$, *defined by the coefficients:*

$$\gamma_{3n+2} = \gamma_2,\ \gamma_{3n+3} = \gamma_3,\ \gamma_{3n+4} = \gamma_4,\ n \geq 0,\ \textit{where}\ \gamma_2 = -\gamma_1 - q\ \textit{and}\ \gamma_3 + \gamma_4 = \gamma_1,$$

for a chosen constant q. Since $p = 0$, from example 3.8, we already know that all the principal component sequences of the correspondent CD are orthogonal. In particular, taking also $b + c = 0$ and $a = 0$, we obtain, for $n \geq 0$:

$$\begin{aligned}
W_{3n}(x) &= P_n(\varpi(x)) \\
W_{3n+1}(x) &= (\gamma_1 - c^2)\gamma_3 R_{n-1}(\varpi(x)) + xR_n(\varpi(x)) - (x^2 - c^2)\gamma_3 R_{n-1}(\varpi(x)) \\
W_{3n+2}(x) &= (c^2 - \gamma_1)R_n(\varpi(x)) - x\gamma_2\gamma_3 R_{n-1}(\varpi(x)) + \left(x^2 - c^2\right) R_n(\varpi(x)),
\end{aligned}$$

where $\varpi(x) = x^3 + qx$, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - \gamma_1\gamma_2\gamma_3$,
$P_{n+3}(x) = xP_{n+2}(x) - \gamma_2\gamma_3\gamma_4 P_{n+1}(x),\ \ n \geq 0$, *and*
$R_{-1}(x) = 0, R_0(x) = 1, R_{n+1}(x) = xR_n(x) - \gamma_2\gamma_3\gamma_4 R_{n-1}(x),\ n \geq 0.$

A cubic decomposition of the Tchebyshev polynomials

The theorem 3.22 characterizes the orthogonal and symmetric sequence such that its CD fulfils: $a_n^1 = a_n^2 = 0,\ \ n \geq 0$. In fact, the recurrence coefficients satisfy:
$\gamma_{3n+2} = -q - \gamma_1$,
$\gamma_{3n}(1 - \delta_{n,0}) + \gamma_{3n+1} = \gamma_1$.

Among the classical polynomials, only the Hermite and the Gegenbauer families are symmetric [8, 32]. Searching what symmetric and classical MOPS have a CD such that $a_n^1 = a_n^2 = 0,\ n \geq 0$, we obtain proposition 3.25.

Proposition 3.25. *The only classical symmetric MOPS admitting a CD such that $a_n^1 = a_n^2 = 0,\ n \geq 0$ is the first kind Tchebyshev MOPS. In this case, the principal components are MOPS and the secondary components fulfil the recurrence relation of second order*

$$B_{n+2}(x) = xB_{n+1}(x) - \frac{1}{4^3}B_n(x),\ n \geq 0.$$

Moreover, $P_n(x) = 4^{-n}T_n(4x)$, $Q_n(x) = \widehat{U}_n\big(\frac{1}{4}(b + c)\big)(x)$, $R_n(x) = \widehat{U}_n(x),\ \ n \geq 0$, where $\widehat{U}_n(x) = 4^{-n}U_n(4x),\ \ n \geq 0$, and $\{T_n\}_{n\geq 0}$ and $\{U_n\}_{n\geq 0}$ represent the first and the second kind Tchebyshev polynomials, respectively. We have also necessarily $\varpi(x) = x^3 - \frac{3}{4}x$.

Proof. From the theorem 3.22, we know that such sequence has the coefficients γ_{3n+2} not depending on n. This fact excludes the Hermite sequence, for which $\gamma_{n+1} = \frac{1}{2}(n+1)$, $\forall n \geq 0$, leaving the Gegenbauer sequence to examine. In this last, we have

$$\gamma_{n+1} = \frac{(n + 1)(n + 2\alpha + 1)}{(2n + 2\alpha + 1)(2n + 2\alpha + 3)},\ \ n \geq 0,\ \alpha \neq -k,\ 2\alpha \neq -k - 1,\ k \geq 1.$$

Then, for some constant M: $\gamma_{3n+2} = M \Leftrightarrow \begin{cases} M = \frac{1}{4} \\ \alpha = \frac{1}{2} \vee \alpha = -\frac{1}{2} \end{cases}$ yielding only two possibilities. For the second kind Tchebyshev polynomials ($\alpha = \frac{1}{2}$) this CD is also impossible, because, for this, we have $\gamma_{n+1} = \frac{1}{4},\ n \geq 0$, which contradicts ($b_{10}$) of theorem 3.22.

For the first kind Tchebyshev polynomials ($\alpha = -\frac{1}{2}$) this CD fits perfectly, because $\gamma_1 = \frac{1}{2},\ \gamma_{n+1} = \frac{1}{4},\ n \geq 1$, fulfilling, from ($b_1$)-($b_{10}$), the following identities:

$$\gamma_{3n}(1-\delta_{n,0}) + \gamma_{3n+1} = \gamma_1;\ \text{since}\ \gamma_{3n+2} = \frac{3}{4} - \gamma_1,\ \ n \geq 0,\ \text{we must have}\ q = -\frac{3}{4};$$

$$P_{n+2}(x) = xP_{n+1}(x) - \frac{1}{4^3}P_n(x),\ \ n \geq 1,\quad P_1(x) = x,\quad P_2(x) = x^2 - \frac{1}{32};$$

$$Q_{n+1}(x) = R_{n+1}(x) - \frac{1}{4}(b+c)R_n(x);\ \ R_{n+1}(x) = xR_n(x) - \frac{1}{4^3}R_{n-1}(x);$$

$$b_n^2(x) = -\frac{1}{4}R_n(x);\ \ b_n^1(x) = aR_n(x) - \frac{1}{4}\Big(a(b+c) - bc - \frac{1}{2}\Big)R_{n-1}(x);$$

$$c_n^2(x) = (b+c)R_n(x) - \frac{1}{16}R_{n-1}(x);\ \ c_n^1(x) = \Big(a(b+c) - bc - \frac{1}{2}\Big)R_n(x) - \frac{a}{16}R_{n-1}(x).$$

Note that as a consequence of theorem 3.2, we saw that if a MOPS is symmetric and we select $p = 0$ and $q = -\frac{3}{4}$, the principal components are all orthogonal.

The recurrence relations of second order of the sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ were already stated above. Using relation (3.53), the initial conditions of theorem 3.2 and the recurrence coefficients of the first kind Tchebyshev polynomials, we obtain the following.

$$Q_{n+2}(x) = xQ_{n+1}(x) - \frac{1}{4^3}Q_n(x),\ \ n \geq 0,\quad Q_1(x) = x - \frac{1}{4}(b+c),\quad Q_0(x) = 1.$$

The secondary components $b_n^1(x),\ b_n^2(x),\ c_n^1(x)$ and $c_n^2(x)$ satisfy the hypotheses of lemma 1.35, therefore, we know that each one fulfils the recurrence relation of second order $B_{n+2}(x) = xB_{n+1}(x) - \frac{1}{4^3}B_n(x),\ n \geq 0$.

Regarding their expressions, indicated above, we can also state the following:

- $\{b_n^2\}_{n\geq 0}$ is an OPS;
- if $b + c \neq 0$, then $\{c_n^2\}_{n\geq 0}$ is an OPS;
- if $b + c = 0$, then $\{c_{n+1}^2\}_{n\geq 0}$ is an OPS;
- if $a \neq 0$, then $\{b_n^1\}_{n\geq 0}$ is an OPS;
- if $a = 0$ and $bc = -\frac{1}{2}$, then $\{b_n^1\}_{n\geq 0}$ vanishes;
- if $a = 0$ and $bc \neq -\frac{1}{2}$, then $\{b_{n+1}^1\}_{n\geq 0}$ is an OPS;
- if $a(b+c) - bc - \frac{1}{2} \neq 0$, then $\{c_n^1\}_{n\geq 0}$ is an OPS;

- if $a(b+c)-bc-\frac{1}{2}=0$ and $a\neq 0$, then $\{c^1_{n+1}\}_{n\geq 0}$ is an OPS;
- if $a(b+c)-bc-\frac{1}{2}=0$ and $a=0$, then $\{c^1_n\}_{n\geq 0}$ vanishes.

In this special CD, where $a^1_n = a^2_n = 0$, $n\geq 0$, the sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be seen as the result of a shift of the first and second kind Tchebyshev polynomials, respectively, and the sequence $\{Q_n\}_{n\geq 0}$ can be seen as a co-recursive sequence of a shift of the second kind Tchebyshev polynomials, as we next show.

Let $\{U_n\}_{n\geq 0}$ denote Tchebyshev of second kind MOPS. Then,

$$U_{n+2}(x) = xU_{n+1}(x) - \frac{1}{4}U_n(x),\ \ n\geq 0,\quad U_0(x)=1,\ U_1(x)=x.$$

Let us define $\widehat{U}_n(x) = A^{-n}U_n(Ax+B),\ \ n\geq 0$. Then, we have:

$$\begin{aligned}\widehat{U}_{n+2}(x) &= A^{-(n+2)}U_{n+2}(Ax+B)\\ &= A^{-(n+2)}\Big((Ax+B)U_{n+1}(Ax+B) - \frac{1}{4}U_n(Ax+B)\Big)\\ &= (x+A^{-1}B)\widehat{U}_{n+1}(x) - \frac{1}{4A^2}\widehat{U}_n(x).\end{aligned}$$

Choosing $A=4$ and $B=0$, we obtain a MOPS $\{\widehat{U}_n\}_{n\geq 0}$ fulfilling:

$$\widehat{U}_{n+2}(x) = x\widehat{U}_{n+1}(x) - \frac{1}{4^3}\widehat{U}_n(x),\ \ \widehat{U}_1(x)=x,\ \ \widehat{U}_0(x)=1.$$

Consequently, $R_n(x) = \widehat{U}_n(x) = 4^{-n}U_n(4x)$ and $Q_n(x) = \widehat{U}_n\big(\frac{1}{4}(b+c)\big)(x),\ \ n\geq 0$, (see definition 1.16).

Let $\{T_n\}_{n\geq 0}$ denote Tchebyshev of first kind MOPS. Then,

$$T_{n+2}(x) = xT_{n+1}(x) - \frac{1}{4}T_n(x),\ \ n\geq 1,\quad T_0(x)=1,\ T_1(x)=x,\ T_2(x)=x^2-\frac{1}{2}.$$

Let us define $\widehat{T}_n(x) = A^{-n}T_n(Ax+B),\ \ n\geq 0$. Similarly, choosing $A=4$ and $B=0$, we obtain a MOPS $\{\widehat{T}_n\}_{n\geq 0}$ fulfilling:

$$\widehat{T}_{n+2}(x) = x\widehat{T}_{n+1}(x) - \frac{1}{4^3}\widehat{T}_n(x),\ \ n\geq 1,$$

where $\widehat{T}_0(x)=1$, $\widehat{T}_1(x) = \frac{1}{4}T_1(4x) = x$ and
$\widehat{T}_2(x) = \frac{1}{4^2}T_2(4x) = \frac{1}{4^2}\big((4x)^2-\frac{1}{2}\big) = x^2 - \frac{1}{32}$.

Consequently, $P_n(x) = 4^{-n}T_n(4x),\ \ n\geq 0$. □

Remark 3.26. *The proposition 3.25 and its demonstration gives us a CD of the first kind Tchebyshev MOPS such that* $a_n^1 = a_n^2 = 0,\ n \geq 0$, *associated to the parameters* a, b, c, $p = 0$, $q = -\frac{3}{4}$ *and* $r = 0$. *The considerations taken in chapter 2 (see page 19) allow us to modify the cubic transformation* $\varpi(x)$.

In fact, given p_0, q_0 *and* r_0 *such that* $q_0 = \dfrac{p_0^2}{3} - \dfrac{3}{4}$ *and* $r_0 = \left(\frac{p_0}{3}\right)^3 - \frac{p_0}{4}$, *we can write a CD of the sequence* $\{W_n\}_{n\geq 0}$ *defined by* $W_n(x) = T_n\left(x + \dfrac{p_0}{3}\right)$, *calculating the CD of* $\{T_n\}_{n\geq 0}$ *for* a, b, c *and* $\varpi(x) = x^3 - \frac{3}{4}x$, *and finally applying the affine transformation* $x \to x + \frac{p_0}{3}$. *The resultant CD has parameters* $a - \frac{p_0}{3}$, $b - \frac{p_0}{3}$, $c - \frac{p_0}{3}$ *and* $\varpi(x) = x^3 + p_0x^2 + q_0 + r_0$.

Corollary 3.27. *The CD such that* $a_n^1 = a_n^2 = 0,\ n \geq 0$, *of the first kind Tchebyshev MOPS, when* $a = 0,\ b = \frac{\sqrt{2}}{2},\ c = -\frac{\sqrt{2}}{2}$ *and* $\varpi(x) = x^3 - \frac{3}{4}x$ *is the following, where* $\{T_n\}_{n\geq 0}$ *and* $\{U_n\}_{n\geq 0}$ *represent the first and the second kind Tchebyshev polynomials, respectively,* $U_{-1}(x) = 0$ *and* $n \geq 0$.

$$\begin{aligned} T_{3n}(x) &= 4^{-n}T_n(4\varpi(x)) \\ T_{3n+1}(x) &= x4^{-n}U_n(4\varpi(x)) - \frac{1}{4}\left(x^2 - \frac{1}{2}\right)4^{-n+1}U_{n-1}(4\varpi(x)) \\ T_{3n+2}(x) &= -\frac{1}{16}x4^{-n+1}U_{n-1}(4\varpi(x)) + \left(x^2 - \frac{1}{2}\right)4^{-n}U_n(4\varpi(x)). \end{aligned}$$

Remark 3.28. *Let us note that the sequence* $\{T_n\}_{n\geq 0}$ *is a* 1*-perturbed sequence of* $\{U_n\}_{n\geq 0}$ *(see definition 1.17), more precisely,* $\{T_n\}_{n\geq 0} = \left\{U_n\left(0;\ \begin{matrix}0\\2\end{matrix}\ ; 1; x\right)\right\}_{n\geq 0}$. *Thus, the presented CD of the first kind Tchebyshev MOPS is completely written in terms of the second kind Tchebyshev MOPS.*

Proof. In the proof of the proposition 3.25, we described the component sequences of the CD of the first kind Tchebyshev MOPS such that $a_n^1 = a_n^2 = 0,\ n \geq 0$, where necessarily $\varpi(x) = x^3 - \frac{3}{4}x$.

For the following choice of the remaining free parameters a, b, and c: $a = 0,\ b = \frac{\sqrt{2}}{2},\ c = -\frac{\sqrt{2}}{2}$, we have, in particular, $b + c = 0$ and $-bc - \frac{1}{2} = 0$, hence

$$P_{n+2}(x) = xP_{n+1}(x) - \frac{1}{64}P_n(x),\ \ n \geq 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{1}{32};$$

$$Q_n(x) = R_n(x),\ \ n \geq 0;$$

$$R_{n+1}(x) = xR_n(x) - \frac{1}{64}R_{n-1}(x),\ \ n \geq 0;$$

$$b_n^2(x) = -\frac{1}{4}R_n(x); \quad b_n^1(x) = 0,\ \ n \geq 0;$$

$$c_n^2(x) = -\frac{1}{16}R_{n-1}(x); \quad c_n^1(x) = 0, \quad n \geq 0.$$

Thus, for $n \geq 0$:

$$\begin{aligned}
T_{3n}(x) &= P_n(\varpi(x)) \\
T_{3n+1}(x) &= xR_n(\varpi(x)) - \frac{1}{4}\left(x^2 - \frac{1}{2}\right) R_{n-1}(\varpi(x)) \\
T_{3n+2}(x) &= -\frac{1}{16}xR_{n-1}(\varpi(x)) + \left(x^2 - \frac{1}{2}\right) R_n(\varpi(x)),
\end{aligned}$$

where $\varpi(x) = x^3 - \frac{3}{4}x$, $R_n(x) = 4^{-n}U_n(4x)$, $P_n(x) = 4^{-n}T_n(4x)$, $n \geq 0$, and $R_{-1}(x) = 0$. In this manner, we have obtained the enunciated CD of the first kind Tchebyshev polynomials. □

Chapter 4

Cubic decomposition of a 2-orthogonal sequence

A d-orthogonal MPS fulfils a $(d+1)$-order recurrence relation [27] (see section 1.3).

If $d = 2$, the third order recurrence relation can be written as follows:

$$W_{n+3}(x) = (x - \beta_{n+2})W_{n+2}(x) - \gamma^1_{n+2}W_{n+1}(x) - \gamma^0_{n+1}W_n(x), \;\; n \geq 0,$$
$$W_0(x) = 1, \;\; W_1(x) = x - \beta_0, \;\; W_2(x) = (x - \beta_1)W_1(x) - \gamma^1_1,$$

where $\gamma^0_{n+1} \neq 0, \;\; n \geq 0$.

Then, the correspondent recurrence coefficients are β_n, γ^1_{n+1}, γ^0_{n+1}, $n \geq 0$.

The aim of this chapter is to study the CD of a 2-orthogonal MPS. We begin with two equivalent characterizations of the CD of a 2-orthogonal MPS, from which we can establish sufficient conditions for the 2-orthogonality of the principal components.

The recurrence coefficients of the 2-orthogonal sequences which admit a diagonal CD are described and we prove that the correspondent principal components are also 2-orthogonal. This particular case is also analysed for a 2-symmetric and 2-orthogonal sequence.

Since the d-orthogonality is preserved by shifts (see proposition 1.23), and attending to the arguments presented in page 19, we will consider in this chapter $r = 0$, without lost of generality.

4.1 Characterizations of the cubic decomposition of a 2-orthogonal sequence

Taking into consideration theorem (2.5) , we begin to give necessary and sufficient relations concerning the decomposed MPS 2-orthogonality.

Theorem 4.1. *[First Characterization of a 2-MOPS CD] A MPS defined by (2.1)-(2.3) is* 2*-orthogonal if and only if the following relations are met, for* $n \geq 0$*, where* $\Theta(x)$ *and* L *are defined by (2.4) and (2.5), and* $c^1_{-1}(x) = c^2_{-1}(x) = R_{-1}(x) = 0$.

(B_0) $b_0^1(x) = a - \beta_0$,

(B_1) $(\beta_{3n+1} - a)b_n^1(x) - \Theta(x)b_{n-1}^2(x) + \gamma_{3n}^0 c_{n-1}^1(x) + c_n^1(x) = -\gamma_{3n+1}^1 P_n(x) - (a-b)(a-c)Q_n(x)$,

(B_2) $\gamma_{3n+1}^1 a_{n-1}^1(x) - b_n^1(x) - Lb_{n-1}^2(x) + \gamma_{3n}^0 c_{n-1}^2(x) + c_n^2(x) = -(\beta_{3n+1} + a - b - c)Q_n(x)$,

(B_3) $\gamma_{3n+1}^1 a_{n-1}^2(x) + (\beta_{3n+1} + b + c + p)b_{n-1}^2(x) = Q_n(x) - \gamma_{3n}^0 R_{n-1}(x) - R_n(x)$,

(B_4) $\gamma_{3n+2}^1 b_n^1(x) + (\beta_{3n+2} - a)c_n^1(x) + (a-b)(a-c)c_n^2(x) = -\gamma_{3n+1}^0 P_n(x) - P_{n+1}(x) + \Theta(x)R_n(x)$,

(B_5) $\gamma_{3n+1}^0 a_{n-1}^1(x) + a_n^1(x) - c_n^1(x) + (\beta_{3n+2} + a - b - c)c_n^2(x) = -\gamma_{3n+2}^1 Q_n(x) + LR_n(x)$,

(B_6) $\gamma_{3n+1}^0 a_{n-1}^2(x) + a_n^2(x) + \gamma_{3n+2}^1 b_{n-1}^2(x) - c_n^2(x) = -(\beta_{3n+2} + b + c + p)R_n(x)$,

(B_7) $(a-b)(a-c)a_n^1(x) - \Theta(x)a_n^2(x) + \gamma_{3n+2}^0 b_n^1(x) + b_{n+1}^1(x) + \gamma_{3n+3}^1 c_n^1(x) =$
$-(\beta_{3n+3} - a)P_{n+1}(x)$,

(B_8) $(\beta_{3n+3} + a - b - c)a_n^1(x) - La_n^2(x) + \gamma_{3n+3}^1 c_n^2(x) = P_{n+1}(x) - \gamma_{3n+2}^0 Q_n(x) - Q_{n+1}(x)$,

(B_9) $a_n^1(x) - (\beta_{3n+3} + b + c + p)a_n^2(x) - \gamma_{3n+2}^0 b_{n-1}^2(x) - b_n^2(x) = \gamma_{3n+3}^1 R_n(x)$.

Proof. The sequence $\{W_n\}_{n\geq 0}$ is 2-orthogonal if and only if its structure coefficients are:

$$\chi_{n+1,n+1} = \gamma_{n+2}^1, \;\; \chi_{n+1,n} = \gamma_{n+1}^0 \neq 0, \text{ and } \chi_{n+1,\nu} = 0, \; 0 \leq \nu < n.$$

Then, theorem 2.5 concludes the proof. □

Theorem 4.2. *[Second Characterization of a 2-MOPS CD] A MPS with CD given by (2.1)-(2.3) is 2-orthogonal if and only if the following relations are fulfilled, for $n \geq 0$:*
identities (B_0), (B_1), (B_2), (B_5), (B_6), (B_7), and (B_9) from theorem 4.1 and the following three, concerning the principal components, where $\Theta(x)$ and L are defined by (2.4) and (2.5).

$$\begin{aligned} P_{n+3}(x) = &\left\{\Theta(x) - \bar{A}_{3n+3}\right\}P_{n+2}(x) - \bar{B}_{3n+3}P_{n+1}(x) - \bar{C}_{3n+3}P_n(x) - \bar{M}_{3n+3}b_n^1(x) \\ &- \bar{K}_{3n+3}b_{n+1}^1(x) - \bar{H}_{3n+3}b_{n+2}^1(x) - \bar{N}_{3n+3}c_n^1(x) - \bar{V}_{3n+3}c_{n+1}^1(x) - \bar{S}_{3n+3}c_{n+2}^1(x), \end{aligned} \tag{4.1}$$

with initial conditions:

$P_0(x) = 1$,
$P_1(x) = \Theta(x) - \gamma_1^1(a - \beta_2) - \gamma_2^1(a - \beta_0) + (a - \beta_0)(a - \beta_1)(a - \beta_2)$
$-(a-b)(a-c)(a + b + c - \beta_0 - \beta_1 - \beta_2)$,
$P_2(x) = \Theta(x)R_1(x) - \gamma_4^0 P_1(x) - (\beta_5 - a)c_1^1(x) - \gamma_5^1 b_1^1(x) - (a-b)(a-c)c_1^2(x)$, *where*
$a_0^2 = -p - \beta_0 - \beta_1 - \beta_2$,

$$a_0^1 = -(a-b)(a-c)+L+(a-\beta_0)(a-\beta_1)-(b+c-\beta_0-\beta_1)(a-b-c+\beta_2)-\gamma_1^1-\gamma_2^1,$$
$$b_0^2 = -(a-b)(a-c)+L+(a-\beta_0)(a-\beta_1)-(b+c-\beta_0-\beta_1)(a-b-c+\beta_2) + (p+\beta_0+\beta_1+\beta_2)(b+c+p+\beta_3)-\gamma_1^1-\gamma_2^1-\gamma_3^1,$$
$$b_1^1(x) = \Theta(x)a_0^2-\gamma_1^0 b_0^1-(\beta_3-a)P_1(x)-\gamma_3^1 c_0^1-(a-b)(a-c)a_0^1,$$
$$c_1^1(x) = \Theta(x)b_0^2-\gamma_3^0 c_0^1-(\beta_4-a)b_1^1(x)-\gamma_4^1 P_1(x)-(a-b)(a-c)Q_1(x),$$
$$c_1^2(x) = b_1^1(x)+Lb_0^2-\gamma_3^0 c_0^2-(\beta_4+a-b-c)Q_1(x)-\gamma_4^1 a_0^1.$$

$$Q_{n+3}(x) = \left\{\Theta(x)-\bar{A}_{3n+4}\right\}Q_{n+2}(x)-\bar{B}_{3n+4}Q_{n+1}(x)-\bar{C}_{3n+4}Q_n(x)-\bar{M}_{3n+4}c_n^2(x) - \bar{K}_{3n+4}c_{n+1}^2(x)-\bar{H}_{3n+4}c_{n+2}^2(x)-\bar{N}_{3n+4}a_n^1(x)-\bar{V}_{3n+4}a_{n+1}^1(x)-\bar{S}_{3n+4}a_{n+2}^1(x), \tag{4.2}$$

with initial conditions:

$Q_0(x) = 1,$

$Q_1(x) = \Theta(x)-\gamma_1^0-\gamma_2^0-\gamma_1^1(b+c-\beta_2-\beta_3)-\gamma_2^1(b+c-\beta_0-\beta_3)-\gamma_3^1(b+c-\beta_0-\beta_1)$
$+(a-\beta_0)(a-\beta_1)(b+c-\beta_2-\beta_3)+(b+c-\beta_0-\beta_1)(a-b-c+\beta_2)(a-b-c+\beta_3)$
$-(a-b)(a-c)(2b+2c-\beta_0-\beta_1-\beta_2-\beta_3)-L(a-b-c+p+\beta_0+\beta_1+\beta_2+\beta_3),$

$Q_2(x) = P_2(x)+La_1^2(x)-\gamma_5^0 Q_1(x)-\gamma_6^1 c_1^2(x)-(\beta_3+a-b-c)a_1^1(x),$ *where*

$$a_1^1(x) = LR_1(x)-\gamma_4^0 a_0^1+c_1^1(x)-(\beta_5-b-c+a)c_1^2(x)-\gamma_5^1 Q_1(x),$$
$$a_1^2(x) = -\gamma_4^0 a_0^2+c_1^2(x)-\gamma_5^1 b_0^2-(\beta_5+b+c+p)R_1(x).$$

$$R_{n+3}(x) = \left\{\Theta(x)-\bar{A}_{3n+5}\right\}R_{n+2}(x)-\bar{B}_{3n+5}R_{n+1}(x)-\bar{C}_{3n+5}R_n(x)-\bar{M}_{3n+5}a_{n+1}^2(x) - \bar{K}_{3n+5}a_{n+1}^2(x)-\bar{H}_{3n+5}a_{n+2}^2(x)-\bar{N}_{3n+5}b_n^2(x)-\bar{V}_{3n+5}b_{n+1}^2(x)-\bar{S}_{3n+5}b_{n+2}^2(x), \tag{4.3}$$

with initial conditions:

$R_0(x) = 1,$

$R_1(x) = \Theta(x)-\gamma_1^0-\gamma_2^0-\gamma_3^0+\gamma_1^1(p+\beta_2+\beta_3+\beta_4)+\gamma_2^1(p+\beta_0+\beta_3+\beta_4)$
$+\gamma_3^1(p+\beta_0+\beta_1+\beta_4)+\gamma_4^1(p+\beta_0+\beta_1+\beta_2)$
$-(p+\beta_0+\beta_1+\beta_2)(b+c+p+\beta_3)(b+c+p+\beta_4)$
$+(b+c-\beta_0-\beta_1)(a-b-c+\beta_2)(a+p+\beta_3+\beta_4)$
$-(a-\beta_0)(a-\beta_1)(p+\beta_2+\beta_3+\beta_4)$
$-(a-b)(a-c)(b+c-p-\beta_0-\beta_1-\beta_2-\beta_3-\beta_4)-L(a+2p+\beta_0+\beta_1+\beta_2+\beta_3+\beta_4),$

$R_2(x) = Q_2(x)-\gamma_6^0 R_1(x)-(\beta_7+b+c+p)b_1^2(x)-\gamma_7^1 a_1^2(x),$ *where*

$$b_1^2(x) = -\gamma_5^0 b_0^2-\gamma_6^1 R_1(x)+a_1^1(x)-(\beta_6+b+c+p)a_1^2(x).$$

We have also:

$$\bar{A}_n = \gamma_{n+2}^0+\gamma_{n+3}^0+\gamma_{n+4}^0+\gamma_{n+3}^1(\beta_{n+2}+2\beta_{n+3}+p) + \gamma_{n+4}^1(2\beta_{n+3}+\beta_{n+4}+p)+(\beta_{n+3}-a)(\beta_{n+3}+a-b-c)(\beta_{n+3}+b+c+p) - (\beta_{n+3}-a)L+(a-b)(a-c)(\beta_{n+3}+b+c+p);$$

$$\bar{B}_n = \gamma^1_{n+1}\gamma^1_{n+2}\gamma^1_{n+3} + \gamma^0_n\gamma^0_{n+2} + \gamma^0_{n+1}(\gamma^0_{n+2} + \gamma^0_{n+3})$$
$$+ \gamma^0_{n+1}\gamma^1_{n+3}(\beta_n + \beta_{n+2} + \beta_{n+3} + p) + \gamma^0_{n+2}\gamma^1_{n+1}(\beta_n + \beta_{n+1} + \beta_{n+3} + p);$$

$$\bar{C}_n = \gamma^0_{n-2}\gamma^0_n\gamma^0_{n+2};$$
$$\bar{M}_n = \gamma^0_{n-1}(\gamma^0_{n+1}\gamma^1_{n+3} + \gamma^0_{n+2}\gamma^1_{n+1}) + \gamma^0_n\gamma^0_{n+2}\gamma^1_{n-1};$$

$$\bar{K}_n = \gamma^0_{n+1}\gamma^1_{n+3} + \gamma^0_{n+3}\gamma^1_{n+2} + \gamma^1_{n+2}\gamma^1_{n+3}(\beta_{n+1} + \beta_{n+2} + \beta_{n+3} + p)$$
$$+ \gamma^0_{n+2}\Big(\gamma^1_{n+1} + \gamma^1_{n+2} + \gamma^1_{n+3} + \gamma^1_{n+4} + (a-b)(a-c) - L$$
$$+ (\beta_{n+3} - a)(\beta_{n+1} + \beta_{n+3} + a + p) + (\beta_{n+1} + a - b - c)(\beta_{n+1} + b + c + p)\Big);$$

$$\bar{H}_n = \gamma^1_{n+3} + \gamma^1_{n+4} + \gamma^1_{n+5} - L + (a-b)(a-c)$$
$$+ (\beta_{n+4} - a)(\beta_{n+3} + \beta_{n+4} + a + p) + (\beta_{n+3} + a - b - c)(\beta_{n+3} + b + c + p);$$
$$\bar{N}_n = \gamma^1_n(\gamma^0_{n+1}\gamma^1_{n+3} + \gamma^0_{n+2}\gamma^1_{n+1}) + \gamma^0_n\gamma^1_{n+2}\gamma^1_{n+3} + \gamma^0_n\gamma^0_{n+2}(\beta_{n-1} + \beta_{n+1} + \beta_{n+3} + p);$$

$$\bar{V}_n = \gamma^0_{n+2}(\beta_{n+1} + \beta_{n+2} + \beta_{n+3} + p) + \gamma^0_{n+3}(\beta_{n+2} + \beta_{n+3} + \beta_{n+4} + p)$$
$$+ \gamma^1_{n+3}\Big(\gamma^1_{n+2} + \gamma^1_{n+3} + \gamma^1_{n+4} + (a-b)(a-c) - L$$
$$+ (\beta_{n+2} + a - b - c)(\beta_{n+2} + b + c + p) + (\beta_{n+3} - a)(\beta_{n+2} + \beta_{n+3} + p + a)\Big);$$

$$\bar{S}_n = \beta_{n+3} + \beta_{n+4} + \beta_{n+5} + p.$$

Remark 4.3. $\bar{C}_n = \gamma^0_{n-2}\gamma^0_n\gamma^0_{n+2} \neq 0$ *(regularity conditions).*

Proof. This result is similar to the theorem 3.2 and the demonstration will also follow the same principal steps. Each one of the identities (4.1), (4.2) and (4.3) begins as a recurrence relation of third order of a principal component and is completed with elements of the only two secondary component sequences that appear in the respective column, when we use matrix notation to write all component sequences of a CD (see the theorem 2.5 proof).

