DYNAMICS OF IMPLICIT OPERATIONS AND TAMENESS OF PSEUDOVARIETIES OF GROUPS

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ABSTRACT. This work gives a new approach to the construction of implicit operations. By considering "higher-dimensional" spaces of implicit operations and implicit operators between them, the projection of idempotents back to one-dimensional spaces produces implicit operations with interesting properties. Besides providing a wealth of examples of implicit operations which can be obtained by these means, it is shown how they can be used to deduce from results of Ribes and Zalesskiĭ, Margolis, Sapir and Weil, and Steinberg that the pseudovariety of p-groups is tame. More generally, for a recursively enumerable extension closed pseudovariety of groups $\mathbf{V}$, if it can be decided whether a finitely generated subgroup of the free group with the pro-$\mathbf{V}$ topology is dense, then $\mathbf{V}$ is tame.

1. INTRODUCTION

The theory of finite semigroups has developed considerably ever since, in the mid-1970's, Eilenberg [16] systematized the connections with the theories of rational languages and finite automata. His approach, which gave a natural framework for several earlier results, led to the notion of a *pseudovariety*, introduced jointly with Schützenberger [17]. Pseudovarieties are classes of finite semigroups which are closed under three of the most simple algebraic constructions, namely taking homomorphic images, subalgebras and finite products. Many results and problems fell then into the general setting of finding an algorithm to test membership in a pseudovariety. In case such an algorithm exists, the pseudovariety is said to be *decidable*. The difficulty lies in the fact that most often pseudovarieties are described not explicitly in terms of verifiable properties of their members but rather implicitly in terms of generators.

An operator on pseudovarieties which has deserved a great deal of attention is the semidirect product. It is intimately connected with the composition of sequential functions and a cascade composition of automata [15]. A central question in the theory has been to determine conditions for the semidirect product $\mathbf{V} * \mathbf{W}$ of two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ to be decidable. Rhodes [33] has shown recently that it does not suffice for the factors to be decidable. Yet, Steinberg and the author [6] have obtained a refined version of decidability, which was later called *tameness*

1991 *Mathematics Subject Classification.* Primary: 20E18, 20M05, 20M07; Secondary: 20F10, 20E07, 20E05.

Key words and phrases. Profinite topology, implicit operation, pseudovariety, free group, extension closed, finite semigroup, semidirect product.

The author gratefully acknowledges support by FCT through the *Centro de Matemática da Universidade do Porto*, by the project Praxis/2/2.1/MAT/63/94 (Portugal), and by NSERC (United Kingdom).

in [7], such that the semidirect product of any finite number of tame pseudovarieties is decidable.

To prove that a pseudovariety is tame is often quite hard and few examples can be found in the literature. For the pseudovariety $\mathbf{G}$ of all finite groups, tameness follows from a seminal paper by Ash [11]. For the pseudovariety $\mathbf{A}$ of all finite aperiodic semigroups, tameness has been announced by J. Rhodes (in a talk given at the International Conference on Algorithmic Problems on Groups and Semigroups, Lincoln, Nebraska, May, 1998 and later submitted for publication as [34]). These two results coupled with the joint result of Steinberg and the author quoted above imply that the Krohn-Rhodes group complexity of finite semigroups is computable, thus solving a long-standing problem in the theory of finite semigroups which has prompted many of its developments. See [7] for further references.

On the other hand, various connections between tameness and other areas of mathematics have been found. In geometric topology, a weak version of tameness has drawn considerable attention (cf. [23, 22]) and tameness of pseudovarieties of groups turns out to have remarkable links with model theory (cf. [25, 5, 4]).

Thus, it appears to be a worthwhile pursuit to obtain further results on tameness of pseudovarieties, the group case being of particular relevance. In the present paper, we address the case of extension closed pseudovarieties of groups $\mathbf{V}$, of which the pseudovariety $\mathbf{G}_p$ of all finite p-groups is an example. We show that if denseness of finitely generated subgroups of the free group with respect to the pro-$\mathbf{V}$ topology is decidable, then $\mathbf{V}$ is tame. In particular, $\mathbf{G}_p$ is tame. A weaker property had been established previously by Steinberg [37] but it also had been observed by Steinberg and the author [6] that $\mathbf{G}_p$ could not be tame with respect to the most common algebraic language. Since such a language had been the only one to play a role so far, a significant new step seemed to be necessary to deal with $\mathbf{G}_p$. Although the details of the argument are somewhat delicate, the idea in the construction of the new language is in fact quite simple, being rather close to standard ideas in dynamical systems.

This work was started while the author was visiting the University of Essex in the Summer of 1999 integrated in a workshop on quasi-crystals. The author wishes to thank several participants in the workshop: Peter M. Higgins for his hospitality and for asking a question on implicit operations which ultimately led to this work; Stuart Margolis and Ben Steinberg for very clear presentations of some of the main ideas in [28] and [37]; Ben Steinberg also for several stimulating discussions, for the observation that, at least without the requirement of computability, it would be possible to enlarge the natural algebraic language to capture closure of rational subsets of the free group with respect to the p-group topology without needing to solve an extra word problem, and for his comments on a preliminary version of this paper. Also thanks to Jon McCammond for his patience in exploring a strange word problem at a stage when details of the proof herein had gone astray.

2. Dynamics of implicit operations

Throughout this section we consider a pseudovariety $\mathbf{V}$ of finite algebras of fixed type. For the applications in the present paper, we will be interested only in the case of pseudovarieties of finite semigroups. But, since the theory is the same in the more general setting, there seems to be no reason to develop it only in that restricted context. We only assume that the algebraic type, or signature, involves only finitely many operations and that the operations have finite arity.

Denote by $\mathbf{S}$ and $\mathbf{M}$ respectively the pseudovarieties of all finite semigroups and all finite monoids.

This paper assumes familiarity with standard notions and basic results on implicit operations. The reader is referred to [2, Chapter 3] and [1, 9] for details.

2.1. **Pro-V topologies and pro-V algebras.** Recall that an algebra S of the same type as the members of $\mathbf{V}$ is said to be *residually in* $\mathbf{V}$ if homomorphisms from S into members of $\mathbf{V}$ suffice to separate the points of S. We always endow finite algebras with the discrete topology. The *pro-*$\mathbf{V}$ *topology* of an algebra S which is residually in $\mathbf{V}$ is the initial topology for the homomorphisms from S into members of $\mathbf{V}$. Such a topology is clearly Hausdorff. Moreover, the basic algebraic operations, being preserved under homomorphisms, are continuous. We say that an algebra endowed with a topology is a *topological algebra* if the basic algebraic operations are continuous. A *compact algebra* is a Hausdorff topological algebra in which the topology is compact. We also say that a topological algebra S is *residually in* $\mathbf{V}$ if continuous homomorphisms into members of $\mathbf{V}$ suffice to separate points of S.

On the other hand, *pro-*$\mathbf{V}$ *algebras* are compact algebras which are residually in $\mathbf{V}$. In case $\mathbf{V}$ consists of all finite algebras, we call them *profinite algebras*. Equivalently, pro-$\mathbf{V}$ algebras are projective limits of (topological) algebras in $\mathbf{V}$. That such projective limits are pro-$\mathbf{V}$ algebras is immediate. For the converse, given a pro-$\mathbf{V}$ algebra S, consider a set of representatives of isomorphism classes of continuous homomorphic images V of S in $\mathbf{V}$ together with specific continuous onto homomorphisms $\varphi_V : S \to V$ and connect them by homomorphisms $\psi : W \to V$ such that $\psi \circ \varphi_W = \varphi_V$. It is an elementary exercise to check that this defines a directed family of algebras and connecting homomorphisms whose projective limit is isomorphic to S as a topological algebra (see [9] for details). It is worth noting that, for some classes of algebras, including semigroups, the separation property in the definition of a profinite algebra is equivalent to zero-dimensionality [1].

We denote by $\overline{\Omega}_A\mathbf{V}$ the free pro-$\mathbf{V}$ algebra on the set A and by $\Omega_A\mathbf{V}$ the free algebra in the (Birkhoff) variety generated by $\mathbf{V}$. Such free objects are characterized by appropriate universal properties. For instance, $\overline{\Omega}_A\mathbf{V}$ comes endowed with a mapping $\iota : A \to \overline{\Omega}_A\mathbf{V}$ such that, for every mapping $\varphi : A \to S$ into a pro-$\mathbf{V}$ algebra S, there exists a unique continuous homomorphism $\hat{\varphi} : \overline{\Omega}_A\mathbf{V} \to S$ such that $\hat{\varphi} \circ \iota = \varphi$.

While the existence of free algebras was shown by Birkhoff in the 1930's and a proof may be found in any introductory textbook on Universal Algebra, the proof of existence of free pro-$\mathbf{V}$ algebras has to take into account the topology, which makes it somewhat more complicated. A specific way to realize $\overline{\Omega}_A\mathbf{V}$ is as the most general projective limit of A-generated algebras in $\mathbf{V}$. See, for instance, [9] for details. However, straight out of the definitions, one can show that the subalgebra of $\overline{\Omega}_A\mathbf{V}$ generated by the set A is a free algebra on A in the variety generated by $\mathbf{V}$. Thus, we will view $\Omega_A\mathbf{V}$ as this subalgebra of $\overline{\Omega}_A\mathbf{V}$. Moreover, the closure of $\Omega_A\mathbf{V}$ in $\overline{\Omega}_A\mathbf{V}$ retains the properties of the free pro-$\mathbf{V}$ algebra on A and therefore $\Omega_A\mathbf{V}$ is dense in $\overline{\Omega}_A\mathbf{V}$. Furthermore, the induced topology on $\Omega_A\mathbf{V}$ is such that every homomorphism $\varphi : \Omega_A\mathbf{V} \to V$ into a member V of $\mathbf{V}$ is continuous: the restriction of such a homomorphism to A extends uniquely to a continuous homomorphism

$\hat{\varphi} : \overline{\Omega}_A\mathbf{V} \to V$ whose restriction to $\Omega_A\mathbf{V}$ must be φ because $\Omega_A\mathbf{V}$ is the subalgebra generated by A. Hence the induced topology on $\Omega_A\mathbf{V}$ contains its pro-$\mathbf{V}$ topology.

Suppose S is a topological algebra which is residually in $\mathbf{V}$. We endow S with a natural *pro-*$\mathbf{V}$ *metric* by letting, for distinct $s_1, s_2 \in S$, $d(s_1, s_2) = 2^{-r(s_1,s_2)}$ where $r(s_1, s_2)$ is the size of the smallest $V \in \mathbf{V}$ for which there is a continuous homomorphism $\varphi : S \to V$ such that $\varphi s_1 \neq \varphi s_2$, and $d(s_1, s_1) = 0$. When convenient, we may wish to represent the functions d and r respectively by d_S and r_S. It is immediate to verify that d is indeed a metric on S, in fact an *ultrametric* in the sense that the triangle inequality is strengthened to the inequality

$$d(s_1, s_3) \leq \max\{d(s_1, s_2), d(s_2, s_3)\}.$$

Since we are only taking continuous homomorphisms into members of $\mathbf{V}$, the topology induced by the metric d is contained in the topology of S. In case the topology of S is the pro-$\mathbf{V}$ topology, the open d-balls in fact form a base of this topology and hence the two topologies coincide. In case S is just a plain algebra which is residually in $\mathbf{V}$, we consider the metric d_S associated with the pro-$\mathbf{V}$ topology.

On the other hand, if S is a pro-$\mathbf{V}$ algebra and s is an element of an open set O, then we may separate s from each element t of the complement $S \setminus O$ by means of a continuous homomorphism $\varphi_t : S \to V_t$ into a member of $\mathbf{V}$. The balls

$$B_t = \{u \in S : d(t, u) < 2^{-|V_t|}\}$$

cover the compact set $S \setminus O$ and none of them contains the point s since $r(t, s) \leq |V_t|$. We may therefore extract from them a finite covering of $S \setminus O$ consisting of balls centered at points $t_1, \ldots, t_n$. Let $V = V_{t_1} \times \cdots \times V_{t_n}$, which is another member of the pseudovariety $\mathbf{V}$. Now the homomorphisms φ_{t_i} induce a continuous homomorphism $S \to V$ which shows that the ball $\{u \in S : d(s, u) < 2^{-|V|}\}$, being disjoint from all the balls B_{t_i} with $i = 1, \ldots, n$, is contained in O. Hence here again the topology of S is induced by the metric d. In particular, the induced topology on $\Omega_A\mathbf{V}$ from that of $\overline{\Omega}_A\mathbf{V}$ is contained in its own pro-$\mathbf{V}$ topology and therefore the two topologies coincide.

