Hopf algebras and Ore extensions

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Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

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Resumo

Extensões de Ore são uma forma de construir novos objetos algébricos a partir de objetos pré-existentes, acrescentando uma nova variável e as relações de comutação que satisfaz. Álgebras de Hopf são álgebras que possuem uma certa estrutura dual adicional. Nesta tese, estudamos a caracaterização sob certas condições das álgebras de Hopf em extensões de Ore de uma álgebra de Hopf, seguindo artigos de Panov e de Brown, O'Hagan, Zhang e Zhuang. A noção de extensão dupla de Ore é uma generalização de extensão de Ore recentemente introduzida por Zhang e Zhang. Abordamos o problema de determinar quais é que são as extensões duplas de Ore sobre um corpo que tem um estrutura de álgebra de Hopf. Este problema está relacionado com o problema de estender uma estrutura de álgebra de Hopf a uma sua extensão dupla de Ore. Fazemos uma separação dos casos possíveis e não possíveis, em função dos parâmetros que determinam a extensão dupla de Ore.

Palavras-chave: extensão de Ore, álgebra de Hopf, extensão de Ore iterada, extensão dupla de Ore, extensão de Hopf Ore.

Abstract

Ore extensions provide a way of constructing new algebraic objects from preexisting ones, by adding a new indeterminate subject to commutation relations. Hopf algebras are algebras which possess a certain additional dual structure. In this thesis, we study a characterization under certain conditions of the Hopf algebra structures on Ore extensions of Hopf algebras, following articles by Panov and by Brown, O'Hagan, Zhang and ZHuang. The notion of double Ore extension is a generalization of Ore extension recently introduced by Zhang and Zhang. We address the problem of determining which are the double Ore extensions of a field that have a Hopf algebra structure. This problem is related to the problem of extending a Hopf algebra structure to a double Ore extension. We split the possible and not possible cases with respect to the data that determines the double Ore extension.

Keywords: Ore extension, Hopf algebra, iterated Ore extension, double Ore extension, Hopf Ore extension.

Contents

Introduction

Hopf algebras are objects which possess both algebra and coalgebra structures, with suitable compatibility between them. From a categorical point of view, they generalize groups. They have ties to quantum mechanics (the so called quantum groups are examples of Hopf algebras) and to noncommutative geometry, making Hopf algebras an active field of research in recent years.

Ore extensions provide a way of building new algebras from given ones. One of our main focuses is the construction of Hopf algebras through Ore extensions of a preexisting Hopf algebra. This task is connected to the classification of Hopf algebras of low Gelfand-Kirillov dimension. A lot of work has been done in these two related topics, for instance in [Pan03; BOZZ15; ZZ08; Zhu13; GZ10]. In 2008, Zhang and Zhang published a paper introducing double Ore extensions, a concept that generalizes the original notion of Ore extension. A few properties have been studied in [ZZ08] and [CLM11] but it is still a recent topic with many questions unanswered. Our goal is to better understand when a Hopf algebra structure can be extended to a double Ore extension.

Chapters 1 and 2 establish the setting for Chapters 3 and 4. In Chapter 1, we introduce Ore extensions and study some of their properties. Most notably a construction is given, the well-definedness of a degree map is proved under certain conditions and the Gelfand-Kirillov dimension of Ore extensions is studied.

In Chapter 2, we study the basics of Hopf algebra theory, starting from the concept of coalgebra and working all the way to Hopf algebras and their properties. Besides several examples and terminology that are introduced, we try to convey the essential ideas and reasonings that are recurrent in Hopf algebra theory and that are applied in later chapters.

In Chapter 3, we focus on the study of Hopf algebra structures on Ore extensions, so called Hopf Ore extensions, following two articles: [Pan03] and [BOZZ15], where the latter greatly expanded on the first. The main result of this chapter is Theorem 3.3.1, in which necessary and sufficient conditions for the existence of a Hopf Ore extension of a given Hopf algebra are given. We write a more detailed version of the proof that appears in [BOZZ15]. We also study the properties of Hopf Ore extensions.

In Chapter 4, the concept of a double Ore extension is introduced, as well as its notation and basic properties. Starting from Section 4.2, we study possible Hopf algebra structures that can be defined on a double Ore extension. First, we reduce the problem to the case of double Ore extensions taken over a field and then we proceed to consider *ad hoc* the several different cases that arise. The results are summarized in Section 4.8 with respect to the data that determines the double Ore extension. Sections 4.2 to 4.8 consist mostly of original work, with some external contributions that are properly acknowledged and of which the author is grateful.

We assume that the reader is at least familiar with linear algebra and basic ring theory at an undergraduate level. Additional necessary background knowledge that an undergraduate may or may not have already is covered in the Section 0.1. Finally some notation and conventions are established in Section 0.2.

Chapter 0

Preliminaries

0.1 Background

Algebras

Let K be a field. An **associative unital algebra** over K is a vector space A over K together with an associative multiplication $\cdot : A \times A \to A$ that is bilinear, i.e.,

$$(\alpha x + \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z),$$
$$x \cdot (\alpha y + \beta z) = \alpha(x \cdot y) + \beta(x \cdot z),$$

for all $x, y, z \in A, \alpha, \beta \in K$ and such there exists a unit 1_A for the multiplication. In other words, $(A, +, \cdot)$ is an associative ring with unit that is also a vector space over K and where scalar multiplication agrees with the multiplication $\cdot$. This means that there exists a linear embedding $\eta: K \to A$ mapping α to $\alpha 1_A$. In general, we identify α with $\alpha 1_A$ and we also drop the symbol $\cdot$, writing xy instead of $x \cdot y$, without further mentions. We will simply write "algebra" instead of "associative algebra over K", when the field K is understood, because we only study associative algebras in this work.

An algebra A is called **commutative** if $xy = yx$, for all $x, y \in A$. An element $x \neq 0$ in A is called **regular** if $xy = 0$ implies $y = 0$, for all $y \in A$. A **zero-divisor** is an element $z \neq 0$ in A that is not regular, i.e., for which there exists $y \neq 0$ such that $yz = 0$ or $zy = 0$. An algebra A without zero-divisors is called a **domain** and if A is commutative, it is called an **integral domain**. Invertibility in an algebra is the same as invertibility in a ring: an element $x \neq 0$ in A is called

invertible if there exists a $y \in A$ such that $xy = yx = 1_A$. Associativity in A implies that such an element y is unique. For that reason, it is called the inverse of x and denoted by x^{-1}. Given an algebra A, the **opposite algebra** A^{op} is the vector space A over K with a new multiplication $*$ given by $x * y = yx$, for all $x, y \in A$. A **direct product of algebras** (or cartesian product) is the direct product of the underlying vector spaces with the algebra structure given by defining multiplication pointwise.

A **subalgebra** B of an algebra A is a subring which is also a vector subspace. A two-sided algebra **ideal** I of A is simply a ring ideal of A and the embedding η implies that it is also a vector subspace of A. Analogously, we define left and right ideals in A. If I is an ideal of A, then the **quotient algebra** A/I is the ring A/I together with a scalar multiplication given by $\alpha(x + I) = \alpha x + I$, for all $\alpha \in K$, $x \in A$. A **maximal ideal** M of A is an ideal such that if I is another ideal of A with $M \subset I \subset A$, then $I = M$ or $I = A$. In a commutative algebra A, an ideal M is maximal if and only if A/M is a field. If $X \subset A$, then the **ideal** $\langle X \rangle$ **generated by** X is the intersection of all the ideals of A that contain X. A **principal ideal** of A is an ideal that is generated by only one element.

If A and B are two algebras over K, then an **algebra homomorphism** is a ring homomorphism that is $f\colon A \to B$ that is also a linear map and such that $f(1_A) = 1_B$. Likewise, we define algebra isomorphisms, algebra endomorphisms and algebra automorphisms as the corresponding ring counterparts that are also linear maps. If $f\colon A \to B$ is a linear map (resp. algebra homomorphism), then the **kernel of** f is the vector subspace (resp. ideal) $\operatorname{Ker} f = \{a \in A\colon f(a) = 0\}$ of A and the **image of** f is the vector subspace (resp. subalgebra) $\operatorname{Im} f = \{f(a)\colon a \in A\}$ of B. We list here the so called **isomorphism theorems**, mostly for future reference.

Proposition 0.1.1. *(i) Let A and B be algebras and $f\colon A \to B$ be an algebra homomorphism. Then $A/\operatorname{Ker} f \simeq \operatorname{Im} f$. Moreover, if f is surjective, then $A/\operatorname{Ker} f \simeq B$.*

(ii) Let A be an algebra, S a subalgebra of A and I an ideal of A. Then $(S + I)/I \simeq S/(S \cap I)$.

(iii) Let A be an algebra and $I \subseteq J$ be two ideals of A. Then $(A/I)/(J/I) \simeq A/J$.

We prove now an elementary result that will be needed later.

Proposition 0.1.2. *In a commutative unital algebra A, if I, J are ideals of A such that $I + J = A$, then $I \cap J = IJ$.*

Proof. By definition of ideal, we always have $IJ \subset I \cap J$. Conversely, let $k \in I \cap J$. Since $I + J = A$, there exist $i \in I$ and $j \in J$ be such that $i + j = 1_A$. Hence we have $k = ki + kj$, where $ki \in JI = IJ$ (A is commutative) and $kj \in IJ$, and thus $I \cap J \subset IJ$. □

An algebra A is called **left noetherian** (resp. **right noetherian**) if it satisfies the ascending chain condition, which states that every chain of left (resp. right) ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

stabilizes, i.e., there exists $n \geq 1$ such that $I_m = I_n$ for all $m \geq n$. An algebra A is called **noetherian** if it is both left and right noetherian.

Let X be a set. The **free algebra over** K **generated by** X is the algebra $K\langle X\rangle$ that satisfies the following **universal property**: there is an embedding $\iota\colon X \hookrightarrow K\langle X\rangle$ and if A is any unital algebra and $\varphi\colon X \to A$ is a map of sets, then there exists a unique algebra homomorphism $\bar{\varphi}\colon K\langle X\rangle \to A$ such that $\bar{\varphi} \circ \iota = \varphi$. A construction of $K\langle X\rangle$ can be found for instance in [Bre14, §6.1] and the uniqueness of $K\langle X\rangle$ up to isomorphism follows from the universal property. If $X = \{x_1, \cdots, x_n\}$ is a finite set, then we also write $K\langle x_1, \cdots, x_n\rangle$ for $K\langle X\rangle$. One particular example is the **polynomial algebra in one indeterminate** $K[x]$, which is commutative. Its universal property means that every algebra homomorphism from $K[x]$ to another algebra A is determined uniquely by the image of x. It is worth mentioning that for n greater than or equal to 2, the free algebra $K\langle x_1, \cdots, x_n\rangle$ does not coincide with the polynomial algebra in n indeterminates $K[x_1, \cdots, x_n]$, as it is not commutative.