This result is accomplished, firstly, by deducing identities (4.1), (4.2) and (4.3) from theorem 4.1 relations (Part II), and conversely, obtaining the three absent relations of the theorem 4.1, (B_3), (B_4) and (B_9), from the ones enunciated (Part III).

In both procedures, it will be useful to have some components of the CD (or algebraic expressions of certain polynomials) written in terms of the elements of one of the columns $(P_n(x), b^1_n(x), c^1_n(x))^T$, $(a^1_{n-1}(x), Q_n(x), c^2_n(x))^T$ and $(a^2_{n-1}(x), b^2_{n-1}(x), R_n(x))^T$, which will be done in Part I.

Part I

Let us begin to write every component in terms of elements of the sequences $\{a^2_{n-1}(x)\}_{n\geq 0}$, $\{b^2_{n-1}(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$.

Let us note (B_9), (B_6) and (B_3) (respectively):

$$a_n^1(x) = (\beta_{3n+3} + b + c + p)a_n^2(x) + \gamma_{3n+2}^0 b_{n-1}^2(x) + b_n^2(x) + \gamma_{3n+3}^1 R_n(x), \quad n \geq 0,$$

$$c_n^2(x) = \gamma_{3n+1}^0 a_{n-1}^2(x) + a_n^2(x) + \gamma_{3n+2}^1 b_{n-1}^2(x) + (\beta_{3n+2} + b + c + p)R_n(x), \quad n \geq 0,$$

$$Q_n(x) = \gamma_{3n+1}^1 a_{n-1}^2(x) + (\beta_{3n+1} + b + c + p)b_{n-1}^2(x) + \gamma_{3n}^0 R_{n-1}(x) + R_n(x), \quad n \geq 0.$$

Using identities (B_5), (B_8) and (B_2) (the first and the last one with $n \leftarrow n+1$), we can also write $c_{n+1}^1(x)$, $P_{n+1}(x)$ and $b_{n+1}^1(x)$ (respectively) as linear combinations of elements of the sequences $\{a_{n-1}^2(x)\}_{n\geq 0}$, $\{b_{n-1}^2(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$, as follows.

$$\begin{aligned}
c_{n+1}^1(x) &= \gamma_{3n+4}^0 a_n^1(x) + a_{n+1}^1(x) + (\beta_{3n+5} + a - b - c)c_{n+1}^2(x) + \gamma_{3n+5}^1 Q_{n+1}(x) - LR_{n+1}(x) \\
&= \gamma_{3n+4}^0 \Big((\beta_{3n+3} + b + c + p)a_n^2(x) + \gamma_{3n+2}^0 b_{n-1}^2(x) + b_n^2(x) + \gamma_{3n+3}^1 R_n(x) \Big) \\
&+ (\beta_{3n+6} + b + c + p)a_{n+1}^2(x) + \gamma_{3n+5}^0 b_n^2(x) + b_{n+1}^2(x) + \gamma_{3n+6}^1 R_{n+1}(x) \\
&+ (\beta_{3n+5} + a - b - c)\Big(\gamma_{3n+4}^0 a_n^2(x) + a_{n+1}^2(x) + \gamma_{3n+5}^1 b_n^2(x) + (\beta_{3n+5} + b + c + p)R_{n+1}(x) \Big) \\
&+ \gamma_{3n+5}^1 \Big(\gamma_{3n+4}^1 a_n^2(x) + (\beta_{3n+4} + b + c + p)b_n^2(x) + \gamma_{3n+3}^0 R_n(x) + R_{n+1}(x) \Big) - LR_{n+1}(x);
\end{aligned}$$

$$\begin{aligned}
\Rightarrow c_{n+1}^1(x) &= \Big(\gamma_{3n+4}^0 (\beta_{3n+3} + \beta_{3n+5} + a + p) + \gamma_{3n+4}^1 \gamma_{3n+5}^1 \Big) a_n^2(x) \\
&+ (\beta_{3n+5} + \beta_{3n+6} + a + p)a_{n+1}^2(x) + \gamma_{3n+2}^0 \gamma_{3n+4}^0 b_{n-1}^2(x) \\
&+ \Big(\gamma_{3n+4}^0 + \gamma_{3n+5}^0 + \gamma_{3n+5}^1 (\beta_{3n+4} + \beta_{3n+5} + a + p) \Big) b_n^2(x) \\
&+ b_{n+1}^2(x) + \Big(\gamma_{3n+3}^0 \gamma_{3n+5}^1 + \gamma_{3n+4}^0 \gamma_{3n+3}^1 \Big) R_n(x) \\
&+ \Big(\gamma_{3n+5}^1 + \gamma_{3n+6}^1 + (\beta_{3n+5} + a - b - c)(\beta_{3n+5} + b + c + p) - L \Big) R_{n+1}(x), \quad n \geq 0.
\end{aligned} \tag{4.4}$$

Let us remark that the relation (4.4) is also fulfilled for $n \geq 0$ after the transformation $n \leftarrow n - 1$.

$$\begin{aligned}
P_{n+1}(x) &= (\beta_{3n+3} + a - b - c)a_n^1(x) - La_n^2(x) + \gamma_{3n+3}^1 c_n^2(x) + \gamma_{3n+2}^0 Q_n(x) + Q_{n+1}(x) \\
&= (\beta_{3n+3} + a - b - c)\Big((\beta_{3n+3} + b + c + p)a_n^2(x) + \gamma_{3n+2}^0 b_{n-1}^2(x) + b_n^2(x) + \gamma_{3n+3}^1 R_n(x) \Big) \\
&- La_n^2(x) + \gamma_{3n+3}^1 \Big(\gamma_{3n+1}^0 a_{n-1}^2(x) + a_n^2(x) + \gamma_{3n+2}^1 b_{n-1}^2(x) + (\beta_{3n+2} + b + c + p)R_n(x) \Big) \\
&+ \gamma_{3n+2}^0 \Big(\gamma_{3n+1}^1 a_{n-1}^2(x) + (\beta_{3n+1} + b + c + p)b_{n-1}^2(x) + \gamma_{3n}^0 R_{n-1}(x) + R_n(x) \Big) \\
&+ \gamma_{3n+4}^1 a_n^2(x) + (\beta_{3n+4} + b + c + p)b_n^2(x) + \gamma_{3n+3}^0 R_n(x) + R_{n+1}(x);
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow P_{n+1}(x) &= \Big(\gamma^0_{3n+1}\gamma^1_{3n+3} + \gamma^0_{3n+2}\gamma^1_{3n+1}\Big)a^2_{n-1}(x) \\
&+ \Big(\gamma^1_{3n+3} + \gamma^1_{3n+4} + (\beta_{3n+3} + a - b - c)(\beta_{3n+3} + b + c + p) - L\Big)a^2_n(x) \\
&+ \Big(\gamma^0_{3n+2}(\beta_{3n+1} + \beta_{3n+3} + a + p) + \gamma^1_{3n+2}\gamma^1_{3n+3}\Big)b^2_{n-1}(x) \\
&+ (\beta_{3n+3} + \beta_{3n+4} + a + p)b^2_n(x) + \gamma^0_{3n}\gamma^0_{3n+2}R_{n-1}(x) \\
&+ \Big(\gamma^1_{3n+3}(\beta_{3n+2} + \beta_{3n+3} + a + p) + \gamma^0_{3n+2} + \gamma^0_{3n+3}\Big)R_n(x) + R_{n+1}(x), \quad n \geq 0.
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
b^1_{n+1}(x) &= \gamma^1_{3n+4}a^1_n(x) - Lb^2_n(x) + \gamma^0_{3n+3}c^2_n(x) + c^2_{n+1}(x) + (\beta_{3n+4} + a - b - c)Q_{n+1}(x) \\
&= \gamma^1_{3n+4}\Big((\beta_{3n+3} + b + c + p)a^2_n(x) + \gamma^0_{3n+2}b^2_{n-1}(x) + b^2_n(x) + \gamma^1_{3n+3}R_n(x)\Big) - Lb^2_n(x) \\
&+ \gamma^0_{3n+3}\Big(\gamma^0_{3n+1}a^2_{n-1}(x) + a^2_n(x) + \gamma^1_{3n+2}b^2_{n-1}(x) + (\beta_{3n+2} + b + c + p)R_n(x)\Big) \\
&+ \gamma^0_{3n+4}a^2_n(x) + a^2_{n+1}(x) + \gamma^1_{3n+5}b^2_n(x) + (\beta_{3n+5} + b + c + p)R_{n+1}(x) \\
&+ (\beta_{3n+4} + a - b - c)\Big(\gamma^1_{3n+4}a^2_n(x) + (\beta_{3n+4} + b + c + p)b^2_n(x) + \gamma^0_{3n+3}R_n(x) + R_{n+1}(x)\Big);
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow b^1_{n+1}(x) &= \gamma^0_{3n+1}\gamma^0_{3n+3}a^2_{n-1}(x) + \Big(\gamma^0_{3n+3} + \gamma^0_{3n+4} + \gamma^1_{3n+4}(\beta_{3n+3} + \beta_{3n+4} + a + p)\Big)a^2_n(x) \\
&+ a^2_{n+1}(x) + \Big(\gamma^0_{3n+2}\gamma^1_{3n+4} + \gamma^0_{3n+3}\gamma^1_{3n+2}\Big)b^2_{n-1}(x) \\
&+ \Big(\gamma^1_{3n+4} + \gamma^1_{3n+5} + (\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p) - L\Big)b^2_n(x) \\
&+ \Big(\gamma^1_{3n+3}\gamma^1_{3n+4} + \gamma^0_{3n+3}(\beta_{3n+2} + \beta_{3n+4} + a + p)\Big)R_n(x) \\
&+ (\beta_{3n+4} + \beta_{3n+5} + a + p)R_{n+1}(x), \quad n \geq 0.
\end{aligned} \tag{4.6}
$$

Let us note that during the deduction of relation (4.4), the identities (B_5), (B_3), (B_6) and (B_9) were used after the transformation $n \leftarrow n+1$; in the deduction of (4.5) only (B_3) was used after the transformation $n \leftarrow n+1$, and in the deduction of (4.6), the relations used after the transformation $n \leftarrow n+1$ were (B_2), (B_3) and (B_6). These considerations will have an important role in Part III.

The next step consists in writing the polynomial

$$\Theta(x)R_{n+1}(x) - (a-b)(a-c)c^2_{n+1}(x) - \Theta(x)P_{n+1}(x)$$

in terms of elements of the sequences $\{P_n(x)\}_{n\geq 0}$, $\{b^1_n(x)\}_{n\geq 0}$ and $\{c^1_n(x)\}_{n\geq 0}$.

Let us note identities (B_7), (B_1) and (B_4) (respectively):

$$\Theta(x)a_n^2(x) = (a-b)(a-c)a_n^1(x)+\gamma_{3n+2}^0 b_n^1(x)+b_{n+1}^1(x)+\gamma_{3n+3}^1 c_n^1(x)+(\beta_{3n+3}-a)P_{n+1}(x), n \geq 0,$$

$$\Theta(x)b_{n-1}^2(x) = (\beta_{3n+1}-a)b_n^1(x)+\gamma_{3n}^0 c_{n-1}^1(x)+c_n^1(x)+\gamma_{3n+1}^1 P_n(x)+(a-b)(a-c)Q_n(x), n \geq 0,$$

$$\Theta(x)R_n(x) = \gamma_{3n+2}^1 b_n^1(x)+(\beta_{3n+2}-a)c_n^1(x)+(a-b)(a-c)c_n^2(x)+\gamma_{3n+1}^0 P_n(x)+P_{n+1}(x), n \geq 0.$$

These identities permit to write, through (B_9) multiplied by $\Theta(x)$, the following:

$$\begin{aligned}
&\Theta(x)a_n^1(x) = (\beta_{3n+3}+b+c+p)\Theta(x)a_n^2(x) + \gamma_{3n+2}^0\Theta(x)b_{n-1}^2(x) + \Theta(x)b_n^2(x) + \gamma_{3n+3}^1\Theta(x)R_n(x)\\
&\Rightarrow \Theta(x)a_n^1(x) = (\beta_{3n+3}+b+c+p)\Big((a-b)(a-c)a_n^1(x) + \gamma_{3n+2}^0 b_n^1(x) + b_{n+1}^1(x)\\
&+ \gamma_{3n+3}^1 c_n^1(x) + (\beta_{3n+3}-a)P_{n+1}(x)\Big)\\
&+ \gamma_{3n+2}^0\Big((\beta_{3n+1}-a)b_n^1(x) + \gamma_{3n}^0 c_{n-1}^1(x) + c_n^1(x) + \gamma_{3n+1}^1 P_n(x) + (a-b)(a-c)Q_n(x)\Big)\\
&+ (\beta_{3n+4}-a)b_{n+1}^1(x) + \gamma_{3n+3}^0 c_n^1(x) + c_{n+1}^1(x) + \gamma_{3n+4}^1 P_{n+1}(x) + (a-b)(a-c)Q_{n+1}(x)\\
&+ \gamma_{3n+3}^1\Big(\gamma_{3n+2}^1 b_n^1(x) + (\beta_{3n+2}-a)c_n^1(x) + (a-b)(a-c)c_n^2(x) + \gamma_{3n+1}^0 P_n(x) + P_{n+1}(x)\Big)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \Theta(x)a_n^1(x) = \Big(\gamma_{3n+2}^0(\beta_{3n+1}+\beta_{3n+3}-a+b+c+p) + \gamma_{3n+2}^1\gamma_{3n+3}^1\Big)b_n^1(x)\\
&+ (\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)b_{n+1}^1(x) + \gamma_{3n}^0\gamma_{3n+2}^0 c_{n-1}^1(x)\\
&+ \Big(\gamma_{3n+3}^1(\beta_{3n+2}+\beta_{3n+3}-a+b+c+p) + \gamma_{3n+2}^0 + \gamma_{3n+3}^0\Big)c_n^1(x) + c_{n+1}^1(x)\\
&+ \Big(\gamma_{3n+2}^0\gamma_{3n+1}^1 + \gamma_{3n+1}^0\gamma_{3n+3}^1\Big)P_n(x)\\
&+ \Big(\gamma_{3n+3}^1 + \gamma_{3n+4}^1 + (\beta_{3n+3}-a)(\beta_{3n+3}+b+c+p)\Big)P_{n+1}(x)\\
&+ (a-b)(a-c)\Big((\beta_{3n+3}+b+c+p)a_n^1(x) + \gamma_{3n+3}^1 c_n^2(x) + \gamma_{3n+2}^0 Q_n(x) + Q_{n+1}(x)\Big).
\end{aligned} \tag{4.7}$$

Using (B_6) multiplied by $\Theta(x)$, we obtain the following:

$$\begin{aligned}
&\Theta(x)c_n^2(x) = \gamma_{3n+1}^0\Theta(x)a_{n-1}^2(x) + \Theta(x)a_n^2(x) + \gamma_{3n+2}^1\Theta(x)b_{n-1}^2(x)\\
&+ (\beta_{3n+2}+b+c+p)\Theta(x)R_n(x)\\
&\Rightarrow \Theta(x)c_n^2(x) = \gamma_{3n+1}^0\Big(\gamma_{3n-1}^0 b_{n-1}^1(x) + b_n^1(x) + \gamma_{3n}^1 c_{n-1}^1(x) + (\beta_{3n}-a)P_n(x)\\
&+ (a-b)(a-c)a_{n-1}^1(x)\Big) + \gamma_{3n+2}^0 b_n^1(x) + b_{n+1}^1(x)\\
&+ \gamma_{3n+3}^1 c_n^1(x) + (\beta_{3n+3}-a)P_{n+1}(x) + (a-b)(a-c)a_n^1(x)\\
&+ \gamma_{3n+2}^1\Big((\beta_{3n+1}-a)b_n^1(x) + \gamma_{3n}^0 c_{n-1}^1(x) + c_n^1(x) + \gamma_{3n+1}^1 P_n(x) + (a-b)(a-c)Q_n(x)\Big)\\
&+ (\beta_{3n+2}+b+c+p)\Big(\gamma_{3n+2}^1 b_n^1(x) + (\beta_{3n+2}-a)c_n^1(x) + (a-b)(a-c)c_n^2(x)\\
&+ \gamma_{3n+1}^0 P_n(x) + P_{n+1}(x)\Big).
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \Theta(x)c_n^2(x) = \gamma_{3n-1}^0\gamma_{3n+1}^0 b_{n-1}^1(x)\\
&+\Big(\gamma_{3n+1}^0+\gamma_{3n+2}^0+\gamma_{3n+2}^1(\beta_{3n+1}+\beta_{3n+2}-a+b+c+p)\Big)b_n^1(x)\\
&+b_{n+1}^1(x)+\Big(\gamma_{3n+1}^0\gamma_{3n}^1+\gamma_{3n}^0\gamma_{3n+2}^1\Big)c_{n-1}^1(x)\\
&+\Big(\gamma_{3n+2}^1+\gamma_{3n+3}^1+(\beta_{3n+2}-a)(\beta_{3n+2}+b+c+p)\Big)c_n^1(x)\\
&+\Big(\gamma_{3n+1}^0(\beta_{3n}+\beta_{3n+2}-a+b+c+p)+\gamma_{3n+1}^1\gamma_{3n+2}^1\Big)P_n(x)\\
&+(\beta_{3n+2}+\beta_{3n+3}-a+b+c+p)P_{n+1}(x)\\
&+(a-b)(a-c)\Big(\gamma_{3n+1}^0a_{n-1}^1(x)+a_n^1(x)+(\beta_{3n+2}+b+c+p)c_n^2(x)+\gamma_{3n+2}^1Q_n(x)\Big),\ \ n\geq 0.
\end{aligned}
\tag{4.8}
$$

Let us note that we have introduced (B_7) with $n \leftarrow n-1$. In fact, considering $b_{-1}^1(x) = c_{-1}^1(x) = 0$ we can see that it is fulfilled for $n \geq 0$.

Let us consider the relation (B_3), with $n \leftarrow n+1$, where $Q_{n+1}(x)$ is replaced by the expression given by (B_8):

$$
\begin{aligned}
R_{n+1}(x) = {} & P_{n+1}(x) - \gamma_{3n+2}^0 Q_n(x) - (\beta_{3n+3}+a-b-c)a_n^1(x) + (L-\gamma_{3n+4}^1)a_n^2(x)\\
& - (\beta_{3n+4}+b+c+p)b_n^2(x) - \gamma_{3n+3}^1 c_n^2(x) - \gamma_{3n+3}^0 R_n(x),
\end{aligned}
$$

and replacing $Q_n(x)$ by the expression given by (B_3), we obtain:

$$
\begin{aligned}
R_{n+1}(x) = {} & P_{n+1}(x) - \gamma_{3n+2}^0\gamma_{3n+1}^1 a_{n-1}^2(x) + (L-\gamma_{3n+4}^1)a_n^2(x)\\
& - \gamma_{3n+2}^0(\beta_{3n+1}+b+c+p)b_{n-1}^2(x) - (\beta_{3n+4}+b+c+p)b_n^2(x) - \gamma_{3n}^0\gamma_{3n+2}^0 R_{n-1}(x)\\
& - (\gamma_{3n+2}^0+\gamma_{3n+3}^0)R_n(x) - (\beta_{3n+3}+a-b-c)a_n^1(x) - \gamma_{3n+3}^1 c_n^2(x).
\end{aligned}
$$

Let us multiply this identity by $\Theta(x)$ and introduce (B_7), (B_1), (B_4), (4.7) and (4.8). We obtain for $n \geq 1$:

$$\Theta(x)\big(R_{n+1}(x) - P_{n+1}(x)\big) = -\gamma^0_{3n+2}\gamma^1_{3n+1}\Big((a-b)(a-c)a^1_{n-1}(x) + \gamma^0_{3n-1}b^1_{n-1}(x)$$
$$+ b^1_n(x) + \gamma^1_{3n}c^1_{n-1}(x) + (\beta_{3n} - a)P_n(x)\Big)$$
$$+ (L - \gamma^1_{3n+4})\Big((a-b)(a-c)a^1_n(x) + \gamma^0_{3n+2}b^1_n(x)$$
$$+ b^1_{n+1}(x) + \gamma^1_{3n+3}c^1_n(x) + (\beta_{3n+3} - a)P_{n+1}(x)\Big)$$
$$- \gamma^0_{3n+2}(\beta_{3n+1} + b + c + p)\Big((\beta_{3n+1} - a)b^1_n(x) + \gamma^0_{3n}c^1_{n-1}(x) + c^1_n(x) + \gamma^1_{3n+1}P_n(x)$$
$$+ (a-b)(a-c)Q_n(x)\Big)$$
$$- (\beta_{3n+4} + b + c + p)\Big((\beta_{3n+4} - a)b^1_{n+1}(x) + \gamma^0_{3n+3}c^1_n(x) + c^1_{n+1}(x) + \gamma^1_{3n+4}P_{n+1}(x)$$
$$+ (a-b)(a-c)Q_{n+1}(x)\Big)$$
$$- \gamma^0_{3n}\gamma^0_{3n+2}\Big(\gamma^1_{3n-1}b^1_{n-1}(x) + (\beta_{3n-1} - a)c^1_{n-1}(x) + (a-b)(a-c)c^2_{n-1}(x) + \gamma^0_{3n-2}P_{n-1}(x) + P_n(x)\Big)$$
$$- (\gamma^0_{3n+2} + \gamma^0_{3n+3})\Big(\gamma^1_{3n+2}b^1_n(x)$$
$$+ (\beta_{3n+2} - a)c^1_n(x) + (a-b)(a-c)c^2_n(x) + \gamma^0_{3n+1}P_n(x) + P_{n+1}(x)\Big)$$
$$- (\beta_{3n+3} + a - b - c)\Big\{\Big(\gamma^0_{3n+2}(\beta_{3n+1} + \beta_{3n+3} - a + b + c + p) + \gamma^1_{3n+2}\gamma^1_{3n+3}\Big)b^1_n(x)$$
$$+ (\beta_{3n+3} + \beta_{3n+4} - a + b + c + p)b^1_{n+1}(x) + \gamma^0_{3n}\gamma^0_{3n+2}c^1_{n-1}(x)$$
$$+ \Big(\gamma^1_{3n+3}(\beta_{3n+2} + \beta_{3n+3} - a + b + c + p) + \gamma^0_{3n+2} + \gamma^0_{3n+3}\Big)c^1_n(x) + c^1_{n+1}(x)$$
$$+ \Big(\gamma^0_{3n+2}\gamma^1_{3n+1} + \gamma^0_{3n+1}\gamma^1_{3n+3}\Big)P_n(x)$$
$$+ \Big(\gamma^1_{3n+3} + \gamma^1_{3n+4} + (\beta_{3n+3} - a)(\beta_{3n+3} + b + c + p)\Big)P_{n+1}(x)$$
$$+ (a-b)(a-c)\Big((\beta_{3n+3} + b + c + p)a^1_n(x) + \gamma^1_{3n+3}c^2_n(x) + \gamma^0_{3n+2}Q_n(x) + Q_{n+1}(x)\Big)\Big\}$$
$$- \gamma^1_{3n+3}\Big\{\gamma^0_{3n-1}\gamma^0_{3n+1}b^1_{n-1}(x)$$
$$+ \Big(\gamma^0_{3n+1} + \gamma^0_{3n+2} + \gamma^1_{3n+2}(\beta_{3n+1} + \beta_{3n+2} - a + b + c + p)\Big)b^1_n(x)$$
$$+ b^1_{n+1}(x) + \Big(\gamma^0_{3n+1}\gamma^1_{3n} + \gamma^0_{3n}\gamma^1_{3n+2}\Big)c^1_{n-1}(x)$$
$$+ \Big(\gamma^1_{3n+2} + \gamma^1_{3n+3} + (\beta_{3n+2} - a)(\beta_{3n+2} + b + c + p)\Big)c^1_n(x)$$
$$+ \Big(\gamma^0_{3n+1}(\beta_{3n} + \beta_{3n+2} - a + b + c + p) + \gamma^1_{3n+1}\gamma^1_{3n+2}\Big)P_n(x)$$
$$+ (\beta_{3n+2} + \beta_{3n+3} - a + b + c + p)P_{n+1}(x)$$
$$+ (a-b)(a-c)\Big(\gamma^0_{3n+1}a^1_{n-1}(x) + a^1_n(x) + (\beta_{3n+2} + b + c + p)c^2_n(x) + \gamma^1_{3n+2}Q_n(x)\Big)\Big\}$$

$$
\begin{aligned}
&\Rightarrow \Theta(x)\big(R_{n+1}(x)-P_{n+1}(x)\big) = -\Big(\gamma^0_{3n-1}\gamma^0_{3n+2}\gamma^1_{3n+1}+\gamma^0_{3n}\gamma^0_{3n+2}\gamma^1_{3n-1}+\gamma^0_{3n-1}\gamma^0_{3n+1}\gamma^1_{3n+3}\Big)b^1_{n-1}(x)\\
&+\Big\{-\gamma^0_{3n+2}\gamma^1_{3n+1}+\gamma^0_{3n+2}(L-\gamma^1_{3n+4})-\gamma^0_{3n+2}(\beta_{3n+1}+b+c+p)(\beta_{3n+1}-a)\\
&-\gamma^1_{3n+2}(\gamma^0_{3n+2}+\gamma^0_{3n+3})\\
&-(\beta_{3n+3}+a-b-c)\Big(\gamma^0_{3n+2}(\beta_{3n+1}+\beta_{3n+3}-a+b+c+p)+\gamma^1_{3n+2}\gamma^1_{3n+3}\Big)\\
&-\gamma^1_{3n+3}\Big(\gamma^0_{3n+1}+\gamma^0_{3n+2}+\gamma^1_{3n+2}(\beta_{3n+1}+\beta_{3n+2}-a+b+c+p)\Big)\Big\}b^1_n(x)\\
&+\Big(L-\gamma^1_{3n+4}-(\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p)-\gamma^1_{3n+3}\\
&-(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+\beta_{3n+4}-a+b+c+p)\Big)b^1_{n+1}(x)\\
&-\Big\{\gamma^0_{3n+2}\gamma^1_{3n+1}\gamma^1_{3n}+\gamma^0_{3n}\gamma^0_{3n+2}(\beta_{3n+1}+b+c+p)+\gamma^0_{3n}\gamma^0_{3n+2}(\beta_{3n-1}-a)\\
&+\gamma^0_{3n}\gamma^0_{3n+2}(\beta_{3n+3}+a-b-c)+\gamma^1_{3n+3}(\gamma^0_{3n+1}\gamma^1_{3n}+\gamma^0_{3n}\gamma^1_{3n+2})\Big\}c^1_{n-1}(x)\\
&+\Big\{\gamma^1_{3n+3}(L-\gamma^1_{3n+4})-\gamma^0_{3n+2}(\beta_{3n+1}+b+c+p)-\gamma^0_{3n+3}(\beta_{3n+4}+b+c+p)\\
&-(\gamma^0_{3n+2}+\gamma^0_{3n+3})(\beta_{3n+2}-a)\\
&-(\beta_{3n+3}+a-b-c)\Big(\gamma^1_{3n+3}(\beta_{3n+2}+\beta_{3n+3}-a+b+c+p)+\gamma^0_{3n+2}+\gamma^0_{3n+3}\Big)\\
&-\gamma^1_{3n+3}\Big(\gamma^1_{3n+2}\gamma^1_{3n+3}+(\beta_{3n+2}-a)(\beta_{3n+2}+b+c+p)\Big)\Big\}c^1_n(x)\\
&-\gamma^0_{3n-2}\gamma^0_{3n}\gamma^0_{3n+2}P_{n-1}(x)-(\beta_{3n+3}+\beta_{3n+4}+a+p)c^1_{n+1}(x)\\
&-\Big\{\gamma^0_{3n+2}\gamma^1_{3n+1}(\beta_{3n}-a)+\gamma^0_{3n+2}\gamma^1_{3n+1}(\beta_{3n+1}+b+c+p)+\gamma^0_{3n}\gamma^0_{3n+2}\\
&+\gamma^0_{3n+1}(\gamma^0_{3n+2}+\gamma^0_{3n+3})+(\beta_{3n+3}+a-b-c)(\gamma^0_{3n+2}\gamma^1_{3n+1}+\gamma^0_{3n+1}\gamma^1_{3n+3})\\
&+\gamma^1_{3n+3}\big(\gamma^1_{3n+1}\gamma^1_{3n+2}+\gamma^0_{3n+1}(\beta_{3n}+\beta_{3n+2}-a+b+c+p)\big)\Big\}P_n(x)\\
&+\Big\{(L-\gamma^1_{3n+4})(\beta_{3n+3}-a)-\gamma^1_{3n+4}(\beta_{3n+4}+b+c+p)-\gamma^0_{3n+2}-\gamma^0_{3n+3}\\
&-(\beta_{3n+3}+a-b-c)\Big(\gamma^1_{3n+3}+\gamma^1_{3n+4}+(\beta_{3n+3}-a)(\beta_{3n+3}+b+c+p)\Big)\\
&-\gamma^1_{3n+3}(\beta_{3n+2}+\beta_{3n+3}-a+b+c+p)\Big\}P_{n+1}(x)\\
&-(a-b)(a-c)\Big\{\gamma^0_{3n+2}\gamma^1_{3n+1}a^1_{n-1}(x)+(\gamma^1_{3n+4}-L)a^1_n(x)+\gamma^0_{3n+2}(\beta_{3n+1}+b+c+p)Q_n(x)\\
&+(\beta_{3n+4}+b+c+p)Q_{n+1}(x)+\gamma^0_{3n}\gamma^0_{3n+2}c^2_{n-1}(x)+(\gamma^0_{3n+2}+\gamma^0_{3n+3})c^2_n(x)\\
&+(\beta_{3n+3}+a-b-c)\Big((\beta_{3n+3}+b+c+p)a^1_n(x)+\gamma^0_{3n+2}Q_n(x)+Q_{n+1}(x)+\gamma^1_{3n+3}c^2_n(x)\Big)\\
&+\gamma^1_{3n+3}\Big(\gamma^0_{3n+1}a^1_{n-1}(x)+a^1_n(x)+\gamma^1_{3n+2}Q_n(x)+(\beta_{3n+2}+b+c+p)c^2_n(x)\Big)\Big\}.
\end{aligned}
$$