By observing that an algebra S which is residually in $\mathbf{V}$ embeds in a product of members of $\mathbf{V}$ as a subalgebra, whose closure is compact by Tychonoff's Theorem, it may be shown that the completion of S with respect to the metric d_S is a pro-$\mathbf{V}$ algebra. In particular, $\overline{\Omega}_A\mathbf{V}$ is the completion of the residually in $\mathbf{V}$ plain algebra $\Omega_A\mathbf{V}$ with respect to the pro-$\mathbf{V}$ topology.

For a more detailed discussion of pro-$\mathbf{V}$ topologies in case $\mathbf{V}$ is a pseudovariety of finite groups, see [28, Section 1]. For later use, we will need the following slight generalization and sharpening of [28, Proposition 1.6]. Say that a subalgebra T of an algebra S is a *retract of* S if there is a homomorphism $\varrho : S \to T$ (called a *retract mapping*) whose restriction to T is the identity. In particular, if G is a group which is the free product of H with another subgroup, then H is a retract of G.

Lemma 2.1. *Let T be a retract of an algebra S which is residually in* $\mathbf{V}$*. Then, for any $u, v \in T$, $d_S(u, v) = d_T(u, v)$. In particular, the pro-*$\mathbf{V}$ *topology of T is the induced topology from the pro-*$\mathbf{V}$ *topology of S.*

Proof. Let $\varrho : S \to T$ be a retract mapping. Let u and v be distinct elements of T. It suffices to show that $r_S(u, v) = r_T(u, v)$. Suppose $\varphi : T \to V$ is a homomorphism into a member of $\mathbf{V}$ such that $\varphi u \neq \varphi v$. Consider the homomorphism $\varphi \circ \varrho : S \to V$. Since the restriction of ϱ to T is the identity, we also have $\varphi\varrho u \neq \varphi\varrho v$. Hence

$r_S(u,v) \le r_T(u,v)$. The reverse inequality is true for any subalgebra, retract or not. Indeed, for a homomorphism $\varphi : S \to V$ such that $\varphi u \ne \varphi v$, the restriction of φ to T still distinguishes u and v. $\square$

The above considerations could be extended without any further effort to the case in which S is not required to be residually in $\mathbf{V}$ leading to a pseudometric d_S and a non-Hausdorff topology. For the applications we have in mind the above special case is sufficient.

2.2. **Implicit operations.** Elements of $\overline{\Omega}_A\mathbf{V}$ are also called *implicit operations.* Since the operational point of view plays an important role in this paper, it is worth spending some time explaining this connection. Given any pro-$\mathbf{V}$ algebra S, we view S^A both as a direct power of S and as the set of all functions from A to S. By the universal property defining $\overline{\Omega}_A\mathbf{V}$, each $\varphi \in S^A$, extends uniquely to a continuous homomorphism $\hat{\varphi} : \overline{\Omega}_A\mathbf{V} \to S$. This allows us to interpret each $\pi \in \overline{\Omega}_A\mathbf{V}$ as an $|A|$-ary operation $\pi_S : S^A \to S$ on S: for each $\varphi \in S^A$, take $\pi_S(\varphi) = \hat{\varphi}(\pi)$. This is called the *natural interpretation* of π as an operation on S. It is easy to check that the natural interpretation of π commutes with continuous homomorphisms in the sense that, for every continuous homomorphism $\rho : S \to T$ between pro-$\mathbf{V}$ algebras and every $\varphi \in S^A$, the equality $\pi_T(\rho \circ \varphi) = \rho(\pi_S(\varphi))$ holds. An $|A|$-ary operation with an interpretation on each member of $\mathbf{V}$ which commutes with homomorphisms is said to be an $|A|$*-ary implicit operation* on $\mathbf{V}$. In particular, through the natural interpretation, to every element of $\overline{\Omega}_A\mathbf{V}$ is associated an $|A|$-ary implicit operation on $\mathbf{V}$. Since $\overline{\Omega}_A\mathbf{V}$ is residually in $\mathbf{V}$ this mapping is injective.

Assuming the set A is finite, the natural interpretation mapping of the preceding paragraph is in fact a bijection. A simple way to establish every implicit operation π is the natural interpretation of some element of $\overline{\Omega}_A\mathbf{V}$ is to construct a suitable Cauchy sequence in $\Omega_A\mathbf{V}$. For each positive integer n, let S_n be the direct product of representatives of isomorphism classes of members of $\mathbf{V}$ with at most n elements. The restriction on the algebraic type assumed at the beginning of the section guarantees that S_n is finite and, therefore, S_n lies in $\mathbf{V}$. Since the free algebra on A in the variety generated by S_n is still finite, there is some $w_n \in \Omega_A\mathbf{V}$ such that $\pi_{S_n} = (w_n)_{S_n}$. It follows that $(w_n)_n$ is a Cauchy sequence with respect to the metric d for certainly $d(w_m, w_n) < 2^{-\min\{m,n\}}$. Let π' be the limit of this sequence in $\overline{\Omega}_A\mathbf{V}$. We claim that the natural interpretation of π' is the original implicit operation π. Indeed, if $S \in \mathbf{V}$, then for all sufficiently large $n \ge |S|$, we have $d(\pi', w_n) < 2^{-|S|}$ and so $\pi'_S = (w_n)_S = \pi_S$. For further details and examples see [2, Chapter 3]. From hereon we will always view the elements of $\overline{\Omega}_A\mathbf{V}$ as implicit operations.

Members of $\Omega_A\mathbf{V}$ are rather special implicit operations as they may be constructed from the component projections using the basic operations of the algebraic signature. In this context they are known as *explicit operations.*

Note that, for sets A and B of the same cardinality, $\overline{\Omega}_A\mathbf{V}$ and $\overline{\Omega}_B\mathbf{V}$ are isomorphic topological algebras via a mapping which extends a bijection between A and B. We will therefore sometimes write $\overline{\Omega}_r\mathbf{V}$ instead of $\overline{\Omega}_A\mathbf{V}$ where $r = |A|$ which may also be viewed as an instance of the preceding remark if the natural number r is regarded as a set as in set theory. This amounts to fixing an ordering for the elements of A and describing explicitly the components on which implicit operations depend, in the form $\pi(x_1, \dots, x_r)$, which is often convenient. We will nevertheless continue to use the more compact notation whenever spelling out components serves no real purpose.

When $A \subseteq B$, we identify $\overline{\Omega}_A\mathbf{V}$ with a closed subalgebra of $\overline{\Omega}_B\mathbf{V}$, namely the image of the unique continuous extension of the inclusion $A \hookrightarrow B$ to a homomorphism $\iota : \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_B\mathbf{V}$. This is possible since ι is injective. Viewing elements of free pro-$\mathbf{V}$ algebras as implicit operations, injectivity of ι is obvious: an element of $\overline{\Omega}_A\mathbf{V}$ is mapped by ι to a $|B|$-ary operation which only depends on the $|A|$ components corresponding to the elements of A, that is the composite of ι with the projection $\overline{\Omega}_B\mathbf{V} \to \overline{\Omega}_A\mathbf{V}$ given by restriction is the identity mapping of $\overline{\Omega}_A\mathbf{V}$.

2.3. **Implicit operators.** Let B and C be finite sets and let S be a pro-$\mathbf{V}$ algebra. For $c \in C$, denote by $x_c : S^C \to S$ the projection in the c-component. We say that a mapping $f : S^B \to S^C$ is an *implicit operator* if, for each $c \in C$, the composite $x_c \circ f$ is of the form π_S for some $\pi \in \overline{\Omega}_B\mathbf{V}$. If $f : S^B \to S^C$ is an implicit operator and $x_c \circ f = (\pi_c)_S$ for every $c \in C$, where $\pi_c \in \overline{\Omega}_B\mathbf{V}$, then we also write $f = \Big((\pi_c)_{c\in C}\Big)_S$. Denote by $\mathcal{O}(S^B, S^C)$ the set of all implicit operators $S^B \to S^C$.

The definition of implicit operator determines an onto function

$$\begin{array}{rcl} \epsilon : (\overline{\Omega}_B\mathbf{V})^C & \to & \mathcal{O}(S^B, S^C) \\ (\pi_c)_{c\in C} & \mapsto & \Big((\pi_c)_{c\in C}\Big)_S \end{array}$$

The function ϵ is in general not injective but it will be injective if $S = \overline{\Omega}_A\mathbf{V}$ with $B \subseteq A$ (or, more generally, if $|B| \le |A|$) since $\pi_S(\varphi) = \pi'$, where $\varphi \in (\overline{\Omega}_A\mathbf{V})^B$ maps each $b \in B$ to x_b and $\pi' \in \overline{\Omega}_A\mathbf{V}$ is obtained from π by ignoring the components in $A \setminus B$.

For a pro-$\mathbf{V}$ algebra S, we endow the set $\mathcal{O}(S^B, S)$ with a metric defined by

$$d(g,h) = \inf\{d(\pi,\rho) : \ g = \pi_S,\ h = \rho_S\}. \tag{2.1}$$

To prove that this function is indeed a metric, we first show that the infimum in its definition is in fact a minimum. To start, we prove that, for each $g \in \mathcal{O}(S^B, S)$ the set K_g of all $\pi \in \overline{\Omega}_B\mathbf{V}$ such that $\pi_S = g$ is closed. Suppose then that $(\pi_n)_n$ is a convergent sequence in K_g with limit π. If $\pi_S \neq g$ then there exists some $\varphi \in S^B$ such that $\pi_S(\varphi) \neq g(\varphi)$ and so, since S is residually in $\mathbf{V}$, there exists some continuous homomorphism $\psi : S \to V$ into some member V of $\mathbf{V}$ such that

$$\psi(\pi_S(\varphi)) \neq \psi(g(\varphi)) \tag{2.2}$$

But, for sufficiently large n, we have $(\pi_n)_V = \pi_V$ which implies

$$\psi(\pi_S(\varphi)) = \pi_V(\psi \circ \varphi) = (\pi_n)_V(\psi \circ \varphi) = \psi((\pi_n)_S(\varphi)) = \psi(g(\varphi))$$

which contradicts (2.2). Next, since the distance function $d : \overline{\Omega}_A\mathbf{V} \times \overline{\Omega}_A\mathbf{V} \to \mathbb{R}$ is continuous the infimum in the image of the restriction of d to the compact set $K_g \times K_h$ must be attained and this is just the value defined above for $d(g,h)$. The proof that d is an ultrametric follows then easily from its definition in terms of an ultrametric on $\overline{\Omega}_A\mathbf{V}$. For instance, if $d(g,h) = 0$ then by the above there exist $\pi \in K_g$ and $\rho \in K_h$ such that $d(\pi,\rho) = 0$ which implies that $\pi = \rho$ and so $g = h$.

In turn, the product operator space $\mathcal{O}(S^B, S^C) \simeq \prod_{c\in C} \mathcal{O}(S^B, S)$ is endowed with an induced metric from the above metric on each factor in a standard way, say by taking the ℓ_1-distance defined by

$$d((g_c)_{c\in C},(h_c)_{c\in C}) = \sum_{c\in C} d(g_c,h_c). \tag{2.3}$$

With respect to this metric, the function ϵ is a contractive mapping and, therefore, continuous. Since $(\overline{\Omega}_B\mathbf{V})^C$ is a compact space, it follows that so is $\mathcal{O}(S^B,S^C)$.