Modules and tensor products

Let A be an algebra over K. A **left** $A-$**module** (or module over an algebra) is a vector space M over K together with a **module action** $. : A \times M \to M$, $(a, m) \mapsto a.m$ that satisfies

$$(\alpha x + \beta y).m = \alpha(x.m) + \beta(y.m),$$

$$x.(m + n) = x.m + x.n,$$

$$(xy).m = x.(y.m),$$

$$1_A.m = m$$

for all $\alpha, \beta \in K$, $x, y \in A$, $m, n \in M$. Equivalently, there exists an algebra homomorphism

$\rho: A \to \mathrm{End}_K(M)$, $a \mapsto [m \mapsto a.m]$, called a **representation** of the vector space M. We can analogously define a **right** $A-$**module structure** in the obvious way. If we do not say otherwise, an $A-$module is always a left $A-$module. The $K-$modules are simply vector spaces over K. The concept of module is more commonly defined for rings, in which case the vector space structure is replaced by the structure of abelian groups. For our purposes however, modules over algebras suffice.

Let M be an $A-$module. An $A-$**submodule** of M (or simply submodule) is a vector subspace N of M such that $a.n \in N$ for all $a \in A, n \in N$. If N is a submodule of M, then the quotient space M/N becomes an $A-$module, called a **quotient module**, via the action $a.(m+N) = a.m + N$, for all $a \in A, m \in M$. If N_1 and N_2 are two submodules of M, then their **sum** $N_1 + N_2$ consists of elements of the form $n_1 + n_2$, with $n_1 \in N_1$ and $n_2 \in N_2$ and it is a submodule of M. An $A-$module M is called **finitely generated** if there exist submodules $N_1, \cdots, N_k$ such that $M = N_1 + \cdots + N_k$.

If $(M_i)_{i\in I}$ is a family of $A-$modules, then the cartesian product $\Pi_{i\in I} M_i$ becomes an $A-$module via pointwise addition and module action. The submodule of $\Pi_{i\in I} M_i$ in which all but finitely many components are zero is called the **direct sum** of the modules $(M_i)_{i\in I}$ and is denoted by $\oplus_{i\in I} M_i$. Its elements are formal sums of the form $m_{i_1} + \cdots + m_{i_n}$ for some $i_1, \cdots, i_n \in I$ and $m_{i_j} \in M_{i_j}$. Given $m \in M$, the **submodule** Am **of** M **generated by** m is $\{a.m : a \in A\}$. An $A-$module M is called **free** if there exists a family $(m_i)_{i\in I}$ of elements of M, called a $A-$**basis**, such that $M = \oplus_{i\in I} Am_i$ and $Am_i \simeq A$, for all $i \in I$. Equivalently, M is free if it is generated by the elements m_i, $i \in I$, and these elements are **linearly independent over** A. Linear independence over A means that if $a_{i_1} m_{i_1} + a_{i_2} m_{i_2} + \cdots + a_{i_k} m_{i_k} = 0$, for some $i_1, \cdots, i_k \in I$ and $a_{i_j} \in A$, then $a_{i_1} = \cdots = a_{i_k} = 0$.

We now introduce an essential concept for our work: the tensor product of vector spaces over a field K. It can be determined by its universal property, which is the approach we choose. An explicit construction can be found in [Bre14, §4.1], for instance. Let U and V be two vector spaces over K. The **tensor product** $U \otimes_K V$ is a vector space over K that satisfies:

(i) There exists a bilinear map $U \times V \to U \otimes_K V$ mapping a pair (u, v) to $u \otimes v$;

(ii) Every element in $U \otimes_K V$ is a sum of elements of the form $u \otimes v$, with $u \in U$, $v \in V$;

(iii) **Universal property of the tensor product**. If W is another vector space over K and

$\varphi: U \times V \to W$ is a bilinear map, then there exists a unique linear map $\bar{\varphi}: U \otimes V \to W$ such that $\bar{\varphi}(u \otimes v) = \varphi(u, v)$, for all $u \in U$ and $v \in V$. In other words, the following diagram commutes

$$\begin{array}{ccc} U \times V & \xrightarrow{\varphi} & W \\ \downarrow & \nearrow_{\exists^1 \bar{\varphi}} & \\ U \otimes V & & \end{array} \tag{0.1.1}$$

From the universal property (iii), it follows that $U \otimes_K V$ is unique up to linear isomorphism. The tensor product can be seen as a tool that turns bilinear maps into linear ones. A standard way to define a linear map on the tensor product $U \otimes V$ is to define the corresponding map in $U \times V$, prove its bilinearity and then apply the universal property. For example, the multiplication map in an algebra A is a bilinear map $\cdot : A \times A \to A$. Hence it induces a linear map $A \otimes A \to A$ that maps $a \otimes b$ to $a \cdot b$. As a matter of fact, this map $A \otimes A \to A$ captures the exact essence of the multiplication being associative and distributive over addition and thus is an equivalent way of defining the multiplication (we will come back to this in Chapter 2). When K is understood, it is common to simply write $U \otimes V$, which we shall do hereinafter.

An element of the form $u \otimes v \in U \otimes V$ is called a **pure tensor** (or simple tensor). Property (ii) tells us that the elements in $U \otimes V$ are sums of pure tensors, but this does not mean that all the elements in $U \otimes V$ are pure tensors themselves. In general, they are not and as a matter of fact, elements are far from being uniquely written as a sum of pure tensors. For instance, in $K[x] \otimes K[x]$, we have $(1+x) \otimes (-1+x) = -1 \otimes 1 + 1 \otimes x - x \otimes 1 + x \otimes x$. Bilinearity also means that we have equalities such as $(\alpha u) \otimes v = u \otimes (\alpha v)$, for $u \in U$, $v \in V$ and $\alpha \in K$, which may seem confusing at first.

Many of the basic properties of tensor products are proved in [Bre14]. We collect here some results which will be useful later on. The tensor product is associative (up to isomorphism), i.e., if U, V and W are vector spaces then $U \otimes (V \otimes W) \simeq (U \otimes V) \otimes W$. The tensor product is commutative (up to isomorphism), i.e., $U \otimes V \simeq V \otimes U$ via the linear isomorphism $[u \otimes v \mapsto v \otimes u]$. The trivial vector space over K, which is K itself can be seen as the identity of the tensor product in the following sense: if U is a vector space over K, then $U \otimes K \simeq U \simeq K \otimes U$. For instance, the linear isomorphism from $U \otimes K$ to U is the one mapping $u \otimes \alpha$ to αu with inverse that maps u to $u \otimes 1$. The other one is analogous.

If $\{u_i\}_{i \in I}$ is a basis of U as a vector space and $\{v_j\}_{j \in J}$ is a basis of V, then $\{u_i \otimes v_j\}_{i \in I, j \in J}$

is a basis of $U \otimes V$. While an element in $U \otimes V$ can be written in many ways as a sum of pure tensors, we can have some control over it if we demand that the left tensorands (resp. the right tensorands) are linearly independent.

Lemma 0.1.3. *Let $U \otimes V$ be a tensor product of vector spaces and let $w = \sum_{i=1}^{n} f_i \otimes g_i \in U \otimes V$. If $\{f_1, \cdots, f_n\}$ (resp. $\{g_1, \cdots, g_n\}$) are linearly independent and $w = 0$, then $g_1 = \cdots = g_n = 0$ (resp. $f_1 = \cdots = f_n = 0$).*

Proof. See [Bre14, Lemmas 4.8 and 4.10]. □

This result means that, given an element of $U \otimes V$, if we can suppose that the sets of either left or right tensorands are linearly independent, then its writing as a sum of pure tensors becomes unique. The good news is that we can always assume that the left and the right tensorands, in the writing of an element as a sum of pure tensors, form linearly independent sets. Indeed, if $w = \sum_{i=1}^{n} f_i \otimes g_i \in U \otimes V$ and we assume that n is minimal then any relation of linear independence in the set $\{f_1, \cdots, f_n\}$ (likewise for $\{g_1, \cdots, g_n\}$) would allow us to rewrite w as a sum of $n-1$ pure tensors, contradicting the minimality of n.

The tensor product of vector spaces can be extended to algebras and this is in fact our main application for it. If A and B are algebras over K, then we can define the tensor product of algebras $A \otimes B$ as the vector space $A \otimes B$ together with the multiplication given by $(a \otimes b)(c \otimes d) = (ac \otimes bd)$. Defining it only in pure tensors is enough because it extends linearly to all the elements in $A \otimes B$, which are sums of pure tensors. The algebra $A \otimes B$ has identity $1_A \otimes 1_B$. The linear isomorphisms that express the associativity, the commutativity and the existence of identity on the tensor product of vector spaces all become algebra isomorphisms.

Lie Algebras

Finally, we introduce a concept that will provide a source of examples for our results in Chapter 3.

Definition 0.1.4. A **Lie algebra** L **over a field** K is a vector space endowed with a bilinear operation $[\cdot,\cdot]\colon L \otimes L \to L$ (called a **Lie bracket** or **commutator**) satisfying $\forall x, y, z \in L$:

(i) $[x, x] = 0$;

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, (Jacobi's identity).

Bilinearity and (i) imply that the bracket $[\cdot,\cdot]$ is anti-commutative, which means that $\forall x, y \in L$:

$$[x, y] = -[y, x]. \tag{0.1.2}$$

Conversely, the anti-commutativity of a bilinear operator implies (i), as long as $\operatorname{char} K \neq 2$. Jacobi's identity reflects the fact that a Lie algebra is not associative, in general.

We call a Lie algebra **abelian** (or trivial) if its Lie bracket is zero. If L and L' are Lie algebras, then a **Lie algebra homomorphism** is a linear map $f\colon L \to L'$ such that $f([x, y]) = [f(x), f(y)]$, for all $x, y \in L$ and moreover, if f is bijective, it is called a **Lie algebra isomorphism**. Given an associative algebra A over K, we can create a Lie algebra structure on the underlying vector space A by defining the Lie bracket as the commutator $[x, y] = xy - yx$, for all $x, y \in A$. The resulting Lie algebra is denoted by $L(A)$.

In dimension one, there is only one Lie algebra, the abelian one, because of condition (i) in Definition 0.1.4. In dimension two, say with basis $x, y \in L$, there are two Lie algebras, up to isomorphism: the abelian one (i.e., $[x, y] = 0$) and another non-isomorphic to the abelian one, in which $[x, y] = x$. The latter Lie algebra is also called the non-trivial Lie algebra of dimension two.

One important notion regarding a Lie algebra is its universal enveloping algebra. Given a Lie algebra L, an **enveloping algebra of** L is an associative algebra A such that $L(A) = L$. The universal enveloping algebra of a Lie algebra, as the name suggests, satisfies a universal property. An explicit construction using the tensor algebra is given in [Kas95, §V.2], but we can also define it by its universal property, which is what we choose to do. The **universal enveloping algebra** $U(L)$ **of a Lie algebra** L is an associative algebra that satisfies:

(i) There exists a linear embedding $\iota\colon L \hookrightarrow U(L)$ such that $\iota([x, y]) = \iota(x)\iota(y) - \iota(y)\iota(x)$, for all $x, y \in L$.