Let us denote the coefficient of $-(a-b)(a-c)$ by $\Psi(x)$. Then,

$$
\begin{aligned}
&\Psi(x) + c^2_{n+1}(x) = \gamma^1_{3n+4}a^1_n(x) + \gamma^0_{3n+3}c^2_n(x) + c^2_{n+1}(x) + (\beta_{3n+4} + a - b - c)Q_{n+1}(x) \\
&+ (\beta_{3n+3} + a - b - c)\Big((\beta_{3n+3} + a - b - c)a^1_n(x) + \gamma^1_{3n+3}c^2_n(x) + Q_{n+1}(x)\Big) \\
&+ \gamma^0_{3n+2}\Big(\gamma^1_{3n+1}a^1_{n-1}(x) + \gamma^0_{3n}c^2_{n-1}(x) + c^2_n(x) + (\beta_{3n+1} + a - b - c)Q_n(x)\Big) \\
&+ \gamma^0_{3n+2}(\beta_{3n+3} + b + c + p)Q_n(x) + (2b + 2c - a + p)Q_{n+1}(x) - La^1_n(x) \\
&+ \gamma^1_{3n+3}\Big(\gamma^0_{3n+1}a^1_{n-1}(x) + a^1_n(x) + (\beta_{3n+2} + a - b - c)c^2_n(x) + \gamma^1_{3n+2}Q_n(x)\Big) \\
&+ (2b + 2c - a + p)\gamma^1_{3n+3}c^2_n + (2b + 2c - a + p)(\beta_{3n+3} + a - b - c)a^1_n(x) \\
&\overset{(B_2),(B_8),(B_5)}{=} b^1_{n+1}(x) + Lb^2_n(x) + (\beta_{3n+3} + a - b - c)\Big(La^2_n(x) + P_{n+1}(x) - \gamma^0_{3n+2}Q_n(x)\Big) \\
&+ \gamma^0_{3n+2}\big(b^1_n(x) + Lb^2_{n-1}(x)\big) + \gamma^0_{3n+2}(\beta_{3n+3} + b + c + p)Q_n(x) + (2b + 2c - a + p)Q_{n+1}(x) \\
&- La^1_n(x) + \gamma^1_{3n+3}\big(LR_n(x) + c^1_n(x)\big) \\
&+ (2b + 2c - a + p)\Big(\gamma^1_{3n+3}c^2_n(x) + (\beta_{3n+3} + a - b - c)a^1_n(x)\Big), \\
&= b^1_{n+1}(x) + \gamma^0_{3n+2}b^1_n(x) + (\beta_{3n+3} + a - b - c)P_{n+1}(x) + \gamma^1_{3n+3}c^1_n(x) \\
&- L\Big(a^1_n(x) - (\beta_{3n+3} + b + c + p)a^2_n(x) - \gamma^0_{3n+2}b^2_{n-1}(x) - b^2_n(x) - \gamma^1_{3n+3}R_n(x)\Big) \\
&+ (2b + 2c - a + p)\Big((\beta_{3n+3} + a - b - c)a^1_n(x) - La^2_n(x) + \gamma^1_{3n+3}c^2_n(x) + \gamma^0_{3n+2}Q_n(x) + Q_{n+1}(x)\Big) \\
&\overset{(B_9),(B_8)}{=} b^1_{n+1}(x) + \gamma^0_{3n+2}b^1_n(x) + (\beta_{3n+3} + a - b - c)P_{n+1}(x) + \gamma^1_{3n+3}c^1_n(x) \\
&+ (2b + 2c - a + p)P_{n+1}(x) \\
&= b^1_{n+1}(x) + \gamma^0_{3n+2}b^1_n(x) + \gamma^1_{3n+3}c^1_n(x) + (\beta_{3n+3} + b + c + p)P_{n+1}(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Theta(x)\big(R_{n+1}(x)-P_{n+1}(x)\big)-(a-b)(a-c)c_{n+1}^{2}(x)\\
&=\Big\{-\gamma_{3n+2}^{0}-\gamma_{3n+3}^{0}-\gamma_{3n+3}^{1}(\beta_{3n+2}+2\beta_{3n+3}+p)-\gamma_{3n+4}^{1}(2\beta_{3n+3}+\beta_{3n+4}+p)\\
&-(\beta_{3n+3}-a)(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)+(\beta_{3n+3}-a)L\\
&-(a-b)(a-c)(\beta_{3n+3}+b+c+p)\Big\}P_{n+1}(x)\\
&-\Big\{\gamma_{3n+1}^{1}\gamma_{3n+2}^{1}\gamma_{3n+3}^{1}+\gamma_{3n}^{0}\gamma_{3n+2}^{0}+\gamma_{3n+1}^{0}(\gamma_{3n+2}^{0}+\gamma_{3n+3}^{0})\\
&+\gamma_{3n+1}^{0}\gamma_{3n+3}^{1}(\beta_{3n}+\beta_{3n+2}+\beta_{3n+3}+p)+\gamma_{3n+2}^{0}\gamma_{3n+1}^{1}(\beta_{3n}+\beta_{3n+1}+\beta_{3n+3}+p)\Big\}P_{n}(x)\\
&-\gamma_{3n-2}^{0}\gamma_{3n}^{0}\gamma_{3n+2}^{0}P_{n-1}(x)\\
&-\Big(\gamma_{3n-1}^{0}(\gamma_{3n+1}^{0}\gamma_{3n+3}^{1}+\gamma_{3n+2}^{0}\gamma_{3n+1}^{1})+\gamma_{3n}^{0}\gamma_{3n+2}^{0}\gamma_{3n-1}^{1}\Big)b_{n-1}^{1}(x)\\
&-\Big\{\gamma_{3n+1}^{0}\gamma_{3n+3}^{1}+\gamma_{3n+3}^{0}\gamma_{3n+2}^{1}+\gamma_{3n+2}^{1}\gamma_{3n+3}^{1}(\beta_{3n+1}+\beta_{3n+2}+\beta_{3n+3}+p)\\
&+\gamma_{3n+2}^{0}\Big(\gamma_{3n+1}^{1}+\gamma_{3n+2}^{1}+\gamma_{3n+3}^{1}+\gamma_{3n+4}^{1}+(a-b)(a-c)-L\\
&+(\beta_{3n+1}-a)(\beta_{3n+1}+b+c+p)+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)\\
&+(\beta_{3n+1}-a)(\beta_{3n+3}+a-b-c)\Big)\Big\}b_{n}^{1}(x)\\
&-\Big\{\gamma_{3n+3}^{1}+\gamma_{3n+4}^{1}-L+(a-b)(a-c)+(\beta_{3n+4}-a)(\beta_{3n+4}+b+c+p)\\
&+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)+(\beta_{3n+4}-a)(\beta_{3n+3}+a-b-c)\Big\}b_{n+1}^{1}(x)\\
&-\Big\{\gamma_{3n}^{1}(\gamma_{3n+1}^{0}\gamma_{3n+3}^{1}+\gamma_{3n+2}^{0}\gamma_{3n+1}^{1})+\gamma_{3n}^{0}\gamma_{3n+2}^{1}\gamma_{3n+3}^{1}\\
&+\gamma_{3n}^{0}\gamma_{3n+2}^{0}(\beta_{3n-1}+\beta_{3n+1}+\beta_{3n+3}+p)\Big\}c_{n-1}^{1}(x)\\
&-\Big\{\gamma_{3n+2}^{0}(\beta_{3n+1}+\beta_{3n+2}+\beta_{3n+3}+p)+\gamma_{3n+3}^{0}(\beta_{3n+2}+\beta_{3n+3}+\beta_{3n+4}+p)\\
&+\gamma_{3n+3}^{1}\Big(\gamma_{3n+2}^{1}+\gamma_{3n+3}^{1}+\gamma_{3n+4}^{1}+(a-b)(a-c)-L\\
&+(\beta_{3n+2}-a)(\beta_{3n+2}+b+c+p)+(\beta_{3n+2}-a)(\beta_{3n+3}+a-b-c)\\
&+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)\Big)\Big\}c_{n}^{1}(x)\\
&-(\beta_{3n+3}+\beta_{3n+4}+a+p)c_{n+1}^{1}(x).
\end{aligned}
\tag{4.9}
$$

Let us note that for every constants M and N the following identity holds:

$$
\begin{aligned}
(M-a)(M+N+a+p)+(N+a-b-c)(N+b+c+p)&=\\
(N-a)(M+N+a+p)+(M+a-b-c)(M+b+c+p)&.
\end{aligned}
\tag{4.10}
$$

Thus, in the relation (4.9), in the coefficient of $b_n^1(x)$ we can replace

$(\beta_{3n+1}-a)(\beta_{3n+1}+\beta_{3n+3}+a+p)+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)$ by $(\beta_{3n+3}-a)(\beta_{3n+1}+\beta_{3n+3}+a+p)+(\beta_{3n+1}+a-b-c)(\beta_{3n+1}+b+c+p)$, and in the coefficient of $c^1_n(x)$ we can replace

$(\beta_{3n+2}-a)(\beta_{3n+2}+\beta_{3n+3}+a+p)+(\beta_{3n+3}+a-b-c)(\beta_{3n+3}+b+c+p)$ by $(\beta_{3n+3}-a)(\beta_{3n+2}+\beta_{3n+3}+a+p)+(\beta_{3n+2}+a-b-c)(\beta_{3n+2}+b+c+p)$.

In brief, for $n\geq 1$, it holds

$$\begin{aligned}\Theta(x)R_{n+1}(x)-(a-b)(a-c)c^2_{n+1}(x)&=\big(\Theta(x)-\bar{A}_{3n}+\gamma^0_{3n+4}\big)P_{n+1}(x)-\bar{B}_{3n}P_n(x)\\&-\bar{C}_{3n}P_{n-1}(x)-\bar{M}_{3n}b^1_{n-1}(x)-\bar{K}_{3n}b^1_n(x)+\big(-\bar{H}_{3n}+\gamma^1_{3n+5}\big)b^1_{n+1}(x)\\&-\bar{N}_{3n}c^1_{n-1}(x)-\bar{V}_{3n}c^1_n(x)+\big(-\bar{S}_{3n}+\beta_{3n+5}-a\big)c^1_{n+1}(x).\end{aligned}\tag{4.11}$$

Let us remark that identity (4.11) was deduced by the use of all the identities of theorem 4.1, where only (B_1), (B_3) and (B_2) were involved after the transformation $n\leftarrow n+1$.

The last step of Part I consists in writing polynomial $P_{n+2}(x)+La^2_{n+1}(x)-\Theta(x)Q_{n+1}(x)$ in terms of elements of the component sequences $\{a^1_{n-1}\}_{n\geq 0}$, $\{c^2_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$.

Let us begin to note identities (B_2), (B_5) and (B_8) (respectively):

$$b^1_n(x)=\gamma^1_{3n+1}a^1_{n-1}(x)-Lb^2_{n-1}(x)+\gamma^0_{3n}c^2_{n-1}(x)+c^2_n(x)+(\beta_{3n+1}+a-b-c)Q_n(x),\ n\geq 0,$$

$$c^1_n(x)=\gamma^0_{3n+1}a^1_{n-1}(x)+a^1_n(x)+(\beta_{3n+2}+a-b-c)c^2_n(x)+\gamma^1_{3n+2}Q_n(x)-LR_n(x),\ n\geq 0,$$

$$P_{n+1}(x)=(\beta_{3n+3}+a-b-c)a^1_n(x)-La^2_n(x)+\gamma^1_{3n+3}c^2_n(x)+\gamma^0_{3n+2}Q_n(x)+Q_{n+1}(x),\ n\geq 0.$$

From (B_3) $(n\leftarrow n+1)$, multiplied by $\Theta(x)$, we have:

$$\begin{aligned}\Theta(x)R_{n+1}(x)&=\Theta(x)Q_{n+1}(x)-\gamma^0_{3n+3}\Theta(x)R_n(x)-\gamma^1_{3n+4}\Theta(x)a^2_n(x)\\&-(\beta_{3n+4}+b+c+p)\Theta(x)b^2_n(x).\end{aligned}$$

Inserting (B_4), (B_7) and (B_1) (with $n\leftarrow n+1$), we obtain:

$$\begin{aligned}&\Theta(x)R_{n+1}(x)=\Theta(x)Q_{n+1}(x)\\&-\gamma^0_{3n+3}\Big(\gamma^1_{3n+2}b^1_n(x)+(\beta_{3n+2}-a)c^1_n(x)+P_{n+1}(x)+\gamma^0_{3n+1}P_n(x)+(a-b)(a-c)c^2_n(x)\Big)\\&-\gamma^1_{3n+4}\Big(\gamma^0_{3n+2}b^1_n(x)+b^1_{n+1}(x)+\gamma^1_{3n+3}c^1_n(x)+(\beta_{3n+3}-a)P_{n+1}(x)+(a-b)(a-c)a^1_n(x)\Big)\\&-(\beta_{3n+4}+b+c+p)\Big((\beta_{3n+4}-a)b^1_{n+1}(x)+\gamma^0_{3n+3}c^1_n(x)\\&+c^1_{n+1}(x)+\gamma^1_{3n+4}P_{n+1}(x)+(a-b)(a-c)Q_{n+1}(x)\Big)\end{aligned}$$

$$\Rightarrow \Theta(x)R_{n+1}(x) = \Theta(x)Q_{n+1}(x) - \Big(\gamma^0_{3n+3}\gamma^1_{3n+2} + \gamma^0_{3n+2}\gamma^1_{3n+4}\Big)b^1_n(x)$$
$$- \Big(\gamma^1_{3n+4} + (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p)\Big)b^1_{n+1}(x)$$
$$- \Big(\gamma^0_{3n+3}(\beta_{3n+2} - a + \beta_{3n+4} + b + c + p) + \gamma^1_{3n+3}\gamma^1_{3n+4}\Big)c^1_n(x)$$
$$- (\beta_{3n+4} + b + c + p)c^1_{n+1}(x) - \gamma^0_{3n+1}\gamma^0_{3n+3}P_n(x)$$
$$- \Big(\gamma^0_{3n+3} + \gamma^1_{3n+4}(\beta_{3n+3} - a + \beta_{3n+4} + b + c + p)\Big)P^1_{n+1}(x)$$
$$- (a - b)(a - c)\Big(\gamma^0_{3n+3}c^2_n(x) + \gamma^1_{3n+4}a^1_n(x) + (\beta_{3n+4} + b + c + p)Q_{n+1}(x)\Big).$$

Let us now introduce the identities (B_2), (B_5) (both with $n \leftarrow n+1$), and (B_8) (with $n \leftarrow n-1$). Notice that the identity (B_8) holds for $n = -1$, considering $c^2_{-1} = Q_{-1} = 0$.

$$\Theta(x)R_{n+1}(x) = \Theta(x)Q_{n+1}(x)$$
$$- \Big(\gamma^0_{3n+3}\gamma^1_{3n+2} + \gamma^0_{3n+2}\gamma^1_{3n+4}\Big)\Big\{\gamma^1_{3n+1}a^1_{n-1}(x)$$
$$- Lb^2_{n-1}(x) + \gamma^0_{3n}c^2_{n-1}(x) + c^2_n(x) + (\beta_{3n+1} + a - b - c)Q_n(x)\Big\}$$
$$- \Big(\gamma^1_{3n+4} + (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p)\Big)\Big\{\gamma^1_{3n+4}a^1_n(x) - Lb^2_n(x)$$
$$+ \gamma^0_{3n+3}c^2_n(x) + c^2_{n+1}(x) + (\beta_{3n+4} + a - b - c)Q_{n+1}(x)\Big\}$$
$$- \Big(\gamma^0_{3n+3}(\beta_{3n+2} - a + \beta_{3n+4} + b + c + p) + \gamma^1_{3n+3}\gamma^1_{3n+4}\Big)\Big\{\gamma^0_{3n+1}a^1_{n-1}(x) + a^1_n(x)$$
$$+ (\beta_{3n+2} + a - b - c)c^2_n(x) - \gamma^1_{3n+2}Q_n(x) + LR_n(x)\Big\}$$
$$- (\beta_{3n+4} + b + c + p)\Big\{\gamma^0_{3n+4}a^1_n(x) + a^1_{n+1}(x) + (\beta_{3n+5} + a - b - c)c^2_{n+1}(x)$$
$$+ \gamma^1_{3n+5}Q_{n+1}(x) - LR_{n+1}(x)\Big\}$$
$$- \gamma^0_{3n+1}\gamma^0_{3n+3}\Big\{(\beta_{3n} + a - b - c)a^1_{n-1}(x) - La^2_{n-1}(x)$$
$$+ \gamma^1_{3n}c^2_{n-1}(x) + \gamma^0_{3n-1}Q_{n-1}(x) + Q_n(x)\Big\}$$
$$- \Big(\gamma^0_{3n+3} + \gamma^1_{3n+4}(\beta_{3n+3} - a + \beta_{3n+4} + b + c + p)\Big)\Big\{(\beta_{3n+3} + a - b - c)a^1_n(x)$$
$$- La^2_n(x) + \gamma^1_{3n+3}c^2_n(x) + \gamma^0_{3n+2}Q_n(x) + Q_{n+1}(x)\Big\}$$
$$- (a - b)(a - c)\Big(\gamma^0_{3n+3}c^2_n(x) + \gamma^1_{3n+4}a^1_n(x) + (\beta_{3n+4} + b + c + p)Q_{n+1}(x)\Big),$$

that is,

$$
\begin{aligned}
&\Theta(x)R_{n+1}(x) = \Theta(x)Q_{n+1}(x)\\
&-\Big(\gamma^0_{3n+1}\gamma^1_{3n+3}\gamma^1_{3n+4} + \gamma^0_{3n+2}\gamma^1_{3n+1}\gamma^1_{3n+4} + \gamma^0_{3n+3}\gamma^1_{3n+1}\gamma^1_{3n+2}\\
&+ \gamma^0_{3n+1}\gamma^0_{3n+3}(\beta_{3n} + \beta_{3n+2} + \beta_{3n+4} + p)\Big)a^1_{n-1}(x)\\
&-\Big\{\gamma^1_{3n+4}\Big(\gamma^1_{3n+3} + \gamma^1_{3n+4} + (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p) + (a-b)(a-c)\\
&+ (\beta_{3n+3} + a - b - c)(\beta_{3n+3} - a + \beta_{3n+4} + b + c + p)\Big) + \gamma^0_{3n+3}(\beta_{3n+2} + \beta_{3n+3} + \beta_{3n+4} + p)\\
&+ \gamma^0_{3n+4}(\beta_{3n+4} + b + c + p)\Big\}a^1_n(x) - (\beta_{3n+4} + b + c + p)a^1_{n+1}(x)\\
&-\Big(\gamma^0_{3n}\gamma^0_{3n+3}\gamma^1_{3n+2} + \gamma^0_{3n}\gamma^0_{3n+2}\gamma^1_{3n+4} + \gamma^0_{3n+1}\gamma^0_{3n+3}\gamma^1_{3n}\Big)c^2_{n-1}(x)\\
&-\Big\{\gamma^0_{3n+3}\Big(\gamma^1_{3n+2} + \gamma^1_{3n+3} + \gamma^1_{3n+4} + (a-b)(a-c) + (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p)\\
&+ (\beta_{3n+2} + a - b - c)(\beta_{3n+2} - a + \beta_{3n+4} + b + c + p)\Big)\\
&+ \gamma^1_{3n+4}\Big(\gamma^0_{3n+2} + \gamma^1_{3n+3}(\beta_{3n+2} + \beta_{3n+3} + \beta_{3n+4} + p)\Big)\Big\}c^2_n(x)\\
&-\Big(\gamma^1_{3n+4} + (\beta_{3n+4} + b + c + p)(\beta_{3n+4} - a + \beta_{3n+5} + a - b - c)\Big)c^2_{n+1}(x)\\
&-\gamma^0_{3n-1}\gamma^0_{3n+1}\gamma^0_{3n+3}Q_{n-1}(x)\\
&-\Big\{\gamma^0_{3n+3}\Big(\gamma^1_{3n+2}(\beta_{3n+1} + \beta_{3n+2} + \beta_{3n+4} + p) + \gamma^0_{3n+1} + \gamma^0_{3n+2}\Big)\\
&+ \gamma^0_{3n+2}\gamma^1_{3n+4}(\beta_{3n+1} + \beta_{3n+3} + \beta_{3n+4} + p) + \gamma^1_{3n+2}\gamma^1_{3n+3}\gamma^1_{3n+4}\Big\}Q_n(x)\\
&-\Big\{\gamma^0_{3n+3} + \gamma^1_{3n+4}(\beta_{3n+3} + 2\beta_{3n+4} + p) + \gamma^1_{3n+5}(\beta_{3n+4} + b + c + p)\\
&+ (\beta_{3n+4} - a)(\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p) + (a-b)(a-c)(\beta_{3n+4} + b + c + p)\Big\}Q_{n+1}(x)\\
&+ L\Big\{\Big(\gamma^0_{3n+3}\gamma^1_{3n+2} + \gamma^0_{3n+2}\gamma^1_{3n+4}\Big)b^2_{n-1}(x)\\
&+ \Big(\gamma^1_{3n+4} + (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p)\Big)b^2_n(x)\\
&+ \Big(\gamma^1_{3n+3}\gamma^1_{3n+4} + \gamma^0_{3n+3}(\beta_{3n+2} - a + \beta_{3n+4} + b + c + p)\Big)R_n(x)\\
&+ \gamma^0_{3n+1}\gamma^0_{3n+3}a^2_{n-1}(x) + (\beta_{3n+4} + b + c + p)R_{n+1}(x)\\
&+ \Big(\gamma^0_{3n+3} + \gamma^1_{3n+4}(\beta_{3n+3} - a + \beta_{3n+4} + b + c + p)\Big)a^2_n(x)\Big\}.
\end{aligned}
\tag{4.12}
$$

Let us now consider identity (B_4) $(n \leftarrow n+1)$, where $b^1_{n+1}(x)$, $c^1_{n+1}(x)$ and $P_{n+1}(x)$ are replaced by the expressions given by (B_2), (B_5) (both with $n \leftarrow n+1$) and (B_8), respectively:

$$P_{n+2}(x) = \Theta(x)R_{n+1}(x) - (a-b)(a-c)c_n^2(x)$$
$$- \gamma_{3n+4}^0\Big((\beta_{3n+3} + a - b - c)a_n^1(x) - La_n^2(x) + \gamma_{3n+3}^1 c_n^2(x) + \gamma_{3n+2}^0 Q_n(x) + Q_{n+1}(x)\Big)$$
$$- \gamma_{3n+5}^1\Big(\gamma_{3n+4}^1 a_n^1(x) - Lb_n^2(x) + \gamma_{3n+3}^0 c_n^2(x) + c_{n+1}^2(x) + (\beta_{3n+4} + a - b - c)Q_{n+1}(x)\Big)$$
$$- (\beta_{3n+5} - a)\Big(\gamma_{3n+4}^0 a_n^1(x) + a_{n+1}^1(x) + (\beta_{3n+5} + a - b - c)c_{n+1}^2(x) + \gamma_{3n+5}^1 Q_{n+1}(x) - LR_{n+1}(x)\Big).$$

Adding the term $La_{n+1}^2(x)$ and replacing $\Theta(x)R_{n+1}(x)$ by (4.12), we obtain:

$$P_{n+2}(x) + La_{n+1}^2(x) = \Theta(x)Q_{n+1}(x)$$
$$- \Big\{\gamma_{3n+1}^0\gamma_{3n+3}^1\gamma_{3n+4}^1 + \gamma_{3n+2}^0\gamma_{3n+1}^1\gamma_{3n+4}^1 + \gamma_{3n+3}^0\gamma_{3n+1}^1\gamma_{3n+2}^1$$
$$+ \gamma_{3n+1}^0\gamma_{3n+3}^0(\beta_{3n} + \beta_{3n+2} + \beta_{3n+4} + p)\Big\}a_{n-1}^1(x)$$
$$- \Big\{\gamma_{3n+4}^1\Big(\gamma_{3n+3}^1 + \gamma_{3n+4}^1 + \gamma_{3n+5}^1 + (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p) + (a-b)(a-c)$$
$$+ (\beta_{3n+3} + a - b - c)(\beta_{3n+3} - a + \beta_{3n+4} + b + c + p)\Big) + \gamma_{3n+3}^0(\beta_{3n+2} + \beta_{3n+3} + \beta_{3n+4} + p)$$
$$+ \gamma_{3n+4}^0(\beta_{3n+3} + \beta_{3n+4} + \beta_{3n+5} + p)\Big\}a_n^1(x) - (\beta_{3n+4} + b + c + p + \beta_{3n+5} - a)a_{n+1}^1(x)$$
$$- \Big(\gamma_{3n}^0\gamma_{3n+3}^0\gamma_{3n+2}^1 + \gamma_{3n}^0\gamma_{3n+2}^0\gamma_{3n+4}^1 + \gamma_{3n+1}^0\gamma_{3n+3}^0\gamma_{3n}^1\Big)c_{n-1}^2(x)$$
$$- \Big\{\gamma_{3n+3}^0\Big(\gamma_{3n+2}^1 + \gamma_{3n+3}^1 + \gamma_{3n+4}^1 + \gamma_{3n+5}^1 + (a-b)(a-c) + (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p)$$
$$+ (\beta_{3n+2} + a - b - c)(\beta_{3n+2} - a + \beta_{3n+4} + b + c + p)\Big)$$
$$+ \gamma_{3n+4}^1\Big(\gamma_{3n+2}^0 + \gamma_{3n+3}^1(\beta_{3n+2} + \beta_{3n+3} + \beta_{3n+4} + p)\Big) + \gamma_{3n+4}^0\gamma_{3n+3}^1\Big\}c_n^2(x)$$
$$- \Big\{\gamma_{3n+4}^1 + \gamma_{3n+5}^1 + (a-b)(a-c) + (\beta_{3n+4} + b + c + p)(\beta_{3n+4} - a + \beta_{3n+5} + a - b - c)$$
$$+ (\beta_{3n+5} - a)(\beta_{3n+5} + a - b - c)\Big\}c_{n+1}^2(x)$$
$$- \gamma_{3n-1}^0\gamma_{3n+1}^0\gamma_{3n+3}^0 Q_{n-1}(x)$$
$$- \Big\{\gamma_{3n+3}^0\Big(\gamma_{3n+2}^1(\beta_{3n+1} + \beta_{3n+2} + \beta_{3n+4} + p) + \gamma_{3n+1}^0 + \gamma_{3n+2}^0\Big) + \gamma_{3n+2}^0\gamma_{3n+4}^0$$
$$+ \gamma_{3n+2}^0\gamma_{3n+4}^1(\beta_{3n+1} + \beta_{3n+3} + \beta_{3n+4} + p) + \gamma_{3n+2}^1\gamma_{3n+3}^1\gamma_{3n+4}^1\Big\}Q_n(x)$$
$$- \Big\{\gamma_{3n+3}^0 + \gamma_{3n+4}^0 + \gamma_{3n+4}^1(\beta_{3n+3} + 2\beta_{3n+4} + p) + \gamma_{3n+5}^1(2\beta_{3n+4} + \beta_{3n+5} + p)$$
$$+ (\beta_{3n+4} - a)(\beta_{3n+4} + a - b - c)(\beta_{3n+4} + b + c + p) + (a-b)(a-c)(\beta_{3n+4} + b + c + p)\Big\}Q_{n+1}(x)$$

$$
\begin{aligned}
&+ L\Big\{\Big(\gamma^0_{3n+3}\gamma^1_{3n+2} + \gamma^0_{3n+2}\gamma^1_{3n+4}\Big)b^2_{n-1}(x) + a^2_{n+1}(x)\\
&+ \Big(\gamma^1_{3n+4} + \gamma^1_{3n+5} + (\beta_{3n+4} - a)(\beta_{3n+4} + b + c + p)\Big)b^2_n(x)\\
&+ \Big(\gamma^1_{3n+3}\gamma^1_{3n+4} + \gamma^0_{3n+3}(\beta_{3n+2} - a + \beta_{3n+4} + b + c + p)\Big)R_n(x)\\
&+ \gamma^0_{3n+1}\gamma^0_{3n+3}a^2_{n-1}(x) + (\beta_{3n+4} + b + c + p + \beta_{3n+5} - a)R_{n+1}(x)\\
&+ \Big(\gamma^0_{3n+3} + \gamma^0_{3n+4} + \gamma^1_{3n+4}(\beta_{3n+3} - a + \beta_{3n+4} + b + c + p)\Big)a^2_n(x)\Big\}.
\end{aligned}
$$

Let us denote by $\Phi(x)$ the polynomial constituted by all the terms with coefficient L. Then,

$$
\begin{aligned}
\Phi(x) &= \gamma^0_{3n+4}a^2_n(x) + a^2_{n+1}(x) + \gamma^1_{3n+5}b^2_n(x) + (\beta_{3n+5} + b + c + p)R_{n+1}(x)\\
&+ (\beta_{3n+4} - a)\Big(\gamma^1_{3n+4}a^2_n(x) + (\beta_{3n+4} + b + c + p)b^2_n(x) + \gamma^0_{3n+3}R_n(x) + R_{n+1}(x)\Big)\\
&+ \gamma^1_{3n+4}\Big((\beta_{3n+3} + b + c + p)a^2_n(x) + \gamma^0_{3n+2}b^2_{n-1}(x) + b^2_n(x) + \gamma^1_{3n+3}R_n(x)\Big)\\
&+ \gamma^0_{3n+3}\Big(\gamma^0_{3n+1}a^2_{n-1}(x) + a^2_n(x) + \gamma^1_{3n+2}b^2_{n-1}(x) + (\beta_{3n+2} + b + c + p)R_n(x)\Big)\\
&\overset{(B_6),(B_3),(B_9)}{=} c^2_{n+1}(x) + (\beta_{3n+4} - a)Q_{n+1}(x) + \gamma^1_{3n+4}a^1_n(x) + \gamma^0_{3n+3}c^2_n(x).
\end{aligned}
$$

Consequently, we have, for $n \geq 0$:

$$
\begin{aligned}
&P_{n+2}(x) + La^2_{n+1}(x) = \big\{\Theta(x) - \bar{A}_{3n+1} + \gamma^0_{3n+5}\big\}Q_{n+1}(x) - \bar{B}_{3n+1}Q_n(x) - \bar{C}_{3n+1}Q_{n-1}(x)\\
&- \bar{N}_{3n+1}a^1_{n-1}(x) - \bar{V}_{3n+1}a^1_n(x) + \big(- \bar{S}_{3n+1} + \beta_{3n+6} + a - b - c\big)a^1_{n+1}(x)\\
&- \bar{M}_{3n+1}c^2_{n-1}(x) - \bar{K}_{3n+1}c^2_n(x) + \big(- \bar{H}_{3n+1} + \gamma^1_{3n+6}\big)c^2_{n+1}(x).
\end{aligned}
\tag{4.13}
$$

Let us note that during the deduction of identity (4.13), the relations (B_1), (B_2), (B_3), (B_4), (B_5) and (B_6) were the only ones involved after the transformation $n \leftarrow n+1$.