Composition of implicit operations is easily described in terms of implicit operators. Namely, given a $|C|$-ary implicit operation $\pi\in\overline{\Omega}_C\mathbf{V}$ and a $|C|$-tuple $\varphi\in(\overline{\Omega}_B\mathbf{V})^C$ of $|B|$-ary implicit operations, the implicit operation $\pi(\varphi)\in\overline{\Omega}_B\mathbf{V}$ is defined by the equality

$$(\pi(\varphi))_V = \pi_V\circ\epsilon(\varphi)$$

determining its interpretation on each $V\in\mathbf{V}$. If an ordering is chosen for the elements of C and the components of C-powers are spelled out, say $\varphi=(\rho_1,\dots,\rho_r)$, the above equality becomes the perhaps more easily readable formula

$$(\pi(\rho_1,\dots,\rho_r))_V = \pi_V((\rho_1)_V,\dots,(\rho_r)_V). \tag{2.4}$$

It is a straightforward exercise in abstract nonsense to check that these formulas define a $|B|$-ary implicit operation on $\mathbf{V}$.

The following inequality holds for implicit operations $\pi,\pi'\in\overline{\Omega}_C\mathbf{V}$ and $\rho_i,\rho_i'\in\overline{\Omega}_B\mathbf{V}$ $(i\in C)$,

$$d(\pi(\rho_1,\dots,\rho_r),\pi'(\rho_1',\dots,\rho_r'))\le\max\{d(\pi,\pi');\ d(\rho_i,\rho_i'):i=1,\dots,r\},$$

where $r=|C|$, in view of the definition of the pro-$\mathbf{V}$ metric: if all the distances $d(\pi,\pi')$ and $d(\rho_i,\rho_i')$ do not exceed 2^{-n}, then $\pi_V=\pi_V'$ and $(\rho_i)_V=(\rho_i')_V$ whenever $V\in\mathbf{V}$ satisfies $|V|<n$ and the equality (2.4) yields $(\pi(\rho_1,\dots,\rho_r))_V=(\pi'(\rho_1',\dots,\rho_r'))_V$. As a consequence we obtain that composition of implicit operations is continuous in the sense that the function

$$\begin{array}{rcl}\overline{\Omega}_C\mathbf{V}\times(\overline{\Omega}_B\mathbf{V})^C & \to & \overline{\Omega}_B\mathbf{V}\\ (\pi,\rho_1,\dots,\rho_r) & \mapsto & \pi(\rho_1,\dots,\rho_r)\end{array}$$

is continuous, as it is even distance-reducing.

Let us now consider the special case when $B=C=A$ where A is a finite set. Denote by $\mathcal{O}(S^A)$ the metric space $\mathcal{O}(S^A,S^A)$. Since composition of implicit operations (of appropriate arities) produces implicit operations, the composite of two elements $g,h\in\mathcal{O}(S^A)$ is again an element of $\mathcal{O}(S^A)$. Hence $\mathcal{O}(S^A)$ is a semigroup. One the other hand, from the continuity of composition of implicit operations and the definition of the metric on implicit operators, see (2.1) and (2.3), it follows that composition of implicit operators is also continuous. In particular, $\mathcal{O}(S^A)$ is a compact monoid. The following is a more precise result.

Proposition 2.2. *The rule $\mathcal{O}_A: S\mapsto\mathcal{O}(S^A)$ defines a functor from the category of pro-$\mathbf{V}$ algebras with onto continuous homomorphisms as morphisms into the category of profinite monoids.*

Proof. We must first define how $\mathcal{O}_A$ transforms onto continuous homomorphisms $\varphi: S\to T$ between pro-$\mathbf{V}$ algebras. Given $g\in\mathcal{O}(S^A)$, there are $\pi_a\in\overline{\Omega}_A\mathbf{V}$ $(a\in A)$ such that $g=((\pi_a)_{a\in A})_S$. We claim that the element $h=((\pi_a)_{a\in A})_T$ of $\mathcal{O}(T^A)$ is independent of the choice of the π_a, which allows us to define $\mathcal{O}_A(\varphi)$ to be the mapping $\mathcal{O}(S^A)\to\mathcal{O}(T^A)$ which sends each g to the operator h defined above.

To establish the claim, we consider an element ψ of T^A. Since φ is onto, there is $\delta \in S^A$ such that $\psi = \varphi \circ \delta$. Since the π_a are implicit operations on $\mathbf{V}$, $h \circ \varphi^A = \varphi^A \circ g$ where $\varphi^A : S^A \to T^A$ is φ component-wise. Hence

$$h(\psi) = h \circ \varphi^A(\delta) = \varphi^A \circ g(\delta)$$

which shows that h depends only on g and not on the choice of the π_a, thereby proving the claim.

From the claim it also follows that $\mathcal{O}_A$ preserves composition of morphisms and identity morphisms. We have thus shown that $\mathcal{O}_A$ is a functor with values in the category of topological monoids. To complete the proof, it remains to show that, for each pro-$\mathbf{V}$ algebra S, the topological monoid $\mathcal{O}(S^A)$ is actually profinite. For this purpose, it suffices to show that, for any two distinct elements g, h of $\mathcal{O}(S^A)$, there is an onto continuous homomorphism $\varphi : S \to T$ into some $T \in \mathbf{V}$ such that $(\mathcal{O}_A(\varphi))(g) \neq (\mathcal{O}_A(\varphi))(h)$. Indeed, by hypothesis there is some $\psi \in S^A$ such that $g(\psi) \neq h(\psi)$. Since S is a pro-$\mathbf{V}$ algebra, it follows that there is some continuous homomorphism $\varphi : S \to T$ into some member of $\mathbf{V}$ such that $\varphi^A(g(\psi)) \neq \varphi^A(h(\psi))$. Without loss of generality, we may assume that φ is onto. Then the operators $(\mathcal{O}_A(\varphi))(g)$ and $(\mathcal{O}_A(\varphi))(h)$ are different since they act differently on $\varphi^A(\psi)$. □

The last part of the above proof amounts to showing that the monoid $\mathcal{O}(S^A)$ is the projective limit of the monoids $\mathcal{O}(T^A)$ where T is a finite homomorphic image of S. We will use this fact below.

2.4. **Polynomial operators and the role of monoid implicit operations.** Denote by $\mathcal{P}(S^B, S^C)$ the subset of $\mathcal{O}(S^B, S^C)$ which is the image of $(\Omega_B \mathbf{V})^C$ under ϵ. We will call its elements the *polynomial operators* (we might also call them *explicit operators*).

We also represent $\mathcal{P}(S^A, S^A)$ by $\mathcal{P}(S^A)$. Note that $\mathcal{P}(S^A)$ is a submonoid of $\mathcal{O}(S^A)$. It is in general not a finitely generated monoid. For instance, let $S = \overline{\Omega}_1 \mathbf{S}$ be the free monogenic profinite semigroup. Then $\mathcal{P}(S)$ consists of all operators of the form $x \mapsto x^r$ ($r \geq 1$) with composition corresponding to exponent multiplication, whence $\mathcal{P}(S)$ is not finitely generated under composition. The same example shows that $\mathcal{O}(S^A)$ is not in general a finitely generated profinite monoid.

If we are willing to go into a higher dimension, that is in a sense we allow for some auxiliary memory, then we can generate implicit operators using composition. Thus the role played by monoid implicit operations in the construction of implicit operators is rather important as the main result of this subsection shows.

Some finiteness assumption seems to be unavoidable in this context. We say that a pseudovariety $\mathbf{V}$ *requires bounded memory for the computation of polynomials* if, for every $n \geq 1$, there is some $k \geq 0$ such that every $p \in \Omega_n \mathbf{V}$ may be computed from the n projections $x_1, \dots, x_n$ using at each step either a projection or one of the basic operations applied to memorized values, and storing in the process values in at most k memory cells.

There are several common examples of pseudovarieties which require bounded memory for the computation of polynomials. First note that this property is preserved under taking subpseudovarieties since the corresponding polynomials are obtained by restriction. The pseudovariety of all finite semigroups (or monoids) has this property since, in computing a word, we need only remember the letters composing it and, reading the word say, from left to right, at each step remember only the value of the prefix read so far. In computing polynomials for groups or inverse semigroups, we may need an extra cell for computing inverses. For rings, we also need an extra memory cell to compute products before adding them to a partial sum. In contrast, groupoids (in the sense of algebras with one binary operation) require an unbounded number of memory cells since all kinds of partial products have to be stored before the end result may be computed.

Proposition 2.3. *Let* $\mathbf{V}$ *be a pseudovariety of finite finitary type which requires bounded memory for the computation of polynomials. Let* S *be a pro-*$\mathbf{V}$ *algebra. Then, for every* n *there is some* $m \geq n$ *such that every element of* $\mathcal{O}(S^n)$ *may be computed from the first* n *components of an implicit operator of the form* $\pi_M(\rho_1, \ldots, \rho_r)$ *with* $r \geq 1$ *bounded, some* $\pi \in \overline{\Omega}_r\mathbf{M}$*, and some* $\rho_1, \ldots, \rho_r \in \mathcal{P}(S^m)$*, where* $M = \mathcal{O}(S^m)$*. The way the computation of an implicit operator depending on* n *variables from one depending on* m *variables takes place is by repeating the first variable* $m - n$ *times at the beginning before applying the operator on* m *variables.*

Proof. We first claim that the hypotheses on $\mathbf{V}$ guarantee that every polynomial operator may be computed as in the statement of the proposition with $\pi \in \Omega_r\mathbf{M}$. Let k be an upper bound on the number of memory cells required to compute polynomials in n variables over $\mathbf{V}$ and let $m = n + k$. For each basic operation f in the algebraic type consider the special polynomial operators such that, for some $i \leq k$, every j-component with $j \neq i$ is the j-projection, while the i-component is f evaluated at the projections in some order. Note that, since the algebraic type is finite and finitary, there are only finitely many such operators. The hypothesis that $\mathbf{V}$ requires only k memory cells for computing polynomials in n variables means that every such polynomial is obtained by applying a composite of the preceding polynomial operators to $(x_1, \ldots, x_1, x_2, \ldots, x_n)$, which proves the claim, where we take r to be the number of special polynomial operators considered above.

To complete the proof, it suffices to observe that every implicit operator is the limit of a sequence of polynomial operators. Indeed, since $\mathcal{O}(S^n)$ is the projective limit of the $\mathcal{O}(T^n) = \mathcal{P}(T^n)$ where T runs over the finite homomorphic images of S, standard techniques show that every element g of $\mathcal{O}(S^n)$ may be approximated by polynomial operators within any desired positive tolerance and whence g is the limit of some sequence of polynomial operators. By the above, all elements of such a sequence may be computed from an expression $(\pi_l)_M(\rho_1, \ldots, \rho_r)$ with $\pi_l \in \Omega_r\mathbf{M}$ and the $\rho_i \in \mathcal{P}(S^m)$ independent of l. By taking a subsequence we may assume that the sequence $(\pi_l)_l$ converges, say to $\pi \in \overline{\Omega}_r\mathbf{M}$. Since the mapping ϵ is continuous, it follows that $g = \pi_M(\rho_1, \ldots, \rho_r)$, as desired. □

Thus to obtain a general implicit operator $f \in \mathcal{O}(S^A)$ on a pro-$\mathbf{V}$ algebra S for a pseudovariety $\mathbf{V}$ which requires bounded memory for the computation of polynomials, it suffices to know monoid implicit operations and explicit operations on $\mathbf{V}$. In particular, taking $S = \overline{\Omega}_A\mathbf{V}$, we conclude that implicit operations on $\mathbf{V}$ are constructed by composing explicit operations in a way described by monoid implicit operations. On the other hand, Proposition 2.3 says nothing interesting in the case of semigroups or monoids.

2.5. **Examples of implicit operators.** Recall that, for an element v of a finite semigroup V, v^ω denotes the unique idempotent power of v. This defines a unary implicit operation $x \mapsto x^\omega$ on finite semigroups (and similarly on finite monoids) which therefore has a natural interpretation on each profinite monoid of the form $\mathcal{O}(S^A)$. This operation now allows us to setup a rather general recursion scheme to construct implicit operations on an arbitrary pseudovariety $\mathbf{V}$ of finite algebras. Note that x^ω is the limit of the sequence $(x^{n!})_n$ of explicit operations.