(ii) **Universal property of universal enveloping algebras.** Given any associative algebra A and any Lie algebra homomorphism $\varphi\colon L \to L(A)$, there exists a unique algebra homomorphism $\bar{\varphi}\colon U(L) \to A$ such that $\bar{\varphi} \circ \iota = \varphi$.

Property (ii) above implies the uniqueness of $U(L)$ up to isomorphism. If L is the (abelian) one-dimensional Lie algebra, then $U(L) \simeq K[x]$ is the polynomial algebra in one variable. More generally, if L is the abelian Lie algebra of dimension n, then $U(L) \simeq K[x_1, \cdots, x_n]$. On the other hand, if L is the non-trivial Lie algebra of dimension two, with $[x, y] = x$, then $U(L)$ is the quotient of the free algebra $K\langle x, y\rangle$ with its ideal generated by $xy - yx - x$. In other words, $U(L)$ is an algebra generated by two elements x, y subject to the relation $xy = yx + x$.

More generally, if L a Lie algebra of dimension n with basis $\{x_1, \cdots, x_n\}$, then we have that $U(L) = K\langle x_1, \cdots, x_n\rangle/\langle x_i x_j - x_j x_i - [x_i, x_j]\rangle_{1\leq i,j\leq n}$.

0.2 Notation and conventions

Throughout this work, we will always assume that K is an algebraically closed field of characteristic zero and by algebra, we always mean a unital associative algebra over K. Also, all tensor products, linear maps and algebraic groups in this work are taken over K. We denote the set of algebra homomorphisms from A to B by $\mathrm{Alg}_K(A, B)$. In contrast, we denote the set of linear maps from A to B by $\mathrm{Hom}_K(A, B)$ and the monoid of linear endomorphisms of A under the composition of maps by $\mathrm{End}_K(A)$. When an algebra quotient A/I is in context, we sometimes use an overline to denote elements in A/I, i.e., we write $\bar{a} = a + I$ for $a \in A$. The restriction of a map with A as its domain to $B \subset A$ is denoted by $\varphi|_B$. The set of invertible elements of an algebra A is denoted $A^\times$. Finally, the symbol $\mathbb{N}$ denotes the set of the natural numbers including 0.

Chapter 1

Ore extensions

1.1 Definition and existence

In this section, we introduce one of the most common and useful tools to build new (not necessarily commutative) algebras from preexisting ones. They are named after Øystein Ore, the influent Norwegian mathematician from the 20th century. The idea is that if we add a new indeterminate to an algebra, it may not commutate with the other elements, but instead, a twisted version of commutation takes place. We start by introducing the notion of a twisted derivation, which plays a role in the definition of Ore extensions.

Definition 1.1.1. Let σ be an algebra endomorphism of R. A (left) $\sigma-$**derivation** is a linear map $\delta\colon R \to R$, which satisfies the $\sigma-$Leibniz rule, i.e.,

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s) \tag{1.1.1}$$

for all $r, s \in R$. We call an Id_R-derivation simply a **derivation** and when we do not wish to specify the endomorphism σ, we say **twisted derivation** instead.

We could accordingly define right twisted derivations, as well, but when we mention twisted derivations hereinafter, we will always refer to the left-sided version. If δ is a $\sigma-$derivation, the $\sigma-$Leibniz rules easily implies that $\delta(1) = 0$. Since δ is linear, this means that $\delta|_K \equiv 0$. A motivating example is $\delta = \dfrac{d}{dx}$ in $R = K[x]$, the map of differentiation with respect to x. It is well known that δ is linear and obeys the Leibniz rule, hence it is a derivation. Moreover, given any algebra endomorphism σ of R, it is straightforward to check that a linear map $\delta\colon R \to R$ is a $\sigma-$derivation if and only if the map from R to $M_{2\times 2}(R)$ sending r to $\left[\begin{smallmatrix}\sigma(r) & \delta(r)\\ 0 & r\end{smallmatrix}\right]$ is an algebra homomorphism. By the universal property of $K[x]$, there is an algebra homomorphism from $K[x]$ to $M_{2\times 2}(R)$ that maps x to $\left[\begin{smallmatrix}\sigma(x) & f\\ 0 & x\end{smallmatrix}\right]$ and hence, we conclude that for all $f \in K[x]$, there exists a unique σ-derivation δ with $\delta(x) = f$.

Definition 1.1.2. Let σ be an algebra endomorphism of R and δ be a σ-derivation of R. A **left Ore extension** $T = R[y;\sigma,\delta]$ of R is an algebra generated by R and by an element $y \in T$, in which the relation

$$yr = \sigma(r)y + \delta(r) \tag{1.1.2}$$

holds for all $r \in R$ and such that T is a free left R-module with basis $\{y^i\}_{i\in N}$. When σ is an automorphism of R, we call $T = R[y;\sigma,\delta]$ simply an **Ore extension**. The elements in $R[y;\sigma,\delta]$ are of the form $f = r_0 + r_1 y + \cdots + r_n y^n$ for some $n \in \mathbb{N}$ and $r_0, \cdots, r_n \in R$. We define a map $\deg\colon R[y;\sigma,\delta] \to \mathbb{N} \cup \{-\infty\}$, called a **degree**, by $\deg f = \max\{i \in \mathbb{N} : r_i \neq 0\}$ for $f \neq 0$ and $\deg 0 = -\infty$. If $\deg f = n$, then the coefficient r_n is called the **leading term** of f. The coefficient r_0 is called the **constant term** of f.

Ore extensions can also be called *skew polynomial algebras*, which emphasizes the noncommutative structure of its multiplication. We will almost always assume that σ is an automorphism, in particular in Chapters 3 and 4. We will see in the next section that the bijectivity of σ determines when multiplication from the right side gives $R[y;\sigma,\delta]$ a structure of free right $R-$module, in addition to the left one. In this introductory chapter however, we study left Ore extensions in general and admit for now that σ is just an endomorphism.

Since $R[y;\sigma,\delta]$ is an algebra, relation (1.1.2) has to be compatible with the distributivity and associativity rules. To satisfy the former, σ and δ must be linear maps. To satisfy the latter, i.e., having $y(rs) = (yr)s$, we must have that

$$\sigma(rs)y + \delta(rs) = (\sigma(r)y + \delta(r))s = \sigma(r)\sigma(s)y + \sigma(r)\delta(s) + \delta(r)s. \tag{1.1.3}$$

Since $\{y^i\}_{i\in\mathbb{N}}$ is a basis of $R[y;\sigma,\delta]$, we can compare the coefficients of y and the constant terms on both sides. It becomes immediately clear that σ needs to be a homomorphism of algebras and that δ needs to be a $\sigma-$derivation, therefore motivating our definition.

We give now some examples of left Ore extensions, which illustrate the concept and notation:

(i) The classic **polynomial algebra** $K[y]$, in which $\sigma = \mathrm{Id}_K$ and $\delta \equiv 0$, which means $K[y]$ is commutative. It is a trivial example, but one that motivates the introduction of Ore extensions. We will model our study of Ore extensions on the study of polynomials in the commutative setting. The polynomial algebra in $n \geq 2$ variables $K[y_1, \cdots, y_n]$ is also an example of Ore extension of $K[y_1, \cdots, y_{n-1}]$.

(ii) A **quantum plane** $K_q[y, z]$, with $q \in K \backslash \{0\}$, is an Ore extension of $K[y]$, in which σ is the algebra endomorphism of R determined by $\sigma(y) = qy$ and $\delta \equiv 0$. In the notation of Ore extensions, we write $K_q[y, z] = K[y][z; \sigma]$, omitting δ because it is zero.

(iii) A **differential operator algebra** $K[y][z; \delta]$, in which $\sigma = \mathrm{Id}_K$ (being omitted for that reason) and δ is simply a derivation. For instance, if $\delta = \dfrac{d}{dy}$, then y and z satisfy the relation $zy = yz + 1$ and $K[y][z; \delta]$ becomes the so called **first Weyl algebra** over K, also denoted $A_1(K)$.

(iv) A first **quantum Weyl algebra** $A_1^q(K) = K[y][z; \sigma, \delta]$, with $q \in K \backslash \{0\}$, where σ is determined by $\sigma(y) = qy$ and $\delta = \dfrac{d}{dy}$. The variables y and z satisfy the relation $zy = qyz + 1$

Despite having these examples, it is not yet clear that for any algebra R, any algebra endomorphism σ of R and any σ-derivation δ, there exists the Ore extension $R[y; \sigma, \delta]$. We will prove this important result in Theorem 1.1.4, which in part illustrates why this construction is so useful. Before doing so, we prove a lemma that helps with computations in Ore extensions, as it gives a general formula for the multiplication.

For $n \in \mathbb{N}$ and $0 \leq k \leq n$, let $S_{n,k}$ be the linear endomorphism of R defined as the sum of all $\binom{n}{k}$ possible compositions of k copies of δ and of $n-k$ copies of σ. In particular, $S_{n,0} = \sigma^n$ and $S_{n,n} = \delta^n$. A recursive formula analogue to the binomial coefficients' one takes place:

$$S_{n+1,k} = S_{n,k} \circ \sigma + S_{n,k-1} \circ \delta, \tag{1.1.4}$$

for $1 \leq k \leq n+1$.

Lemma 1.1.3. *Let $R[y; \sigma, \delta]$ be a left Ore extension and let $f = \sum_{i=0}^{n} r_i y^i$ and $g = \sum_{i=0}^{m} s_i y^i$ in $R[y; \sigma, \delta]$. Write $fg = \sum_{i \geq 0} t_i y^i$, for some $t_i \in R$. Then, for all $i \geq 0$, we have*

$$t_i = \sum_{p=0}^{i} r_p \sum_{k=0}^{p} S_{p,k}(s_{i-p+k}) \tag{1.1.5}$$

and for all $r \in R$ and $n \in \mathbb{N}$, we have

$$y^n r = \sum_{k=0}^{n} S_{n,k}(r) y^{n-k}. \tag{1.1.6}$$

Proof. Equation (1.1.5) follows from equation (1.1.6). The latter follows by induction on n. For $n = 0$, it is just equation (1.1.2). The induction step follows from the relations (1.1.2) and (1.1.4). □

Let $R[y]$ be the free left R-module with basis $\{y^i\}_{i\in\mathbb{N}}$. It consists of elements of the form $f = r_0 + r_1 y + \cdots + r_n y^n$ for some $n \in \mathbb{N}$ and $r_0, \cdots, r_n \in R$. This $R-$module is our candidate to become the desired Ore extension $R[y; \sigma, \delta]$, provided that we can give it an algebra structure with the suitable multiplication, i.e., one which respects (1.1.2). Retain the notions of $\deg$, leading term and constant term, given in Definition 1.1.2.