Part II

($\Rightarrow$) To deduce (4.3), we consider (B_4), with $n \leftarrow n+1$, and we replace $b^1_{n+1}(x)$, $c^1_{n+1}(x)$, $P_{n+2}(x)$ and $P_{n+1}(x)$ by (4.6), (4.4) and (4.5), respectively, and $c^2_{n+1}(x)$ by (B_6) (with $n \leftarrow n+1$). To identify the relation (4.3) we just need to apply (4.10) in the coefficient of $a^2_{n+1}(x)$.

Let us again consider (B_4), with $n \leftarrow n+1$,

$$
P_{n+2}(x) = \Theta(x)R_{n+1}(x) - (a-b)(a-c)c^2_{n+1}(x) - \gamma^1_{3n+5}b^1_{n+1}(x) - (\beta_{3n+5} - a)c^1_{n+1}(x) - \gamma^0_{3n+4}P_{n+1}(x).
$$

To obtain (4.1), we only need to substitute $\Theta(x)R_{n+1}(x) - (a-b)(a-c)c^2_{n+1}(x)$ by the expression given by (4.11).

To obtain (4.2), we take (B_8), with $n \leftarrow n+1$, and we replace $P_{n+2}(x) + La^2_{n+1}(x)$ by the expression given by (4.13).

We must remark that the relation (4.1) is fulfilled for $n \geq 1$, the relation (4.2) is fulfilled for $n \geq 0$ (because (B_8) is satisfied for $n = -1$), and the relation (4.3) is fulfilled for $n \geq 0$.

Finally, the polynomials $P_1(x)$, $Q_1(x)$, $R_1(x)$, $P_2(x)$, $Q_2(x)$ and $R_2(x)$ presented were calculated using *Mathematica 6* software. These polynomials are easily computed using the procedure described in chapter 5, based on theorem 2.5, which yields theorem 4.1 when $\{W_n\}_{n\geq 0}$ is 2-orthogonal.

Part III ($\Leftarrow$) Let us suppose now the enunciated relations. We will demonstrate relations (B_3), (B_4) and (B_8) of theorem 4.1 by induction over n.

With respect to the initial conditions, we can easily observe the following:

- (B_3) is obviously satisfied for $n = 0$;
- using the polynomial $P_1(x)$ and the relations (B_0), (B_1), (B_2) we get (B_4) for $n = 0$;
- using polynomials $P_1(x)$ and $Q_1(x)$, and the relations (B_5), (B_6) and (B_2), we get (B_8) for $n = 0$.

In fact, since the relations (B_0), (B_1), (B_2), (B_5), (B_6), (B_7) and (B_9) are satisfied for all $n \geq 0$, and introducing the initial conditions $P_1(x)$, $Q_1(x)$, $R_1(x)$, $P_2(x)$, $Q_2(x)$ and $R_2(x)$ (whose expressions came from theorem 4.1), we can recursively confirm that (B_3) is satisfied for $n = 0, 1, 2$, and (B_4) and (B_8) are fulfilled for $n = 0, 1$.

Let us take as induction hypotheses that the relations (B_4) and (B_8) are fulfilled for $n \leq k$ and (B_3) is fulfilled for $n \leq k+1$, for some $k \geq 1$. Next, we will show that relations (B_4) and (B_8) are fulfilled for $n = k+1$, and (B_3) is fulfilled for $n = k+2$, which concludes the proof.

Let us begin with (B_3) for $n = k+2$. Inserting (4.3) we obtain:

$$\begin{aligned}
&R_{k+2}(x) + \gamma^0_{3k+6} R_{k+1}(x) - Q_{k+2}(x) + \gamma^1_{3k+7} a^2_{k+1}(x) + (\beta_{3k+7} + b + c + p) b^2_{k+1}(x) \\
&= \left\{\Theta(x) - \bar{A}_{3k+2}\right\} R_{k+1}(x) - \bar{B}_{3k+2} R_k(x) - \bar{C}_{3k+2} R_{k-1}(x) \\
&- \bar{M}_{3k+2} a^2_{k-1}(x) - \bar{K}_{3k+2} a^2_k(x) - \bar{H}_{3k+2} a^2_{k+1}(x) - \bar{N}_{3k+2} b^2_{k-1}(x) - \bar{V}_{3k+2} b^2_k(x) \\
&- \bar{S}_{3k+2} b^2_{k+1}(x) + \gamma^0_{3k+6} R_{k+1}(x) - Q_{k+2}(x) + \gamma^1_{3k+7} a^2_{k+1}(x) + (\beta_{3k+7} + b + c + p) b^2_{k+1}(x).
\end{aligned} \tag{4.14}$$

We will now write the difference $\Theta(x)R_{k+1}(x) - Q_{k+2}(x)$ in terms of elements of the sequences $\{R_n(x)\}_{n\geq 0}$, $\{a^2_{n-1}(x)\}_{n\geq 0}$ and $\{b^2_{n-1}(x)\}_{n\geq 0}$, using only the list of hypotheses.

By hypothesis, (B_3) is fulfilled for $n = k+1$, hence we begin to consider it, multiplied by $\Theta(x)$:

$$\Theta(x)R_{k+1}(x) + \gamma^0_{3k+3}\Theta(x)R_k(x) = \Theta(x)Q_{k+1}(x) - \gamma^1_{3k+4}\Theta(x)a^2_k(x) - (\beta_{3k+4} + b + c + p)\Theta(x)b^2_k(x).$$

Let us replace $\Theta(x)R_k(x)$, $\Theta(x)Q_{k+1}(x)$, $\Theta(x)a^2_k(x)$ and $\Theta(x)b^2_k(x)$ by (B_4), with $n = k$, (4.2), (B_7), with $n = k$, and (B_1), with $n = k+1$, respectively, yielding:

$$
\begin{aligned}
&\Theta(x)R_{k+1}(x) - Q_{k+2}(x) = \bar{N}_{3k+1}a^1_{k-1}(x) + \Big(\bar{V}_{3k+1} - (a-b)(a-c)\gamma^1_{3k+4}\Big)a^1_k(x) \\
&+ \bar{S}_{3k+1}a^1_{k+1}(x) - \Big(\gamma^0_{3k+3}\gamma^1_{3k+2} + \gamma^0_{3k+2}\gamma^1_{3k+4}\Big)b^1_k(x) \\
&- \Big(\gamma^1_{3k+4} + (\beta_{3k+4} - a)(\beta_{3k+4} + b + c + p)\Big)b^1_{k+1}(x) \\
&- \Big((\beta_{3k+2} - a)\gamma^0_{3k+3} + \gamma^1_{3k+3}\gamma^1_{3k+4} + \gamma^0_{3k+3}(\beta_{3k+4} + b + c + p)\Big)c^1_k(x) \\
&- (\beta_{3k+4} + b + c + p)c^1_{k+1}(x) + \bar{M}_{3k+1}c^2_{k-1}(x) + \Big(\bar{K}_{3k+1} - (a-b)(a-c)\gamma^0_{3k+3}\Big)c^2_k(x) \\
&+ \bar{H}_{3k+1}c^2_{k+1}(x) - \gamma^0_{3k+1}\gamma^0_{3k+3}P_k(x) \\
&- \Big(\gamma^0_{3k+3} + \gamma^1_{3k+4}(\beta_{3k+3} - a + \beta_{3k+4} + b + c + p)\Big)P_{k+1}(x) \\
&+ \bar{C}_{3k+1}Q_{k-1}(x) + \bar{B}_{3k+1}Q_k(x) + \Big(\bar{A}_{3k+1} - (a-b)(a-c)(\beta_{3k+4} + b + c + p)\Big)Q_{k+1}(x).
\end{aligned}
$$

We will now proceed with the following replacements, recalling that $k \geq 1$:

-$a^1_{k-1}(x)$, $a^1_k(x)$ and $a^1_{k+1}(x)$ by (B_9);

-$b^1_k(x)$ and $b^1_{k+1}(x)$ by (4.6);

-$c^1_k(x)$ and $c^1_{k+1}(x)$ by (4.4);

-$c^2_{k-1}(x)$, $c^2_k(x)$ and $c^2_{k+1}(x)$ by the expression given by (B_6);

-$P_k(x)$, $P_{k+1}(x)$ by (4.5); and

-$Q_{k-1}(x)$, $Q_k(x)$ and $Q_{k+1}(x)$ by the expression given by (B_3).

Therefore, the right member of identity (4.14) is fully expressed in terms of $a^2_{k-2}(x)$, $a^2_{k-1}(x)$, $a^2_k(x)$, $a^2_{k+1}(x)$, $b^2_{k-2}(x)$, $b^2_{k-1}(x)$, $b^2_k(x)$, $b^2_k(x)$, $b^2_{k+1}(x)$, $R_{k-2}(x)$, $R_{k-1}(x)$, $R_k(x)$ and $R_{k+1}(x)$, where we can see the following:

- $a^2_{k-2}(x)$ coefficient is:

$$
\begin{aligned}
&-\gamma^0_{3k-2}\gamma^0_{3k}\left(\gamma^0_{3k+3}\gamma^1_{3k+2} + \gamma^0_{3k+2}\gamma^1_{3k+4}\right) + \gamma^0_{3k-2}M_{3k+1} \\
&-\gamma^0_{3k+1}\gamma^0_{3k+3}\left(\gamma^0_{3k-2}\gamma^1_{3k} + \gamma^0_{3k-1}\gamma^1_{3k-2}\right) + \gamma^1_{3k-2}C_{3k+1};
\end{aligned}
$$

- $a_{k-1}^2(x)$ coefficient is:

$$-M_{3k+2} + N_{3k+1}(b+c+p+\beta_{3k})$$
$$-(\gamma_{3k+3}^0\gamma_{3k+2}^1 + \gamma_{3k+2}^0\gamma_{3k+4}^1)\Big(\gamma_{3k}^0 + \gamma_{3k+1}^0 + \gamma_{3k+1}^1(\beta_{3k} + \beta_{3k+1} + a + p)\Big)$$
$$-\gamma_{3k+1}^0\gamma_{3k+3}^0\Big(\gamma_{3k+4}^1 + (\beta_{3k+4} - a)(b+c+p+\beta_{3k+4})\Big) - \Big(\gamma_{3k+3}^0(-a+b+c+p$$
$$+\beta_{3k+2} + \beta_{3k+4}) + \gamma_{3k+3}^1\gamma_{3k+4}^1\Big)\Big(\gamma_{3k+1}^0(a+p+\beta_{3k}+\beta_{3k+2}) + \gamma_{3k+1}^1\gamma_{3k+2}^1\Big)$$
$$+M_{3k+1} + \gamma_{3k+1}^0(K_{3k+1} - (a-b)(a-c)\gamma_{3k+3}^0)$$
$$-\gamma_{3k+1}^0\gamma_{3k+3}^0\Big(\gamma_{3k}^1 + \gamma_{3k+1}^1 - L + (a-b-c+\beta_{3k})(b+c+p+\beta_{3k})\Big)$$
$$-\Big(\gamma_{3k+2}^0\gamma_{3k+1}^1 + \gamma_{3k+1}^0\gamma_{3k+3}^1\Big)\Big(\gamma_{3k+3}^0 + (-a+b+c+p+\beta_{3k+3}+\beta_{3k+4})\,\gamma_{3k+4}^1\Big)$$
$$+B_{3k+1}\gamma_{3k+1}^1;$$

- $a_k^2(x)$ coefficient is:

$$K_{3k+1} - K_{3k+2} - (a-b)(a-c)\gamma_{3k+3}^0 + H_{3k+1}\gamma_{3k+4}^0$$
$$-\gamma_{3k+3}^0\gamma_{3k+2}^1 + \Big(A_{3k+1} - (a-b)(a-c)(b+c+p+\beta_{3k+4})\Big)\gamma_{3k+4}^1$$
$$-\gamma_{3k+2}^0\gamma_{3k+4}^1 + (b+c+p+\beta_{3k+3})\Big(V_{3k+1} - (a-b)(a-c)\gamma_{3k+4}^1\Big)$$
$$-\Big((\beta_{3k+4}-a)(b+c+p+\beta_{3k+4}) + \gamma_{3k+4}^1\Big)\Big(\gamma_{3k+3}^0 + \gamma_{3k+4}^0 + (a+p+\beta_{3k+3}+\beta_{3k+4})\gamma_{3k+4}^1\Big)$$
$$-\Big(\gamma_{3k+3}^1 + \gamma_{3k+4}^1 - L + (a-b-c+\beta_{3k+3})(b+c+p+\beta_{3k+3})\Big)\Big(\gamma_{3k+3}^0$$
$$+(-a+b+c+p+\beta_{3k+3}+\beta_{3k+4})\gamma_{3k+4}^1\Big)$$
$$-(a+p+\beta_{3k+2}+\beta_{3k+3})\Big((-a+b+c+p+\beta_{3k+2}+\beta_{3k+4})\gamma_{3k+3}^0 + \gamma_{3k+3}^1\gamma_{3k+4}^1\Big)$$
$$-(b+c+p+\beta_{3k+4})\Big((a+p+\beta_{3k+3}+\beta_{3k+5})\gamma_{3k+4}^0 + \gamma_{3k+4}^1\gamma_{3k+5}^1\Big);$$

- $a_{k+1}^2(x)$ coefficient is:

$$H_{3k+1} - H_{3k+2} - (\beta_{3k+4}-a)(b+c+p+\beta_{3k+4}) + S_{3k+1}(b+c+p+\beta_{3k+6})$$
$$-(b+c+p+\beta_{3k+4})(a+p+\beta_{3k+5}+\beta_{3k+6}) - \gamma_{3k+4}^1 + \gamma_{3k+7}^1;$$

- $b_{k-2}^2(x)$ coefficient is:

$$C_{3k+1}(b+c+p+\beta_{3k-2}) + N_{3k+1}\gamma_{3k-1}^0 + M_{3k+1}\gamma_{3k-1}^1$$
$$-\gamma_{3k+1}^0\gamma_{3k+3}^0\Big((a+p+\beta_{3k}+\beta_{3k-2})\gamma_{3k-1}^0 + \gamma_{3k}^1\gamma_{3k-1}^1\Big)$$
$$-(\gamma_{3k}^0\gamma_{3k-1}^1 + \gamma_{3k-1}^0\gamma_{3k+1}^1)(\gamma_{3k+3}^0\gamma_{3k+2}^1 + \gamma_{3k+2}^0\gamma_{3k+4}^1)$$
$$-\gamma_{3k-1}^0\gamma_{3k+1}^0\Big((-a+b+c+p+\beta_{3k+2}+\beta_{3k+4})\gamma_{3k+3}^0 + \gamma_{3k+3}^1\gamma_{3k+4}^1\Big);$$

• $b^2_{k-1}(x)$ coefficient is:

$$N_{3k+1} - N_{3k+2} + B_{3k+1}(b+c+p+\beta_{3k+1}) - (a+p+\beta_{3k}+\beta_{3k+1})\gamma^0_{3k+1}\gamma^0_{3k+3}$$
$$-(b+c+p+\beta_{3k+4})\gamma^0_{3k+2}\gamma^0_{3k+4} + \Big(K_{3k+1} - (a-b)(a-c)\gamma^0_{3k+3}\Big)\gamma^1_{3k+2}$$
$$+\gamma^0_{3k+2}\Big(V_{3k+1} - (a-b)(a-c)\gamma^1_{3k+4}\Big) - \Big((a+p+\beta_{3k+1}+\beta_{3k+3})\gamma^0_{3k+2}$$
$$+\gamma^1_{3k+2}\gamma^1_{3k+3}\Big)\Big(\gamma^0_{3k+3} + (-a+b+c+p+\beta_{3k+3}+\beta_{3k+4})\gamma^1_{3k+4}\Big)$$
$$-\Big(\gamma^1_{3k+1} + \gamma^1_{3k+2} - L + (a-b-c+\beta_{3k+1})(b+c+p+\beta_{3k+1})\Big)\Big(\gamma^0_{3k+3}\gamma^1_{3k+2} + \gamma^0_{3k+2}\gamma^1_{3k+4}\Big)$$
$$-\Big((\beta_{3k+4}-a)(b+c+p+\beta_{3k+4}) + \gamma^1_{3k+4}\Big)\Big(\gamma^0_{3k+3}\gamma^1_{3k+2} + \gamma^0_{3k+2}\gamma^1_{3k+4}\Big)$$
$$-\Big(\gamma^0_{3k+1} + \gamma^0_{3k+2} + (a+p+\beta_{3k+1}+\beta_{3k+2})\gamma^1_{3k+2}\Big)\Big((-a+b+c+p$$
$$+\beta_{3k+2}+\beta_{3k+4})\gamma^0_{3k+3} + \gamma^1_{3k+3}\gamma^1_{3k+4}\Big);$$

• $b^2_k(x)$ coefficient is:

$$V_{3k+1} - V_{3k+2} + (b+c+p+\beta_{3k+4})\Big(A_{3k+1} - (a-b)(a-c)(b+c+p+\beta_{3k+4})\Big)$$
$$-(-a+b+c+p+\beta_{3k+2}+\beta_{3k+4})\gamma^0_{3k+3} + S_{3k+1}\gamma^0_{3k+5}$$
$$-(a-b)(a-c)\gamma^1_{3k+4} - \gamma^1_{3k+3}\gamma^1_{3k+4}$$
$$-(a+p+\beta_{3k+3}+\beta_{3k+4})\Big(\gamma^0_{3k+3} + (-a+b+c+p+\beta_{3k+3}+\beta_{3k+4})\gamma^1_{3k+4}\Big)$$
$$+H_{3k+1}\gamma^1_{3k+5} - \Big((\beta_{3k+4}-a)(b+c+p+\beta_{3k+4})$$
$$+\gamma^1_{3k+4}\Big)\Big(\gamma^1_{3k+4} + \gamma^1_{3k+5} - L + (a-b-c+\beta_{3k+4})(b+c+p+\beta_{3k+4})\Big)$$
$$-(b+c+p+\beta_{3k+4})\Big(\gamma^0_{3k+4} + \gamma^0_{3k+5} + (a+p+\beta_{3k+4}+\beta_{3k+5})\gamma^1_{3k+5}\Big);$$

• $b^2_{k+1}(x)$ coefficient is:

$$S_{3k+1} - S_{3k+2} - \beta_{3k+4} + \beta_{3k+7};$$

• $R_{k-2}(x)$ coefficient is:

$$C_{3k+1}\gamma^0_{3k-3} - \gamma^0_{3k-3}\gamma^0_{3k-1}\gamma^0_{3k+1}\gamma^0_{3k+3};$$

• $R_{k-1}(x)$ coefficient is:

$$C_{3k+1} - C_{3k+2} + M_{3k+1}(b+c+p+\beta_{3k-1}) + B_{3k+1}\gamma^0_{3k}$$
$$+N_{3k+1}\gamma^1_{3k} - \gamma^0_{3k+1}\gamma^0_{3k+3}\Big(\gamma^0_{3k} + \gamma^0_{3k-1} + (a+p+\beta_{3k}+\beta_{3k-1})\gamma^1_{3k}\Big)$$
$$-\gamma^0_{3k}\gamma^0_{3k+2}\Big(\gamma^0_{3k+3} + (-a+b+c+p+\beta_{3k+3}+\beta_{3k+4})\gamma^1_{3k+4}\Big)$$
$$-\Big((a+p+\beta_{3k-1}+\beta_{3k+1})\gamma^0_{3k} + \gamma^1_{3k}\gamma^1_{3k+1}\Big)(\gamma^0_{3k+3}\gamma^1_{3k+2} + \gamma^0_{3k+2}\gamma^1_{3k+4})$$
$$-(\gamma^0_{3k+1}\gamma^1_{3k} + \gamma^0_{3k}\gamma^1_{3k+2})\Big((-a+b+c+p+\beta_{3k+2}+\beta_{3k+4})\gamma^0_{3k+3} + \gamma^1_{3k+3}\gamma^1_{3k+4}\Big);$$

- $R_k(x)$ coefficient is:

$$
\begin{aligned}
&B_{3k+1}-B_{3k+2}+\Big(A_{3k+1}-(a-b)(a-c)(b+c+p+\beta_{3k+4})\Big)\gamma^0_{3k+3}\\
&-\gamma^0_{3k+1}\gamma^0_{3k+3}+(b+c+p+\beta_{3k+2})\Big(K_{3k+1}-(a-b)(a-c)\gamma^0_{3k+3}\Big)\\
&+\gamma^1_{3k+3}\Big(V_{3k+1}-(a-b)(a-c)\gamma^1_{3k+4}\Big)\\
&-\Big(\gamma^0_{3k+2}+\gamma^0_{3k+3}+(a+p+\beta_{3k+2}+\beta_{3k+3})\gamma^1_{3k+3}\Big)\Big(\gamma^0_{3k+3}\\
&+(-a+b+c+p+\beta_{3k+3}+\beta_{3k+4})\gamma^1_{3k+4}\Big)\\
&-(a+p+\beta_{3k+1}+\beta_{3k+2})(\gamma^0_{3k+3}\gamma^1_{3k+2}+\gamma^0_{3k+2}\gamma^1_{3k+4})\\
&-\Big((\beta_{3k+4}-a)(b+c+p+\beta_{3k+4})+\gamma^1_{3k+4}\Big)\Big((a+p+\beta_{3k+2}+\beta_{3k+4})\gamma^0_{3k+3}+\gamma^1_{3k+3}\gamma^1_{3k+4}\Big)\\
&-\Big(\gamma^1_{3k+2}+\gamma^1_{3k+3}-L+(a-b-c+\beta_{3k+2})(b+c+p+\beta_{3k+2}\Big)\Big(\gamma^1_{3k+3}\gamma^1_{3k+4}\\
&+(-a+b+c+p+\beta_{3k+2}+\beta_{3k+4})\gamma^0_{3k+3}\Big)\\
&-(b+c+p+\beta_{3k+4})(\gamma^0_{3k+4}\gamma^1_{3k+3}+\gamma^0_{3k+3}\gamma^1_{3k+5});
\end{aligned}
$$

- $R_{k+1}(x)$ coefficient is:

$$
\begin{aligned}
&A_{3k+1}-A_{3k+2}-(a-b)(a-c)(b+c+p+\beta_{3k+4})+H_{3k+1}(b+c+p+\beta_{3k+5})\\
&-\gamma^0_{3k+3}+\gamma^0_{3k+6}-(-a+b+c+p+\beta_{3k+3}+\beta_{3k+4})\gamma^1_{3k+4}\\
&-(a+p+\beta_{3k+4}+\beta_{3k+5})\Big((\beta_{3k+4}-a)(b+c+p+\beta_{3k+4})+\gamma^1_{3k+4}\Big)+S_{3k+1}\gamma^1_{3k+6}\\
&-(b+c+p+\beta_{3k+4})\Big(\gamma^1_{3k+5}+\gamma^1_{3k+6}-L+(a-b-c+\beta_{3k+5})(b+c+p+\beta_{3k+5})\Big).
\end{aligned}
$$

All the coefficients are null, as we can confirm using *Mathematica 6*.

Let us follow with identity (B_4), with $n=k+1$, and let us insert (4.1), as follows.

$$
\begin{aligned}
&P_{k+2}(x)+\gamma^0_{3k+4}P_{k+1}(x)-\Theta(x)R_{k+1}(x)+\gamma^1_{3k+5}b^1_{k+1}(x)+(\beta_{3k+5}-a)c^1_{k+1}(x)\\
&+(a-b)(a-c)c^2_{k+1}(x)\\
&=\Theta(x)P_{k+1}(x)-\Theta(x)R_{k+1}(x)+\gamma^0_{3k+4}P_{k+1}(x)-\bar{A}_{3k}P_{k+1}(x)-\bar{B}_{3k}P_k(x)-\bar{C}_{3k}P_{k-1}(x)\\
&-\bar{M}_{3k}b^1_{k-1}(x)-\bar{K}_{3k}b^1_k(x)-\bar{H}_{3k}b^1_{k+1}(x)-\bar{N}_{3k}c^1_{k-1}(x)-\bar{V}_{3k}c^1_k(x)-\bar{S}_{3k}c^1_{k+1}(x)\\
&+\gamma^1_{3k+5}b^1_{k+1}(x)+(\beta_{3k+5}-a)c^1_{k+1}(x)+(a-b)(a-c)c^2_{k+1}(x)\\
&=-\Big(\Theta(x)(R_{k+1}(x)-P_{k+1}(x))-(a-b)(a-c)c^2_{k+1}(x)\Big)-\bar{M}_{3k}b^1_{k-1}(x)-\bar{K}_{3k}b^1_k(x)\\
&+\big(\gamma^1_{3k+5}-\bar{H}_{3k}\big)b^1_{k+1}(x)-\bar{N}_{3k}c^1_{k-1}(x)-\bar{V}_{3k}c^1_k(x)+\big(\beta_{3k+5}-a-\bar{S}_{3k}\big)c^1_{k+1}(x)\\
&-\bar{C}_{3k}P_{k-1}(x)-\bar{B}_{3k}P_k(x)+\Big(\gamma^0_{3k+4}-\bar{A}_{3k}\Big)P_{k+1}(x).
\end{aligned}
\tag{4.15}
$$

We will now replace $\Theta(x)(R_{k+1}(x) - P_{k+1}(x)) - (a-b)(a-c)c^2_{k+1}(x)$ by the expression given by (4.11). Let us remark that the relation (4.11) was deduced by identities in the list of hypotheses considered, more precisely, only the relations (B_1), (B_2) and (B_3), were used after the transformation $n \leftarrow n+1$.

Therefore, in (4.15), we see that, for $k \geq 1$,

$$P_{k+2}(x) + \gamma^0_{3k+4}P_{k+1}(x) - \Theta(x)R_{k+1}(x) + \gamma^1_{3k+5}b^1_{k+1}(x) + (\beta_{3k+5} - a)c^1_{k+1}(x) \\ +(a-b)(a-c)c^2_{k+1}(x) = 0.$$

Finally, let us consider identity (B_8), with $n = k+1$, and let us insert (4.2).

$$\begin{aligned}
&Q_{k+2}(x) + \gamma^0_{3k+5}Q_{k+1}(x) - P_{k+2}(x) + (\beta_{3k+6} + a - b - c)a^1_{k+1}(x) - La^2_{k+1}(x) + \gamma^1_{3k+6}c^2_{k+1}(x) \\
&= \Theta(x)Q_{k+1}(x) - \bar{A}_{3k+1}Q_{k+1}(x) - \bar{B}_{3k+1}Q_k(x) - \bar{C}_{3k+1}Q_{k-1}(x) \\
&- \bar{M}_{3k+1}c^2_{k-1}(x) - \bar{K}_{3k+1}c^2_k(x) - \bar{H}_{3k+1}c^2_{k+1}(x) - \bar{N}_{3k+1}a^1_{k-1}(x) - \bar{V}_{3k+1}a^1_k(x) - \bar{S}_{3k+1}a^1_{k+1}(x) \\
&+ \gamma^0_{3k+5}Q_{k+1}(x) - P_{k+2}(x) + (\beta_{3k+6} + a - b - c)a^1_{k+1}(x) - La^2_{k+1}(x) + \gamma^1_{3k+6}c^2_{k+1}(x) \\
&= -\Big(P_{k+2}(x) + La^2_{k+1}(x) - \Theta(x)Q_{k+1}(x)\Big) - \bar{N}_{3k+1}a^1_{k-1}(x) - \bar{V}_{3k+1}a^1_k(x) \\
&+ \Big((\beta_{3k+6} + a - b - c) - \bar{S}_{3k+1}\Big)a^1_{k+1}(x) - \bar{M}_{3k+1}c^2_{k-1}(x) - \bar{K}_{3k+1}c^2_k(x) \\
&+ \Big(\gamma^1_{3k+6} - \bar{H}_{3k+1}\Big)c^2_{k+1}(x) - \bar{C}_{3k+1}Q_{k-1}(x) - \bar{B}_{3k+1}Q_k(x) + \Big(\gamma^0_{3k+5} - \bar{A}_{3k+1}\Big)Q_{k+1}(x).
\end{aligned} \tag{4.16}$$

Replacing, in (4.16), $P_{k+2}(x) + La^2_{k+1}(x) - \Theta(x)Q_{k+1}(x)$ by the expression (4.13), we see that, for $k \geq 1$:

$$Q_{k+2}(x) + \gamma^0_{3k+5}Q_{k+1}(x) - P_{k+2}(x) + (\beta_{3k+6} + a - b - c)a^1_{k+1}(x) - La^2_{k+1}(x) + \gamma^1_{3k+6}c^2_{k+1}(x) = 0.$$

Let us note that the application of (4.13) is adequate. In fact, we have already proved (B_3) and (B_4), and (B_8) was not used after the transformation $n \leftarrow n+1$ in the deduction of (4.13). □

Remark 4.4. *In reference [11] we find theorem 4.2 relations for $\{W_n\}_{n\geq 0}$ defined by (2.1)-(2.3), 2-orthogonal and 2-symmetric, with $a = b = c = 0$ and $\varpi(x) = x^3$. In this particular case, the principal components $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are also 2-orthogonal. Next, we will see that this particular CD is the only possible for a 2-orthogonal and 2-symmetric MPS.*

The following result gives us sufficient conditions for the 2-orthogonality of each principal component of a 2-orthogonal sequence CD. As we saw in the orthogonal case (corollary 3.4), those relations consist in having the secondary components of the corresponding column null.