Proposition 2.4. *Let A be a finite set, S a pro-$\mathbf{V}$ algebra, π_a an element of $\overline{\Omega}_A\mathbf{V}$ $(a \in A)$, and s_a an element of S $(a \in A)$. Define recursively a sequence $((u_{n,a})_{a\in A})_{n\geq 0}$ by*

$$u_{0,a} = s_a, \ u_{n+1,a} = \pi_a((u_{n,b})_{b\in A}).$$

a) Each of the sequences $(u_{n!,a})_n$ converges in S.

b) The function g sending each $(s_a)_{a\in A}$ to $(\lim_n u_{n!,a})_{a\in A}$ belongs to $\mathcal{O}(S^A)$ and is precisely the idempotent $((\pi_a)_{a\in A})_S^\omega$.

Proof. It suffices to observe that the correspondence $(s_a)_a \mapsto (\pi_b((s_a)_a))_b$ is precisely the function $h = ((\pi_a)_a)_S : S^A \to S^A$. Since $h^\omega = \lim_n h^{n!}$, the result follows. □

We say that an implicit operation $\pi \in \overline{\Omega}_A\mathbf{V}$ is *computable* if there is an algorithm which given $V \in \mathbf{V}$ and $\varphi \in V^A$, outputs the value $\pi_V(\varphi)$.

We denote by $\circ_a^\omega((\pi_b)_b)$ the component $x_a \circ g$ of the function g of Proposition 2.4(b). An immediate corollary of the above proposition is the following result which in computer science might be called a fixed point theorem.

Corollary 2.5. *Let $\pi_1, \ldots, \pi_r \in \overline{\Omega}_r\mathbf{V}$. Then $\circ_i^\omega(\pi_1, \ldots, \pi_r)$ $(i = 1, \ldots, r)$ is also a member of $\overline{\Omega}_r\mathbf{V}$. Moreover, if each π_i is a computable operation, then so is each $\circ_i^\omega(\pi_1, \ldots, \pi_r)$.* □

Here are several examples of application of the above scheme. Denote by $\mathbf{S}$ the pseudovariety of all finite semigroups and, for a prime p, by $\mathbf{G}_p$ the pseudovariety of all finite p-groups. Recall that a *pseudoidentity* (over $\mathbf{V}$) is a formal equality $\pi = \rho$ between implicit operations (on $\mathbf{V}$) of the same arity. A (pro-$\mathbf{V}$) algebra S *satisfies* the pseudoidentity $\pi = \rho$ if $\pi_S = \rho_S$. The class of all members of $\mathbf{V}$ which satisfy all pseudoidentities in a given set Σ is said to be *defined by* Σ. Reiterman [32] showed that every subpseudovariety of $\mathbf{V}$ is defined by some set of pseudoidentities.

Examples 2.6. 1) For $\pi \in \overline{\Omega}_1\mathbf{V}$, denote $\circ_1^\omega(\pi)$ by $\pi^{\circ\omega}$. For example, for $\mathbf{V} = \mathbf{S}$ and $\pi(x) = x^n$, we obtain the implicit operation

$$\pi^{\circ\omega}(x) = \lim_k x^{n^{k!}}$$

which we naturally denote by x^{n^ω}. By the elementary Euler congruence theorem, the pseudoidentity $x^{n^\omega} = x$ is valid in $\mathbf{G}_p$ if n is not divisible by p while the pseudoidentity $x^{n^\omega} = 1$ is valid in $\mathbf{G}_p$ otherwise. In fact, $\mathbf{G}_p$ is defined by the pseudoidentity $x^{p^\omega} = 1$.

2) Let $\pi \in \overline{\Omega}_r\mathbf{V}$ and $\rho, \rho_2, \ldots, \rho_r \in \overline{\Omega}_t\mathbf{V}$. Define recursively a sequence $(u_n)_n$ by letting $u_0 = \rho$ and $u_{n+1} = \pi(u_n, \rho_2, \ldots, \rho_r)$. Then the sequence $(u_{n!})_n$ converges, namely to

$$\circ_1^\omega(\pi, x_2, \ldots, x_r)(\rho, \rho_2, \ldots, \rho_r).$$

Particular cases for $\mathbf{V} = \mathbf{S}$ are

$$\begin{aligned} x^\omega &= \circ_1^\omega(x_1x_2, x_2)(1, x) \\ x^{\omega-1} &= \circ_1^\omega(x_1x_2, x_3, x_3)(1, 1, x). \end{aligned}$$

3) For recurrence depending on a bounded number of previous steps, say

$$u_0 = \rho_1, \ldots, \ u_{k-1} = \rho_k, \ u_{n+k} = \pi(u_n, \ldots, u_{n+k-1}) \ (n \geq 0),$$

we get

$$\lim_n u_{n!} = \circ_1^\omega(x_2, x_3, \ldots, x_k, \pi)(\rho_1, \ldots, \rho_k).$$

This gimmick allows us to delay the start of a recursion which really only depends on the previous step. For instance, for $\mathbf{V} = \mathbf{M}$, we obtain the following implicit operation:

$$x^{r^{\omega-1}} = \circ_1^\omega(x_2, x_2^r)(1, x).$$

For instance, if G is a finite group with exponent 12, then G satisfies the pseudoidentities $x^{2^\omega} = x^4$ and $x^{2^{\omega-1}} = x^8$. Note that, in $\overline{\Omega}_1\mathbf{M}$, we have the equalities $\left(x^{r^{\omega-1}}\right)^r = x^{r^\omega} = \left(x^{r^\omega}\right)^{r^\omega}$. More generally, $\overline{\Omega}_1\mathbf{M}$ may be viewed as a profinite semi-ring whose addition is the semigroup operation and whose multiplication is composition. Then x^{r^ω} is just the multiplicative ω-power of x^r in this semi-ring.

4) A more general example is obtained by introducing parameters as in (2). For the recurrence defined by

$$u_0 = \rho_1, \ldots,\ u_{k-1} = \rho_k,\ u_{n+k} = \pi(u_n, \ldots, u_{n+k-1}, \tau_1, \ldots, \tau_s)\ (n \geq 0),$$

we obtain the implicit operation

$$\lim_n u_{n!} = \circ_1^\omega(x_2, x_3, \ldots, x_k, \pi, x_{k+1}, \ldots, x_{k+s})(\rho_1, \ldots, \rho_k, \tau_1, \ldots, \tau_s). \quad \square$$

Rather than taking the ω-power of an implicit operator, we may wish to take another power, say with exponent ε where $x^\varepsilon \in \overline{\Omega}_1\mathbf{S}$. Say, if $T \in \mathcal{O}(S^n)$ is the implicit operator $T = (\pi_1, \ldots, \pi_n)$, then we will also write $\circ_i^\varepsilon(\pi_1, \ldots, \pi_n)$ instead of $x_i \circ T^\varepsilon$. The operations of the form $\circ_i^{\omega-1}(\pi_1, \ldots, \pi_n)$ play an important role in Section 6 as a convenient technical refinement of the operations of the form $\circ_i^\omega(\pi_1, \ldots, \pi_n)$. We have, of course, the following equality:

$$\circ_i^\omega(\pi_1, \ldots, \pi_n) = (\circ_i^{\omega-1}(\pi_1, \ldots, \pi_n)) \circ (\pi_1, \ldots, \pi_n).$$

But, again applying the delay trick, we obtain the equality

$$\circ_i^{\omega-1}(\pi_1, \ldots, \pi_n) = \circ_{n+1}^\omega(\pi_1, \ldots, \pi_n, x_i)$$

so that each operator $\circ_i^{\omega-1}$ and $\circ_j^\omega$ could be defined in terms of the other.

All the following examples concern finite semigroups.

Examples 2.7. 1) Define, recursively,

$$[x, {}_1y] = [x, y] = x^{\omega-1}y^{\omega-1}xy$$
$$[x, {}_{n+1}y] = [[x, {}_ny], y].$$

Then the sequence $([x, {}_{n!}y])_n$ converges to

$$[x, {}_\omega y] = \circ_1^\omega([x_1, x_2], x_2)(x, y).$$

Groups satisfying the identity $[x, {}_ny] = 1$ for some $n \geq 1$ are called (right) Engel groups (where, for an infinite group, $x^{\omega-1}$ is interpreted as the inverse of x). By a theorem of Zorn, the finite Engel groups are the finite nilpotent groups. Denoting the pseudovariety of all such finite groups by $\mathbf{G}_{nil}$, we conclude that it is defined by the pseudoidentity $[x, {}_\omega y] = 1$.

2) Consider the implicit operation

$$u_\omega = \circ_1^\omega([[x_1, x_2], [x_1, x_3]], x_2, x_3)([x, y], x, y).$$

Clearly every finite solvable group satisfies the pseudoidentity $u_\omega = 1$. E. Plotkin proposed in the Semigroup Conference held in St Petersburg in July, 1999, a conjecture which is equivalent to stating that the pseudoidentity $u_\omega = 1$ defines the pseudovariety $\mathbf{G}_{sol}$ of all finite solvable groups (see [24] for motivation and some evidence towards the conjecture). Since the pseudoidentity $u_\omega = 1$ only involves two variables, the validity of this conjecture would entail a result due to Thompson [38] stating that a finite group is solvable if and only if all its 2-generated subgroups are solvable. While Thompson derived this result as a corollary of his complete classification of simple groups whose proper subgroups are solvable, a proof of which extends over 410 published pages, Flavell [18] obtained a direct short and elementary proof of the same corollary.

3) The implicit operation

$$v_\omega = \circ_1^\omega((x_2(x_1x_3)^\omega)^\omega, x_2, x_3)(x^\omega, x, y)$$

has been considered by M. V. Volkov and the author in unpublished work showing that $v_\omega y v_\omega$ and $v_\omega x(v_\omega y)^\omega v_\omega$ describe two arbitrary $\mathcal{H}$-equivalent group elements of a finite semigroup. □

3. Tame pseudovarieties

Steinberg and the author introduced in [7] the notion of a tame pseudovariety as a reformulation of an earlier notion used in [6]. Before introducing these and some related notions which are the object of the main results in this paper, we recall the following theorem which is extracted from [7, Theorem 12.4] which in turn is a simplified reformulation of [6, Theorem 5.10], and is the basic motivation for the notion of tameness. Here, a pseudovariety is said to be *decidable* if there is an algorithm to test membership in it.

Theorem 3.1. *Let $\mathbf{V}_1, \dots, \mathbf{V}_n$ be tame pseudovarieties of semigroups. Then their semidirect product $\mathbf{V}_1 * \cdots * \mathbf{V}_n$ is decidable.*

This remarkable result is in contrast with the general situation. Rhodes [33] has shown that there is for instance a decidable pseudovariety $\mathbf{V}$ such that $\mathbf{Sl} * \mathbf{V}$ is undecidable, where $\mathbf{Sl}$ stands for the pseudovariety of all finite semilattices. Like any finitely generated pseudovariety, $\mathbf{Sl}$ is tame, so Rhodes' pseudovariety $\mathbf{V}$ is not tame by Theorem 3.1.

As it is perhaps to be expected, the notion of tameness is somewhat technical. It envolves several ingredients underlying which lies a suitable choice of an enlarged signature.

By an *implicit signature* we mean a set σ of implicit operations over finite semigroups which contains the basic semigroup operation of multiplication. For shortness, as in [7, Section 8], we will say that σ is *highly computable* if it is a countable recursively enumerable set and its members are computable operations. In the following we assume that σ is a highly computable implicit signature. Profinite semigroups are viewed as σ-algebras under the natural interpretation of semigroup implicit operations.

An important example of an implicit signature is the *canonical signature* κ consisting of the basic multiplication operation and the unary operation $x^{\omega-1}$. The word "canonical" is not used here in any technical sense but it rather just reflects the fact that the signature κ pervades most of finite semigroup theory. Clearly κ is highly computable.