Theorem 1.1.4. *Given an algebra endomorphism σ of R and a σ-derivation δ of R, there exists a unique algebra structure on $R[y]$ that turns it into the left Ore extension $R[y; \sigma, \delta]$.*

Proof. This proof follows the one in [Kas95, Theorem I.7.1]. As Lemma 1.1.3 shows, the multiplication in an Ore extension is uniquely determined by the relation $yr = \sigma(r)y + \delta(r)$. Hence, the uniqueness of the respective algebra structure on $R[y]$ is clear.

In order to prove the existence, we will construct a linear embedding of $R[y]$ into a certain algebra, which induces the desired algebra structure back on $R[y]$. Denote by $\mathrm{End}_K(R)$ the algebra of linear endomorphisms of R under composition.

Let $\mathcal{M}$ be the associative algebra of the row finite and column finite $\mathbb{N} \times \mathbb{N}$ matrices over $\mathrm{End}_K(R)$. This means that $\mathcal{M}$ is the algebra consisting of infinite matrices $(f_{ij})_{i,j\in\mathbb{N}}$, where each f_{ij} is a linear endomorphism of R, such that in each row and in each column there are only finitely many nonzero entries and where the operation is matrix multiplication. This operation is well defined because row and column finiteness means that we only face a finite number of nonzero summands when computing an entry in a product of infinite matrices.

The identity of $\mathcal{M}$ is the infinite diagonal matrix I with Id_R in the diagonal. Given an element $r \in R$, we denote by $\widehat{r} \in \mathrm{End}_K(R)$ the left multiplication by r. Hence, we can embed R into $\mathcal{M}$ by mapping r to $\widehat{r}I$. Since σ is an algebra endomorphism and δ is a σ-derivation, they are in particular linear endomorphisms which satisfy additional relations in $\mathrm{End}_K(R)$,

$$\sigma\widehat{r} = \widehat{\sigma(r)}\sigma \quad \text{and} \quad \delta\widehat{r} = \widehat{\delta(r)} + \widehat{\sigma(r)}\delta. \tag{1.1.7}$$

Consider the following infinite matrix in $\mathcal{M}$,

$$Y = \begin{bmatrix} \delta & 0 & 0 & 0 & \cdots \\ \sigma & \delta & 0 & 0 & \cdots \\ 0 & \sigma & \delta & 0 & \cdots \\ 0 & 0 & \sigma & \delta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and define a linear map $\Phi \colon R[y] \to \mathcal{M}$ by

$$\Phi\left(\sum_{i=0}^{n} r_i y^i\right) = \sum_{i=0}^{n} (\widehat{r_i} I)\, Y^i. \tag{1.1.8}$$

We now prove that Φ is injective. Let e_i be the infinite column vector with entries in R, whose i-th entry equals 1 and the all the others are zero, for $i \in \mathbb{N}$. Then $\{e_i\}_{i\in\mathbb{N}}$ is a basis for the $R-$module of infinite column vectors with finitely many nonzero entries in R. Elements of $\mathcal{M}$ act on infinite column vectors just like finite matrices of endomorphisms act on finite column vectors. If we apply Y to each e_i, recalling that $\sigma(1) = 1$ and $\delta(1) = 0$, we get

$$Y(e_i) = e_{i+1}. \tag{1.1.9}$$

Let $f = \sum_{i=1}^{n} r_i y^i \in R[y]$ be such that $\Phi(f) = 0$. In light of (1.1.9), applying $\Phi(f)$ to e_1 yields

$$0 = \Phi(f)(e_1) = \sum_{i=0}^{n} (\widehat{r_i} I)\, Y^i(e_1) = \sum_{i=0}^{n} r_i e_{i+1}.$$

Hence, from the equation above, we conclude that $r_i = 0$, for all $1 \le i \le n$ and therefore $f = 0$.

From relations (1.1.7), it follows that in $\mathcal{M}$ we have

$$Y(\widehat{r}I) = (\widehat{\sigma(r)}I)Y + \widehat{\delta(r)}I, \tag{1.1.10}$$

for all $r \in R$. Let S be the subalgebra of $\mathcal{M}$ generated by the elements Y and $\widehat{r}I$, for all $r \in R$. Because of equation (1.1.10), every element in S can be written in the form $\sum_{i=0}^{n} (\widehat{r_i}I)Y^i$ and thus S is the image of $R[y]$ by Φ. Since Φ is injective, we have an isomorphism between $R[y]$ and S. Hence, we can define the algebra structure on $R[y]$ induced by S and because of (1.1.10), the multiplication in $R[y]$ satisfies $yr = \sigma(r)y + \delta(r)$, for all $r \in R$, like we wanted. □

1.2 Properties of Ore extensions

With the existence of left Ore extensions assured, we turn our focus to some of their properties. We recall that, in Definition 1.1.2, we introduced a map $\deg\colon R[y;\sigma,\delta] \to \mathbb{N} \cup \{-\infty\}$, given by $\deg f = \max\{i \in \mathbb{N} : r_i \neq 0\}$, for $f = \sum_{i\geq 0} r_i y^i \neq 0$, and by $\deg 0 = -\infty$. Note that the elements of R are exactly those with degree 0 plus the element 0 itself, which has degree $-\infty$. Also, in $\mathbb{N} \cup \{-\infty\}$, we set $-\infty + n = n - \infty = -\infty$ by convention, for all $n \in \mathbb{N}$. One desirable property in $R[y;\sigma,\delta]$ would be that the map deg satisfies $\deg fg = \deg f + \deg g$, for all $f, g \in R[y;\sigma,\delta]$, just as happens in the classical polynomial algebra. This property implies that $R[y;\sigma,\delta]$ is a domain, since if $f, g \neq 0$, then $\deg f, \deg g \geq 0$ and thus, $\deg fg \geq 0$, which means that fg cannot be zero. In the next proposition, we study necessary and sufficient conditions for this situation to occur.

Proposition 1.2.1. *Let $R[y;\sigma,\delta]$ be a left Ore extension and the map $\deg\colon R[y;\sigma,\delta] \to \mathbb{N} \cup \{-\infty\}$ be as in Definition 1.1.2. Then, the following conditions are equivalent*

(i) $R[y;\sigma,\delta]$ is a domain in which $\deg fg = \deg f + \deg g$ for all $f, g \in R[y]$;.

(ii) R is a domain and σ is injective.

Proof. (i)⇒ (ii) Since $R[y;\sigma,\delta]$ is a domain, then in particular so is R. Seeking a contradiction, assume that σ is not injective, that is, there exists $r \neq 0$ such that $\sigma(r) = 0$. Then $yr = \sigma(r)y + \delta(r) = \delta(r) \in R$ and hence yr has degree less than 1. On the other hand, $\deg yr = \deg y + \deg r = 1$ by hypothesis, which gives the desired contradiction.

(ii) ⇒ (i) As discussed in the paragraph preceding this proposition, it is enough to show that the formula $\deg fg = \deg f + \deg g$ holds for all $f, g \in R[y;\sigma,\delta]$. Let $f = \sum_{i=0}^{n} r_i y^i$ and $g = \sum_{i=0}^{m} s_i y^i$ in $R[y;\sigma,\delta]$ with $r_n, s_m \neq 0$, i.e., $\deg f = n$ and $\deg g = m$. Write $fg = \sum_{i\geq 0} t_i y^i$. By formula (1.1.5), we have

$$t_i = \sum_{p=0}^{i} r_p \sum_{k=0}^{p} S_{p,k}(s_{i-p+k}).$$

Assume that $i > n+m$. Observe that if $p > n$, then $r_p = 0$. On the other hand, if $p \leq n$, then $i-p+k > m+k \geq m$ and thus $s_{i-p+k} = 0$. In either case, we have that $t_i = 0$ for $i > n+m$. Assume now that $i = n+m$. Then, if $p > n$, we have $r_p = 0$ as previously and if $p \leq n$, then $i-p+k \geq m+k \geq m$ with equality if and only if $p = n$ and $k = 0$. Thus, the only nonzero summand in the formula for t_{n+m} is

$$r_n S_{n,0}(s_m),$$

where $S_{n,0} = \sigma^n$ is an injective homomorphism by hypothesis. Thus, since R is a domain and $r_n, s_m \neq 0$, it follows that $t_{n+m} \neq 0$ and $\deg fg = n+m$, like we wanted to prove. □

The following lemma comprises two technical observations about invertibility in Ore extensions that are useful to know.

Lemma 1.2.2. *In a left Ore extension $R[y;\sigma,\delta]$, the element y is not left invertible. If, in addition, R is a domain and σ is injective, then the only invertible elements in $R[y;\sigma,\delta]$ are in R.*

Proof. Assume that y has a left inverse f, which we can write as $f = r_0 + r_1 y + \cdots - r_n y^n$. Having $1 = fy$ is equivalent to $1 - r_0 y - \cdots - r_n y^{n+1} = 0$, which contradicts the freeness of the basis $\{y^i\}_{i\in\mathbb{N}}$ as a left $R-$module. Assume now that R is a domain. By Proposition 1.2.1, if $f \in R[y;\sigma,\delta]$ has an inverse g, then we have $\deg f + \deg g = \deg 1 = 0$, from where it follows that $\deg f = \deg g = 0$. □

We defined left Ore extensions, because we required the $R-$module structure on $R[y;\sigma,\delta]$ to be left-sided but we could have made the equivalent construction using the right-sided versions of twisted derivations, of equation (1.1.2) and of free $R-$modules. In an Ore extension (i.e., when σ is an automorphism), it would make sense that both left and right versions apply and were compatible and indeed this happens. We use the concept of opposite algebra because, in principle, right-sided statements can be seen as left-sided in the opposite algebra. We recall that if A is an algebra, the opposite algebra A^{op} has the same underlying vector space structure as A together with a multiplication $*$ defined by $a * b = ba$, for all $a, b \in A$.

Proposition 1.2.3. *Let $R[y;\sigma,\delta]$ be an Ore extension. Then, the following holds:*

(i) *$R[y;\sigma,\delta]$ is also a right free R-module with the same basis $\{y^i\}_{i\in\mathbb{N}}$;*

(ii) The opposite algebra $(R[y;\sigma,\delta])^{op}$ is also an Ore extension. More precisely,

$$(R[y;\sigma,\delta])^{op} = R^{op}[y;\sigma^{-1}, -\delta\circ\sigma^{-1}]. \tag{1.2.1}$$

Proof. (i) Since σ is invertible, we have by formula (1.1.6) in Lemma 1.1.3 that

$$y^n\sigma^{-n}(r) = ry^n + \text{lower-degree terms}, \tag{1.2.2}$$

for all $n\in\mathbb{N}$. Hence, we can prove by induction that $\{y^i\}_{i\in\mathbb{N}}$ generates $R[y;\sigma,\delta]$ as a right $R-$module. That is, any $f\in R[y;\sigma,\delta]$ can be written as $f=\sum_{i=0}^{n} y^i r_i$ where $n=\deg f$ is the degree defined on the left module structure. If $n=0$, there is nothing to do. Assume that $n>0$ and that the result holds in degree less than n. Let f be an element of degree n. Then, by (1.2.2), we have

$$f = ry^n + \text{lower-degree terms} = y^n\sigma^{-n}(r) + \text{lower-degree terms}$$

and the claim follows by the induction hypothesis. It remains to prove that $\{y^i\}_{i\in\mathbb{N}}$ is free. Seeking a contradiction, assume that

$$r_0 + yr_1 + \cdots + y^n r_n = 0$$

with $r_n\neq 0$. Thus, again by formula (1.1.6), we have

$$\sigma^n(r_n)y^n + \text{ lower-degree terms} = 0$$

which implies that $\sigma^n(r_n)=0$ since $\{y^i\}_{i\in\mathbb{N}}$ is a basis of $R[y;\sigma,\delta]$ as a left module. This is a contradiction because $r_n\neq 0$ and σ is an automorphism, by hypothesis.