Corollary 4.5. *Let $\{W_n\}_{n\geq 0}$ be a 2-orthogonal MPS given by (2.1)-(2.3). Then*

a) $b_n^1 = c_n^1 = 0,\ n \geq 0 \quad \Rightarrow \quad \{P_n\}_{n\geq 0}$ *is a* 2*-orthogonal MPS;*

b) $c_n^2 = a_n^1 = 0,\ n \geq 0 \quad \Rightarrow \quad \{Q_n\}_{n\geq 0}$ *is a* 2*-orthogonal MPS;*

c) $a_n^2 = b_n^2 = 0,\ n \geq 0 \quad \Rightarrow \quad \{R_n\}_{n\geq 0}$ *is a* 2*-orthogonal MPS.*

Proof. Considering identity (4.1) and $b_n^1 = c_n^1 = 0, n \geq 0$, we obtain a third order recurrence relation for $\{P_n\}_{n\geq 0}$. Similarly, from identities (4.2) and (4.3), hypotheses b) and c) imply a third order recurrence relation for $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, respectively. □

4.2 The diagonal cubic decomposition of a 2-orthogonal sequence

Let us now study the diagonal CD of a 2-orthogonal MPS. Unlike the orthogonal case, the diagonal CD is possible for a 2-orthogonal MPS, as the next result points out.

Theorem 4.6. *Let* $\{W_n\}_{n\geq 0}$ *be a MPS defined by (2.1)-(2.3), such that*

$$a_n^1 = a_n^2 = b_n^1 = b_n^2 = c_n^1 = c_n^2 = 0, \quad n \geq 0.$$

Then $\{W_n\}_{n\geq 0}$ *is* 2*-orthogonal if and only if the following relations are met, where* $\Theta(x)$ *is defined by (2.4).*

(b_1) $L = bc - q - (b + c + p)(b + c) = 0,$

(b_2) $(a - b)(a - c) = 0,$

(b_3) $\beta_{3n} = a, \quad n \geq 0,$

(b_4) $\beta_{3n+1} = b + c - a, \quad n \geq 0,$

(b_5) $\beta_{3n+2} = -(b + c + p), \quad n \geq 0,$

(b_6) $\gamma_n^1 = 0, \quad n \geq 1,$

(b_7) $P_{n+1}(x) = Q_{n+1}(x) + \gamma_{3n+2}^0 Q_n(x), \quad n \geq 0,$

(b_8) $Q_{n+1}(x) = R_{n+1}(x) + \gamma_{3n+3}^0 R_n(x), \quad n \geq 0,$

(b_9)

$$\begin{aligned}
&R_{n+3}(x) = \{\Theta(x) - \gamma_{3n+7}^0 - \gamma_{3n+8}^0 - \gamma_{3n+9}^0\} R_{n+2}(x) \\
&- \{\gamma_{3n+5}^0 \gamma_{3n+7}^0 + \gamma_{3n+6}^0(\gamma_{3n+7}^0 + \gamma_{3n+8}^0)\} R_{n+1}(x) - \gamma_{3n+3}^0 \gamma_{3n+5}^0 \gamma_{3n+7}^0 R_n(x), \quad n \geq 0, \\
&R_0(x) = 1, \quad R_1(x) = \Theta(x) - \gamma_1^0 - \gamma_2^0 - \gamma_3^0, \\
&R_2(x) = \{\Theta(x) - \gamma_4^0 - \gamma_5^0 - \gamma_6^0\} R_1(x) - \gamma_2^0 \gamma_4^0 - \gamma_3^0(\gamma_4^0 + \gamma_5^0).
\end{aligned}$$

Proof. Let us consider the relations of theorem 4.1, with $a_n^1 = a_n^2 = b_n^1 = b_n^2 = c_n^1 = c_n^2 = 0, \;\; n \geq 0$.

$(\tilde{B}_0)$ $\beta_0 = 0$,

$(\tilde{B}_1)$ $\gamma_{3n+1}^1 P_n(x) + (a-b)(a-c)Q_n(x) = 0$,

$(\tilde{B}_2)$ $\beta_{3n+1} + a - b - c = 0$,

$(\tilde{B}_3)$ $Q_n(x) = \gamma_{3n}^0 R_{n-1}(x) + R_n(x)$,

$(\tilde{B}_4)$ $P_{n+1}(x) + \gamma_{3n+1}^0 P_n(x) = \Theta(x) R_n(x)$,

$(\tilde{B}_5)$ $\gamma_{3n+2}^1 Q_n(x) = L R_n(x)$,

$(\tilde{B}_6)$ $\beta_{3n+2} + b + c + p = 0$,

$(\tilde{B}_7)$ $\beta_{3n+3} = a$,

$(\tilde{B}_8)$ $P_{n+1}(x) = \gamma_{3n+2}^0 Q_n(x) + Q_{n+1}(x)$,

$(\tilde{B}_9)$ $\gamma_{3n+3}^1 = 0, \;\; n \geq 0$.

Notice that the relations $(\tilde{B}_2)$, $(\tilde{B}_3)$, $(\tilde{B}_6)$ and $(\tilde{B}_8)$ correspond to identities (b_4), (b_8), (b_5) and (b_7). Also, $(\tilde{B}_7)$ and $(\tilde{B}_0)$ correspond to (b_3).

Let us replace the term $Q_n(x)$ of $(\tilde{B}_5)$ by the expression given by $(\tilde{B}_3)$:

$$\gamma_{3n+2}^1 \Big(\gamma_{3n}^0 R_{n-1}(x) + R_n(x) \Big) = L R_n(x)$$

$$\Rightarrow \gamma_{3n+2}^1 \gamma_{3n}^0 R_{n-1}(x) + \big(\gamma_{3n+2}^1 - L \big) R_n(x) = 0$$

$$\Rightarrow \gamma_{3n+2}^1 = 0 \wedge L = 0.$$

Let us now consider $(\tilde{B}_1)$, with $n \leftarrow n+1$, and replace $P_{n+1}(x)$ by the correspondent expression given by $(\tilde{B}_8)$:

$$\gamma_{3n+4}^1 \Big(\gamma_{3n+2}^0 Q_n(x) + Q_{n+1}(x) \Big) + (a-b)(a-c) Q_n(x) = 0$$

$$\Rightarrow \gamma_{3n+4}^1 \gamma_{3n+2}^0 + (a-b)(a-c) = 0 \wedge \gamma_{3n+4}^1 = 0$$

$$\Rightarrow (a-b)(a-c) = 0 \wedge \gamma_{3n+4}^1 = 0.$$

Regarding $(\tilde{B}_1)$ with $n = 0$, and since $(a-b)(a-c) = 0$, we obtain $\gamma_{3n+1}^1 = 0, \;\; n \geq 0$. Therefore, we get $\gamma_n^1 = 0, \;\; n \geq 0$, $L = 0$ and $(a-b)(a-c) = 0$.

Identities $(\tilde{B}_3)$ and $(\tilde{B}_8)$ allow us to write $P_{n+1}(x)$ in terms of elements of the sequence $\{R_n\}_{n \geq 0}$, as follows.

$$P_{n+1}(x) = \gamma_{3n+2}^0 \Big(\gamma_{3n}^0 R_{n-1}(x) + R_n(x) \Big) + \gamma_{3n+3}^0 R_n(x) + R_{n+1}(x),$$

that is,

$$P_{n+1}(x) = \gamma_{3n}^0 \gamma_{3n+2}^0 R_{n-1}(x) + \Big(\gamma_{3n+2}^0 + \gamma_{3n+3}^0 \Big) R_n(x) + R_{n+1}(x). \qquad (4.17)$$

Let us take $(\tilde{B}_4)$, with $n \leftarrow n+1$, and replace $P_{n+2}(x)$ and $P_{n+1}(x)$ by the correspondent expression given by (4.17):

$$\begin{aligned}
&\gamma^0_{3n+3}\gamma^0_{3n+5}R_n(x) + \Big(\gamma^0_{3n+5} + \gamma^0_{3n+6}\Big)R_{n+1}(x) + R_{n+2}(x) \\
&+\gamma^0_{3n+4}\Big\{\gamma^0_{3n}\gamma^0_{3n+2}R_{n-1}(x) + \Big(\gamma^0_{3n+2} + \gamma^0_{3n+3}\Big)R_n(x) + R_{n+1}(x)\Big\} = \Theta(x)R_{n+1}(x) \\
&\Leftrightarrow R_{n+2}(x) = \Big\{\Theta(x) - \gamma^0_{3n+4} - \gamma^0_{3n+5} - \gamma^0_{3n+6}\Big\}R_{n+1}(x) \\
&-\Big\{\gamma^0_{3n+3}\gamma^0_{3n+5} + \gamma^0_{3n+4}\Big(\gamma^0_{3n+2} + \gamma^0_{3n+3}\Big)\Big\}R_n(x) - \gamma^0_{3n}\gamma^0_{3n+2}\gamma^0_{3n+4}R_{n-1}(x), \quad n \geq 0.
\end{aligned}$$

To determine $R_1(x)$, we observe the following calculations: from $(\tilde{B}_4)$, with $n = 0$, we have $P_1(x) = \Theta(x) - \gamma^0_1$; from $(\tilde{B}_8)$, with $n = 0$, we have $P_1(x) = \gamma^0_2 + Q_1(x)$, that is, $Q_1(x) = \Theta(x) - \gamma^0_1 - \gamma^0_2$; and from $(\tilde{B}_3)$, with $n = 1$, we have $R_1(x) = \Theta(x) - \gamma^0_1 - \gamma^0_2 - \gamma^0_3$.

Conversely, let us suppose the relations (b_0)-(b_9) and $a^1_n = a^2_n = b^1_n = b^2_n = c^1_n = c^2_n = 0, \ n \geq 0$.

The sequence $\{W_n\}_{n\geq 0}$ is 2-orthogonal if and only if the relations (B_0)-(B_9) of theorem 4.1 are fulfilled. Introducing the hypotheses $a^1_n = a^2_n = b^1_n = b^2_n = c^1_n = c^2_n = 0, \ n \geq 0$, we have to prove that $(\tilde{B}_0)$-$(\tilde{B}_9)$ are fulfilled.

Since $\gamma^1_n = 0$, $n \geq 0$, $L = 0$ and $(a-b)(a-c) = 0$, we obviously obtain $(\tilde{B}_1)$, $(\tilde{B}_5)$ and $(\tilde{B}_9)$. Identities (b_4), (b_5) and (b_3) correspond to $(\tilde{B}_2)$, $(\tilde{B}_6)$, $(\tilde{B}_7)$ and $(\tilde{B}_0)$. Also, (b_8) and (b_7) correspond to $(\tilde{B}_3)$ and $(\tilde{B}_8)$.

With respect to identity $(\tilde{B}_4)$, notice that (b_9) is equivalent to the following identity:

$$\begin{aligned}
&\gamma^0_{3n+3}\gamma^0_{3n+5}R_n(x) + \Big(\gamma^0_{3n+5} + \gamma^0_{3n+6}\Big)R_{n+1}(x) + R_{n+2}(x) \\
&+\gamma^0_{3n+4}\Big\{\gamma^0_{3n}\gamma^0_{3n+2}R_{n-1}(x) + \Big(\gamma^0_{3n+2} + \gamma^0_{3n+3}\Big)R_n(x) + R_{n+1}(x)\Big\} = \Theta(x)R_{n+1}(x) \\
&\Leftrightarrow \gamma^0_{3n+5}\Big(\gamma^0_{3n+3}R_n(x) + R_{n+1}(x)\Big) + \gamma^0_{3n+6}R_{n+1}(x) + R_{n+2}(x) \\
&+\gamma^0_{3n+4}\Big\{\gamma^0_{3n+2}\Big(\gamma^0_{3n}R_{n-1}(x) + R_n(x)\Big) + \gamma^0_{3n+3}R_n(x) + R_{n+1}(x)\Big\} = \Theta(x)R_{n+1}(x).
\end{aligned}$$

Inserting (b_8), we get

$$\gamma^0_{3n+5}Q_{n+1}(x) + Q_{n+2}(x) + \gamma^0_{3n+4}\Big(\gamma^0_{3n+2}Q_n(x) + Q_{n+1}(x)\Big) = \Theta(x)R_{n+1}(x),$$

and inserting (b_7), we obtain $(\tilde{B}_4)$, for $n \geq 1$.

Regarding the following calculations, we prove $(\tilde{B}_4)$ for $n = 0$.

$$\begin{aligned}
P_1(x) + \gamma^0_1 - \Theta(x) \quad &\overset{(\tilde{B}_8)}{=} \gamma^0_2 + Q_1(x) + \gamma^0_1 - \Theta(x) \\
&\overset{(\tilde{B}_3)}{=} \gamma^0_2 + \gamma^0_3 + R_1(x) + \gamma^0_1 - \Theta(x) = 0.
\end{aligned}$$

□

When a 2-orthogonal MPS admits a diagonal CD, we know that the principal sequence $\{R_n\}_{n\geq 0}$ fulfils the recurrence relation (b_9) of theorem 4.6. Considering theorem 4.2, we can also write the following recurrence relations of third order for the remaining principal sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$.

$$\begin{aligned} P_{n+3}(x) &= \{\Theta(x) - \gamma^0_{3n+5} - \gamma^0_{3n+6} - \gamma^0_{3n+7}\}P_{n+2}(x) \\ &\quad - \{\gamma^0_{3n+3}\gamma^0_{3n+5} + \gamma^0_{3n+4}(\gamma^0_{3n+5} + \gamma^0_{3n+6})\}P_{n+1}(x) - \gamma^0_{3n+1}\gamma^0_{3n+3}\gamma^0_{3n+5}P_n(x), \quad n \geq 0, \end{aligned} \tag{4.18}$$

$$\begin{gathered} P_0(x) = 1, \quad P_1(x) = \Theta(x) - \gamma^0_1, \\ P_2(x) = \{\Theta(x) - \gamma^0_2 - \gamma^0_3 - \gamma^0_4\}P_1(x) - \gamma^0_1(\gamma^0_2 + \gamma^0_3); \end{gathered}$$

$$\begin{aligned} Q_{n+3}(x) &= \{\Theta(x) - \gamma^0_{3n+6} - \gamma^0_{3n+7} - \gamma^0_{3n+8}\}Q_{n+2}(x) \\ &\quad - \{\gamma^0_{3n+4}\gamma^0_{3n+6} + \gamma^0_{3n+5}(\gamma^0_{3n+6} + \gamma^0_{3n+7})\}Q_{n+1}(x) - \gamma^0_{3n+2}\gamma^0_{3n+4}\gamma^0_{3n+6}Q_n(x), \quad n \geq 0, \end{aligned} \tag{4.19}$$

$$\begin{gathered} Q_0(x) = 1, \quad Q_1(x) = \Theta(x) - \gamma^0_1 - \gamma^0_2, \\ Q_2(x) = \{\Theta(x) - \gamma^0_3 - \gamma^0_4 - \gamma^0_5\}Q_1(x) - \gamma^0_1\gamma^0_3 - \gamma^0_2(\gamma^0_3 + \gamma^0_4). \end{gathered}$$

In brief, given a 2-orthogonal MPS $\{W_n\}_{n\geq 0}$, with respect to (w_0, w_1), defined by (2.1)-(2.3), such that $a^1_n = a^2_n = b^1_n = b^2_n = c^1_n = c^2_n = 0, \quad n \geq 0$, the principal components are also 2-orthogonal, with respect to (u_0, u_1), (v_0, v_1) and (r_0, r_1), respectively.

Theorems 2.7 and 1.22 allow us to write the forms u_0, u_1, v_0, v_1, r_0 and r_1 in terms of the forms w_0 and w_1, as we next explain.

In fact, by theorem 2.7, we know that:

$$u_0 = \sigma_\varpi(w_0), \quad u_1 = \sigma_\varpi(w_3),$$

$$v_0 = \sigma_\varpi((x-a)w_1), \quad v_1 = \sigma_\varpi((x-a)w_4),$$

$$r_0 = \sigma_\varpi((x-b)(x-c)w_2), \text{ and } r_1 = \sigma_\varpi((x-b)(x-c)w_5).$$

Remark 4.7. *In reference [11] we also find these relations, stated in item (c) of theorem 2.7, for $\{W_n\}_{n\geq 0}$ defined by (2.1)-(2.3), d-orthogonal and d-symmetric, with $a = b = c = 0$ and $\varpi(x) = x^3$.*

Applying theorem 1.22, there are polynomials $\Lambda^\mu(n, \nu)$, $0 \leq \mu \leq 1$, such that

$$\begin{aligned} w_2 &= \Lambda^0(1,0)w_0 + \Lambda^1(1,0)w_1, \\ w_3 &= \Lambda^0(1,1)w_0 + \Lambda^1(1,1)w_1, \end{aligned}$$

$$w_4 = \Lambda^0(2,0)w_0 + \Lambda^1(2,0)w_1,$$
$$w_5 = \Lambda^0(2,1)w_0 + \Lambda^1(2,1)w_1,$$
where $\deg \Lambda^0(1,0) = 1$, $\deg \Lambda^1(1,0) = 0$, $\deg \Lambda^0(1,1) \leq 1$, $\deg \Lambda^1(1,1) = 1$, $\deg \Lambda^0(2,0) = 2$, $\deg \Lambda^1(2,0) \leq 1$, $\deg \Lambda^0(2,1) \leq 2$ and $\deg \Lambda^1(2,1) = 2$.

On the other hand, theorem 4.6 puts in evidence two relations of finite type. In fact, identities (b_7) and (b_8) yield the following relation:

$$P_n(x) = R_n(x) + \left(\gamma^0_{3n-1} + \gamma^0_{3n}\right) R_{n-1}(x) + \gamma^0_{3n-3}\gamma^0_{3n-1} R_{n-2}(x), \quad n \geq 2,$$

or, $P_n(x) = \displaystyle\sum_{\nu=n-2}^{n} \lambda_{n,\nu} R_\nu(x), \quad n \geq 2$, where $\lambda_{n,n-2} = \gamma^0_{3n-3}\gamma^0_{3n-1} \neq 0$, $\lambda_{n,n-1} = \gamma^0_{3n-1} + \gamma^0_{3n}$ and $\lambda_{n,n} = 1$. In other words, there is a strictly finite-type relation between sequences $\{R_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$, with respect to $\Phi(x) = 1$ (see definition 1.30).

Similarly, identity (b_8) is the following strictly finite-type relation between sequences $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, with respect to $\Phi(x) = 1$.

$$Q_n(x) = \sum_{\nu=n-1}^{n} \lambda_{n,\nu} R_\nu(x), \quad n \geq 1, \quad \text{where} \quad \lambda_{n,n-1} = \gamma^0_{3n} \neq 0, \quad \lambda_{n,n} = 1.$$

Thus, theorem 1.32 establishes the following relations between the elements of the dual sequences of the principal components, for the diagonal CD of a 2-orthogonal MPS.

$$r_m = u_m + \left(\gamma^0_{3m+2} + \gamma^0_{3m+3}\right) u_{m+1} + \gamma^0_{3m+3}\gamma^0_{3m+5} u_{m+2}, \quad m \geq 0;$$

$$r_m = v_m + \gamma^0_{3m+3} v_{m+1}, \quad m \geq 0.$$

Next, we present the possible choices of the parameters of a CD of a 2-orthogonal and 2-symmetric MPS in the diagonal case, that is, when all the secondary component sequences vanish.

Corollary 4.8. *Let $\{W_n\}_{n\geq 0}$ be a 2-orthogonal and 2-symmetric MPS defined by (2.1)-(2.3), such that $a^1_n = a^2_n = b^1_n = b^2_n = c^1_n = c^2_n = 0, \;\; n \geq 0$. Then $a = b = c = p = q = 0$.*

Proof. Attending to the relations of theorem 4.6 and since $\beta_n = 0, \;\; n \geq 0$, we obtain:

$$bc - q - (b+c+p)(b+c) = 0$$
$$(a-b)(a-c) = 0$$
$$a = 0$$
$$b + c - a = 0$$
$$-(b+c+p) = 0$$

which implies $a = b = c = p = q = 0$. □

Chapter 5

Symbolic computations in the *Mathematica* language

In the present chapter, we use symbolic computations in order to compute the first elements of each one of the nine component sequences of a CD, of a given MPS $\{W_n\}_{n\geq 0}$, and to investigate some properties of these sequences. This possibility is sustained by the relations of theorem 2.5 (page 21) which allow recursive computation of the polynomials of the nine component sequences. The software chosen to accomplish this purpose is *Mathematica* [39].

From a proper set of data, concerning the structure coefficients of the MPS $\{W_n\}_{n\geq 0}$ and the six parameters of the CD, the implementation of the relations $(Z_0) - (Z_9)$ of theorem 2.5 gives us the first elements of the nine component sequences explicitly.

Once achieved the explicit first polynomials of a component sequence, we can determine the correspondent first structure coefficients and, consequently, investigate its orthogonal character, or other. Let us notice that, with respect to the secondary components, this investigation requires previous calculations concerning the linear independence of the sequence and the normalization of each polynomial.

The *Mathematica* commands Collect, Factor or FullSimplify are often applied to the definitions, in order to obtain the results in a convenient form, that is, to get them written in the canonical basis with factorized, or mostly simplified, coefficients.

In chapter three, we considered a MOPS $\{W_n\}_{n\geq 0}$ and we discussed sufficient conditions for the orthogonality of the principal components of the correspondent CD. One of the results obtained is related to the coefficients A_n, B_n, K_n, H_n, V_n and S_n of theorem 3.2 (page 44). In fact, we can, also, define these coefficients in *Mathematica* and obtain them easily for every given MOPS.

The following sections explain each step of this path, by presenting the commands and definitions used and examples. A special attention is given to examples of symmetric sequences, for specific choices of the six parameters of the CD. From this set of examples, we formulate a list of conjectures, some of them can be proved and become results in the last section.

We remark that due to restrictions of space, we present only a sample of the results for each example. The conclusions are often taken from a larger amount of output.

In this chapter, it will be used the following list of commands and definitions of *Mathematica* [39].

Exponent[expr, x]
gives the maximum power with which x appears in the expanded form of *expr*.

Coefficient [expr, x, n]
gives the coefficient of x^n in the polynomial *expr*.

Collect[expr, x]
collects together terms involving the same powers of objects matching x.

Collect[expr,var,h]
applies h to the expression that forms the coefficient of each term obtained.

Factor[poly]
factors a polynomial over the integers.

FullSimplify[expr]
tries a wide range of transformations on *expr* involving elementary and special functions, and returns the simplest form it finds.

PolynomialRemainder[p,q,x]
gives the remainder from dividing p by q, treated as polynomials in x.

PolynomialQuotient[p,q,x]
gives the quotient of p and q, treated as polynomials in x, with any remainder dropped.

Do[expr, $\{i, i_{min}, i_{imax}, di\}$]
evaluates *expr* with the variable i successively taking on the values i_{min} through i_{imax}, in steps of di.

If[condition,t,f]
gives t if condition evaluates to True, and f if it evaluates to False.

For[start,test,incr,body]
executes *start*, then repeatedly evaluates *body* and *incr* until *test* fails to give True.

lhs=!=rhs
yields True if the expression lhs is not identical to rhs, and yields False otherwise.

lhs===rhs
yields True if the expression lhs is identical to rhs, and yields False otherwise.

expr1 $\|$ **expr2** $\| \ldots$
is the logical OR function. It evaluates its arguments in order, giving True immediately if any of them are True, and False if they are all False.

expr1 && **expr2** && ...
is the logical AND function. It evaluates its arguments in order, giving False

immediately if any of them are False, and True if they are all True.

patt /; test
is a pattern which matches only if the evaluation of test yields True.

IntegerQ[expr]
gives True if expr is an integer, and False otherwise.

Return[expr]
returns the value expr from a function.

5.1 Recursive computation of all component sequences

The relations $(Z_0) - (Z_9)$ of the constructive theorem 2.5 have been implemented in order to compute recursively the first *nmax* elements of the nine component sequences of any CD of $\{W_n\}_{n\geq 0}$, where *nmax* has a fixed value like 5 or 10, for example.

The data that must be furnished is:

- the polynomial $\varpi(x)$, by its coefficients p, q and r;

- the zeros, a, b and c, of the auxiliary polynomials;

- the structure coefficients definitions of $\{\beta_n\}_{n\geq 0}$ and $\{\chi_{n,\nu}\}_{0\leq\nu\leq n,\, n\geq 0}$ for every n, or their first elements $\{\beta_n\}_{n=0,\ldots,3nmax+1}$ and $\{\chi_{n,\nu}\}_{0\leq\nu\leq n,\, n=0,\ldots,3nmax}$, for *nmax* fixed.

The above relations $(Z_0) - (Z_9)$ are implemented as *Mathematica* functions definitions that remember values that they find, following the next syntax: $a1_{n\_,x\_} := a1_{n,x} = \cdots$, or $P_{n\_,x\_} := P_{n,x} = \cdots$, for example. This means that any computed object is automatically stored in memory and does not need to be recomputed. As mentioned before, we use the *Mathematica* commands Collect and FullSimplify, in order to get the component sequences elements written in a adequate form.

Assembling the nine component sequences in the following matrix

$$M_n(x) = \begin{pmatrix} L_{1,n}(x) \\ L_{2,n}(x) \\ L_{3,n}(x) \end{pmatrix} = \begin{pmatrix} P_n(x) & a^1_{n-1}(x) & a^2_{n-1}(x) \\ b^1_n(x) & Q_n(x) & b^2_{n-1}(x) \\ c^1_n(x) & c^2_n(x) & R_n(x) \end{pmatrix},$$

the implementation gives the first *nmax* matrices $M_0, M_1, \ldots, M_{nmax}$. Remark that we must call the definitions for fixed values of n ($n = 0, 1, 2, \ldots, nmax$). The definitions do not work for n symbolic, therefore, they do not give the direct closed formulas for the component sequences.

For each example (for each set of data), we may observe, for the first elements of component sequences, the following aspects: existence of null secondary components, the degrees of secondary components, the symmetric character, among other aspects. We note that this implementation is prepared to receive the structure coefficients of every MPS, not necessarily orthogonal. We next cite two examples of nonsymmetric sequences, where $a = b = c = p = q = r = 0$.

Introducing the structure coefficients definitions of the Laguerre sequence, with parameter $\alpha = 0$, which are: $\beta_n = 2n+1$, $\chi_{n,n} = (n+1)^2$ and $\chi_{n,\nu} = 0,\ 0 \leq \nu < n,\ n \geq 0$, we obtain for $n = 0, 1, 2$,

$$M_0(x) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix},$$

$$M_1(x) = \begin{pmatrix} x-6 & 18 & -9 \\ -16x+24 & x-96 & 72 \\ 200x-120 & -25x+600 & x-600 \end{pmatrix},$$

$$M_2(x) =$$
$$\begin{pmatrix} x^2-2400x+720 & 450x-4320 & -36x+5400 \\ -49x^2+29400x-5040 & x^2-7350x+35280 & 882x-52920 \\ 1568x^2-376320x+40320 & -64x^2+117600x-322560 & x^2-18816x+564480 \end{pmatrix},$$

respectively.

Introducing the structure coefficients definitions of the Bessel sequence, with parameter $\alpha = 1$, which are: $\beta_0 = -1$, $\beta_{n+1} = 0$, $\chi_{n,n} = -\dfrac{1}{(2n+1)(2n+3)}$ and $\chi_{n,\nu} = 0,\ 0 \leq \nu < n,\ n \geq 0$, we obtain for $n = 0, 1, 2$,

$$M_0(x) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/3 & 1 & 1 \end{pmatrix},$$

$$M_1(x) = \begin{pmatrix} x+1/15 & 2/5 & 1 \\ x+1/105 & x+2/21 & 3/7 \\ 4x/9+1/945 & x+1/63 & x+1/9 \end{pmatrix},$$

$$M_2(x) = \begin{pmatrix} x^2+\frac{4x}{33}+\frac{1}{10395} & \frac{5x}{11}+\frac{1}{495} & x+\frac{2}{99} \\ x^2+\frac{10x}{429}+\frac{1}{135135} & x^2+\frac{5x}{39}+\frac{4}{19305} & \frac{6x}{13}+\frac{2}{715} \\ \frac{7}{15}x^2+\frac{2x}{585}+\frac{1}{2027025} & x^2+\frac{x}{39}+\frac{4}{225225} & x^2+\frac{2x}{15}+\frac{2}{6435} \end{pmatrix}, \text{ respectively.}$$

For these two examples, observing $M_n(x),\ n = 0, \ldots, 6$, we can remark that there are not secondary components null and that $\deg a^1_{n-1}(x) = \deg a^2_{n-1}(x) = \deg b^2_{n-1}(x) = n-1$ and $\deg b^1_n(x) = \deg c^1_n(x) = \deg c^2_n(x) = n$.