Let $\mathbf{V}$ be a pseudovariety of semigroups. The free object over a set A in the variety of σ-algebras generated by $\mathbf{V}$ is denoted by $\Omega_A^\sigma \mathbf{V}$. It is the σ-subalgebra of $\overline{\Omega}_A \mathbf{V}$ generated by the component projections x_a ($a \in A$). In particular, we view $\Omega_A^\sigma \mathbf{V}$ as a topological σ-algebra. By the discussion in Section 2, the topology of $\Omega_A^\sigma \mathbf{V}$ induced from that of $\overline{\Omega}_A \mathbf{V}$ is the initial topology for all homomorphisms of σ-algebras into members of $\mathbf{V}$. We say that $\mathbf{V}$ is *σ-recursive* if the word problem for $\Omega_A^\sigma \mathbf{V}$ is decidable. By [6, Theorem 3.1] $\mathbf{V}$ is σ-recursive if and only if the variety of σ-algebras generated by $\mathbf{V}$ has a recursively enumerable basis of identities.

By a *graph* we mean what is usually called in the literature a directed multigraph given by a set of vertices V, a disjoint set of edges E, and adjacency functions $\alpha, \omega : E \to V$. A *labeling* of a graph $G = V \cup E$ by a semigroup S is a function $\varphi : G \to S^1$ such that $\varphi E \subseteq S$, where S^1 denotes the smallest monoid containing S as a subsemigroup. The labeling φ is said to be *consistent* if

$$(\varphi\alpha e)(\varphi e) = \varphi\omega e$$

for every edge e.

A *relational morphism* of semigroups is a relation $\mu : S \to T$ between two semigroups S and T with domain S such that, as a subset of $S \times T$, μ is a subsemigroup. In this case, $\mu' = \mu \cup \{(1,1)\}$ is a relational morphism $S^1 \to T^1$. If φ and ψ are labelings of a graph G respectively by the semigroups S and T, we say that they are *μ-related* by a relational morphism $\mu : S \to T$ if $(\varphi g, \psi g) \in \mu'$ for every $g \in G$.

We say that a labeling φ of a graph G by a semigroup S is *μ-inevitable* for a relational morphism $\mu : S \to T$ if there is a μ-related consistent labeling of G by T. A labeling of a finite graph by a finite semigroup S is said to be *$\mathbf{V}$-inevitable* if it is μ-inevitable for every relational morphism $\mu : S \to T$ into a member T of $\mathbf{V}$. We say that a relational morphism $\mu : S \to T$ *characterizes $\mathbf{V}$-inevitability* if, for every labeling of a finite graph by S, the labeling is $\mathbf{V}$-inevitable if and only if it is μ-inevitable.

Given a finite A-generated semigroup S, the author [3, Proposition 3] has shown that, for every onto continuous homomorphism $\varphi : \overline{\Omega}_A \mathbf{S} \to S$ and every labeling γ of a finite graph by S, there is a labeling δ of the same graph by $\overline{\Omega}_A \mathbf{S}$ such that $\varphi \circ \gamma = \delta$ and $p_{\mathbf{V}} \circ \delta$ is consistent where $p_{\mathbf{V}}$ is the natural projection $\overline{\Omega}_A \mathbf{S} \to \overline{\Omega}_A \mathbf{V}$ given by restriction of implicit operations. (The continuity assumption is incorrectly left out in the statement of [3, Proposition 3].) In the language of relational morphisms, this property may be reformulated by saying that the closed relational morphism $\mu_{S,\mathbf{V}} : S \to \overline{\Omega}_A \mathbf{V}$ generated by the pairs (a, a) with $a \in A$ characterizes $\mathbf{V}$-inevitability since, by the universal property of $\overline{\Omega}_A \mathbf{S}$, a pair $(s, \pi) \in S \times \overline{\Omega}_A \mathbf{V}$ lies in $\mu_{S,\mathbf{V}}$ if and only if there is some $\rho \in \overline{\Omega}_A \mathbf{S}$ such that $\varphi(\rho) = s$ and $p_{\mathbf{V}}(\rho) = \pi$. This is a sort of compactness theorem but it is not directly usable to derive computability results since $\overline{\Omega}_A \mathbf{V}$ is in general far too large as it is uncountable. The basic idea in [6] is to work with a computable part of $\overline{\Omega}_A \mathbf{V}$, namely a suitable relatively free σ-algebra $\Omega_A^\sigma \mathbf{V}$.

Following [6, Section 4] but again with a similar reformulation in the language of relational morphisms, we say that $\mathbf{V}$ is *σ-reducible* if for every finite semigroup S the relational morphism $\nu_{S,\mathbf{V}}^\sigma : S \to \Omega_A^\sigma \mathbf{V}$ generated by the pairs (a, a) with $a \in A$ characterizes $\mathbf{V}$-inevitability. On the other hand, as in [7, Section 8], we say that $\mathbf{V}$ is *weakly σ-reducible* if, for every finite semigroup S, the intersection of $\mu_{S,\mathbf{V}}$ with $S \times \Omega_A^\sigma \mathbf{V}$ characterizes $\mathbf{V}$-inevitability. Note that, by elementary point set topology, since $\Omega_A^\sigma\mathbf{V}$ is endowed with the induced topology of $\overline{\Omega}_A\mathbf{V}$, the closure $\overline{\nu_{S,\mathbf{V}}^\sigma}$ of $\nu_{S,\mathbf{V}}^\sigma$ in $S\times\overline{\Omega}_A\mathbf{V}$ coincides with $\mu_{S,\mathbf{V}}\cap(S\times\Omega_A^\sigma\mathbf{V})$.

A recursively enumerable pseudovariety $\mathbf{V}$ of semigroups is said to be *tame* if it is σ-recursive and σ-reducible for some highly computable implicit signature σ. If the pseudovariety $\mathbf{V}$ is tame with respect to the implicit signature σ then we also say that it is *σ-tame*.

The most important example of a tame (in fact κ-tame) pseudovariety to date is the pseudovariety $\mathbf{G}$ of all finite groups. This was proved by Ash [11] up to two translations. The need for one of these translations comes from the fact that Ash considered edge-labelings of finite graphs by finite monoids, consistency in groups being replaced by every undirected circuit having trivial label, where the label of an undirected path is obtained by successively multiplying the labels of its edges, inverted if the edge is traversed by the path in the opposite of its direction. That this leads to an equivalent property follows from [3, Section 4]. The other translation is required because Ash did not use the natural relational morphism $\nu_{S,\mathbf{V}}^\kappa$ but rather the submonoid of $S\times\Omega_A^\kappa\mathbf{G}$ generated by all pairs (a,a) together with all the pairs $(s,a^{\omega-1})$ such that $sas=s$. However, the equality between the two relational morphisms was proved in [6, Lemma 4.8]. Delgado [14] proved that Ash's relational morphism is closed in $S\times\Omega_A^\kappa\mathbf{G}$. Hence $\overline{\nu_{S,\mathbf{G}}^\kappa}=\nu_{S,\mathbf{G}}^\kappa$, a circumstance which is rather special.

Another equally important example of a κ-tame pseudovariety which has been announced by J. Rhodes [34] is the pseudovariety $\mathbf{A}$ of all finite aperiodic semigroups. See McCammond [30] and Zhiltsov [39] for two proofs of κ-recursiveness. Using Theorem 3.1, it follows that the Krohn-Rhodes complexity of finite semigroups (see the discussion following Corollary 6.3) is computable.

Steinberg proved in [37] that the pseudovariety $\mathbf{G}_p$ is weakly κ-reducible and it is obviously κ-recursive since $\Omega_A^\kappa\mathbf{G}_p$ is the free group on the set A, which follows from a result of Baumslag [12] proving that the free group is residually a finite p-group. However, by Baumslag's result, $\mathbf{G}_p$ may not be defined by identities in the signature κ and so, by [6, Proposition 4.2], $\mathbf{G}_p$ is not κ-reducible. So, the question of whether κ might be enlarged to some highly computable implicit signature σ so that $\mathbf{G}_p$ is σ-tame remained open. In Section 6 we show how to apply the techniques developed in Section 2 to obtain such a signature. In the next section, we introduce some preliminary results which we require for that purpose.

4. The subgroup theorem

We say that a pseudovariety of groups $\mathbf{V}$ is *closed under extension* if, for any group G and normal subgroup N such that N and G/N both lie in $\mathbf{V}$, G also belongs to $\mathbf{V}$. Equivalently, the pseudovariety $\mathbf{V}$ is an idempotent for semidirect product. Throughout this section we assume that $\mathbf{V}$ is a pseudovariety of groups closed under extension. See [28, Sections 1 and 2] for a more systematic study of pro-$\mathbf{V}$ topologies with respect to pseudovarieties of groups.

The subgroup theorem states that an open subgroup of a free pro-$\mathbf{V}$ group is itself a free pro-$\mathbf{V}$ group (cf. [19, Proposition 15.27]). Since a refined version of this result will be used in Section 6, it seems appropriate to present here a rather short and elementary proof. The proof depends on the following lemma which is easily extracted from the proof of [28, Proposition 1.9], which in turn reformulates

[20, Lemma 3.1] where an elementary four-line proof attributed to J. Poland is presented. The proof found in [28] has the advantage of leading to better estimates.

Lemma 4.1. *Let G be a group which is residually in $\mathbf{V}$ endowed with the pro-$\mathbf{V}$ topology. Let H be a clopen subgroup of G. Then every normal subgroup U of H such that $H/U \in \mathbf{V}$ contains a normal subgroup U_G of G such that $G/U_G \in \mathbf{V}$ and $(G : U_G) \leq (G : U)!$.*

Proof. In [28], the group U_G is taken to be the *core* of U, that is the intersection of all conjugates of U in G and, using the hypothesis that $\mathbf{V}$ is extension closed, it is shown that $G/U_G \in \mathbf{V}$. As it is well known, say by letting G act on right cosets of U by right translation, the kernel of this action is contained in U_G and the desired inequality follows. □

It is deduced in [28, Proposition 1.9] that the pro-$\mathbf{V}$ topology of H is the induced topology from the pro-$\mathbf{V}$ topology of G. The estimate added above allows us to prove the following finer result.

Lemma 4.2. *Let G be a group which is residually in $\mathbf{V}$ endowed with the pro-$\mathbf{V}$ topology. Let H be a clopen subgroup of G. Then the pro-$\mathbf{V}$ metric d_H and the restriction to H of the pro-$\mathbf{V}$ metric d_G have the same Cauchy sequences.*

Proof. Denote by d and r respectively the restriction of d_G and r_G to H. We claim that the following inequalities of functions hold in the sense that they hold whenever the functions are evaluated at the same element of $H \times H$:

$$r_H \leq r \leq ((G : H) \cdot r_H)! \tag{4.1}$$

To prove the claim, we have already observed in the proof of Lemma 2.1 that the first inequality holds in a much more general setting. For the second inequality, suppose that $u, v \in H$ are distinct and let $\varphi : H \to K$ be a homomorphism into a member of $\mathbf{V}$ such that $\varphi u \neq \varphi v$. Then the kernel U of φ is a normal subgroup of H and φ induces a one-to-one homomorphism $\varphi' : H/U \to K$. In particular, $H/U \in \mathbf{V}$ and so, by Lemma 4.1, U contains a normal subgroup U_G of G such that $G/U_G \in \mathbf{V}$ and $(G : U_G) \leq (G : U)!$. By taking $|K|$ minimum, so that $(H : U) = r_H(u, v)$, we deduce that

$$r(u, v) \leq (G : U_G) \leq (G : U)! = ((G : H) \cdot (H : U))! = ((G : H) \cdot r_H(u, v))!$$

which proves the claim.