(ii) Denote by $*$ the multiplication in the opposite algebras R^{op} and $(R[y;\sigma,\delta])^{op}$. Since σ is an automorphism of R, it is also an automorphism of R^{op} and thus, it is clear that relations (1.1.2) are equivalent to $y*r=\sigma^{-1}(r)*y-\delta(\sigma^{-1}(r))$, for all $r\in R$. We now check that $-\delta\circ\sigma^{-1}$ is a $(\sigma^{-1})-$derivation in R^{op}. Let $r,s\in R$ and compute

$$\begin{aligned}-\delta(\sigma^{-1}(r*s)) = -\delta(\sigma^{-1}(sr)) &= -\delta(\sigma^{-1}(s))\sigma^{-1}(r) - \sigma(\sigma^{-1}(s))\delta(\sigma^{-1}(r))\\ &= -\delta(\sigma^{-1}(r))*s - \sigma^{-1}(r)*\delta(\sigma^{-1}(s)).\end{aligned}$$

Finally, left $R-$module actions in $(R[y;\sigma,\delta])^{op}$ correspond to right ones on $R[y;\sigma,\delta]$ and vice-versa, because $r * y = yr$ for all $r \in R$. Since $R[y;\sigma,\delta]$ is free as right $R-$module (by the previous item), then $(R[y;\sigma,\delta])^{op}$ is free as a left $R-$module. Hence, $(R[y;\sigma,\delta])^{op}$ is an Ore extension. □

Essentially, even though we are not giving an explicit definition of right Ore extensions, what Proposition 1.2.3(ii) tells us is that left Ore extensions over R correspond to right Ore extensions over R^{op} and vice-versa. We will almost always require that σ is an automorphism, unless stated otherwise. Hence, we stop discussing left-sided or right-sided Ore extensions from this point on, even if some of the results below may hold in these more general situations.

The next result is a non-commutative version of the well-known Hilbert's basis theorem.

Proposition 1.2.4. *Let $T = R[y;\sigma,\delta]$ be an Ore extension (i.e., σ is an automorphism). If R is noetherian (resp. left noetherian, right noetherian), then so is T.*

Proof. For the proof of the left noetherian version, see [Kas95, Theorem I.8.3]. The right noetherian version follows from the left one and (1.2.3)(ii). □

We collect now two results that will be needed in the following chapters.

Proposition 1.2.5. *If R is a domain and φ is either an automorphism or an anti-automorphism of $T = R[y;\sigma,\delta]$ such that $\varphi(R) \subseteq R$, then $\varphi(y) = ay + b$, where $a \in R^{\times}, b \in R$.*

Proof. Suppose that φ is an automorphism. First, any element $r_0 + r_1 y + \cdots r_n y^n$ in T is mapped by φ into

$$\varphi(r_0) + \varphi(r_1)\varphi(y) + \cdots + \varphi(r_n)\varphi(y)^n, \tag{1.2.3}$$

because φ is a homomorphism.

Let us see that $\varphi(y)$ cannot be in R. As we are assuming that $\varphi(R) \subseteq R$, if on top of that we have $\varphi(y) \in R$, then by (1.2.3), we conclude that $\varphi(T) \subseteq R$. This is absurd, because φ is bijective and therefore, $\varphi(T) = T$. Hence, we have that $\deg \varphi(y) \geq 1$.

We want to prove in the following that $\deg \varphi(y) \leq 1$. Seeking a contradiction, assume that $\deg \varphi(y) > 1$ and consider the element $\varphi^{-1}(y)$. Certainly, we have $\varphi^{-1}(y) \notin R$, since φ^{-1} is a bijective as well. Thus, it is possible to write

$$\varphi^{-1}(y) = r_0 + r_1 y + \cdots + r_n y^n$$

for a certain $n \geq 1$ with each $r_i \in R$ and $r_n \neq 0$. Applying φ to this expression yields

$$y = \varphi(r_0) + \varphi(r_1)\varphi(y) + \cdots + \varphi(r_n)\varphi(y)^n,$$

with $\varphi(r_n) \neq 0$, since φ is injective. Hence, as $\varphi(R) \subseteq R$, the properties of the degree in T imply that y has degree $n \deg \varphi(y) > n \geq 1$. This gives the desired contradiction because y has degree 1. We conclude therefore that $\varphi(y) = ay + b$ for certain $a, b \in R$ with $a \neq 0$. We show next that a is invertible.

Applying the same reasoning to the automorphism φ^{-1}, there exist $c, d \in R$ with $c \neq 0$ such that $\varphi^{-1}(y) = cy + d$. Thus, we have

$$\begin{aligned} y = \varphi(\varphi^{-1}(y)) = \varphi(cy + d) &= \varphi(c)(ay + b) + \varphi(d) \\ &= (\varphi(c)a)\, y + (\varphi(c)b + \varphi(d)) \end{aligned}$$

and

$$\begin{aligned} y = \varphi^{-1}\varphi(y) = \varphi^{-1}(ay + b) &= \varphi^{-1}(a)(cy + d) + \varphi^{-1}(b) \\ &= \left(\varphi^{-1}(a)c\right) y + \left(\varphi^{-1}(a)d + \varphi^{-1}(b)\right). \end{aligned}$$

Since T is free as a $R-$module, we compare the coefficients of y, yielding $\varphi(c)a = 1 = \varphi^{-1}(a)c$. Applying φ to the second equation, we have $1 = a\varphi(c)$ and hence, a is invertible with inverse $\varphi(c)$.

Suppose now that φ is an anti-automorphism, which means that it can be seen as an isomorphism $\varphi\colon T \to T^{op}$. By Proposition 1.2.3(ii), $T^{op} = R^{op}[y; \sigma^{-1}, -\delta\sigma^{-1}]$ is also an Ore extension over a domain and in particular, is a free module over R^{op} and has a degree map satisfying Proposition 1.2.1. Hence, the argument from the automorphism case still applies and the result follows. □

The automorphism τ of $K[x, y]$ determined by $\tau(x) = y$ and $\tau(y) = x$, gives a simple example of how Proposition 1.2.5 fails to hold, if $\varphi(R) \not\subseteq R$.

We wish to establish a universal property for Ore extension, a result that tells us when homomorphisms from R to another algebra A can be extended to homomorphisms from $R[y; \sigma, \delta]$ to A. This property will be useful later, in Chapter 3.

Proposition 1.2.6. *Let $R[y; \sigma, \delta]$ be an Ore extension, A be an algebra and $\varphi\colon R \to A$ be an algebra homomorphism. Fix an element $a \in A$. There exists an algebra homomorphism $\bar{\varphi}\colon R[y; \sigma, \delta] \to A$ ex-*

tending the original one and such that $\varphi(y) = a$ *if and only if* a *satisfies* $a\varphi(r) = \varphi(\sigma(r))a + \varphi(\delta(r))$, *for all* $r \in R$.

Proof. The 'only if' part is trivial, just apply φ to the defining equation (1.1.2). Suppose now that $a\varphi(r) = \varphi(\sigma(r))a + \varphi(\delta(r))$, for every $r \in R$. Define $\bar{\varphi}\colon R[y;\sigma,\delta] \to A$ by mapping $f = \sum_{i=0}^{n} r_i y^i$ to $\bar{\varphi}(f) = \sum_{i=0}^{n} \varphi(r_i)a^i \in A$. It is well defined because in $R[y;\sigma,\delta]$ every element can be written uniquely in the form $\sum_{i=0}^{n} r_i y^i$. Evidently, it extends φ and it is also straightforward to check that it is linear over K, because φ is too. If $f = \sum_{i=0}^{n} r_i y^i$ and $g = \sum_{i=0}^{m} s_i y^i$ are elements in $R[y;\sigma,\delta]$, then $fg = \sum_{i\geq 0} t_i y^i$, where t_i is given by formula (1.1.5) in Lemma 1.1.3. Just as this formula was uniquely determined by the equation $yr = \sigma(r)y + \delta(r)$, the same argument via induction using formula $a\varphi(r) = \varphi(\sigma(r))a + \varphi(\delta(r))$ gives that $\bar{\varphi}(f)\bar{\varphi}(g) = \sum_{i\geq 0} b_i a^i$ where

$$b_i = \sum_{p=0}^{i} \varphi(r_p) \sum_{k=0}^{p} \varphi(S_{p,k}(s_{i-p+k})).$$

Since φ is an algebra homomorphism, it follows that $b_i = \varphi(t_i)$ and hence, we have

$$\bar{\varphi}(f)\bar{\varphi}(g) = \sum_{i\geq 0} \varphi(t_i)a^i = \bar{\varphi}(fg),$$

which proves $\bar{\varphi}$ is an algebra homomorphism. □

1.3 Gelfand-Kirillov dimension of an Ore extension

We finish this chapter by introducing a concept that is an important invariant of an algebra, the Gelfand-Kirillov dimension (or in short GK dimension). Among the many different notions of dimension, this particular one is useful in the classification of Ore extensions and Hopf algebras, which we introduce in the next chapter, as the articles [BOZZ15; GZ10; Zhu13] show.

Definition 1.3.1. Let A be a finitely generated algebra over K, that is, A contains a finite dimensional vector subspace V that generates A as an algebra. The **Gelfand-Kirillov dimension of** A is

$$\operatorname{GKdim} A = \limsup_{n\to\infty} \frac{\log \dim_K(V^n)}{\log n}. \tag{1.3.1}$$

One of the first questions to consider is if this definition depends on the vector subspace V chosen. It does not, as is proved in [KL00, Lemma 1.1.]. The notion is extended to algebras A that are not finitely generated as follows:

$$\operatorname{GKdim} A = \sup\{\operatorname{GKdim} B : B \text{ is a finitely generated subalgebra of } A\}. \tag{1.3.2}$$

Algebras that are finite dimensional (as vector spaces) have GK dimension equal to 0. For integral domains (i.e., a commutative algebra without zero-divisors) which are finitely generated, the GK dimension is equal to the transcendence degree, i.e., the maximal number of algebraically independent elements of the algebra. For instance, the GK dimension of the polynomial algebra in n indeterminates $K[x_1, \cdots, x_n]$ is precisely n. In an Ore extension $R[y; \sigma, \delta]$, we would like to know if adding a new indeterminate y increases the GK dimension by one. The following proposition tells us that the answer is affirmative under a certain condition.