The above implementation which generates the component sequences of a precise CD of a MPS $\{W_n\}_{n\geq 0}$ can be verified, up to a certain degree, either for symbolic structure coefficients, or for the structure coefficients of a concrete sequence (see in another case [36]). The following commands establish this verification; here we begin to define the construction of the sequence $\{W_n\}_{n\geq 0}$ through its structure coefficients.

$$W_{0,x\_} := W_{0,x} = 1;$$
$$W_{1,x\_} := W_{1,x} = x - \beta_0;$$
$$W_{n\_/;And[IntegerQ[n],n>1],\,x\_} := W_{n,x} = (x - \beta_{n-1})W_{n-1,x} - \sum_{\nu=0}^{n-2} \chi_{n-2,\nu} W_{\nu,x};$$

$$\begin{aligned}
&\text{verificationCD}_{nmax_/;And[IntegerQ[nmax],nmax>0]} := \text{verificationCD}_{nmax} = \text{Module}[\\
&\quad \{n,\ answer = \text{Table}[True, \{n, 0, nmax\}]\},\\
&\quad \text{Table}[\ \text{FullSimplify}\ [\\
&\quad W_{3\,n,x} - \Big(P_{n,\varpi[x]} + (x-a)\, a1_{n-1,\varpi[x]} + (x-b)\,(x-c)\, a2_{n-1,\varpi[x]}\Big)\ ] === 0\ \ \&\&\\
&\qquad \text{FullSimplify}[\\
&\quad W_{3\,n+1,x} - \Big(b1_{n,\varpi[x]} + (x-a)\, Q_{n,\varpi[x]} + (x-b)\,(x-c)\, b2_{n-1,\varpi[x]}\Big)\ ] === 0\ \ \&\&\\
&\qquad \text{FullSimplify}\ [\\
&\quad W_{3\,n+2,x} - \Big(c1_{n,\varpi[x]} + (x-a)\, c2_{n,\varpi[x]} + (x-b)\,(x-c)\, R_{n,\varpi[x]}\Big)\ ] === 0\ \ \&\&\\
&\quad , \{n, 0, nmax\}](*end\ of\ \text{Table}*)\\
&\qquad === answer]; (*end\ of\ \text{Module}*)
\end{aligned}$$

where, for instance, $a1_{n-1,x}$ denotes the generic element of the secondary component $\{a^1_{n-1}\}_{n\geq 0}$.

The calling statement to get this verification up to 5, for example, is the following

$$\text{verificationCD}_5$$

and we obtain the answer $True$.

5.2 The principal components

Given a MPS, the previous implementation of the relations of theorem 2.5 gives us the first elements of every component sequence, written in the canonical basis, thus, we can construct a command that investigates if a principal component sequence fulfils a recurrence relation of second order of type (1.9-1.10), that is, if a principal component sequence is orthogonal, using the command PolynomialRemainder.

In order to explain the command for the sequence $\{\zeta_{n,x}\}_{n\geq 0}$, let us divide $\zeta_{n+2,x}$ by $\zeta_{n+1,x}$ and denote by $rem1_{\zeta,n,x}$ the correspondent remainder, that is,

$$rem1_{\zeta\_,n\_/;And[IntegerQ[n],n<0],x\_} := rem1_{\zeta,n,x} = 0;$$

$rem1_{\zeta\_,n\_,x\_} := rem1_{\zeta,n,x} =$

Collect[FullSimplify[PolynomialRemainder $[\zeta_{n+2,x},\ \zeta_{n+1,x},\ x]$], x, Factor];

Considering also the following remainder:

$rem2_{\zeta\_,n\_/;And[IntegerQ[n],n<0],x\_} := rem2_{\zeta,n,x} = 0;$

$rem2_{\zeta\_,n\_,x\_} := rem2_{\zeta,n,x} =$

Collect[FullSimplify[PolynomialRemainder $[rem1_{\zeta,n,x},\ \zeta_{n,x},\ x]$], x, Factor];

the recurrence relation

$$\zeta_{n+2}(x) = (x - \beta^{\zeta}_{n+1})\zeta_{n+1}(x) - \gamma^{\zeta}_{n+1}\zeta_n(x) \tag{5.1}$$

is fulfilled if and only if the degree of $rem1_{\zeta,n,x}$ is equal to n and $rem2_{\zeta,n,x} = 0$.

Given $nmax > 0$, the following command called OrthoPCdirectTest$_{\zeta,nmax}$ determines if the above remainders fulfil the requested for $n = 0, \ldots, nmax$, giving as output the message "$\{\zeta_n\}_{n\geq 0}$ is not orthogonal" or " $\{\zeta_n\}_{n\geq 0}$ satisfies the orthogonal recurrence relation of second order up to $nmax$".

Next, we present the command, remarking that it is valid for any MPS $\{\zeta_{n,x}\}_{n\geq 0}$, although it will be applied only to the principal components.

```
OrthoPCdirectTest_{ζ_,nmax_/;And[IntegerQ[nmax],nmax>0]} := Module[{x},
  Do[ If [Exponent [rem1_{ζ,n,x}, x] < n || rem2_{ζ,n,x} =! = 0,
       Print ["{", ζ, "_n}_{n≥0} is not orthogonal "] && Goto [end] ,
      Continue[ ] ], {n, 0, nmax}
      ]; (*end of Do*)
  Print ["{", ζ, "_n}_{n≥0} satisfies the orthogonal recurrence relation
  of second order up to", nmax ];
  Label[end];
]; (*end of Module*)
```

Obviously, we can not conclude that a principal component is orthogonal, because this iterative process is finite; nevertheless we can conclude that a principal component is not orthogonal, for the chosen numeric set of parameters a, b, c, p, q, r.

In general, given a MPS $\{\zeta_{n,x}\}_{n\geq 0}$ not orthogonal, we can calculate its structure coefficients defined by (1.1-1.2). In fact, in the majority of the sequences and parameters tested, we conclude that the three principal components are not orthogonal, therefore, the following commands were implemented in order to calculate the structure coefficients of any MPS.

Given $n \geq 0$, we define, as follows, βSC$_{\zeta,n}$ which calculates the coefficient β^{ζ}_n and χSC$_{\zeta,n,\nu}$ which corresponds to the coefficient $\chi^{\zeta}_{n,\nu}$. The last command, called PrintSC$_{\zeta,nmax}$, only prints the set of structure coefficients β_n^ζ and $\{\chi_{n,\nu}^\zeta,\ 0 \leq \nu \leq n\}$ of the MPS $\{\zeta_{n,x}\}_{n\geq 0}$, for $n = 0$ to $n = nmax$, where $nmax$ is a given non-negative integer.

$$\begin{aligned}&\beta\mathrm{SC}_{\zeta\_,n\_/;And[IntegerQ[n],n\geq 0]} := \beta\mathrm{SC}_{\zeta,n} = \mathrm{Module}[\{x\},\\ &\quad \mathrm{Return}[\mathrm{Factor}[\mathrm{FullSimplify}[x - \mathrm{PolynomialQuotient}[\zeta_{n+1,x},\ \zeta_{n,x},\ x]\,]\,]\,]\\ &];\end{aligned} \tag{5.2}$$

$$\begin{aligned}&\chi\mathrm{SC}_{\zeta\_,n\_/;And[IntegerQ[n],n\geq 0],\nu\_} := \chi\mathrm{SC}_{\zeta,n,\nu} = \mathrm{Module}[\{x\},\\ &\mathrm{Return}[-\mathrm{Factor}[\ \mathrm{FullSimplify}[\ \mathrm{PolynomialQuotient}[rem1_{\zeta,n,x}\\ &\qquad\qquad + \sum_{i=\nu+1}^{n} \chi\mathrm{SC}_{\zeta,n,i}\ \zeta_{i,x},\ \zeta_{\nu,x},\ x\ ]\,]\,]\,];\\ &];\end{aligned} \tag{5.3}$$

$$\begin{aligned}&\mathrm{PrintSC}_{\zeta\_,nmax\_/;And[IntegerQ[nmax],nmax\geq 0]} := \mathrm{Module}[\{\},\\ &\mathrm{Do}[\mathrm{Print}[\beta_n^\zeta, " = ", \beta\mathrm{SC}_{\zeta,n}], \{n, 0, nmax\}];\\ &\mathrm{Do}[\mathrm{Print}[\chi_{n,\nu}^\zeta, " = ", \chi\mathrm{SC}_{\zeta,n,\nu}], \{n, 0, nmax\}, \{\nu, 0, n\}]\\ &];\end{aligned} \tag{5.4}$$

If a MPS $\{\zeta_{n,x}\}_{n\geq 0}$ satisfies the recurrence relation of second order (5.1) for $n = 0, \ldots, nmax$, then we might determine more efficiently its recurrence coefficients, using the following commands.

$$\begin{aligned}&\gamma\mathrm{RC}_{\zeta\_,n\_/;And[IntegerQ[nmax],nmax\geq 1]} := \gamma\mathrm{RC}_{\zeta,n} = \mathrm{Module}[\{x\},\\ &\mathrm{Return}[\,-\mathrm{Factor}\,[\mathrm{PolynomialQuotient}\,[rem1_{\zeta,n-1,x}, \zeta_{n-1,x},\ x]]\,]\,];\end{aligned}$$

$$\begin{aligned}&\mathrm{PrintRC}_{\zeta\_,nmax\_/;And[IntegerQ[nmax],nmax\geq 0]} := \mathrm{Module}[\{\},\\ &\mathrm{Do}[\mathrm{Print}[\beta_n^\zeta, " = ", \beta\mathrm{SC}_{\zeta,n}], \{n, 0, nmax\}];\\ &\mathrm{Do}[\mathrm{Print}[\gamma_n^\zeta, " = ", \gamma\mathrm{RC}_{\zeta,n}], \{n, 1, nmax\}]\\ &];\end{aligned} \tag{5.5}$$

If the given MPS is orthogonal, we can also search for conditions that assure the principal components orthogonality by computing the coefficients A_n, B_n, K_n, H_n, V_n and S_n

of theorem 3.2. Recall that if $K_n = H_n = V_n = S_n = 0,\ n \geq 0$, then all the principal components are orthogonal. If we are able to give the definitions of the structure coefficients, for every n, then we can compute these coefficients for all n, that is, for symbolic n, and not only for the first $nmax$ values. Hence, the implementation of these coefficients allow us to obtain the corresponding closed formulas, for each MOPS. We try to write them in a convenient form using the commands FullSimplify and Factor.

Finally, let us remark that the commands presented in this section can also be the subject of a verification similar to the one described in the previous section for the general implementation of theorem 2.5 relations.

5.3 The secondary components

The secondary components are not necessarily free sequences, that is, the degree of a^1_{n-1}, a^2_{n-1} and b^2_{n-1} can be less than $n-1$; and the degree of b^1_n, c^1_n and c^2_n can be less than n. Therefore, properties like orthogonality will only make sense for the ones that are PSs. In order to investigate this aspect, we consider, in *Mathematica*, the following definition:

$$\deg_{\zeta\_,n\_} := \deg_{\zeta,n} = \text{Module}[\{x\}, \text{Exponent}\,[\text{FullSimplify}[\zeta_{n,x}], x]\,]; \tag{5.6}$$

where the argument ζ will be replaced by a^1, a^2, b^1, b^2, c^1 and c^2. Notice that, for the command Exponent, we have Exponent$[0, x] = -\infty$, which is helpful to distinguish a non null constant polynomial from the null polynomial. When, for example, the polynomial(s) of degree zero of the sequence in study is null, we do not have a free sequence.

For some secondary components we have: $\deg_{\zeta,0} = -\infty$, $\deg_{\zeta,1} = 0$, $\deg_{\zeta,2} = 1$, $\deg_{\zeta,3} = 2$, for instance, giving us the idea that the sequence

$$l\zeta_{n,x} = \zeta_{n+1,x},\ n \geq 0,$$

might be a free sequence. Hence, we define these $l\zeta$ sequences, and the degree of each element, of these new sequences, is reported by the definition (5.6), considering $\zeta = la^1, la^2, lb^1, lb^2, lc^1$ and lc^2.

In order to list the first values of (5.6), we define a command called SCDegreeTest$_{\zeta,nmax}$ that, given a non-negative integer $nmax$, prints, for i from 0 to $nmax$, the constant $\deg_{\zeta,i}$.

When the application of the command SCDegreeTest$_{\zeta,nmax}$ indicates that the sequence $\{\zeta_{n,x}\}_{n\geq 0}$ might be free, that is, the elements $\zeta_{0,x}, \zeta_{1,x}, \ldots, \zeta_{nmax,x}$ constitute a basis of $\mathcal{P}_{nmax}$ (vectorial space of the polynomial functions of maximum degree $nmax$), we can normalize the sequence and investigate some properties, like we did before for the principal components.

Thus, we define the leading coefficient of a polynomial $\zeta_{n,x}$, using the command Coefficient[expr, x, n] as follows.

$Lcoef_{\zeta\_,n\_/;And[IntegerQ[n],n<0]} := Lcoef_{\zeta,n} = 0;$

$Lcoef_{\zeta\_,n\_} := Lcoef_{\zeta,n} = \text{Module}\ [\{x\}, \text{FullSimplify}[\text{Coefficient}\,[\zeta_{n,x}, x, n]]\,];$

The normalized (monic) sequences are called $m\zeta$, where

$$\zeta = a^1, a^2, b^1, b^2, c^1, c^2,\ la^1, la^2, lb^1, lb^2, lc^1 \text{ and } lc^2,$$

and defined by:

$$m\zeta_{n\_,x\_} := m\zeta_{n,x} = \text{Collect}\left[\text{FullSimplify}\left[\frac{1}{Lcoef_{\zeta,n}}\zeta_{n,x}\right], x, \text{Factor}\right];$$

Consequently, assuming that a sequence $\{m\zeta_{n,x}\}_{n\geq 0}$ is a MPS, we can calculate the correspondent structure coefficients using the commands (5.2 - 5.4), introduced in the previous section.

5.4 Some orthogonal examples

In this section, we present the results of the implementation of the commands described in the previous sections for several orthogonal sequences. For this reason, we will abandon the indication $\chi_{n,\nu} = 0,\ 0 \leq \nu < n,\ n \geq 0$, and only point out the recurrence coefficients β_n and $\gamma_{n+1},\ \ n \geq 0$. In fact, if we deal exclusively with orthogonal sequences, we can take the constructive implementation of theorem 3.1 instead of theorem 2.5 relations, having no changes of results and making possible the systematic use of the recurrence coefficients β_n and $\gamma_{n+1},\ \ n \geq 0$, as input data, and also, improving the time execution. Furthermore, in numerical cases, the use of the constructive implementation of theorem 3.1 is crucial in order to minimize round-off effects.

5.4.1 Orthogonality of all the principal components

When the given sequence $\{W_n\}_{n\geq 0}$ is orthogonal, from a corollary of theorem 3.2, we know that if $K_n = H_n = V_n = S_n = 0,\ n \geq 0$, then all the principal components are orthogonal. In fact, the observation of their expressions can reveal what values of the parameters imply the orthogonality of the three principal components by vanishing these four coefficients.

Next, we present the coefficients of theorem 3.2, for all parameters a, b, c, p, q and r, and for some widely known orthogonal sequences.

- The Hermite sequence, for which $\beta_n = 0$ and $\gamma_{n+1} = \frac{1}{2}(n+1)$, $n \geq 0$:

$$\begin{aligned}
&\Theta(x) - A_n = -\frac{1}{2}(7+2n)p - r + x,\\
&B_n = \frac{1}{8}(1+n)(2+n)(3+n),\\
&K_n = \frac{1}{4}(2+n)(3+n)p,\\
&H_n = \frac{1}{2}(12+3n+2q),\\
&V_n = \frac{1}{4}(3+n)(9+3n+2q),\\
&S_n = p.
\end{aligned}$$

- The first kind Tchebyshev sequence, for which $\beta_n = 0$, $\gamma_1 = \frac{1}{2}$, $\gamma_{n+2} = \frac{1}{4}$, $n \geq 0$:

$$\begin{aligned}
&\Theta(x) - A_n = -\frac{p}{2} - r + x,\\
&B_0 = \frac{1}{32}, \quad B_n = \frac{1}{64}, \; n \geq 1,\\
&K_n = \frac{p}{16},\\
&H_n = \frac{1}{4}(3+4q),\\
&V_n = \frac{1}{16}(3+4q),\\
&S_n = p.
\end{aligned}$$

- The second kind Tchebyshev sequence, for which $\beta_n = 0$, $\gamma_{n+1} = \frac{1}{4}$, $n \geq 0$:

$$\begin{aligned}
&\Theta(x) - A_n = -\frac{p}{2} - r + x,\\
&B_n = \frac{1}{64},\\
&K_n = \frac{p}{16},\\
&H_n = \frac{1}{4}(3+4q),\\
&V_n = \frac{1}{16}(3+4q),\\
&S_n = p.
\end{aligned}$$

- The monic modified Lommel sequence, for which, $\beta_n = 0$, $\gamma_{n+1} = \frac{1}{4(n+\alpha+1)(n+\alpha)}$, $n \geq 0$, $\alpha \neq -n$:

$$\begin{aligned}
\Theta(x) - A_n &= -\frac{p + 2(r-x)\big(8 + n^2 + 6\alpha + \alpha^2 + 2n(3+\alpha)\big)}{2\big(8 + n^2 + 6\alpha + \alpha^2 + 2n(3+\alpha)\big)},\\
B_n &= \frac{1}{64(n+\alpha)(1+n+\alpha)^2(2+n+\alpha)^2(3+n+\alpha)},\\
K_n &= \frac{p}{16(1+n+\alpha)(2+n+\alpha)^2(3+n+\alpha)},\\
H_n &= \frac{3 + 40q + 28nq + 4n^2q + 28q\alpha + 8nq\alpha + 4q\alpha^2}{4(2+n+\alpha)(5+n+\alpha)},\\
V_n &= \frac{3 + 16q + 20nq + 4n^2q + 20q\alpha + 8nq\alpha + 4q\alpha^2}{16(1+n+\alpha)(2+n+\alpha)(3+n+\alpha)(4+n+\alpha)},\\
S_n &= p.
\end{aligned}$$

In the above symmetric examples, only the Tchebyshev sequences yielded a choice of parameters that can guarantee the principal components orthogonality, as we referred in example 3.8, that is, fixing $p = 0$ and $q = -\frac{3}{4}$.

Setting $p = 0$ and $q = -\frac{3}{4}$, we see also that the first matrices M_n indicate two secondary components null, more precisely, the secondary components $a^1_{n-1}(x)$ and $a^2_{n-1}(x)$ for the first kind Tchebyshev sequence (see proposition 3.25) and the secondary components $a^2_{n-1}(x)$ and $b^2_{n-1}(x)$ for the second kind Tchebyshev sequence. Still in this section we will present more aspects of the CD of the second kind Tchebyshev sequence for this choice of parameters and other ones. We present next the matrices M_0 and M_1 for the first kind and the second kind Tchebyshev sequences, respectively.

$$M_0(x) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ -bc + a(b+c) - \frac{1}{2} & b+c & 1 \end{pmatrix},$$

$$M_1(x) = \begin{pmatrix} x - r & 0 & 0 \\ \frac{1}{8}(2bc - 2a(b+c+4r)+1) + ax & \frac{1}{4}(-b-c-4r) + x & -\frac{1}{4} \\ -(b+c)ra - \frac{a}{16} + \left(bc + \frac{1}{2}\right) r + \left(-bc + a(b+c) - \frac{1}{2}\right) x & -(b+c)r + (b+c)x - \frac{1}{16} & x - r \end{pmatrix};$$

$$M_0(x) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ -bc + a(b+c) - \frac{1}{4} & b+c & 1 \end{pmatrix},$$

$$M_1(x) = \begin{pmatrix} \frac{1}{4}(a - 4r) + x & \frac{1}{4} & 0 \\ -ar + ax + \frac{1}{16} & x - r & 0 \\ \frac{1}{4}(4bc - 4a(b+c)+1)r + \left(-bc + a(b+c) - \frac{1}{4}\right) x & (b+c)x - (b+c)r & x - r \end{pmatrix}.$$

Let us now consider the MOPS $\{W_n\}_{n\geq 0}$ such that $\beta_n = \beta$, $\gamma_{n+1} = \alpha$, $n \geq 0$, $\alpha \neq 0$. Notice that this sequence is a shift of the second kind Tchebychev polynomials. More precisely, $W_n(x) = A^{-n}U_n(Ax+B)$, $n \geq 0$, where $A = \frac{1}{2\sqrt{\alpha}}$, $B = -\frac{\beta}{2\sqrt{\alpha}}$ and $\{U_n\}_{n\geq 0}$ denotes the second kind Tchebyshev polynomials. Then,

$$\begin{aligned}
&\Theta(x) - A_n = x - r - 2p\alpha - q\beta - 6\alpha\beta - p\beta^2 - \beta^3,\\
&B_n = \alpha^3,\\
&K_n = \alpha^2(p+3\beta),\\
&H_n = q + 3\alpha + 2p\beta + 3\beta^2,\\
&V_n = \alpha(q + 3\alpha + 2p\beta + 3\beta^2),\\
&S_n = p + 3\beta.
\end{aligned}$$

Consequently, if $p = -3\beta$ and $q = -3\alpha + 3\beta^2$, then $K_n = H_n = V_n = S_n = 0$ and by theorem 3.2, the principal components are orthogonal. Also, we can write precisely the recurrence coefficients of the principal components (see theorem 3.2), as follows.

$$\begin{aligned}
&\beta_0^P = r - a\alpha - 2\alpha\beta + \beta^3, \;\; \beta_n^P = r - 3\alpha\beta + \beta^3, \;\; n \geq 1,\\
&\beta_n^Q = \beta_n^R = r - 3\alpha\beta + \beta^3, \;\; n \geq 0,\\
&\gamma_{n+1}^P = \gamma_{n+1}^Q = \gamma_{n+1}^R = \alpha^3, \;\; n \geq 0.
\end{aligned}$$

5.4.2 Orthogonality tests and structure coefficients of the component sequences

In this subsection we consider some orthogonal sequences and particular choices of the CD parameters, like, for example, choose $a = b = c = 0$ and leave the parameters p, q and r free, or choose $p = q = r = 0$ and leave the parameters a, b and c free, and present the results obtained concerning the commands OrthoPCdirectTest$_{\zeta,nmax}$, PrintSC$_{\zeta,nmax}$ and SCDegreeTest$_{\zeta,nmax}$, which were explained in the sections 5.2 and 5.3. The examples taken are the Hermite and the second kind Tchebyshev polynomials that are symmetric, and we analyse only one nonsymmetric sequence: a shift of the second kind Tchebyshev polynomials, presented above.

Hermite sequence

The recurrence coefficients of the Hermite sequence are: $\beta_n = 0$, $\gamma_{n+1} = \frac{1}{2}(n+1)$ $n \geq 0$.

- Let us set **a=b=c=p=q=r=0**. Then, for $n = 0, 1$, we have:

$$M_0(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix},$$

$$M_1(x) = \left(\begin{array}{lll} x & -\frac{3}{2} & 0 \\ \frac{3}{4} & x & -3 \\ -5x & \frac{15}{4} & x \end{array} \right).$$

The principal components are not orthogonal:

OrthoPCdirectTest$_{P,3} \to \{P_n\}_{n\geq 0}$ is not orthogonal
OrthoPCdirectTest$_{Q,3} \to \{Q_n\}_{n\geq 0}$ is not orthogonal
OrthoPCdirectTest$_{R,3} \to \{R_n\}_{n\geq 0}$ is not orthogonal

Calling **PrintSC**$_{P,4}$, we obtain:

$\beta_0^P = 0$, $\beta_1^P = 0$, $\beta_2^P = 0$, $\beta_3^P = 0$, $\beta_4^P = 0$,
$\chi_{0,0}^P = \frac{15}{8}$,
$\chi_{1,0}^P = 0$, $\chi_{1,1}^P = \frac{1245}{8}$,
$\chi_{2,0}^P = \frac{178605}{64}$, $\chi_{2,1}^P = 0$, $\chi_{2,2}^P = 1575$,
$\chi_{3,0}^P = 0$, $\chi_{3,1}^P = \frac{36205785}{32}$, $\chi_{3,2}^P = 0$, $\chi_{3,3}^P = \frac{61215}{8}$,
$\chi_{4,0}^P = \frac{37380368025}{512}$, $\chi_{4,1}^P = 0$, $\chi_{4,2}^P = \frac{2630673045}{64}$, $\chi_{4,3}^P = 0$, $\chi_{4,4}^P = \frac{203385}{8}$.

Calculating **PrintSC**$_{Q,4}$, we obtain:

$\beta_0^Q = 0$, $\beta_1^Q = 0$, $\beta_2^Q = 0$, $\beta_3^Q = 0$, $\beta_4^Q = 0$,
$\chi_{0,0}^Q = \frac{105}{8}$,
$\chi_{1,0}^Q = 0$, $\chi_{1,1}^Q = \frac{3045}{8}$,
$\chi_{2,0}^Q = \frac{2236815}{64}$, $\chi_{2,1}^Q = 0$, $\chi_{2,2}^Q = \frac{11295}{4}$,
$\chi_{3,0}^Q = 0$, $\chi_{3,1}^Q = \frac{278513235}{64}$, $\chi_{3,2}^Q = 0$, $\chi_{3,3}^Q = \frac{23595}{2}$,
$\chi_{4,0}^Q = \frac{684175667175}{512}$, $\chi_{4,1}^Q = 0$, $\chi_{4,2}^Q = \frac{859864005}{8}$, $\chi_{4,3}^Q = 0$, $\chi_{4,4}^Q = \frac{71715}{2}$.

Calculating **PrintSC**$_{R,4}$, we obtain:

$\beta_0^R = 0$, $\beta_1^R = 0$, $\beta_2^R = 0$, $\beta_3^R = 0$, $\beta_4^R = 0$,
$\chi_{0,0}^R = \frac{105}{2}$,
$\chi_{1,0}^R = 0$, $\chi_{1,1}^R = \frac{3255}{4}$,
$\chi_{2,0}^R = \frac{15062355}{64}$, $\chi_{2,1}^R = 0$, $\chi_{2,2}^R = \frac{38115}{8}$,
$\chi_{3,0}^R = 0$, $\chi_{3,1}^R = \frac{910945035}{64}$, $\chi_{3,2}^R = 0$, $\chi_{3,3}^R = \frac{140595}{8}$,
$\chi_{4,0}^R = \frac{3298283863875}{256}$, $\chi_{4,1}^R = 0$, $\chi_{4,2}^R = \frac{8290937655}{32}$, $\chi_{4,3}^R = 0$, $\chi_{4,4}^R = 49470$.

We see also that for each principal component, we have $\beta_n^\zeta = 0$ and $\chi_{2n+1,2\nu}^\zeta = 0,\ 0 \leq \nu \leq n$, $\chi_{2n,2\nu+1}^\zeta = 0,\ 0 \leq \nu \leq n-1$, where $\zeta = P, Q$ and R.

Further results could be presented for the secondary components; nonetheless, we will exemplify the analysis of those components for the more general choices of parameters $a = b = c = 0$ and $p = q = r = 0$.

• Let us set only **a=b=c=0**. Then, for $n = 0, 1$, we have:

$$M_0(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix},$$

$M_1(x) =$

$$\begin{pmatrix} x - r & -q - \frac{3}{2} & -p \\ pr - px + \frac{3}{4} & pq - r + x & p^2 - q - 3 \\ (-p^2 + q + 5)\, r + (p^2 - q - 5)\, x & q\,(-p^2 + q + 5) + pr - px + \frac{15}{4} & -p^3 + 2qp + 5p - r + x \end{pmatrix}.$$

Calling $\mathbf{PrintSC}_{P,2}$, we obtain:

$\beta_0^P = r,$

$\beta_1^P = \frac{1}{2}(-15p + 2p^3 - 4pq + 2r),$

$\beta_2^P = \frac{1}{2}(-57p + 6p^3 - 8pq + 2r),$

$\chi_{0,0}^P = \frac{15}{8},$

$\chi_{1,0}^P = \frac{15}{16}p(-57 + 6p^2 - 8q),$

$\chi_{1,1}^P = \frac{1}{8}(1245 + 954p^2 - 264p^4 + 16p^6 + 756q + 264p^2q - 40p^4q + 144q^2 + 16p^2q^2 + 8q^3),$

$\chi_{2,0}^P = \frac{15}{64}(11907 + 9990p^2 - 2568p^4 + 144p^6 + 5184q + 2016p^2q - 280p^4q + 648q^2$
$+96p^2q^2 + 24q^3),$

$\chi_{2,1}^P = \frac{1}{16}p\big(- 295785 - 41994p^2 + 37224p^4 - 5088p^6 + 208p^8 - 183096q - 11016p^2q$
$+9720p^4q - 688p^6q - 39852q^2 - 528p^2q^2 + 592p^4q^2 - 3576q^3 + 16p^2q^3 - 112q^4\big),$

$\chi_{2,2}^P = \frac{1}{4}\big(6300+4995p^2-1284p^4+72p^6+2592q+1008p^2q-140p^4q+324q^2+48p^2q^2+12q^3\big).$

Calling $\mathbf{PrintSC}_{Q,2}$, we obtain:

$\beta_0^Q = -pq + r,$

$\beta_1^Q = \frac{1}{2}(-21p + 2p^3 - 6pq + 2r)$

$\beta_2^Q = \frac{1}{2}(-69p + 6p^3 - 10pq + 2r),$

$\chi_{0,0}^Q = \frac{1}{8}(105 + 210q + 84q^2 + 8q^3),$

$\chi_{1,0}^Q = \frac{1}{16}p(-69 + 6p^2 - 8q)(105 + 210q + 84q^2 + 8q^3),$

$\chi_{1,1}^Q = \frac{1}{8}(3045 + 1638p^2 - 348p^4 + 16p^6 + 2310q + 528p^2q - 64p^4q + 456q^2 + 40p^2q^2 + 24q^3),$

$\chi^Q_{2,0} = \frac{1}{64}\big(2236815+1502550p^2-323820p^4+15120p^6+5290110q+3337740p^2q-687120p^4q$
$+30240p^6q + 3670380q^2 + 1884960p^2q^2 - 338016p^4q^2 + 12096p^6q^2 + 1128960q^3$
$+415872p^2q^3 - 56256p^4q^3 + 1152p^6q^3 + 170448q^4 + 39456p^2q^4 - 3008p^4q^4 + 12192q^5$
$+1344p^2q^5 + 320q^6\big),$

$\chi^Q_{2,1} = \frac{1}{16}p\big(-792855-87570p^2+60924p^4-6504p^6+208p^8-605430q-23028p^2q+19032p^4q$
$-1072p^6q - 151632q^2 - 1152p^2q^2 + 1408p^4q^2 - 14928q^3 + 32p^2\ q^3 - 496q^4\big),$

$\chi^Q_{2,2} = \frac{1}{4}\big(11295 + 7155p^2 - 1542p^4 + 72p^6 + 5175q + 1584p^2q - 188p^4q + 666q^2 + 84p^2q^2$
$+24q^3\big).$

Calling $\mathbf{PrintSC}_{R,2}$, we get:

$\beta^R_0 = -5p + p^3 - 2pq + r,$

$\beta^R_1 = -23p + 3p^3 - 4pq + r,$

$\beta^R_2 = \frac{1}{2}(-109p + 12p^3 - 12pq + 2r)$

$\chi^R_{0,0} = \frac{1}{2}(105 + 125p^2 - 48p^4 + 4p^6 + 105q + 48p^2q - 10p^4q + 28q^2 + 4p^2q^2 + 2q^3),$

$\chi^R_{1,0} = \frac{1}{16}p\big(- 93555 - 16900p^2 + 21088p^4 - 3864p^6 + 208p^8 - 84540q - 5368p^2q$
$+7416p^4q - 688p^6q - 25008q^2 - 280p^2q^2 + 592p^4q^2 - 2888q^3 + 16p^2q^3 - 112q^4\big),$

$\chi^R_{1,1} = \frac{1}{4}(3255+3344p^2-1050p^4+72p^6+1770q+828p^2q-140p^4q+274q^2+48p^2q^2+12q^3),$

$\chi^R_{2,0} = \frac{1}{64}\big(15062355 + 51933420p^2 - 7347720p^4 - 8117648p^6 + 2722496p^8 - 307200p^{10}$
$+11840p^{12} + 20020770q + 46994940p^2q - 6950448p^4q - 3643760p^6q + 854112p^8q$
$-47936p^{10}q + 9552060q^2 + 15586224p^2q^2 - 2090832p^4q^2 - 493760p^6q^2 + 61376p^8q^2$
$+2179296q^3 + 2366096p^2q^3 - 242944p^4q^3 - 20160p^6q^3 + 255264q^4 + 165504p^2q^4$
$-9600p^4q^4 + 14656q^5 + 4288p^2q^5 + 320q^6\big),$

$\chi^R_{2,1} = \frac{1}{16}p\big(-2900205-713064p^2+586204p^4-89232p^6+4080p^8-1630728q-206136p^2q+$
$140592p^4q-10912p^6q-316476q^2-18640p^2q^2+8016p^4q^2-25104q^3-512p^2q^3-688q^4\big),$

$\chi^R_{2,2} = \frac{1}{8}\big(38115 + 35574p^2 - 10200p^4 + 632p^6 + 14058q + 6552p^2q - 1000p^4q + 1524q^2$
$+288p^2q^2 + 48q^3\big).$

The observation of the above results allows us to conclude that if **p=0**, then for each principal component, we have:

$\beta^\zeta_n = r$; $\chi^\zeta_{2n+1,2\nu} = 0,\ 0 \le \nu \le n,\ \chi_{2n,2\nu+1} = 0,\ 0 \le \nu \le n-1$, where $\zeta = P, Q$ and R. Recall that these conditions characterize a symmetric MPS, if $r = 0$.