To finish the proof, in view of the first inequality in (4.1), every sequence of H which is Cauchy with respect to the metric d_H is also Cauchy for d. For the converse, let $f(n) = ((G : H) \cdot n)!$. Since f is an increasing function, a little calculation shows that, for any $\varepsilon > 0$,

$$d \leq 2^{-f(\lceil -\log_2 \varepsilon \rceil)} \Rightarrow d_H \leq \varepsilon$$

which implies that Cauchy sequences for d are also Cauchy sequences for d_H. □

Theorem 4.3. *Let H be a subgroup of a free group G in the variety generated by $\mathbf{V}$ and suppose H is a free factor of a clopen subgroup in the pro-$\mathbf{V}$ topology of G. Then the completion of H with respect to the restriction of the pro-$\mathbf{V}$ metric d_G of G is a free pro-$\mathbf{V}$ group.*

Proof. If $\mathbf{V}$ is the trivial pseudovariety, consisting only of singleton groups, then the result is trivial. Otherwise, $\mathbf{V}$ contains some $\mathbb{Z}/p\mathbb{Z}$ and therefore also $\mathbf{G}_p$ for some prime p. Since the free group is residually a finite p-group, we may assume that the group G is free. Hence H is also a free group by the Nielsen-Schreier theorem. The completion of H with respect to the metric d_H is therefore a free pro-$\mathbf{V}$ group. Since, by Lemmas 2.1 and 4.2, d_H and the restriction of d_G to H have the same Cauchy sequences, the completion of H with respect to this restriction is the same topological group and so it is a free pro-$\mathbf{V}$ group. □

This brings us to the subgroup theorem which is now a simple corollary of a special case of Theorem 4.3.

Corollary 4.4. *Let H be a clopen subgroup of a free pro-$\mathbf{V}$ group G. Then H is itself a free pro-$\mathbf{V}$ group.*

Proof. Let X be a free generating set for G. Then the subgroup G' generated by X is dense in G and it is free in the variety generated by $\mathbf{V}$. Let $H' = H \cap G'$. Since H is clopen in G, H' is clopen in G'. Moreover, since H is open, H' is dense in H. Hence H is the completion of H' with the respect to the restriction of the metric d_G. Since H' is trivially a free factor of itself, by Theorem 4.3 it follows that H is a free pro-$\mathbf{V}$ group. □

5. Computing closures of subgroups

We start this section by recalling some constructions from geometric group theory which we require. See [28, Section 2.1] for details.

For a subset Y of a group G, let $\langle Y \rangle$ denote the subgroup generated by Y. As say in [36, 28], if G is a free group on a finite set A, then we associate to Y a finite inverse automaton as follows:

- take a linear graph labeled by each word in Y, with individual (directed) edges labeled by elements of A so that the label (in the sense of Section 3) of one of the (undirected) simple paths traversing the whole graph is the original word;
- glue these graphs together by identifying all their ends to a single vertex v_0 which is declared to be the unique initial and final state;
- fold edges so that the resulting automaton becomes inverse, that is the transformations defined by the labels are partial bijections: whenever there are two edges with the same label leaving from the same state or arriving at the same state, identify the two edges and repeat this procedure until no further identifications of this kind are possible.

This automaton is *reduced* in the sense that it has a unique initial and final state, from which every state is accessible, and there is no state of degree 1 except possibly the initial and final state. The automaton recognizes precisely the reduced group words in the alphabet A which lie in $\langle Y \rangle$. Conversely, every finite reduced inverse automaton $\mathcal{A}$ over the alphabet A recognizes a finitely generated subgroup of the free group on A whose associated automaton is $\mathcal{A}$, namely the fundamental group of the underlying graph. It can be shown that the automaton $\mathcal{A}$ depends only on the subgroup $\langle Y \rangle$ and not on the specific generating set Y [29]. For a finitely generated subgroup H of the free group G on a set A, we represent this automaton by $\mathcal{A}(H)$.

A *congruence* on an inverse automaton $\mathcal{A}$ is an equivalence relation $\sim$ on its set of states which is compatible with the action of the corresponding input alphabet A, both in the forward and backward directions, so that an induced action as partial bijections for the elements of A is obtained on the set of $\sim$-classes of states. The resulting *quotient automaton* is denoted $\mathcal{A}/{\sim}$.

As in [28, Section 2.1], let H and K be finitely generated subgroups of a free group G such that $H \subseteq K$, and consider an associated congruence $\sim_{H,K}$ on $\mathcal{A}(H)$ defined as follows. In the automaton $\mathcal{A}(H)$, fix for each state p a reduced word u_p labeling a path from the initial state to p. The congruence $\sim_{H,K}$ identifies two states p and q if $u_p u_q^{-1} \in K$. Let L be the finitely generated subgroup of G such that $\mathcal{A}(L) = \mathcal{A}(H)/{\sim_{H,K}}$. The definition of the congruence $\sim_{H,K}$ is made precisely so that the quotient $\mathcal{A}(L)$ of $\mathcal{A}(H)$ embeds in $\mathcal{A}(K)$, from which it follows that L is a finitely generated free factor of K by [28, Proposition 2.6].

Let $\mathbf{V}$ be a nontrivial pseudovariety of groups closed under extension and consider the free group $F = \Omega_n^\kappa \mathbf{V}$ endowed with the pro-$\mathbf{V}$ topology. The following result is implicit in [35, 28] and is presented here in order to mention a basic idea underlying most of what follows.

Proposition 5.1. *Suppose a finitely generated subgroup H of F is not dense and say $\varphi : F \to G$ is a homomorphism onto a member of $\mathbf{V}$ such that $\varphi H \subsetneqq G$. Then one can compute from φ a closed finitely generated free factor L of some clopen subgroup K of F such that $\mathcal{A}(L)$ is a quotient of the automaton $\mathcal{A}(H)$.*

Proof. Let $K = \varphi^{-1}\varphi H$. Then K is a proper clopen subgroup of F, hence finitely generated, containing H. Consider the congruence $\sim_{H,K}$ on $\mathcal{A}(H)$ as defined above and again let L be the finitely generated subgroup of F such that $\mathcal{A}(L) = \mathcal{A}(H)/{\sim_{H,K}}$. Then, as observed above, L is a free factor of K. Moreover, by [35, Corollary 3.8] L is a closed subgroup of F. This completes the proof since all the constructions are effective. □

Let p be a prime integer. We consider on the free group $F = \Omega_n^\kappa \mathbf{G}_p$ the pro-$\mathbf{G}_p$ topology, whose open subgroups have index which is a power of p.

Ribes and Zalesskiĭ [35] have shown how to compute the closure in F of a finitely generated subgroup H. Margolis, Sapir and Weil [28] further refined that result by giving an efficient formulation of the algorithm. As motivation for the main ideas in the proof of the key result in the next section and in preparation for that proof, we now present the refined formulation of the algorithm of which Proposition 5.1 is one of the essential ingredients.

For each reduced group word $w \in F$ and each $x \in X$, let $|w|_x$ denote the sum of the exponents in the occurrences of the letter x in the word w. For each m-tuple $(w_1, \ldots, w_m)$ of elements of F, let $A_p(w_1, \ldots, w_m)$ be the $m \times n$ matrix with entries in $\mathbb{Z}/p\mathbb{Z}$ whose (i,j)-entry is the mod p-class of $|w_i|_{x_j}$.

Proposition 5.2 ([28]). *Let H be a finitely generated subgroup of the free group F. If H is dense, then H contains a subgroup of rank n which is dense in F. Moreover, if $\{w_1, \ldots, w_n\}$ is a generating set of the subgroup H, then H is dense in F if and only if the matrix $A_p(w_1, \ldots, w_n)$ is invertible over the field $\mathbb{Z}/p\mathbb{Z}$.*

Proposition 5.3 ([28]). *Suppose $\{w_1, \ldots, w_m\}$ is a generating set of the subgroup H of the free group F such that the matrix $A_p(w_1, \ldots, w_m)$ has rank less than n over the field $\mathbb{Z}/p\mathbb{Z}$. Then the restriction homomorphism $\varphi : F = \Omega_n^\kappa \mathbf{G}_p \to \Omega_n^\kappa \mathbf{V}(\mathbb{Z}/p\mathbb{Z})$, where $\mathbf{V}(\mathbb{Z}/p\mathbb{Z})$ is the pseudovariety of all finite Abelian groups of exponent p, is such that $\varphi H \subsetneqq \Omega_n^\kappa \mathbf{V}(\mathbb{Z}/p\mathbb{Z})$.*

One may then proceed by finding a finite set $\{v_1, \ldots, v_r\}$ of generators for a free factor K of a clopen subgroup of F according to Proposition 5.1. Then express the w_i in terms of the v_j and repeat the procedure until a closed subgroup K is found in which H is dense, that is K is the closure of H. Since each time this construction is applied the number of states goes down strictly or at the next step we find a dense subgroup, the process will stop in a finite number of steps. Moreover, by Lemmas 2.1 and 4.2, the induced topology on each successive closed subgroup approximating the closure of H coincides with its own pro-$\mathbf{V}$ topology, so that the algorithm does indeed produce the closure of H in $\Omega_n^\kappa\mathbf{V}$. A more detailed analysis of this procedure further allowed Margolis, Sapir and Weil to deduce that a set of generators for the closure of H is actually obtained in polynomial time.

Let us put these results into a more general framework. Say that *denseness is decidable* for the pseudovariety of groups $\mathbf{V}$ if $\mathbf{V}$ is recursively enumerable and there is an algorithm to test whether, given a finite subset Y of the free group $\Omega_n^\kappa\mathbf{V}$, the subgroup $\langle Y\rangle$ is dense. By the above, $\mathbf{G}_p$ is an example of such a pseudovariety. For the purpose of finding an efficient algorithm, Proposition 5.3 plays a role in the case of p-groups. But, just for decidability purposes, it is superfluous as the next trivial observation shows.

Proposition 5.4. *If $\mathbf{V}$ is a recursively enumerable pseudovariety of groups with decidable denseness, then there is an algorithm to find for each non-dense finitely generated subgroup H of $\Omega_n^\kappa\mathbf{V}$ a homomorphism $\varphi : \Omega_n^\kappa\mathbf{V} \to G$ onto some $G \in \mathbf{V}$ such that $\varphi H \subsetneqq G$.*

Proof. That the subgroup is not dense is equivalent to the existence of such a homomorphism. Knowing that one exists, since $\mathbf{V}$ is recursively enumerable, we can just by brute force try successively each onto homomorphism $\varphi : \Omega_n^\kappa\mathbf{V} \to G$ with $G \in \mathbf{V}$ until one is found with the required property. □

The above discussion proves most of the following result. Missing details may be found in [35, 28].

Theorem 5.5. *Let $\mathbf{V}$ be an extension closed pseudovariety of groups. Let H be a finitely generated subgroup of the free group $\Omega_n^\kappa\mathbf{V}$ and let $\overline{H}$ be its closure.*

a) *The group $\overline{H}$ is a finitely generated subgroup of $\Omega_n^\kappa\mathbf{V}$ which is a free factor of a clopen subgroup.*
b) *The rank of $\overline{H}$ does not exceed that of H.*
c) *The automaton $\mathcal{A}(\overline{H})$ is a quotient of $\mathcal{A}(H)$.*
d) *If $\mathbf{V}$ is recursively enumerable and has decidable denseness then there is an algorithm to construct a finite set of generators of $\overline{H}$.* □

Moreover, in view of Theorem 4.3, we obtain the following result.

Theorem 5.6. *Let $\mathbf{V}$ be a nontrivial extension closed pseudovariety of groups and let H be a closed subgroup of the free group $F = \Omega_n^\kappa\mathbf{V}$. Then the completion of H with respect to the restriction of the metric d_F to H is a free pro-$\mathbf{V}$ group.* □

A pseudovariety $\mathbf{V}$ of groups is called *arborescent* in [8] if $(\mathbf{V}\cap\mathbf{Ab}) * \mathbf{V} = \mathbf{V}$, where $\mathbf{Ab}$ denotes the pseudovariety of all finite Abelian groups. The name is justified by the result that arborescent pseudovarieties $\mathbf{V}$ are those for which the profinite Cayley graph of each $\Omega_n^\kappa\mathbf{V}$ is a profinite tree [21, 8]. In particular, an extension closed pseudovariety of groups is arborescent.

On the other hand, a pseudovariety of groups is called an $\overline{\mathcal{RZ}}$*-pseudovariety* in [37] if the closure of a finitely generated subgroup of $\Omega_n^\kappa\mathbf{V}$ is itself finitely generated and the product of closed finitely generated subgroups is closed. By [35, Proposition 3.4 and Theorem 5.1], extension closed pseudovarieties of groups are $\overline{\mathcal{RZ}}$-pseudovarieties.