Proposition 1.3.2 (C. Huh and C. Kim)**.** *Let R be a finitely generated algebra and $R[y; \sigma, \delta]$ be an Ore extension of R. Let V be a finite dimensional vector subspace of R that generates R as an algebra and suppose that $\sigma(V) = V$. Then,* $\operatorname{GKdim} R[y; \sigma, \delta] = \operatorname{GKdim} R + 1$.

Proof. See [HK96, Lemma 2.2]. □

There are known examples of $\operatorname{GKdim} R[y; \sigma, \delta] = \operatorname{GKdim} R + 1$ failing to hold when there is no finite dimensional space V invariant under σ. Indeed, the difference $\operatorname{GKdim} R[y; \sigma, \delta] - \operatorname{GKdim} R$ can be any natural number or even be infinite. This is discussed, for instance, in [KL00, §12.3] and further examples are given in [MR01, Chapter 8].

Let R be an algebra and α an algebra endomorphism of R. We say that α is **locally algebraic** if for every $r \in R$, the set $\{\alpha^n(r)\}_{n \geq 1}$ is contained in a finite dimensional vector subspace of R. We have the following corollary of Proposition 1.3.2.

Corollary 1.3.3. *Let R be a finitely generated algebra and $R[y; \sigma, \delta]$ be an Ore extension of R. If σ is locally algebraic, then* $\operatorname{GKdim} R[y; \sigma, \delta] = \operatorname{GKdim} R + 1$.

Proof. Let U be a finite dimensional vector subspace of R that generates R as an algebra and let $u_1, \cdots, u_k$ be a basis of U. Then by hypothesis, $\{\sigma^n(u_i)\}_{n \in \mathbb{N}} \subseteq V_i$ for some finite dimensional subspace V_i of R, for $1 \leq i \leq k$. Hence $V := \sum_{n \in \mathbb{N}} \sigma^n(U) \subseteq \sum_{i=1}^{k} V_i$ and hence V is a finite dimensional vector subspace that generates R as an algebra, since it contains U. By construction. $\sigma(V) \subseteq V$ and therefore Proposition 1.3.2 applies. □

Chapter 2

Hopf algebras

In an associative unital algebra A over a field K, we have a multiplication map $\mu\colon A \otimes A \to A$, which is linear and a unit map, an embedding $\eta\colon K \to A$ that maps λ to $\lambda 1_A$. Together with the K-vector space structure of A, these two maps completely determine the algebra structure on A, which we can denote by (A, μ, η), for this reason. The property of associativity on A and the compatibility between the multiplication and the scalar action in A are expressed by the following commutating diagrams

$$\begin{CD} A \otimes A \otimes A @>{\mu \otimes \mathrm{Id}}>> A \otimes A \\ @V{\mathrm{Id} \otimes \mu}VV @VV{\mu}V \\ A \otimes A @>{\mu}>> A \end{CD} \tag{Assoc}$$

and

$$\begin{array}{ccccc} K \otimes A & \xrightarrow{\eta \otimes \mathrm{Id}} & A \otimes A & \xleftarrow{\mathrm{Id} \otimes \eta} & A \otimes K \\ & \searrow^{\simeq} & \downarrow{\scriptstyle \mu} & {}^{\simeq}\swarrow & \\ & & A & & \end{array} \tag{Un}$$

respectively. The property of distributivity on A of the multiplication over the addition is mirrored in the definition of μ on the tensor product $A \otimes A$, rather than on $A \times A$. The isomorphisms $A \simeq K \otimes A$ and $A \simeq A \otimes K$ in the diagram (Un) are the canonical ones. For instance, in $A \simeq K \otimes A$, $a \in A$ is mapped to $1 \otimes a$ and conversely, $\lambda \otimes a \in K \otimes A$ is mapped to $\eta(\lambda)a$ (which we simply write λa).

2.1 Coalgebras

The dual concept of an algebra arises naturally when we reverse all the arrows in the diagrams (Assoc) and (Un). In this section, we will introduce coalgebras and study some of their properties. Our focus is not on coalgebra theory on its own, but rather as an ingredient to define Hopf algebras later on. For clarity nonetheless, we prove some results in this section that only depend on the coalgebra structure.

Definition 2.1.1. A **coalgebra** is a triple (C, Δ, ε) where C is a K-vector space and $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to K$ are linear maps that make the following diagrams commute:

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{\mathrm{Id}\otimes\Delta} & C \otimes C \\ {\scriptstyle \Delta\otimes\mathrm{Id}}\uparrow & & \uparrow{\scriptstyle \Delta} \\ C \otimes C & \xleftarrow[\Delta]{} & C \end{array} \tag{Coassoc}$$

and

$$\begin{array}{ccccc} K \otimes C & \xleftarrow{\varepsilon\otimes\mathrm{Id}} & C \otimes C & \xrightarrow{\mathrm{Id}\otimes\varepsilon} & C \otimes K \\ & {\scriptstyle\simeq}\nwarrow & \uparrow{\scriptstyle\Delta} & \nearrow{\scriptstyle\simeq} & \\ & & C & & \end{array} \tag{Coun}$$

The map Δ is called a **comultiplication** of C and the map ε is called the **counit** of C. Together, they are called the **structure maps** of the coalgebra C. The commutation of the diagram (Coassoc) is called the **coassociativity axiom**, while the commutation of (Coun) is called the **counit axiom**.

The reason why ε is called *the* counit map is because it is indeed unique. Suppose that ε_1 and ε_2 are two maps satisfying (Coun) and let $c \in C$. Via the canonical algebra isomorphism $K \simeq K \otimes K$, we can identify $\varepsilon_1(c)$ with $\varepsilon_1(c) \otimes 1 = (\varepsilon_1 \otimes \mathrm{Id})(c \otimes 1)$. On the other hand, via the counit axiom for ε_2, we have

$$c \otimes 1 = (\mathrm{Id} \otimes \varepsilon_2)(\Delta(c)).$$

Hence, we can identify via isomorphism, $\varepsilon_1(c)$ with $(\varepsilon_1 \otimes \varepsilon_2)(\Delta(c))$. But, by an analogous argument, we also can identify $\varepsilon_2(c)$ via an algebra isomorphism with the same element $(\varepsilon_1 \otimes \varepsilon_2)(\Delta(c))$. Since the only algebra isomorphism on K is the identity map, it follows that $\varepsilon_1(c) = \varepsilon_2(c)$. This argument might seem a bit unnatural at first specially regarding all the identifications via isomorphisms. However, we will have a chance to rewrite it in a clearer way when we introduce Sweedler's notation later in this section. We can also remark that the comultiplication map in a coalgebra is always injective. If $c \in C$ is such that $\Delta(c) = 0$, then $c \otimes 1 = 0$ by the counit axiom and hence, $c = 0$.

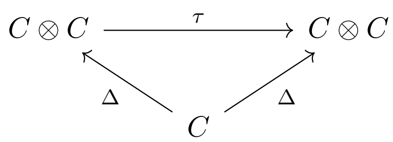
Many basic concepts of algebras such as homomorphisms, ideals and tensor products find their analogues in coalgebra theory. One such concept is that of a homomorphism of coalgebras, i.e., a linear map that respects coalgebra structures. Very much in the spirit of this whole section, it is the dual concept of algebra homomorphism. Given algebras (A, μ_A, η_A) and (B, μ_B, η_B), an (unital) algebra homomorphism $f\colon A \to B$ is a linear map such that $f \circ \mu_A = (f \otimes f) \circ \mu_B$ and $f \circ \eta_A = \eta_B$. With this in mind, it is clear what the definition should be.

Definition 2.1.2. Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras. A **coalgebra homomorphism** is a linear map $\varphi\colon C \to D$ such that

$$\Delta_D \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_C, \quad \varepsilon_D \otimes \varphi = \varepsilon_C. \tag{2.1.1}$$

Not all algebras are commutative, but commutativity is an important property on its own and like associativity, it can be represented by a diagram. It should come as no surprise that there exists a dual concept for coalgebras, which is accordingly called cocommutativity. From the diagram below, writing the original diagram expressing commutativity for algebras may be an interesting challenge for a beginner.

Definition 2.1.3. Let $\tau\colon C \otimes C \to C \otimes C$ be the linear map, called the **flip**, such that $\tau(a \otimes b) = b \otimes a$, for all $a, b \in C$. A coalgebra (C, Δ, μ) is called **cocommutative** if $\tau \circ \Delta = \Delta$, i.e., the following diagram commutes



(Cocom)

Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras. A **coalgebra antihomomorphism** is a linear map $\varphi\colon C \to D$ such that

$$\Delta_D \circ \varphi = (\varphi \otimes \varphi) \circ \tau \circ \Delta_C, \quad \varepsilon_D \otimes \varphi = \varepsilon_C. \tag{2.1.2}$$

If C is cocommutative, then a linear map $\varphi\colon C \to D$ is a coalgebra homomorphism if and only if it is a coalgebra antihomomorphism.

Notation 2.1.4 (**Sweedler's notation** (or Sigma notation))**.** Named after Moss E. Sweedler, who introduced it in his pioneering book [Swe69], it can be very useful to denote the comultiplication of an element c in a coalgebra C by $\Delta(c) = \sum_{(c)} c_1 \otimes c_2$ or simply $\Delta(c) = \sum c_1 \otimes c_2$.

Contrary to multiplication in an algebra, in which two elements combine to yield only one element, the comultiplication in a coalgebra produces a finite sum of pairs of elements (pure tensors) from a single element. As a consequence, computations on coalgebras tend to get very complicated in only a few steps. This generic notation makes it possible to perform computations involving arbitrary elements. In more advanced references and throughout the literature (e.g., in [BOZZ15]), it is common to find an even lighter third notation, dropping the sum sign altogether and writing only $\Delta(c) = c_1 \otimes c_2$.

We will stick to the second notation, i.e., $\Delta(c) = \sum c_1 \otimes c_2$, following [BG02]. Even though carrying the sum sign at all times is not the lightest option, we feel that it gives the right balance between being practical and being welcoming to beginners in the subject. It mostly serves to remind that $\Delta(c)$ is not usually a pure tensor: in general, there are several different tensors $c_1 \otimes c_2$ appearing in the sum! One has to be careful with this notation, since $c_1 \otimes c_2$ does not represent any fixed unique summand in $\Delta(c)$ but instead denotes a generic one, in which c_1 refers to the left tensorand and c_2 to the right tensorand. If, like the author, the reader finds this notation a bit confusing at first, we hope that by the end of this chapter, its usefulness becomes apparent.