Let us now focus on the **secondary components**, when **a=b=c=p=0**.

Calculating $\mathbf{SCDegreeTest}_{a^1,4}$, we obtain:

$$\deg_{a^1,0}=0,\;\; \deg_{a^1,1}=1,\;\; \deg_{a^1,2}=2,\;\; \deg_{a^1,3}=3,\;\; \deg_{a^1,4}=4.$$

Remark 5.1. *Having choose the parameters a, b, c and p, the structure coefficients, that follows, will depend only on the coefficients q and r. If we specify these latest, we must apply again the command SCDegreeTest, in order to confirm that the assumption of linear independence of the first polynomials still holds for the precise set of parameters taken.*

Then, we can proceed with the computation of the structure coefficients, for the normalized sequence. Calling $\mathbf{PrintSC}_{ma^1,2}$, we obtain:

$\beta_0^{ma^1}=r,$

$\beta_1^{ma^1}=r,$

$\beta_2^{ma^1}=r,$

$\chi_{0,0}^{ma^1}=\Big(945+2520q+1512q^2+288q^3+16q^4\Big)/\Big(48(6+q)\Big),$

$\chi_{1,0}^{ma^1}=0,$

$\chi_{1,1}^{ma^1}=\Big(904365+1066905q+427104q^2+75924q^3+6096q^4+176q^5\Big)/\Big((48(6+q)(33+4q)\Big),$

$\chi_{2,0}^{ma^1}=\Big(32168418975+101974014450q+105441349860q^2+53950344840q^3+15832635840q^4$
$+2833911360q^5+314038080q^6+20967552q^7$
$+768768q^8+11776q^9\Big)/\Big(3840(6+q)(21+2q)(33+4q)\Big),$

$\chi_{2,1}^{ma^1}=0,$

$\chi_{2,2}^{ma^1}=\Big(7827435+5758830q+1532916q^2+188136q^3+10704q^4+224q^5\Big)/\Big(16(21$
$+2q)(33+4q)\Big).$

Calculating $\mathbf{SCDegreeTest}_{a^2,4}$, we obtain:

$$\deg_{a^2,0}=-\infty,\;\; \deg_{a^2,1}=0,\;\; \deg_{a^2,2}=1,\;\; \deg_{a^2,3}=2,\;\; \deg_{a^2,4}=3.$$

Calculating $\mathbf{SCDegreeTest}_{la^2,4}$, we obtain:

$$\deg_{la^2,0}=0,\;\; \deg_{la^2,1}=1,\;\; \deg_{la^2,2}=2,\;\; \deg_{la^2,3}=3,\;\; \deg_{la^2,4}=4,$$

and by $\mathbf{PrintSC}_{mla^2,2}$, we get:

$\beta_0^{mla^2}=r,$

$\beta_1^{mla^2}=r,$

$\beta_2^{mla^2}=r,$

$\chi_{0,0}^{mla^2} = \Big(31185 + 51975q + 27720q^2 + 5940q^3 + 528q^4 + 16q^5\Big)/\Big(12(495 + 132q + 8q^2)\Big),$

$\chi_{1,0}^{mla^2} = 0,$

$\chi_{1,1}^{mla^2} = \Big(3958779825 + 4659226110q + 2159374140q^2 + 516839400q^3 + 69466320q^4 + 5263200q^5 + 208512q^6 + 3328q^7\Big)/\Big(120(495 + 132q + 8q^2)(819 + 168q + 8q^2)\Big),$

$\chi_{2,0}^{mla^2} = \Big(470037549556575 + 1090951349588100q + 1033994833571700q^2 + 545289090386040q^3 + 181270180143360q^4 + 40457765446560q^5 + 6264083548800q^6 + 681727052160q^7 + 51978060288q^8 + 2716319232q^9 + 92565504q^{10} + 1849344q^{11} + 16384q^{12}\Big)/\Big(1920(306 + 51q + 2q^2)(495 + 132q + 8q^2)(819 + 168q + 8q^2)\Big),$

$\chi_{2,1}^{mla^2} = 0,$

$\chi_{2,2}^{mla^2} = \Big(4776076305 + 3949050105q + 1320072390q^2 + 232677900q^3 + 23430960q^4 + 1349856q^5 + 41184q^6 + 512q^7\Big)/\Big(40(306 + 51q + 2q^2)(819 + 168q + 8q^2)\Big).$

Calculating $\mathbf{SCDegreeTest}_{b^1,4}$, we obtain:

$$\deg_{b^1,0} = -\infty,\ \ \deg_{b^1,1} = 0,\ \ \deg_{b^1,2} = 1,\ \ \deg_{b^1,3} = 2,\ \ \deg_{b^1,4} = 3.$$

Calculating $\mathbf{SCDegreeTest}_{lb^1,4}$, we obtain:

$$\deg_{b^1,0} = 0,\ \ \deg_{b^1,1} = 1,\ \ \deg_{b^1,2} = 2,\ \ \deg_{b^1,3} = 3,\ \ \deg_{b^1,4} = 4,$$

and by $\mathbf{PrintSC}_{mlb^1,2}$, we get:

$\beta_0^{mlb^1} = r,$

$\beta_1^{mlb^1} = r,$

$\beta_2^{mlb^1} = r,$

$\chi_{0,0}^{mlb^1} = \frac{315}{16(105+30q+2q^2)},$

$\chi_{1,0}^{mlb^1} = 0,$

$\chi_{1,1}^{mlb^1} = \frac{56081025+72825480q+38911320q^2+10862280q^3+1703520q^4+150240q^5+6912q^6+128q^7}{48(105+30q+2q^2)(715+156q+8q^2)},$

$\chi_{2,0}^{mlb^1} = \frac{273(456080625+459248130q+180864090q^2+36709200q^3+4189680q^4+270720q^5+9216q^6+128q^7)}{256(105+30q+2q^2)(273+48q+2q^2)(715+156q+8q^2)},$

$\chi_{2,1}^{mlb^1} = 0,$

$\chi_{2,2}^{mlb^1} = \frac{13(70945875+64428210q+23898105q^2+4691115q^3+526680q^4+33840q^5+1152q^6+16q^7)}{30(273+48q+2q^2)(715+156q+8q^2)}.$

Calculating $\mathbf{SCDegreeTest}_{b^2,4}$, we obtain:

$$\deg_{b^2,0} = 0, \ \ \deg_{b^2,1} = 1, \ \ \deg_{b^2,2} = 2, \ \ \deg_{b^2,3} = 3, \ \ \deg_{b^2,4} = 4,$$

and by $\mathbf{PrintSC}_{mb^2,2}$, we get:

$\beta_0^{mb^2} = r,$

$\beta_1^{mb^2} = r,$

$\beta_2^{mb^2} = r,$

$\chi_{0,0}^{mb^2} = \Big(4725 + 6300q + 2520q^2 + 360q^3 + 16q^4\Big)/\Big(24(15 + 2q)\Big),$

$\chi_{1,0}^{mb^2} = 0,$

$\chi_{1,1}^{mb^2} = \Big(5712525 + 4914810q + 1529100q^2 + 218520q^3 + 14448q^4 + 352q^5\Big)/\Big(48(15+ 2q)(39 + 4q)\Big),$

$\chi_{2,0}^{mb^2} = \Big(182249817750+333796287825q+239198532300q^2+91140498540q^3+20740992840q^4 +2958444000q^5 + 266582880q^6 + 14708160q^7 + 451968q^8 +5888q^9\Big)/\Big(1920(12 + q)(15 + 2q)(39 + 4q)\Big),$

$\chi_{2,1}^{mb^2} = 0,$

$\chi_{2,2}^{mb^2} = \Big(8918910 + 5332041q + 1186848q^2 + 124020q^3 + 6096q^4 + 112q^5\Big)/\Big(16(12 + q)(39 +4q)\Big).$

Calculating $\mathbf{SCDegreeTest}_{c^1,4}$, we obtain:

$$\deg_{c^1,0} = 0, \ \ \deg_{c^1,1} = 1, \ \ \deg_{c^1,2} = 2, \ \ \deg_{c^1,3} = 3, \ \ \deg_{c^1,4} = 4,$$

and by $\mathbf{PrintSC}_{mc^1,2}$, we get:

$\beta_0^{mc^1} = r,$

$\beta_1^{mc^1} = r,$

$\beta_2^{mc^1} = r,$

$\chi_{0,0}^{mc^1} = \frac{105}{32(7+q)},$

$\chi_{1,0}^{mc^1} = 0,$

$\chi_{1,1}^{mc^1} = \frac{479325+456750q+166320q^2+28160q^3+2208q^4+64q^5}{32(7+q)(55+6q)},$

$\chi_{2,0}^{mc^1} = \frac{1155(626535+450117q+116406q^2+13848q^3+768q^4+16q^5)}{128(7+q)(55+6q)(91+8q)},$

$\chi_{2,1}^{mc^1} = 0,$

$\chi_{2,2}^{mc^1} = \frac{11(716625+472815q+117810q^2+13848q^3+768q^4+16q^5)}{4(55+6q)(91+8q)}$.

Calculating $\textbf{SCDegreeTest}_{c^2,4}$, we obtain:
$\deg_{c^2,0} = -\infty$, $\deg_{c^2,1} = 0$, $\deg_{c^2,2} = 1$, $\deg_{c^2,3} = 2$, $\deg_{c^2,4} = 3$.
Calculating $\textbf{SCDegreeTest}_{lc^2,4}$, we obtain:

$$\deg_{c^2,0} = 0,\ \deg_{c^2,1} = 1,\ \deg_{c^2,2} = 2,\ \deg_{c^2,3} = 3,\ \deg_{c^2,4} = 4,$$

and by $\textbf{PrintSC}_{mlc^2,2}$, we get:

$\beta_0^{mlc^2} = r$,

$\beta_1^{mlc^2} = r$,

$\beta_2^{mlc^2} = r$,

$\chi_{0,0}^{mlc^2} = \Big(10395 + 34650q + 27720q^2 + 7920q^3 + 880q^4 + 32q^5\Big)/\Big(48(165 + 55q + 4q^2)\Big)$,

$\chi_{1,0}^{mlc^2} = 0$,

$\chi_{1,1}^{mlc^2} = \Big(905269365 + 1449187740q + 865168920q^2 + 258433560q^3 + 42374640q^4 +3845248q^5 + 179584q^6 + 3328q^7\Big)/\Big(48(165 + 55q + 4q^2)(3003 + 728q + 40q^2)\Big)$,

$\chi_{2,0}^{mlc^2} = \Big(200523563665425+791563006409175q+1077842826210150q^2+761731192322100q^3 +326192784077160q^4 + 91326416116320q^5 + 17394380609760q^6 + 2293463652480q^7 +209232481920q^8 + 12946326272q^9 + 517553152q^{10} + 12026880q^{11} +122880q^{12}\Big)/\Big(3840(357 + 68q + 3q^2)(165 + 55q + 4q^2)(3003 + 728q + 40q^2)\Big)$,

$\chi_{2,1}^{mlc^2} = 0$,

$\chi_{2,2}^{mlc^2} = \Big(12109041945 + 12269852595q + 4908148245q^2 + 1018623060q^3 + 119240940q^4 +7897624q^5 + 274224q^6 + 3840q^7\Big)/\Big(40(357 + 68q + 3q^2)(3003 + 728q + 40q^2)\Big)$.

From the data obtained, we suspect that the sequences $\{ma^1\}_{n\geq 0}$, $\{mla^2\}_{n\geq 0}$, $\{mlb^1\}_{n\geq 0}$, $\{mb^2\}_{n\geq 0}$, $\{mc^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are MPS, not orthogonal, and they satisfy:
$\beta_n^{\zeta} = r$; $\chi_{2n+1,2\nu}^{\zeta} = 0$, $0 \leq \nu \leq n$, $\chi_{2n,2\nu+1} = 0$, $0 \leq \nu \leq n-1$, where $\zeta = ma^1, mla^2, mlb^1, mb^2, mc^1$ and mlc^2, that is, they are symmetric for $r = 0$.

- Let us set **p=q=r=0**.

Calculating $\textbf{PrintSC}_{P,2}$, we obtain:

$\beta_0^P = \frac{3a}{2}$,

$\beta_1^P = 6a$,

$\beta_2^P = \frac{21a}{2}$,
$\chi_{0,0}^P = \frac{3}{8}(5 + 24a^2 - 30ab - 30ac + 30bc)$,

$\chi_{1,0}^P = \frac{27}{16}a(115 + 70a^2 - 144ab - 144ac + 144bc)$,
$\chi_{1,1}^P = \frac{3}{8}(415 + 210a^2 - 222ab - 222ac + 222bc)$,

$\chi_{2,0}^P = \frac{1215}{64}\big(147 + 848a^2 + 160a^4 - 858ab - 436a^3b + 164a^2b^2 - 858ac - 436a^3c$
$+858bc + 764a^2bc - 328ab^2c + 164a^2c^2 - 328abc^2 + 164b^2c^2\big)$,
$\chi_{2,1}^P = \frac{405}{8}a\big(217 + 40a^2 - 69ab - 69ac + 69bc\big)$,
$\chi_{2,2}^P = \frac{9}{4}\big(700 + 120a^2 - 123ab - 123ac + 123bc\big)$.

Calling $\mathbf{PrintSC}_{Q,2}$, we obtain:
$\beta_0^Q = 3(b + c)$,
$\beta_1^Q = \frac{15(b+c)}{2}$,
$\beta_2^Q = 12(b + c)$,
$\chi_{0,0}^Q = \frac{15}{8}(7 + 12b^2 + 24bc + 12c^2)$,
$\chi_{1,0}^Q = \frac{189}{16}(b + c)(85 + 32b^2 + 64bc + 32c^2)$,
$\chi_{1,1}^Q = \frac{21}{8}(145 + 48b^2 + 96bc + 48c^2)$,

$\chi_{2,0}^Q = \frac{8505}{64}\big(263 + 632b^2 + 88b^4 + 1264bc + 352b^3c + 632c^2 + 528b^2c^2 + 352bc^3 + 88c^4\big)$,
$\chi_{2,1}^Q = \frac{2835}{8}(b + c)\big(79 + 11b^2 + 22bc + 11c^2\big)$,
$\chi_{2,2}^Q = \frac{45}{4}\big(251 + 33b^2 + 66bc + 33c^2\big)$.

Calling $\mathbf{PrintSC}_{R,2}$, we get:
$\beta_0^R = 0$, $\beta_1^R = 0$, $\beta_2^R = 0$,
$\chi_{0,0}^R = \frac{105}{2}$, $\chi_{1,0}^R = 0$, $\chi_{1,1}^R = \frac{3255}{4}$,
$\chi_{2,0}^R = \frac{15062355}{64}$, $\chi_{2,1}^R = 0$, $\chi_{2,2}^R = \frac{38115}{8}$.

Notice that the sequence $\{R_n\}_{n\geq 0}$ appears to be symmetric. Also, the symmetric conditions seem to be fulfilled by $\{P_n\}_{n\geq 0}$ if $a = 0$, and by $\{Q_n\}_{n\geq 0}$ if $b + c = 0$.

Let us now focus on the **secondary components**.
Choosing **p=q=r=0**, it seems that the sequences $\{mla^2\}_{n\geq 0}$ and $\{mb^2\}_{n\geq 0}$ are MPS and satisfy the symmetry conditions, as we can see next.

Calling $\mathbf{PrintSC}_{mla^2,2}$, we obtain:

$\beta_0^{mla^2} = 0,\ \beta_1^{mla^2} = 0,\ \beta_2^{mla^2} = 0,$

$\chi_{0,0}^{mla^2} = \frac{21}{4},\ \chi_{1,0}^{mla^2} = 0,\ \chi_{1,1}^{mla^2} = \frac{651}{8},$

$\chi_{2,0}^{mla^2} = \frac{505197}{256},\ \chi_{2,1}^{mla^2} = 0,\ \chi_{2,2}^{mla^2} = \frac{7623}{16}.$

Calling $\mathbf{PrintSC}_{mb^2,2}$, we get:

$\beta_0^{mb^2} = 0,\ \beta_1^{mb^2} = 0,\ \beta_2^{mb^2} = 0,$

$\chi_{0,0}^{mb^2} = \frac{105}{8},\ \chi_{1,0}^{mb^2} = 0,\ \chi_{1,1}^{mb^2} = \frac{3255}{16},$

$\chi_{2,0}^{mb^2} = \frac{3461535}{256},\ \chi_{2,1}^{mb^2} = 0,\ \chi_{2,2}^{mb^2} = \frac{38115}{32}.$

Choosing **p=q=r=0** and **a=0**, it seems that the sequences $\{mlb^1\}_{n\geq 0}$ and $\{mc^1\}_{n\geq 0}$ are MPS and satisfy the symmetry conditions, as we can see next.

Calling $\mathbf{PrintSC}_{mlb^1,2}$, we obtain:

$\beta_0^{mlb^1} = 0,\ \beta_1^{mlb^1} = 0,\ \beta_2^{mlb^1} = 0,$

$\chi_{0,0}^{mlb^1} = \frac{21(1+10bc)}{16(7+bc)},\ \chi_{1,0}^{mlb^1} = 0,\ \chi_{1,1}^{mlb^1} = \frac{21(4565+4726bc+620b^2c^2)}{16(7+bc)(55+4bc)},$

$\chi_{2,0}^{mlb^1} = \frac{18711(11375+101683bc+42880b^2c^2+2960b^3c^3)}{256(7+bc)(55+4bc)(91+4bc)},\ \chi_{2,1}^{mlb^1} = 0,\ \chi_{2,2}^{mlb^1} = \frac{693(9100+3353bc+220b^2c^2)}{8(55+4bc)(91+4bc)}.$

Calculating $\mathbf{PrintSC}_{mc^1,2}$, we get:

$\beta_0^{mc^1} = 0,\ \beta_1^{mc^1} = 0,\ \beta_2^{mc^1} = 0,$

$\chi_{0,0}^{mc^1} = \frac{105(1+8bc)}{16(14+bc)},\ \chi_{1,0}^{mc^1} = 0,\ \chi_{1,1}^{mc^1} = \frac{105(4565+3584bc+248b^2c^2)}{16(14+bc)(55+2bc)},$

$\chi_{2,0}^{mc^1} = \frac{93555(7735+57494bc+17662b^2c^2+644b^3c^3)}{64(14+bc)(55+2bc)(91+2bc)},\ \chi_{2,1}^{mc^1} = 0,\ \chi_{2,2}^{mc^1} = \frac{3465(2275+619bc+22b^2c^2)}{4(55+2bc)(91+2bc)}.$

Choosing **p=q=r=0** and **b+c=0**, it seems that the sequences $\{ma^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are MPS and satisfy the symmetry conditions, as we can see next.

Calling $\mathbf{PrintSC}_{ma^1,2}$, we obtain:

$\beta_0^{ma^1} = 0,\ \beta_1^{ma^1} = 0,\ \beta_2^{ma^1} = 0,$

$\chi_{0,0}^{ma^1} = \frac{105}{32},\ \chi_{1,0}^{ma^1} = 0,\ \chi_{1,1}^{ma^1} = \frac{3045}{32},$

$\chi_{2,0}^{ma^1} = \frac{1031535}{512},\ \chi_{2,1}^{ma^1} = 0,\ \chi_{2,2}^{ma^1} = \frac{11295}{16}.$

Calling $\mathbf{PrintSC}_{mlc^2,2}$, we get:

$\beta_0^{mlc^2} = 0,\ \beta_1^{mlc^2} = 0,\ \beta_2^{mlc^2} = 0,$

$\chi_{0,0}^{mlc^2} = \frac{21}{16},\ \chi_{1,0}^{mlc^2} = 0,\ \chi_{1,1}^{mlc^2} = \frac{609}{16},$

$\chi_{2,0}^{mlc^2} = \frac{75573}{256},\ \chi_{2,1}^{mlc^2} = 0,\ \chi_{2,2}^{mlc^2} = \frac{2259}{8}.$

Second kind Tchebyshev sequence

The recurrence coefficients of the second kind Tchebyshev sequence are: $\beta_n = 0$, $\gamma_{n+1} = \frac{1}{4}$ $n \geq 0$.

• Let us set **a=b=c=p=q=r=0**. The matrices $M_0(x)$, $M_1(x)$, $M_2(x)$ were presented above, and the principal components are not orthogonal:

OrthoPCdirectTest$_{P,3}$ $\rightarrow \{P_n\}_{n\geq 0}$ is not orthogonal
OrthoPCdirectTest$_{Q,3}$ $\rightarrow \{Q_n\}_{n\geq 0}$ is not orthogonal
OrthoPCdirectTest$_{R,3}$ $\rightarrow \{R_n\}_{n\geq 0}$ is not orthogonal

Calling **PrintSC**$_{P,4}$, we obtain:
$\beta_0^P = 0,\ \beta_1^P = 0,\ \beta_2^P = 0,\ \beta_3^P = 0,\ \beta_4^P = 0,$
$\chi_{0,0}^P = \frac{1}{64},$
$\chi_{1,0}^P = 0,\ \chi_{1,1}^P = \frac{19}{64},$
$\chi_{2,0}^P = \frac{63}{4096},\ \chi_{2,1}^P = 0,\ \chi_{2,2}^P = 1,$
$\chi_{3,0}^P = 0,\ \chi_{3,1}^P = \frac{2637}{4096},\ \chi_{3,2}^P = 0,\ \chi_{3,3}^P = \frac{17}{8},$
$\chi_{4,0}^P = \frac{9333}{131072},\ \chi_{4,1}^P = 0,\ \chi_{4,2}^P = \frac{4725}{1024},\ \chi_{4,3}^P = 0,\ \chi_{4,4}^P = \frac{235}{64}.$

Calling **PrintSC**$_{Q,4}$, we obtain:
$\beta_0^Q = 0,\ \beta_1^Q = 0,\ \beta_2^Q = 0,\ \beta_3^Q = 0,\ \beta_4^Q = 0,$
$\chi_{0,0}^Q = \frac{1}{16},$
$\chi_{1,0}^Q = 0,\ \chi_{1,1}^Q = \frac{31}{64},$
$\chi_{2,0}^Q = \frac{333}{4096},\ \chi_{2,1}^Q = 0,\ \chi_{2,2}^Q = \frac{85}{64},$
$\chi_{3,0}^Q = 0,\ \chi_{3,1}^Q = \frac{5607}{4096},\ \chi_{3,2}^Q = 0,\ \chi_{3,3}^Q = \frac{83}{32},$
$\chi_{4,0}^Q = \frac{30897}{65536},\ \chi_{4,1}^Q = 0,\ \chi_{4,2}^Q = \frac{15687}{2048},\ \chi_{4,3}^Q = 0,\ \chi_{4,4}^Q = \frac{137}{32}.$

Calling **PrintSC**$_{R,4}$, we obtain:
$\beta_0^R = 0,\ \beta_1^R = 0,\ \beta_2^R = 0,\ \beta_3^R = 0,\ \beta_4^R = 0,$
$\chi_{0,0}^R = \frac{5}{32},$
$\chi_{1,0}^R = 0,\ \chi_{1,1}^R = \frac{23}{32},$
$\chi_{2,0}^R = \frac{531}{2048},\ \chi_{2,1}^R = 0,\ \chi_{2,2}^R = \frac{109}{64},$
$\chi_{3,0}^R = 0,\ \chi_{3,1}^R = \frac{5355}{2048},\ \chi_{3,2}^R = 0,\ \chi_{3,3}^R = \frac{199}{64},$
$\chi_{4,0}^R = \frac{487197}{262144},\ \chi_{4,1}^R = 0,\ \chi_{4,2}^R = \frac{49599}{4096},\ \chi_{4,3}^R = 0,\ \chi_{4,4}^R = \frac{79}{16}.$

We see also that for each principal component, we have $\beta_n^\zeta = 0$ and

$\chi^{\zeta}_{2n+1,2\nu} = 0,\ 0 \leq \nu \leq n,\ \chi^{\zeta}_{2n,2\nu+1} = 0,\ 0 \leq \nu \leq n-1$, where $\zeta = P, Q$ and R.

• Let us set **a=b=c=0**. Then, the structure coefficients of the principal components are similar to the structure coefficients presented above for the Hermite sequence. We also notice that if $p = 0$, then, for each principal component, we have, for $\zeta = P, Q$ and R:

$$\beta^{\zeta}_n = r, \quad \chi^{\zeta}_{2n+1,2\nu} = 0,\ 0 \leq \nu \leq n, \quad \chi^{\zeta}_{2n,2\nu+1} = 0,\ 0 \leq \nu \leq n-1.$$

We obtained similar results for the secondary components, when $a = b = c = p = 0$; thus, we suspect that the sequences $\{ma^1\}_{n\geq 0}, \{mla^2\}_{n\geq 0}, \{mlb^1\}_{n\geq 0}, \{mb^2\}_{n\geq 0}, \{mc^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are MPS, not orthogonal, and they satisfy:

$\beta^{\zeta}_n = r$; $\chi^{\zeta}_{2n+1,2\nu} = 0,\ 0 \leq \nu \leq n,\ \chi_{2n,2\nu+1} = 0,\ 0 \leq \nu \leq n-1$, where $\zeta = ma^1, mla^2, mlb^1, mb^2, mc^1$ and mlc^2.

• Let us set **p=q=r=0**. Then, the structure coefficients of the principal components are similar to the ones obtained for the same choice of parameters in the Hermite sequence.

Here, we also notice that the sequence $\{R_n\}_{n\geq 0}$ appears to be symmetric. Also, the symmetric conditions seem to be fulfilled for $\{P_n\}_{n\geq 0}$ if $a = 0$, and for $\{Q_n\}_{n\geq 0}$ if $b+c = 0$.

With respect to the secondary components, we can point out the following aspects.

Choosing $p = q = r = 0$, it seems that the sequences $\{mla^2\}_{n\geq 0}$ and $\{mb^2\}_{n\geq 0}$ are MPS and satisfy the symmetry conditions.

Choosing $p = q = r = 0$ and $b + c = 0$, it seems that the sequences $\{ma^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are MPS and satisfy the symmetry conditions.

Choosing $p = q = r = 0$ and $a = 0$, it appears that the sequences $\{mlb^1\}_{n\geq 0}$ and $\{mc^1\}_{n\geq 0}$ are MPS and satisfy the symmetry conditions.

• Let us now set **p=0** and **q=-3/4**.

The principal components are orthogonal, and we can write precisely their recurrence coefficients using the coefficients of theorem 3.2 (presented in subsection 5.4.1) as follows.