The following extension of Ash's result is due to Steinberg [37, Theorem 11.12].

Theorem 5.7. *Every arborescent $\overline{\mathcal{RZ}}$-pseudovariety is weakly κ-reducible.*

In particular, an extension closed pseudovariety of groups is weakly κ-reducible.

6. Tameness of extension closed pseudovarieties of groups

Throughout this section we consider a nontrivial pseudovariety $\mathbf{V}$ of groups which is closed under extension. We always consider on the free group $\Omega_n^\kappa\mathbf{V}$ the pro-$\mathbf{V}$ topology.

Consider the implicit signature σ (depending on $\mathbf{V}$) obtained by adding to the canonical signature κ all implicit operations of the form

$$\circ_j^{\omega-1}(w_1,\dots,w_n)(v_1,\dots,v_n) \tag{6.1}$$

with $j = 1,\dots,n$, the w_i κ-terms such that the subgroup $\langle w_1,\dots,w_n\rangle$ is dense in $\Omega_n^\kappa\mathbf{V}$, and the v_i κ-terms such that each v_i induces the same element of the free group $\Omega_n^\kappa\mathbf{V}$ as w_i.

The aim of this section is to establish the following result.

Theorem 6.1. *Let $\mathbf{V}$ be a nontrivial extension closed pseudovariety of groups.*

- a) *If $\mathbf{V}$ has decidable denseness, then the implicit signature σ is highly computable.*
- b) *The group $\Omega_n^\sigma\mathbf{V}$ is the free group of rank n. More precisely, $\Omega_n^\sigma\mathbf{V} = \Omega_n^\kappa\mathbf{V}$ with each operation in $\sigma\setminus\kappa$ restricting to a component projection on $\mathbf{V}$.*
- c) *The pseudovariety $\mathbf{V}$ is σ-reducible.*

Note that (a) is the only part of Theorem 6.1 with a computability hypothesis (and thesis). Yet, the other two parts seem to be worthless on their own.

Before proceeding with the proof of Theorem 6.1, it is worth drawing as a consequence the main theorem of this paper.

Theorem 6.2. *Every recursively enumerable extension closed pseudovariety of groups which has decidable denseness is tame.*

Proof. Let $\mathbf{V}$ be a pseudovariety satisfying the hypothesis of the theorem and consider the associated implicit signature σ. Since σ is highly computable by Theorem 6.1(a), it suffices to show that $\mathbf{V}$ is σ-tame. That $\mathbf{V}$ is σ-recursive follows from Theorem 6.1(b) as the solution of the word problem in the free group also solves the word problem in $\Omega_n^\sigma\mathbf{V}$. Finally, σ-reducibility of $\mathbf{V}$ is given by Theorem 6.1(c). $\square$

As an immediate corollary, in view of Proposition 5.2, we obtain the following result.

Corollary 6.3. *The pseudovariety $\mathbf{G}_p$ is tame.* $\square$

It follows from the Krohn-Rhodes decomposition theorem [26] that every finite semigroup S divides a wreath product in which the factors are alternately finite aperiodic semigroups and finite groups; moreover, the group factors may be taken from the extension closed pseudovariety of groups generated by the subgroups of S. The *group complexity* of a finite semigroup is the least number of group factors without any further restriction on them. When raising the question of computability of group complexity, Krohn and Rhodes already proposed refined versions of the complexity function by restricting its domain to finite semigroups with subgroups in a given extension closed pseudovariety $\mathbf{H}$ of groups and considering only group factors from $\mathbf{H}$ (cf. [26, 27]). The cases of the pseudovarieties of p-groups and solvable groups naturally deserve special attention.

In view of Theorem 3.1, the computability of the complexity function $c_{\mathbf{H}}$ with respect to a certain extension closed pseudovariety $\mathbf{H}$ of groups follows from tameness of the pseudovarieties $\mathbf{H}$ and $\mathbf{A}$. Thus, Corollary 6.3 immediately yields the following corollary.

Corollary 6.4. *If $\mathbf{A}$ is tame, then the p-group complexity of finite semigroups in which all subgroups are p-groups is computable.* □

The remainder of the section is dedicated to the proof of Theorem 6.1.

Proof of Theorem 6.1(a). By successively enumerating all pairs $w_1, \dots, w_n$ and $v_1, \dots, v_n$ of n-tuples of κ-terms in the alphabet $\{x_1, \dots, x_n\}$, over all positive integers n, and for each of them testing whether the subgroup $\langle w_1, \dots, w_n \rangle$ is dense in $\Omega_n^\kappa \mathbf{V}$ and whether $w_i = v_i$ in $\Omega_n^\kappa \mathbf{V}$, discarding those pairs of n-tuples for which one of these tests fails, we effectively enumerate the elements of the implicit signature σ, except possibly for those in κ. Hence σ is recursively enumerable. Since $x^{\omega-1} = \lim_{n\to\infty} x^{n!-1}$, by the analogue of Corollary 2.5 for $(\omega-1)$-powers every element of σ is a computable implicit operation. Hence σ is highly computable. □

The following simple observation contains the central idea in the whole paper.

Lemma 6.5. *Let $w_1, \dots, w_n$ be κ-terms over the alphabet $\{x_1, \dots, x_n\}$ such that the subgroup $\langle w_1, \dots, w_n \rangle$ is dense in $\Omega_n^\kappa \mathbf{V}$. Then*

$$(w_1, \dots, w_n)^\omega = \mathrm{id} = (x_1, \dots, x_n) \quad \text{in } \mathcal{O}((\overline{\Omega}_n \mathbf{V})^n). \tag{6.2}$$

Proof. By definition of the pro-$\mathbf{V}$ topology, given any $G \in \mathbf{V}$ and any homomorphism $\varphi : \Omega_n^\kappa \mathbf{V} \to G$, the restriction of φ to the subgroup H generated by $\{w_1, \dots, w_n\}$ is onto. In other words, the images φx_i of the free generators may be expressed by means of a (group) word in terms of the images φw_j of the generators. This means that the operator $(w_1, \dots, w_n) \in \mathcal{O}(G^n)$ is invertible. In particular, since the monoid $\mathcal{O}(G^n)$ is finite, it follows that

$$(w_1, \dots, w_n)^\omega = \mathrm{id} = (x_1, \dots, x_n) \quad \text{in } \mathcal{O}(G^n). \tag{6.3}$$

Let $(\pi_1, \dots, \pi_n) = (w_1, \dots, w_n)^\omega$. By (6.3), every $G \in \mathbf{V}$ satisfies the pseudoidentities $\pi_i = x_i$ $(i = 1, \dots, n)$, that is $\mathbf{V}$ satisfies them, which establishes (6.2). □

Proof of Theorem 6.1(b). Consider an implicit operation of the form (6.1) with the w_i κ-terms such that the subgroup $\langle w_1, \dots, w_n \rangle$ is dense in $\Omega_n^\kappa \mathbf{V}$ and the v_i κ-terms such that $v_i = w_i$ in $\Omega_n^\kappa \mathbf{V}$. Then the operation coincides in $\mathcal{O}((\overline{\Omega}_n \mathbf{V})^n)$ with $\circ_j^\omega(w_1, \dots, w_n)$ and this operation is the projection x_j by Lemma 6.5. □

Proof of Theorem 6.1(c). The departure point for the proof that the pseudovariety $\mathbf{V}$ is σ-reducible is that $\mathbf{V}$ is weakly κ-reducible (Theorem 5.7). Indeed, from that result it follows that it suffices to show that, given any finite n-generated semigroup S, any element $s \in S$, and any group word $w \in \Omega_n^\kappa \mathbf{V}$ belonging to $\overline{\nu_{S,\mathbf{V}}^\kappa} s$, there is some $\pi \in \Omega_n^\sigma \mathbf{S}$ such that $\pi = w$ in $\mathbf{V}$ and $\varphi\pi = s$, where $\varphi : \overline{\Omega}_A \mathbf{S} \to S$ is the canonical homomorphism determined by the choice of generators of S. In other words, in view of Theorem 6.1(b), it suffices to show that

$$\overline{\nu_{S,\mathbf{V}}^\kappa} \subseteq \nu_{S,\mathbf{V}}^\sigma. \tag{6.4}$$

The first step is to reformulate the problem in terms of rational languages. Let

$$L_s = (\varphi^{-1}s) \cap \{x_1, \dots, x_n\}^+.$$

Then L_s is a rational language whose closure $\overline{L_s}$ in the free group $\Omega_n^\kappa \mathbf{V}$ is the set $\overline{\nu_{S,\mathbf{V}}^\kappa s}$. Denote by ψ the canonical projection $\overline{\Omega}_n\mathbf{S} \to \overline{\Omega}_n\mathbf{V}$ given by restriction of implicit operations to $\mathbf{V}$. Since ψ is continuous and $\overline{\Omega}_n\mathbf{S}$ is compact, the mapping ψ is closed. Denoting the closure of L_s in $\overline{\Omega}_n\mathbf{S}$ by $\mathrm{Cl}(L_s)$, we deduce that

$$\overline{\nu_{S,\mathbf{V}}^\kappa s} = \overline{L_s} = \psi(\mathrm{Cl}(L_s)) \cap \Omega_n^\kappa \mathbf{V}.$$

Hence, for a group word $w \in \Omega_n^\kappa \mathbf{V}$, we have

$$w \in \overline{\nu_{S,\mathbf{V}}^\kappa s} \text{ if and only if } \mathrm{Cl}(L_s) \cap \psi^{-1}w \neq \emptyset.$$

Thus, to show that a given $w \in \overline{\nu_{S,\mathbf{V}}^\kappa s}$ belongs to $\nu_{S,\mathbf{V}}^\sigma s$, it suffices to establish that

$$\mathrm{Cl}(L_s) \cap (\psi^{-1}w) \cap \Omega_n^\sigma \mathbf{S} \neq \emptyset.$$

Dropping reference to the finite semigroup S and the element s, it suffices to show that, for any rational language L over the alphabet $\{x_1, \dots, x_n\}$ and any group word w,

$$w \in \overline{L} \Rightarrow \mathrm{Cl}(L) \cap (\psi^{-1}w) \cap \Omega_n^\sigma \mathbf{S} \neq \emptyset. \tag{6.5}$$

The next step consists in a reduction to the case in which the rational language L is a subsemigroup of $\{x_1, \dots, x_n\}^+$. The essential ingredient is the Pin and Reutenauer [31] rules to compute the closure of a rational language in $\Omega_n^\kappa \mathbf{V}$: for rational languages L and K,

(1) $\overline{L} = L$ if L is finite;
(2) $\overline{L \cup K} = \overline{L} \cup \overline{K}$;
(3) $\overline{LK} = \overline{L} \cdot \overline{K}$;
(4) $\overline{L^+} = \overline{\langle L \rangle}$.

(In fact it suffices for this purpose to assume that $\mathbf{V}$ is an $\overline{\mathcal{RZ}}$-pseudovariety, cf. [35, Theorem 5.5] and [37, Theorem 9.4]). If a language L is finite, then any w in $\overline{L}$ is also an element of $\mathrm{Cl}(L) \cap \psi^{-1}w \cap \Omega_n^\sigma \mathbf{S}$ and so finite languages satisfy (6.5). We now show that union and concatenation preserve property (6.5). So, suppose L and K are rational languages verifying (6.5). If $w \in \overline{L \cup K} = \overline{L} \cup \overline{K}$, then at least one of the sets $\mathrm{Cl}(L) \cap (\psi^{-1}w) \cap \Omega_n^\sigma \mathbf{S}$ and $\mathrm{Cl}(K) \cap (\psi^{-1}w) \cap \Omega_n^\sigma \mathbf{S}$ is nonempty and, therefore, so is their union $\mathrm{Cl}(L \cup K) \cap (\psi^{-1}w) \cap \Omega_n^\sigma \mathbf{S}$, which shows that $L \cup K$ still satisfies (6.5). On the other hand, if $w \in \overline{LK} = \overline{L} \cdot \overline{K}$, say $w = uv$ with $u \in \overline{L}$ and $v \in \overline{K}$, then the sets $P = \mathrm{Cl}(L) \cap (\psi^{-1}u) \cap \Omega_n^\sigma \mathbf{S}$ and $Q = \mathrm{Cl}(K) \cap (\psi^{-1}v) \cap \Omega_n^\sigma \mathbf{S}$ are both nonempty. Since

$$(\psi^{-1}u)(\psi^{-1}v) \subseteq \psi^{-1}(uv) = \psi^{-1}w \tag{6.6}$$

and $\mathrm{Cl}(LK) = \mathrm{Cl}(L) \cdot \mathrm{Cl}(K)$, as multiplication on a compact semigroup is a closed mapping, it follows that the set $\mathrm{Cl}(LK) \cap (\psi^{-1}w) \cap \Omega_n^\sigma \mathbf{S}$ contains PQ and, therefore, it is also nonempty. Hence LK again satisfies (6.5). Thus it remains to show that the plus operation $L \mapsto L^+$ also preserves property (6.5).