As examples, we list some of the properties mentioned so far, using their Sweedler's notation form.

(i) **Coassociativity**: given $c \in C$, we write $(\Delta \otimes \mathrm{Id})(\Delta(c)) = \sum \Delta(c_1) \otimes c_2 = \sum (c_{1_1} \otimes c_{1_2}) \otimes c_2$ and likewise, $(\mathrm{Id} \otimes \Delta)(\Delta(c)) = \sum c_1 \otimes c_{2_1} \otimes c_{2_2}$. The coassociativity axiom means that these two sums are equal and as such, we write them simply as $\sum c_1 \otimes c_2 \otimes c_3$. Expanding on this notation, if we apply $(\mathrm{Id} \otimes \Delta \otimes \mathrm{Id})$ to $\sum c_1 \otimes c_2 \otimes c_3$, for instance, we write the result as $\sum c_1 \otimes c_2 \otimes c_3 \otimes c_4$ and so on.

(ii) **Counit axiom**: given $c \in C$, we write $(\mathrm{Id} \otimes \varepsilon)(\Delta(c)) = \sum c_1 \otimes \varepsilon(c_2) = c \otimes 1$. Since we can identify $C \otimes K$ with C by a linear isomorphism, the equality above can yet be written as

$\sum c_1\varepsilon(c_2) = c$. This form is preferred and we will stick to it henceforth. Likewise, the other counit axiom equality can be written as $\sum \varepsilon(c_1)c_2 = c$.

(iii) **Uniqueness of the counit map**: if ε_1 and ε_2 are two maps $C \to K$ satisfying the counit axiom (Coun), then given $c \in C$, we have

$$\varepsilon_1(c) = \varepsilon_1\left(\sum c_1\varepsilon_2(c_2)\right) = \sum \varepsilon_1(c_1)\varepsilon_2(c_2) = \varepsilon_2\left(\sum \varepsilon_1(c_1)c_2\right) = \varepsilon_2(c) \tag{2.1.3}$$

using in this order, the counit axiom, the linearity of ε_1 and of ε_2 and finally, the counit axiom again. The simplicity of this argument contrasts with the one given previously.

(iv) **Cocommutativity**: the cocommutativity axiom in a cocommutative coalgebra C can be written as $\Delta(c) = \sum c_1 \otimes c_2 = \sum c_2 \otimes c_1 = \tau(\Delta(c))$, for all $c \in C$.

We now present some examples of coalgebras, which we will carry on to next sections where we introduce bialgebras and Hopf algebras (some of the examples will have these additional structures, others will not):

(i) Let S be a set. Call $K[S]$ the vector space over K with S as a basis. Its elements are of the form $\sum_{s\in S} \lambda_s s$, with finitely many $\lambda_s \in K\backslash\{0\}$. Given $s \in S$, define

$$\Delta(s) = s \otimes s, \qquad \varepsilon(s) = 1 \tag{2.1.4}$$

and extend Δ and ε linearly to $K[S]$. Then, it is straightforward to check that $(K[S], \Delta, \varepsilon)$ is a coalgebra, which is cocommutative. Particular cases of this coalgebra that we will study later are the monoid algebra and group algebra, when S is a monoid or a group, respectively.

(ii) In the polynomial algebra $K[x]$, on top of the coalgebra structure defined in the previous item (note that $K[x]$ can be seen as the monoid algebra $K[\mathbb{N}]$), we can define another coalgebra structure. Set

$$\Delta(x) = 1 \otimes x + x \otimes 1, \qquad \varepsilon(x) = 0. \tag{2.1.5}$$

Since x generates $K[x]$ as an algebra, the images $\Delta(x)$ and $\varepsilon(x)$ determine unique algebra homomorphisms $\Delta\colon K[x] \to K[x] \otimes K[x]$ and $\varepsilon\colon K[x] \to K$. For the same reason, it is enough to check the coassociativity and counit axioms for x. We have

$$(\Delta \otimes \mathrm{Id})(\Delta(x)) = 1 \otimes 1 \otimes x + 1 \otimes x \otimes 1 + x \otimes 1 \otimes 1 = (\mathrm{Id} \otimes \Delta)(\Delta(x)),$$
$$(\varepsilon \otimes \mathrm{Id})(\Delta(x)) = \varepsilon(1) \otimes x + \varepsilon(x) \otimes 1 = 1 \otimes x, \tag{2.1.6}$$
$$(\mathrm{Id} \otimes \varepsilon)(\Delta(x)) = 1 \otimes \varepsilon(x) + x \otimes \varepsilon(1) = x \otimes 1.$$

Hence, $K[x]$ is a cocommutative coalgebra.

(iii) The **field** K, being a particular case of a group algebra (of the trivial group). The comultiplication map is the canonical linear isomorphism $K \simeq K \otimes K$ and the counit map is the identity Id_K. Given any coalgebra (C, Δ, ε), we have that $\varepsilon \colon C \to K$ is a coalgebra homomorphism. For this reason, K together with the structure maps mentioned above is called the **ground coalgebra**.

(iv) Let (C, Δ, ε) be a coalgebra structure. Then $(C, \tau \circ \Delta, \varepsilon)$ is also a coalgebra structure called the **co-opposite coalgebra** and it is denoted by C^{cop}. A coalgebra C is cocommutative if and only if $C = C^{cop}$.

We will see some examples of non-cocommutative coalgebras in Chapter 3.

Just like the tensor product of algebras has a natural algebra structure (with pointwise multiplication), so does the tensor product of coalgebras. The comultiplication and counit we can define on the latter are the duals of the multiplication and unit maps defined on the former. The next proposition yields a new class of examples of coalgebras and will be necessary later.

Proposition 2.1.5. *Let* $(C, \Delta_C, \varepsilon_C)$ *and* $(D, \Delta_D, \varepsilon_D)$ *be coalgebras. Denote by* τ *the linear map* $C \otimes D \to D \otimes C$ *such that* $\tau(c \otimes d) = d \otimes c$*, for all* $c \in C$ *and* $d \in D$ *and denote by* μ_K *the multiplication of the field* K*. Then the maps* $\Delta_{C\otimes D} := (\mathrm{Id}_C \otimes \tau \otimes \mathrm{Id}_D) \circ (\Delta_C \otimes \Delta_D)$ *and* $\varepsilon_{C\otimes D} := \mu_K \circ (\varepsilon_C \otimes \varepsilon_D)$ *give the tensor product* $C \otimes D$ *a coalgebra structure. In Sweedler's notation, given* $c \in C$ *and* $d \in D$*, we have*

$$\Delta_{C\otimes D}(c \otimes d) = \sum (c_1 \otimes d_1) \otimes (c_2 \otimes d_2) \in (C \otimes D) \otimes (C \otimes D), \tag{2.1.7}$$
$$\varepsilon_{C\otimes D}(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d) \in K \tag{2.1.8}$$

Proof. The coassociativity and counit axioms for $\Delta_{C\otimes D}$ are straightforward to check, using the respective properties for both C and D. □

We finish this section with the concepts of subcoalgebra and coideal. The latter is the dual concept of an ideal in an algebra.

Definition 2.1.6. Let (C, Δ, ε) be a coalgebra. A **subcoalgebra** D of C is a vector subspace of C such that $\Delta(D) \subseteq D \otimes D$. A **coideal** I of C is a vector subspace of C such that $\Delta(I) \subseteq I \otimes C + C \otimes I$ and $\varepsilon(I) = 0$.

As expected, coideals are the subspaces of C such that the respective quotient space has a coalgebra structure induced by C.

Proposition 2.1.7. *If I is a coideal of a coalgebra (C, Δ, ε), then the quotient space $(C/I, \bar{\Delta}, \bar{\varepsilon})$ is a coalgebra, called **quotient coalgebra**, where the maps $\bar{\Delta}$ and $\bar{\varepsilon}$ are given by*

$$\bar{\Delta}(c + I) = \Delta(c) + (I \otimes C + C \otimes I) \in C/I \otimes C/I, \tag{2.1.9}$$

$$\bar{\varepsilon}(c + I) = \varepsilon(c) \in K, \tag{2.1.10}$$

for all $c \in C$.

Proof. The important part is to prove that the maps $\bar{\Delta}$ and $\bar{\varepsilon}$ are well defined. Then the coassociativity and counit axioms C/I follow at once from the original ones on C. Since $\varepsilon(I) = 0$ by hypothesis, the well-definedness of $\bar{\varepsilon}$ is clear. Let $\rho: C \otimes C \to C/I \otimes C/I$ be the surjective linear map sending $c \otimes d$ to $(c + I) \otimes (d + I)$. We claim that $\operatorname{Ker} \rho = I \otimes C + C \otimes I$.

It is clear that $C \otimes I + I \otimes C$ is a vector subspace of $C \otimes C$ and is contained in $\operatorname{Ker} \rho$. Conversely, let $\{u_i + I\}_{i \in \Lambda}$ be a basis of C/I and let $\{v_{i'}\}_{i' \in \Lambda'}$ be a basis of I. Then it is direct to check that $\{u_i\}_{i \in \Lambda} \cup \{v_{i'}\}_{i' \in \Lambda'}$ is a basis of C. From this, we can construct a basis of $C \otimes C$ consisting of pure tensors of the forms $u_i \otimes u_j$, $u_i \otimes v_{j'}$, $v_{i'} \otimes u_j$ and $v_{i'} \otimes v_{j'}$, with $i, j \in \Lambda$ and $i', j' \in \Lambda'$. Notice that those elements of the forms $u_i \otimes v_{j'}$, $v_{i'} \otimes u_j$ and $v_{i'} \otimes v_{j'}$ are in $I \otimes C + C \otimes I$. Therefore, given $c \in C \otimes C$, we can write

$$c = \sum_{i,j \in \Lambda} \lambda_{ij} u_i \otimes u_j + c'$$

for some $\lambda_{ij} \in K$, finitely many nonzero, and some $c' \in I \otimes C + C \otimes I$. It follows that

$$\rho(c) = \sum_{i,j \in \Lambda} \lambda_{ij} (u_i + I) \otimes (u_j + I).$$

Hence, if $c \in \operatorname{Ker} \rho$, it is clear all the $\lambda_{ij} = 0$ because $\{(u_i + I) \otimes (u_j + I)\}_{i,j \in \Lambda}$ constitutes a basis of $C/I \otimes C/I$. As such, $c = c' \in I \otimes C + C \otimes I$ and this proves that $\operatorname{Ker} \rho = I \otimes C + C \otimes I$. Since ρ is surjective, it yields an isomorphism between $(C \otimes C)/(I \otimes C + C \otimes I)$ and $C/I \otimes C/I$. Consider now the composition map $\rho \circ \Delta\colon C \to C/I \otimes C/I$. By hypothesis, $\Delta(I) \subseteq \operatorname{Ker} \rho$ or equivalently, $I \subseteq \operatorname{Ker} \rho \circ \Delta$. Therefore $\rho \circ \Delta$ factors through C/I to the desired map $\bar{(}\Delta)$. □

2.2 Bialgebras and convolutions

As the name suggests, a bialgebra is simultaneously an algebra and a coalgebra, in such a way that these two structures are compatible. What we mean by compatible, can be understood in one of two ways which turn out to be equivalent, as the next proposition shows.