$\beta^P_0 = \frac{1}{4}\big(-a + 4r\big),\ \ \beta^P_n = r,\ \ n \geq 1, \quad \beta^Q_n = \beta^R_n = r,\ \ n \geq 0,$

$\gamma^P_{n+1} = \gamma^Q_{n+1} = \gamma^R_{n+1} = \frac{1}{64},\ \ n \geq 0.$

Let us now focus on the secondary components. It seems that the sequences $\{a^2\}_{n\geq 0}$ and $\{b^2\}_{n\geq 0}$ vanish. The sequences $\{ma^1\}_{n\geq 0}$, $\{mb^1\}_{n\geq 0}$, $\{mc^1\}_{n\geq 0}$ and $\{mc^2\}_{n\geq 0}$ appear to be MOPS. In fact, calculating $\mathbf{PrintSC}_{ma^1,2}$, we obtain:

$\beta^{ma^1}_0 = \beta^{ma^1}_1 = \beta^{ma^1}_2 = r,$

$\chi^{ma^1}_{0,0} = \frac{1}{64},\ \chi^{ma^1}_{1,0} = 0,\ \chi^{ma^1}_{1,1} = \frac{1}{64},\ \chi^{ma^1}_{2,0} = 0,\ \chi^{ma^1}_{2,1} = 0,\ \chi^{ma^1}_{2,2} = \frac{1}{64}.$

Calculating $\mathbf{PrintSC}_{mb^1,2}$, we obtain:

$\beta^{mb^1}_0 = \frac{-1+16ar}{16a},\ \beta^{mb^1}_1 = \beta^{mb^1}_2 = r,$

$\chi^{mb^1}_{0,0} = \frac{1}{64},\ \chi^{mb^1}_{1,0} = 0,\ \chi^{mb^1}_{1,1} = \frac{1}{64},\ \chi^{mb^1}_{2,0} = 0,\ \chi^{mb^1}_{2,1} = 0,\ \chi^{mb^1}_{2,2} = \frac{1}{64}.$

Calculating $\mathbf{PrintSC}_{mc^1,2}$, we obtain:

$\beta_0^{mc^1} = \beta_1^{mc^1} = \beta_2^{mc^1} = r$,

$\chi_{0,0}^{mc^1} = \frac{1}{64}$, $\chi_{1,0}^{mc^1} = 0$, $\chi_{1,1}^{mc^1} = \frac{1}{64}$, $\chi_{2,0}^{mc^1} = 0$, $\chi_{2,1}^{mc^1} = 0$, $\chi_{2,2}^{mc^1} = \frac{1}{64}$.

Calculating $\mathbf{PrintSC}_{mc^2,2}$, we obtain:

$\beta_0^{mc^2} = \beta_1^{mc^2} = \beta_2^{mc^2} = r$,

$\chi_{0,0}^{mc^2} = \frac{1}{64}$, $\chi_{1,0}^{mc^2} = 0$, $\chi_{1,1}^{mc^2} = \frac{1}{64}$, $\chi_{2,0}^{mc^2} = 0$, $\chi_{2,1}^{mc^2} = 0$, $\chi_{2,2}^{mc^2} = \frac{1}{64}$.

A shift of the second kind Tchebychev polynomials (a nonsymmetric example)

Let us now consider the MOPS $\{W_n\}_{n\geq 0}$ such that $\beta_n = \beta$, $\gamma_{n+1} = \alpha$, $n \geq 0$, $\alpha \neq 0$, already considered at the end of subsection 5.4.1.

- Let us set $a = b = c = p = q = r = 0$. We obtain, for $n = 0, 1$:

$$M_0(x) = \begin{pmatrix} 1 & 0 & 0 \\ -\beta & 1 & 0 \\ \beta^2 - \alpha & -2\beta & 1 \end{pmatrix},$$

$$M_1(x) = \begin{pmatrix} -\beta^3 + 2\alpha\beta + x & 3\beta^2 - 2\alpha & -3\beta \\ \beta^4 - 3\alpha\beta^2 - 4x\beta + \alpha^2 & -4\beta^3 + 6\alpha\beta + x & 6\beta^2 - 3\alpha \\ x\left(10\beta^2 - 4\alpha\right) - \beta\left(\beta^4 - 4\alpha\beta^2 + 3\alpha^2\right) & 5\beta^4 - 12\alpha\beta^2 - 5x\beta + 3\alpha^2 & x + 2\beta\left(6\alpha - 5\beta^2\right) \end{pmatrix}.$$

The principal components are not orthogonal:

$\mathbf{OrthoPCdirectTest}_{P,3} \rightarrow \{P_n\}_{n\geq 0}$ is not orthogonal

$\mathbf{OrthoPCdirectTest}_{Q,3} \rightarrow \{Q_n\}_{n\geq 0}$ is not orthogonal

$\mathbf{OrthoPCdirectTest}_{R,3} \rightarrow \{R_n\}_{n\geq 0}$ is not orthogonal.

Calling $\mathbf{PrintSC}_{P,2}$, we obtain:

$\beta_0^P = -\beta(2\alpha - \beta^2)$,

$\beta_1^P = -\beta(18\alpha - 19\beta^2)$,

$\beta_2^P = -4\beta(9\alpha - 16\beta^2)$,

$\chi_{0,0}^P = \alpha^3 + 30\alpha^2\beta^2 - 51\alpha\beta^4 + 18\beta^6$,

$\chi_{1,0}^P = -3\beta(23\alpha^4 + 251\alpha^3\beta^2 - 1141\alpha^2\beta^4 + 1257\alpha\beta^6 - 378\beta^8)$,

$\chi_{1,1}^P = 19\alpha^3 + 516\alpha^2\beta^2 - 1725\alpha\beta^4 + 1197\beta^6$,

$\chi_{2,0}^P = 3(21\alpha^6 + 1837\alpha^5\beta^2 + 9853\alpha^4\beta^4 - 116642\alpha^3\beta^6 + 277842\alpha^2\beta^8$
$-237285\alpha\beta^{10} + 63342\beta^{12})$,

$\chi_{2,1}^P = -3\beta(695\alpha^4 + 8600\alpha^3\beta^2 - 66913\alpha^2\beta^4 + 124272\alpha\beta^6 - 66825\beta^8)$,

$\chi_{2,2}^P = 2(32\alpha^3 + 987\alpha^2\beta^2 - 5061\alpha\beta^4 + 5292\beta^6)$.

Calling $\mathbf{PrintSC}_{Q,2}$, we obtain:

$\beta_0^Q = -2\beta(3\alpha - 2\beta^2)$,
$\beta_1^Q = -\beta(24\alpha - 31\beta^2)$,
$\beta_2^Q = -\beta(42\alpha - 85\beta^2)$,
$\chi_{0,0}^Q = 4\alpha^3 + 114\alpha^2\beta^2 - 252\alpha\beta^4 + 117\beta^6$,
$\chi_{1,0}^Q = -9\beta(36\alpha^4 + 404\alpha^3\beta^2 - 2262\alpha^2\beta^4 + 3060\alpha\beta^6 - 1167\beta^8)$,
$\chi_{1,1}^Q = 31\alpha^3 + 870\alpha^2\beta^2 - 3420\alpha\beta^4 + 2772\beta^6$,
$\chi_{2,0}^Q = 9(37\alpha^6 + 3462\alpha^5\beta^2 + 18918\alpha^4\beta^4 - 270719\alpha^3\beta^6 + 758301\alpha^2\beta^8 -775872\alpha\beta^{10} + 257121\beta^{12})$,
$\chi_{2,1}^Q = -9\beta(450\alpha^4 + 5875\alpha^3\beta^2 - 52278\alpha^2\beta^4 + 110322\alpha\beta^6 - 67815\beta^8)$,
$\chi_{2,2}^Q = 85\alpha^3 + 2760\alpha^2\beta^2 - 15822\alpha\beta^4 + 18414\beta^6$.

Calling $\mathbf{PrintSC}_{R,2}$, we obtain:

$\beta_0^R = -2\beta(6\alpha - 5\beta^2)$,
$\beta_1^R = -2\beta(15\alpha - 23\beta^2)$,
$\beta_2^R = -\beta(48\alpha - 109\beta^2)$,
$\chi_{0,0}^R = 10\alpha^3 + 270\alpha^2\beta^2 - 747\alpha\beta^4 + 432\beta^6$,
$\chi_{1,0}^R = -9\beta(103\alpha^4 + 1210\alpha^3\beta^2 - 8082\alpha^2\beta^4 + 12969\alpha\beta^6 - 5967\beta^8)$,
$\chi_{1,1}^R = 46\alpha^3 + 1350\alpha^2\beta^2 - 6111\alpha\beta^4 + 5670\beta^6$,
$\chi_{2,0}^R = 9(118\alpha^6 + 11856\alpha^5\beta^2 + 66634\alpha^4\beta^4 - 1124196\alpha^3\beta^6 + 3620025\alpha^2\beta^8 -4305582\alpha\beta^{10} + 1691928\beta^{12})$,
$\chi_{2,1}^R = -9\beta(791\alpha^4 + 10892\alpha^3\beta^2 - 109242\alpha^2\beta^4 + 258228\alpha\beta^6 - 178497\beta^8)$,
$\chi_{2,2}^R = 109\alpha^3 + 3726\alpha^2\beta^2 - 23625\alpha\beta^4 + 30294\beta^6$.

With respect to the **secondary components**, it seems that the sequences $\{ma^1\}_{n\geq 0}$, $\{ma^2\}_{n\geq 0}$, $\{mb^1\}_{n\geq 0}$, $\{mb^2\}_{n\geq 0}$, $\{mc^1\}_{n\geq 0}$ and $\{mc^2\}_{n\geq 0}$ are MPS, if we suppose $\beta \neq 0$.

• Let us set $p = -3\beta$ and $q = -3\alpha + 3\beta^2$.

As we saw in subsection 5.4.1, for this choice of parameters, the principal components are orthogonal, and we can write precisely their recurrence coefficients.

Let us now focus on the secondary components. It seems that the sequences $\{a^2\}_{n\geq 0}$ and $\{b^2\}_{n\geq 0}$ vanish. The sequences $\{ma^1\}_{n\geq 0}$, $\{mb^1\}_{n\geq 0}$, $\{mc^1\}_{n\geq 0}$ and $\{mc^2\}_{n\geq 0}$ appear to be MOPS. In fact, calculating $\mathbf{PrintSC}_{ma^1,2}$, we obtain:

$\beta_0^{ma^1} = \beta_1^{ma^1} = \beta_2^{ma^1} = r - 3\alpha\beta + \beta^3$,

$\chi_{0,0}^{ma^1} = \alpha^3$, $\chi_{1,0}^{ma^1} = 0$, $\chi_{1,1}^{ma^1} = \alpha^3$, $\chi_{2,0}^{ma^1} = 0$, $\chi_{2,1}^{ma^1} = 0$, $\chi_{2,2}^{ma^1} = \alpha^3$.

Calculating $\mathbf{PrintSC}_{mb^1,2}$, we obtain:

$\beta_0^{mb^1} = \frac{-ar+\alpha^2+r\beta+3a\alpha\beta-3\alpha\beta^2-a\beta^3+\beta^4}{\beta-a}$,

$\beta_1^{mb^1} = \beta_2^{mb^1} = r - 3\alpha\beta + \beta^3$,

$\chi_{0,0}^{mb^1} = \alpha^3$, $\chi_{1,0}^{mb^1} = 0$, $\chi_{1,1}^{mb^1} = \alpha^3$, $\chi_{2,0}^{mb^1} = 0$, $\chi_{2,1}^{mb^1} = 0$, $\chi_{2,2}^{mb^1} = \alpha^3$.

Calculating $\mathbf{PrintSC}_{mc^1,2}$, we obtain:

$\beta_0^{mc^1} = \beta_1^{mc^1} = \beta_2^{mc^1} = r - 3\alpha\beta + \beta^3$,

$\chi_{0,0}^{mc^1} = \alpha^3$, $\chi_{1,0}^{mc^1} = 0$, $\chi_{1,1}^{mc^1} = \alpha^3$, $\chi_{2,0}^{mc^1} = 0$, $\chi_{2,1}^{mc^1} = 0$, $\chi_{2,2}^{mc^1} = \alpha^3$.

Calculating $\mathbf{PrintSC}_{mc^2,2}$, we obtain:

$\beta_0^{mc^2} = \beta_1^{mc^2} = \beta_2^{mc^2} = r - 3\alpha\beta + \beta^3$,

$\chi_{0,0}^{mc^2} = \alpha^3$, $\chi_{1,0}^{mc^2} = 0$, $\chi_{1,1}^{mc^2} = \alpha^3$, $\chi_{2,0}^{mc^2} = 0$, $\chi_{2,1}^{mc^2} = 0$, $\chi_{2,2}^{mc^2} = \alpha^3$.

These results allow us to infer the following:

$$\beta_n^{\zeta} = r - 3\alpha\beta + \beta^3,\ n \geq 0,\quad \zeta = ma^1, mc^1, mc^2,$$

$$\beta_0^{mb^1} = \frac{-ar + \alpha^2 + r\beta + 3a\alpha\beta - 3\alpha\beta^2 - a\beta^3 + \beta^4}{\beta - a},\quad \beta_n^{mb^1} = r - 3\alpha\beta + \beta^3,\ n \geq 0,$$

$$\chi_{n,n}^{\zeta} = \alpha^3,\quad \chi_{n,\nu}^{\zeta} = 0,\quad 0 \leq \nu < n,\quad \zeta = ma^1, mb^1, mc^1, mc^2.$$

5.5 Some conjectures about symmetry

In the symmetric examples studied in the preceding section (Hermite and second kind Tchebyshev polynomials), the component sequences revealed several aspects with respect to symmetry. The same properties were observed in the following sequences: modified Lommel with $\alpha = 1$ (see subsection 5.4.1) and Tricomi-Carlitz with $\alpha = 1$ and $\alpha = 2$. Let us recall that the Tricomi-Carlitz sequence is defined by the following recurrence relations: $\beta_n = 0$, $\gamma_{n+1} = \frac{n+1}{(n+1+\alpha)(n+\alpha)}$, $n \geq 0$, $\alpha \neq -n$.

Next, we enumerate the properties found for these four symmetric sequences. The following statements are now presented as conjectures, although some of the ones for which $p = q = r = 0$ will be proved in general in the next section.

1. When **a=b=c=p=q=r=0**, each principal component seems to be symmetric. Also, the sequences $\{ma^1\}_{n\geq0}, \{mla^2\}_{n\geq0}, \{mlb^1\}_{n\geq0}, \{mb^2\}_{n\geq0}, \{mc^1\}_{n\geq0}$ and $\{mlc^2\}_{n\geq0}$ seem to be symmetric MPSs.

2. When **a=b=c=0**, then, it appears that setting **p=0**, we obtain, for $\zeta = P, Q$ and R, the following:

$$\beta_n^{\zeta} = r, \quad \chi_{2n+1,2\nu}^{\zeta} = 0,\ 0 \le \nu \le n, \quad \chi_{2n,2\nu+1}^{\zeta} = 0,\ 0 \le \nu \le n-1.$$

   We suspect, also, that the sequences $\{ma^1\}_{n\geq 0}, \{mla^2\}_{n\geq 0}, \{mlb^1\}_{n\geq 0}, \{mb^2\}_{n\geq 0}, \{mc^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are MPSs and they satisfy these same identities.

3. When **p=q=r=0**, then it appears that $\{R_n\}_{n\geq 0}$, $\{mla^2\}_{n\geq 0}$ and $\{mb^2\}_{n\geq 0}$ are symmetric MPSs, with structure coefficients not depending of the parameters a, b and c.

   (i) If **p=q=r=0** and **a=0**, then it seems that we have also: $\{P_n\}_{n\geq 0}$ is symmetric; the sequences $\{ma^1\}_{n\geq 0}, \{mla^2\}_{n\geq 0}, \{mlb^1\}_{n\geq 0}, \{mb^2\}_{n\geq 0}, \{mc^1\}_{n\geq 0}$ and $\{mc^2\}_{n\geq 0}$ are MPSs; and the sequences $\{mla^2\}_{n\geq 0}, \{mlb^1\}_{n\geq 0}, \{mb^2\}_{n\geq 0}$ and $\{mc^1\}_{n\geq 0}$ are symmetric MPSs.

   (ii) If **p=q=r=0** and **b+c=0**, then it seems that we have also: $\{Q_n\}_{n\geq 0}$ is symmetric; the sequences $\{ma^1\}_{n\geq 0}, \{mla^2\}_{n\geq 0}, \{mb^1\}_{n\geq 0}, \{mb^2\}_{n\geq 0}, \{mc^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are MPSs; and the sequences $\{ma^1\}_{n\geq 0}, \{mla^2\}_{n\geq 0}, \{mb^2\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are symmetric MPSs. Let us, also, notice that the determined structure coefficients of $\{Q_n\}_{n\geq 0}$, $\{ma^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ do not depend of the parameters a and c.

5.6 Some results about symmetry

The analytical approach to the conjectures advanced in the previous section was taken, although with some success only for the case when $p = q = r = 0$ and $\{W_n\}_{n\geq 0}$ is symmetric. In the next result we aim to clarify, for that case, the symmetric character obtained during the experimental essay in *Mathematica* of the previous section. For the secondary components, and since they are not necessarily PSs, we can characterize their parity, saying that they are even or odd, meaning that they are even or odd functions, that is, fulfilling one of the conditions $f(-x) = f(x)$ or $f(-x) = -f(x)$, respectively. In order to simplify the presentation of the following result, we will commit a slight abuse of language by saying that a polynomial sequence $\{F_n\}_{n\geq 0}$, such that $\deg F_n \le n,\ \ n \ge 0$, is symmetric if it fulfils $F_n(-x) = (-1)^n F_n(x),\ n \ge 0$.

Theorem 5.2. *Let $\{W_n\}_{n\geq 0}$ be a symmetric MPS defined by (2.1)-(2.3), where $p = q = r = 0$. Then, we have:*

- *$\{R_n\}_{n\geq 0}$, $\{la_n^2\}_{n\geq 0}$ and $\{b_n^2\}_{n\geq 0}$ are symmetric;*
- *if $a = 0$, then $\{P_n\}_{n\geq 0}$, $\{lb_n^1\}_{n\geq 0}$ and $\{c_n^1\}_{n\geq 0}$ are symmetric;*
- *if $b + c = 0$, then $\{Q_n\}_{n\geq 0}$, $\{a_n^1\}_{n\geq 0}$ and $\{lc_n^2\}_{n\geq 0}$ are symmetric.*

Proof. Writing every component sequence in terms of the canonical sequence, we have:

$$W_n(x)=\sum_{k=0}^{n} w_{n,k}x^k,\quad P_n(x)=\sum_{k=0}^{n} p_{n,k}x^k,\quad Q_n(x)=\sum_{k=0}^{n} q_{n,k}x^k,\ R_n(x)=\sum_{k=0}^{n} r_{n,k}x^k,$$

$$a^1_{n-1}(x)=\sum_{k=0}^{n-1} a^1_{n-1,k}x^k,\quad a^2_{n-1}(x)=\sum_{k=0}^{n-1} a^2_{n-1,k}x^k,$$

$$b^1_n(x)=\sum_{k=0}^{n} b^1_{n,k}x^k,\quad b^2_{n-1}(x)=\sum_{k=0}^{n-1} b^2_{n-1,k}x^k,$$

$$c^1_n(x)=\sum_{k=0}^{n} c^1_{n,k}x^k,\quad c^2_n(x)=\sum_{k=0}^{n} c^2_{n,k}x^k,$$

where, by convention, $\sum_{k=0}^{-1} . = 0$.

The MPS $\{W_n\}_{n\geq 0}$ fulfils $W_n(-x)=(-1)^n W_n(x),\quad n\geq 0$, therefore, the identities (2.1)-(2.3), can be written as follows, considering $p=q=r=0$ and depending of the parity of n. If n is even, then we have:

$$\begin{aligned}\sum_{k=0}^{(3n)/2} w_{3n,2k}x^{2k} &= \sum_{k=0}^{n} p_{n,k}x^{3k} - a\sum_{k=0}^{n-1} a^1_{n-1,k}x^{3k} + bc\sum_{k=0}^{n-1} a^2_{n-1,k}x^{3k} \\ &\quad + \sum_{k=0}^{n-1} a^1_{n-1,k}x^{3k+1} - (b+c)\sum_{k=0}^{n-1} a^2_{n-1,k}x^{3k+1} + \sum_{k=0}^{n-1} a^2_{n-1,k}x^{3k+2}\end{aligned} \tag{5.7}$$

$$\begin{aligned}\sum_{k=0}^{(3n)/2} w_{3n+1,2k+1}x^{2k+1} &= \sum_{k=0}^{n} b^1_{n,k}x^{3k} - a\sum_{k=0}^{n} q_{n,k}x^{3k} + bc\sum_{k=0}^{n-1} b^2_{n-1,k}x^{3k} \\ &\quad + \sum_{k=0}^{n} q_{n,k}x^{3k+1} - (b+c)\sum_{k=0}^{n-1} b^2_{n-1,k}x^{3k+1} + \sum_{k=0}^{n-1} b^2_{n-1,k}x^{3k+2}\end{aligned} \tag{5.8}$$

$$\begin{aligned}\sum_{k=0}^{(3n+3)/2} w_{3n+2,2k}x^{2k} &= \sum_{k=0}^{n} c^1_{n,k}x^{3k} - a\sum_{k=0}^{n} c^2_{n,k}x^{3k} + bc\sum_{k=0}^{n} r_{n,k}x^{3k} \\ &\quad + \sum_{k=0}^{n} c^2_{n,k}x^{3k+1} - (b+c)\sum_{k=0}^{n} r_{n,k}x^{3k+1} + \sum_{k=0}^{n} r_{n,k}x^{3k+2}.\end{aligned} \tag{5.9}$$

If n is odd, then we have:

$$\sum_{k=0}^{(3n-1)/2} w_{3n,2k+1}x^{2k+1} = \sum_{k=0}^{n} p_{n,k}x^{3k} - a\sum_{k=0}^{n-1} a^1_{n-1,k}x^{3k} + bc\sum_{k=0}^{n-1} a^2_{n-1,k}x^{3k} + \sum_{k=0}^{n-1} a^1_{n-1,k}x^{3k+1} - (b+c)\sum_{k=0}^{n-1} a^2_{n-1,k}x^{3k+1} + \sum_{k=0}^{n-1} a^2_{n-1,k}x^{3k+2} \quad (5.10)$$

$$\sum_{k=0}^{(3n+1)/2} w_{3n+1,2k}x^{2k} = \sum_{k=0}^{n} b^1_{n,k}x^{3k} - a\sum_{k=0}^{n} q_{n,k}x^{3k} + bc\sum_{k=0}^{n-1} b^2_{n-1,k}x^{3k} + \sum_{k=0}^{n} q_{n,k}x^{3k+1} - (b+c)\sum_{k=0}^{n-1} b^2_{n-1,k}x^{3k+1} + \sum_{k=0}^{n-1} b^2_{n-1,k}x^{3k+2} \quad (5.11)$$

$$\sum_{k=0}^{(3n+1)/2} w_{3n+2,2k+1}x^{2k+1} = \sum_{k=0}^{n} c^1_{n,k}x^{3k} - a\sum_{k=0}^{n} c^2_{n,k}x^{3k} + bc\sum_{k=0}^{n} r_{n,k}x^{3k} + \sum_{k=0}^{n} c^2_{n,k}x^{3k+1} - (b+c)\sum_{k=0}^{n} r_{n,k}x^{3k+1} + \sum_{k=0}^{n} r_{n,k}x^{3k+2}. \quad (5.12)$$

Let us remark that the terms x^{3k}, x^{3n+1} and x^{3m+2} are all different for every set of positive integers k, n and m, and, also, that: $3k$ is even if and only if k is even; $3k+1$ is even if and only if k is odd; $3k+2$ is even if and only if k is even. These two properties are subjacent to the following arguments.

Looking carefully to identities (5.7) and (5.10) and the correspondent terms of type x^{3k+2}, we conclude that $la^2_n(-x) = (-1)^n la^2_n(x),\ n \geq 0$; and analysing the part written in terms of x^{3k+1}, we get that if $b+c=0$, then $a^1_{n-1}(x)$ is odd, when n is even, and $a^1_{n-1}(x)$ is even, when n is odd, $n \geq 0$, that is, $a^1_n(-x) = (-1)^n a^1_n(x),\ n \geq 0$.

Let us suppose that $a=0$ and analyse the part written in terms of x^{3k}. From identities (5.7) and (5.10), and since we have $la^2_n(-x) = (-1)^n la^2_n(x),\ n \geq 0$, we can conclude that $\{P_n\}_{n\geq 0}$ is symmetric.

Let us now focus in identities (5.8) and (5.11). We can easily establish that $\{b^2_n\}_{n\geq 0}$ is symmetric. Also, analysing the part written in terms of x^{3k+1}, we get that if $b+c=0$, then $\{Q_n\}_{n\geq 0}$ is symmetric. From the part written in terms of x^{3k} we get that if $a=0$, then $b^1_n(x)$ is odd, when n is even, and $b^1_n(x)$ is even, when n is odd, $n \geq 0$, that is, $lb^1_n(-x) = (-1)^n lb^1_n(x),\ n \geq 0$.

In the same manner, considering identities (5.9) and (5.12), the remaining conclusions are easily obtained. □

An analogous approach to the parity of the component sequences of a symmetric MPS $\{W_n\}_{n\geq 0}$, when $a=b=c=0$ and $\varpi(x) = x^3 + qx + r$ $(p=0)$, is inefficacious, since the polynomial $(\varpi(x))^k$ has odd and even powers of x for each value of k. Even considering

$r = 0$, yielding that $(\varpi(x))^k$ is even if and only if k is even, this direct analysis is insufficient, leaving us, for now, with the conjectures taken previously.

Finally, we present two tables that organize the results and conjectures advanced above, for each choice of parameters taken. The symmetric character announced for the choices involving $p = q = r = 0$ is due to theorem 5.2.

Table 5.1: Hermite, second kind Tchebyshev, modified Lommel and Tricomi-Carlitz sequences

results	conjectures
$a=b=c=0$ and $p=q=r=0$	
The principal components are symmetric and not orthogonal. $\{ma^1\}_{n\geq 0}$, $\{mla^2\}_{n\geq 0}$, $\{mlb^1\}_{n\geq 0}$, $\{mb^2\}_{n\geq 0}$, $\{mc^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are symmetric.	$\{ma^1\}_{n\geq 0}$, $\{mla^2\}_{n\geq 0}$, $\{mlb^1\}_{n\geq 0}$, $\{mb^2\}_{n\geq 0}$, $\{mc^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are MPSs.
$a=b=c=0$ and $p=0$	
	$\{ma^1\}_{n\geq 0}$, $\{mla^2\}_{n\geq 0}$, $\{mlb^1\}_{n\geq 0}$, $\{mb^2\}_{n\geq 0}$, $\{mc^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are MPSs. For $\zeta = P, Q, R, ma^1, mla^2, mlb^1, mb^2, mc^1$ and mlc^2, we have: $\beta_n^\zeta = r, \quad \chi_{2n+1,2\nu}^\zeta = 0,\ 0 \leq \nu \leq n,$ $\chi_{2n,2\nu+1}^\zeta = 0,\ 0 \leq \nu \leq n-1.$
$p=q=r=0$	
$\{R_n\}_{n\geq 0}$ is symmetric; $\{mla^2\}_{n\geq 0}$ and $\{mb^2\}_{n\geq 0}$ are symmetric.	$\{mla^2\}_{n\geq 0}$ and $\{mb^2\}_{n\geq 0}$ are MPSs.
$p=q=r=0$ and $a=0$	
$\{P_n\}_{n\geq 0}$ is symmetric; $\{mlb_n^1\}_{n\geq 0}$ and $\{mc_n^1\}_{n\geq 0}$ are symmetric.	$\{ma_n^1\}_{n\geq 0}$, $\{mlb^1\}_{n\geq 0}$, $\{mc^1\}_{n\geq 0}$ and $\{mc^2\}_{n\geq 0}$ are MPSs.
$p=q=r=0$ and $b+c=0$	
$\{Q_n\}_{n\geq 0}$ is symmetric; $\{ma_n^1\}_{n\geq 0}$ and $\{mlc_n^2\}_{n\geq 0}$ are symmetric.	$\{ma_n^1\}_{n\geq 0}$, $\{mb^1\}_{n\geq 0}$, $\{mc^1\}_{n\geq 0}$ and $\{mlc^2\}_{n\geq 0}$ are MPSs.

Table 5.2: A shift of the second kind Tchebyshev sequence: $\beta_n = \beta$, $\gamma_{n+1} = \alpha$, $n \geq 0$, $\alpha \neq 0$.

results	conjectures
$p = -3\beta$ and $q = -3\alpha + 3\beta^2 \quad (\beta \neq a)$	
The principal components are orthogonal, with recurrence coefficients: $\beta_0^P = r - a\alpha - 2\alpha\beta + \beta^3$, $\beta_n^P = r - 3\alpha\beta + \beta^3,\ \ n \geq 1$, $\beta_n^Q = \beta_n^R = r - 3\alpha\beta + \beta^3,\ \ n \geq 0$, $\gamma_{n+1}^P = \gamma_{n+1}^Q = \gamma_{n+1}^R = \alpha^3,\ \ n \geq 0$. (In virtue of Theorem 3.2)	$\{a^2\}_{n\geq 0}$ and $\{b^2\}_{n\geq 0}$ vanish; $\{ma_n^1\}_{n\geq 0}$, $\{mb^1\}_{n\geq 0}$, $\{mc^1\}_{n\geq 0}$ and $\{mc^2\}_{n\geq 0}$ are M**O**PSs, with recurrence coefficients: $\beta_n^\zeta = r - 3\alpha\beta + \beta^3,\ n \geq 0$, $\zeta = ma^1, mc^1, mc^2$, $\beta_0^{mb^1} = \frac{-ar+\alpha^2+r\beta+3a\alpha\beta-3\alpha\beta^2-a\beta^3+\beta^4}{\beta-a}$, $\beta_n^{mb^1} = r - 3\alpha\beta + \beta^3,\ n \geq 0$, $\gamma_{n+1}^\zeta = \alpha^3,\ \ \zeta = ma^1, mb^1, mc^1, mc^2$.

Chapter 6

Conclusions

The path pursued in this thesis begins with the general description of the elements of a CD, called component sequences, and the constructive determination of them.

We remark that the structure used to present the CD, namely the matrix presentation of the nine component sequences, and the relations that determine these sequences, can be generalized to a decomposition of higher order. Therefore, considering a decomposition of order n, having the $n^2 + 1$ constructive relations, which allow the computation of the n^2 component sequences, all procedures presented might have a straightforward accommodation.

Having treated, in this thesis, the CD of an orthogonal and a 2-orthogonal sequence, it is our belief that the study of the CD of a d-orthogonal sequence, where $d > 2$, can be pursuit using the techniques presented here for the cases $d = 1$ and $d = 2$, although, perhaps with a considerable increase of calculations.

On the other hand, the tools constructed in *Mathematica*, which are the kernel of the fifth chapter, can be an efficient method of testing some future ideas, avoiding, in a few cases, the extensive analytical calculations that are involved in this kind of decomposition.

To finalize, we would like to remark that some of the main results established in this thesis are present in the reference [34]. The content of the fifth chapter will be the subject of a forthcoming publication [35] with a notebook called CubicDecomposition2010.nb attached, where all the examples and procedures performed in the *Mathematica* will be available.

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