A further useful step is provided by the following result which is an adaptation of Anissimov and Seifert's theorem stating that the rational subgroups of a group are the finitely generated subgroups [10]. For a proof one can follow closely say the presentation given by Berstel [13] and so we will only describe the little required changes.

Lemma 6.6. *Let $L \subseteq \{x_1, \dots, x_n\}^+$ be a rational language. Then there is a (computable) finite subset Y of $\Omega_n^\kappa \mathbf{S}$ such that:*

i) $\mathrm{Cl}(Y^+) \subseteq \mathrm{Cl}(L^+)$;
ii) $\langle Y \rangle = \langle L \rangle$.

Proof. Consider first a subset of $\Omega_n^\kappa \mathbf{S}$ of the form

$$K = w_0 T_1^* w_1 \cdots T_r^* w_r \tag{6.7}$$

where the w_i are elements and the T_i are subsets of $\Omega_n^\kappa \mathbf{S}$. Set

$$\begin{array}{rcll} u_i & = & w_0 w_1 \cdots w_i & (i = 0, \dots, r) \\ v_i & = & w_{i+1} \cdots w_r & (i = 0, \dots, r-1) \\ v_r & = & 1 & \\ S_i & = & u_i T_i v_i u_r^{\omega-1} & (i = 1, \dots, r) \\ P & = & \{u_r\} \cup S_1 \cup \ldots \cup S_r. & \end{array}$$

Since $u_r^{\omega-1} = \lim_{k\to\infty} u_r^{k!-1}$, we have $u_r^{\omega-1} \in \mathrm{Cl}(K^+)$ and so $\mathrm{Cl}(P^+) \subseteq \mathrm{Cl}(K^+)$. Since, in the free group $\Omega_n^\kappa \mathbf{V}$, $S_i = u_i T_i u_i^{-1}$, a little calculation shows that $\langle P \rangle = \langle K \rangle$.

The proof may now proceed by induction on the star-height of the rational language L, the above step allowing us to reduce star-height by 1 whenever it is positive. Since star-height 0 languages are finite languages, this concludes the proof of the lemma. □

Applying Lemma 6.6 and the observation that $\overline{L^+} = \overline{\langle L \rangle}$, we obtain that, to show that a rational language of the form L^+ satisfies (6.5), it suffices to establish that, for any finite subset Y of $\Omega_n^\kappa \mathbf{S}$,

$$w \in \overline{\langle Y \rangle} \Rightarrow \mathrm{Cl}(Y^+) \cap (\psi^{-1} w) \cap \Omega_n^\sigma \mathbf{S} \neq \emptyset. \tag{6.8}$$

This brings us to the core of the proof of Theorem 6.1 involving the calculation of the pro-$\mathbf{V}$ closure of finitely generated subgroups of the free group $\Omega_n^\kappa \mathbf{V}$. A further remark which is useful at this point is that not all group words $w \in \Omega_n^\kappa \mathbf{V}$ need to be considered in (6.8). Indeed, since both $\mathrm{Cl}(Y^+)$ and $\Omega_n^\sigma \mathbf{S}$ are κ-subsemigroups of $\overline{\Omega}_n \mathbf{S}$, using (6.6) we conclude that it suffices to show

$$\mathrm{Cl}(Y^+) \cap (\psi^{-1} w) \cap \Omega_n^\sigma \mathbf{S} \neq \emptyset \tag{6.9}$$

for any w in a generating set for the closure in $\Omega_n^\kappa \mathbf{V}$ of the subgroup $\langle Y \rangle$.

In the special case in which the finitely generated subgroup $\langle Y \rangle$ is dense in $\Omega_n^\kappa \mathbf{V}$, we may readily conclude the proof. Indeed then there are elements $w_1, \dots, w_n$ of $\langle Y \rangle$ such that the subgroup $\langle w_1, \dots, w_n \rangle$ is dense in $\Omega_n^\kappa \mathbf{V}$. Since $\mathrm{Cl}(Y^+)$ is a κ-subsemigroup of $\overline{\Omega}_n \mathbf{S}$, we may take $w_1, \dots, w_n \in \mathrm{Cl}(Y^+)$ with the same property. By Lemma 6.5, setting

$$(\pi_1, \dots, \pi_n) = (w_1, \dots, w_n)^\omega,$$

the restriction of each π_i to $\mathbf{V}$ coincides with the component projection x_i. Since each component of the finite power $(w_1, \dots, w_n)^k$ belongs to $\mathrm{Cl}(Y^+)$, each π_i is an element of $\mathrm{Cl}(Y^+)$. Hence $\pi_i \in \mathrm{Cl}(Y^+) \cap (\psi^{-1} x_i) \cap \Omega_n^\sigma \mathbf{S}$, which proves (6.9) for $w = x_i$. Moreover, note that each generator x_i is obtained as the projection of a σ-term in $\mathrm{Cl}(Y^+)$ which uses only one operation symbol from $\sigma \setminus \kappa$, namely

$$x_i = \circ_i^{\omega-1}(w_1, \dots, w_n)(w_1, \dots, w_n).$$

Hence, every element of $\Omega_n^\kappa\mathbf{V}$ can be obtained as a σ-term in $\mathrm{Cl}(Y^+)$ which only uses symbols of $\sigma\setminus\kappa$ and without nesting them.

It remains to treat the case in which the finitely generated subgroup $H=\langle Y\rangle$ is not dense in $\Omega_n^\kappa\mathbf{V}$. Set $Y=\{w_1,\dots,w_m\}$. By Theorem 5.5, there is a finite set $Z=\{z_1,\dots,z_r\}$ of elements of $\Omega_n^\kappa\mathbf{S}$ such that Z generates the closure $K=\overline{H}$, which is a free factor of a clopen subgroup of $\Omega_n^\kappa\mathbf{V}$, and $\mathcal{A}(K)$ is a quotient of the automaton $\mathcal{A}(H)$. Since the subgroup K contains H, one may rewrite the elements of Y, viewed as elements of $\Omega_n^\kappa\mathbf{V}$, as reduced group words $w'_1,\dots,w'_m$ (and therefore as κ-terms) in the elements of Z.

Remark. A word of warning concerning notation. As we have done with words, we are viewing κ-terms in various contexts: either as abstract κ-terms or as representing implicit operations either on all finite semigroups or just on $\mathbf{V}$. This is clearly an abuse of notation but the context should make it clear at which level the κ-term is being taken. Each element of the free group $\Omega_n^\kappa\mathbf{V}$ has many representations as a κ-term and even as the restriction to $\mathbf{V}$ of elements of $\Omega_n^\kappa\mathbf{S}$. We do not claim that the z_i have liftings $\hat{z}_i$ to $\Omega_n^\kappa\mathbf{S}$ such that the elements of Y, viewed as elements of $\Omega_n^\kappa\mathbf{S}$, have expressions as κ-terms in the $\hat{z}_i$. It is easy to give examples in which this is not possible. On the other hand, whenever we talk about a reduced group word w, we associate with it the unique κ-term obtained from w by replacing every exponent -1 with $\omega-1$.

Set $Y'=\{w'_1,\dots,w'_m\}$, viewed as a subset of $\Omega_r^\kappa\mathbf{S}$. We claim that, if

$$w\in\overline{\langle Y'\rangle}\Rightarrow \mathrm{Cl}((Y')^+)\cap(\psi^{-1}w)\cap\Omega_r^\sigma\mathbf{S}\neq\emptyset. \tag{6.10}$$

then (6.8) also holds. Here, another abuse of notation is taking place as we are still representing by ψ the restriction mapping $\overline{\Omega}_r\mathbf{S}\to\overline{\Omega}_r\mathbf{V}$ and by $\mathrm{Cl}(\_)$ the pro-$\mathbf{V}$ closure in $\Omega_r^\kappa\mathbf{V}$. Suppose then that $v\in\overline{H}$. There exists a sequence of group words $(u_k)_k$ such that

$$v=\lim_{k\to\infty}u_k(w_1,\dots,w_m).$$

Consider the sequence $(u_k(w'_1,\dots,w'_m))_k$. Since this sequence has some convergent subsequence in the compact group $\overline{\Omega}_r\mathbf{V}$, we may without loss of generality assume that the sequence itself is convergent. Denote the limit by v'. Set

$$\begin{array}{rcl}\theta:\overline{\Omega}_r\mathbf{V} & \to & \overline{\Omega}_n\mathbf{V}\\ x_i & \mapsto & z_i \quad (i=1,\dots,r)\end{array}$$

to be a continuous homomorphism. Then, applying θ to v' we obtain v. Now, since Z generates a closed subgroup of $\Omega_n^\kappa\mathbf{V}$ and $\mathbf{V}$ is extension closed, by Theorem 5.6 the mapping θ is an isomorphism of topological groups onto the closure of K in $\overline{\Omega}_n\mathbf{V}$. In particular, θ is one-to-one. Therefore $v'\in\Omega_r^\kappa\mathbf{V}$ so that $v'\in\overline{\langle Y'\rangle}$. By (6.10), there is some $\pi'\in\mathrm{Cl}((Y')^+)\cap\Omega_r^\sigma\mathbf{S}$ whose restriction to $\mathbf{V}$ is v'.

To establish the claim, it remains to construct a σ-term $\pi\in\mathrm{Cl}(Y^+)$ whose restriction to $\mathbf{V}$ is v. By the special case, and induction on the number of states of the automaton $\mathcal{A}(H)$, we may assume that the σ-term π' has no nested operation symbols from $\sigma\setminus\kappa$ and, more precisely, that π' is a κ-term in σ-terms of the form

$$\circ_j^{\omega-1}(t'_1,\dots,t'_r)(s'_1,\dots,s'_r)\quad\text{with } t'_i,s'_i\in\Omega_r^\kappa\mathbf{S} \text{ such that } \psi t'_i=\psi s'_i \tag{6.11}$$

where the s'_i belong to Y'. Since the operation symbols in κ, namely product and the $(\omega-1)$-power, stay within the closed subsemigroup of $\overline{\Omega}_n\mathbf{S}$ generated by the arguments, and since $\kappa \subseteq \sigma$, it suffices to deal with operations of the form (6.11). By the same reasoning, the operation $\circ_j^{\omega-1}(t_1',\dots,t_r')$ also preserves membership in closed subsemigroups of $\overline{\Omega}_n\mathbf{S}$. Summarizing, we want to modify an operation of the form (6.11), by replacing the s_i' by $s_i \in \Omega_n^\kappa\mathbf{S}$ so that $\psi s_i = \psi\theta s_i'$, and $s_i \in \mathrm{Cl}(Y^+)$. But this is easy: Y' was obtained from Y by writing the elements of this set, viewed as group words, as κ-terms in the new generators $z_1,\dots,z_r$. This implies that to each s_i' corresponds some $s_i \in Y$ such that $\psi s_i = \psi\theta s_i'$ $(= \psi\theta t_i')$. Thus we may take

$$\pi = \circ_j^{\omega-1}(\theta t_1',\dots,\theta t_r')(s_1,\dots,s_r).$$

In this way we conclude the verification of the remaining case in the proof of Theorem 6.1(c). □

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