Proposition 2.2.1. *Let (H, μ, η) and (H, Δ, ε) be algebra and coalgebra structures in the same vector space H. The following conditions are equivalent:*

(i) Δ *and* ε *are algebra homomorphisms;*

(ii) μ *and* η *are coalgebra homomorphisms.*

Proof. This proof follows [Kas95, Theorem III.2.1]. It is convenient to start by recalling the algebra and coalgebra structures in the tensor product $H \otimes H$. The algebra structure is given by the multiplication map $\mu_{H\otimes H} := (\mu \otimes \mu) \circ (\mathrm{Id} \otimes \tau \otimes \mathrm{Id})\colon (H \otimes H) \otimes (H \otimes H) \to H \otimes H$ and by the unit map $\eta_{H\otimes H} := (\eta \otimes \eta) : K \simeq K \otimes K \to H \otimes H$. The coalgebra structure is given by the comultiplication map $\Delta_{H\otimes H} := (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \circ (\Delta \otimes \Delta)\colon H \otimes H \to (H \otimes H) \otimes (H \otimes H)$ and by the counit map $\varepsilon_{H\otimes H} := (\varepsilon \otimes \varepsilon)\colon H \otimes H \to K \otimes K \simeq K$. We also recall that the multiplication map $\mu_K\colon K \otimes K \to K$ is a linear isomorphism and hence, K has algebra structure $(K, \mu_K, \mathrm{Id}_K)$ and coalgebra structure $(K, \mu_K^{-1}, \mathrm{Id}_K)$.

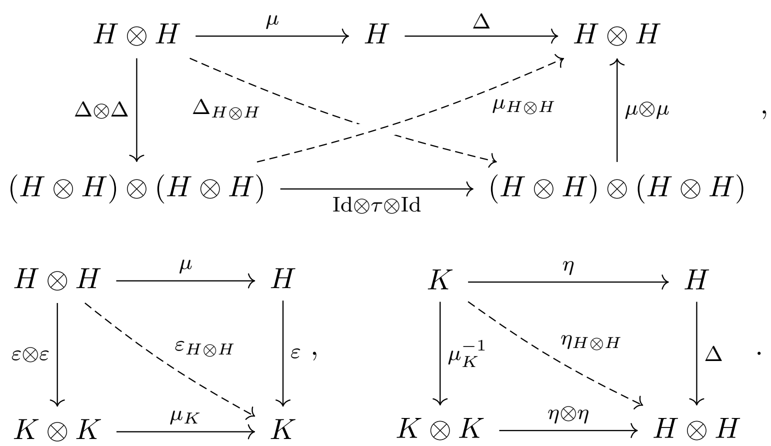
According to Definition 2.1.2, the maps $\mu\colon H \otimes H \to H$ and $\eta\colon K \to H$ are homomorphisms of coalgebras if and only if

$$\begin{cases} (\mu \otimes \mu) \circ \Delta_{H\otimes H} = \Delta \circ \mu & \\ \varepsilon_{H\otimes H} = \varepsilon \circ \mu & \text{(for } \mu\text{)} \\ (\eta \otimes \eta) \circ \mu_K^{-1} = \Delta \circ \eta & \\ \mathrm{Id}_K = \varepsilon \circ \eta & \text{(for } \eta\text{)} \end{cases} . \tag{2.2.1}$$

On the other hand, the maps $\Delta\colon H \to H \otimes H$ and $\varepsilon\colon H \to K$ are homomorphisms of algebras if and only if

$$\begin{cases} \Delta \circ \mu = \mu_{H\otimes H} \circ (\Delta \otimes \Delta) \\ \Delta \circ \eta = \eta_{H\otimes H} \end{cases} \text{(for } \Delta\text{)} \qquad \begin{cases} \mu_K \circ (\varepsilon \otimes \varepsilon) = \varepsilon \circ \mu \\ \varepsilon \circ \eta = \mathrm{Id}_K \end{cases} \text{(for } \varepsilon\text{)} . \tag{2.2.2}$$

The last equality in both (2.2.1) and (2.2.2) is the same. We now write the remaining equalities into the following three diagrams using the definition of $\mu_{H\otimes H}$ and $\Delta_{H\otimes H}$.



It becomes clear that the systems (2.2.1) and (2.2.2) are indeed equivalent. □

Definition 2.2.2. A **bialgebra** is a tuple $(H, \mu, \eta, \Delta, \varepsilon)$ where (H, μ, η) is a algebra, (H, Δ, ε) is a coalgebra and such that one of the equivalent conditions in Proposition 2.2.1 holds. When the structure maps are understood, we just write that H is a bialgebra.

We now take another look at the list of examples that we introduced in the previous section:

(i) If G is a monoid, then $K[G]$, the vector space of K with G as a basis, has a coalgebra structure given by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, for $g \in G$, as we saw previously. It also has a unital algebra structure induced by the group operation and with 1_G as its identity element. Given $g, h \in G$, we have that $\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g)\Delta(h)$ and $\Delta(1_G) = 1_G \otimes 1_G$ and also that $\varepsilon(gh) = 1 = \varepsilon(g)\varepsilon(h)$ and $\varepsilon(1_G) = 1$. Thus Δ and ε are algebra homomorphisms and therefore $K[G]$ is a bialgebra, called the **monoid algebra**. It is always cocommutative and it is commutative if and only if G is abelian. If G is a group, then $K[G]$ is called the **group algebra** of G.

(ii) In the polynomial algebra $K[x]$, we have of course an algebra structure and also a coalgebra structure, as we have seen before. We also argued that the maps Δ and ε determined by $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\varepsilon(x) = 0$ are algebra homomorphisms. Thus $K[x]$ is a bialgebra.

(iii) Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Then we have the following bialgebras: the **opposite bialgebra** $H^{op} = (H, \mu \circ \tau, \eta, \Delta, \varepsilon)$, the **co-opposite bialgebra** $H^{cop} = (H, \mu, \eta, \tau \circ \Delta, \varepsilon)$ and the **opposite co-opposite bialgebra** $H^{opcop} = (H, \mu \circ \tau, \eta, \tau \circ \Delta, \varepsilon)$.

(iv) If H and H' are bialgebras, then $H \otimes H'$ has a bialgebra structure, which combines the structures of the tensor product of algebras and the tensor product of coalgebras (see Proposition 2.1.5). The maps $\Delta_{H\otimes H'} := (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \circ (\Delta_H \otimes \Delta_{H'})$ and $\varepsilon_{H\otimes H'} := \mu_K \circ (\varepsilon_H \otimes \varepsilon_{H'})$ are algebra homomorphisms because they are the composition of algebra homomorphisms.

We now introduce an important concept that is required to define Hopf algebras. Given a coalgebra (C, Δ, ε) and an algebra (A, μ, η), denote by $\mathrm{Hom}_K(C, A)$ the K-vector space of linear maps from C to K. Using Δ and μ, we can define a convolution operation on $\mathrm{Hom}_K(C, A)$.

Definition 2.2.3. Let $f, g \in \mathrm{Hom}_K(C, A)$. The **convolution** of f and g is the linear map $f \star g := \mu \circ (f \otimes g) \circ \Delta \colon C \to A$, i.e., in Sweedler's notation

$$(f \star g)(c) = \sum f(c_1)g(c_2), \tag{2.2.3}$$

for all $c \in C$. The **convolution product** is the map $\star \colon \mathrm{Hom}_K(C, A) \times \mathrm{Hom}_K(C, A) \to \mathrm{Hom}_K(C, A)$, that sends a pair (f, g) to $f \star g$.

One particular case in which we will focus afterwards is when $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra and the convolution product is considered between linear endomorphisms of H. We prove now some properties about the convolution product.

Proposition 2.2.4. *Let (C, Δ, ε) be a coalgebra and (A, μ, η) be an algebra. Then the convolution product $\star$ on $\mathrm{Hom}_K(C, A)$*

(i) is bilinear;

(ii) is associative;

(iii) has a convolution identity element, which is $\eta \circ \varepsilon \in \mathrm{Hom}_K(C, A)$.

These properties mean that $\mathrm{Hom}_K(C, A)$ *with* $\star$ *as multiplication is a unital associative algebra.*

Proof. (i) follows from the fact that μ and the tensor product of maps are bilinear.

(ii) follows from the associative properties of μ and the tensor product of maps, as well as from the coassociativity of Δ. Given $f, g, h \in \mathrm{Hom}_K(C, A)$ and $c \in C$, we write in Sweedler's notation

$$\begin{aligned} ((f \star g) \star h)(c) &= \sum (f \star g)(c_1)h(c_2) \\ &= \sum f(c_1)g(c_2)h(c_3) \\ &= \sum f(c_1)(g \star h)(c_2) \\ &= (f \star (g \star h))(c), \end{aligned}$$

which proves that $(f \star g) \star h = f \star (g \star h)$. Hence, $\star$ is associative on $\mathrm{Hom}_K(C, A)$.

(iii) Let $f \in \mathrm{Hom}_K(C, A)$ and $c \in C$. Then we compute in Sweedler's notation

$$(f \star (\eta \circ \varepsilon))(c) = \sum f(c_1)\eta(\varepsilon(c_2)) = f\left(\sum c_1\varepsilon(c_2)\right) = f(c), \tag{2.2.4}$$

by identifying $\eta \circ \varepsilon$ with ε, because f is linear and finally, by the counit axiom. Likewise, we prove that $(\eta \circ \varepsilon) \star f = f$. □

If $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra, then by Proposition 2.2.4 we have that $\mathrm{End}_K(H)$ has an algebra structure different from the one given by composition of maps. As a matter of fact, Id_H is not the convolution identity, but it turns out to play an important role in defining what a Hopf algebra is.

Definition 2.2.5. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. A linear endomorphism S is called an **antipode** if it is the convolution inverse of Id_H, i.e., $S \star \mathrm{Id}_H = \mathrm{Id}_H \star S = \eta \circ \varepsilon$, or in Sweedler's notation:

$$\sum S(h_1)h_2 = \varepsilon(h)1_H = \sum h_1 S(h_2), \tag{2.2.5}$$

for all $h \in H$. Relation (2.2.5) is called the **antipode property** of H.

An antipode may not always exist in a bialgebra but when it does, it is unique by the uniqueness of inverses in an algebra.

2.3 Hopf algebras

The existence of an antipode in a bialgebra is what turns it into a Hopf algebra.

Definition 2.3.1. A **Hopf algebra** is a bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ which has an antipode S. We denote a Hopf algebra by the tuple $(H, \mu, \eta, \Delta, \varepsilon, S)$, or when the structure maps are understood, simply by H.

Let us recover the examples that we studied in the previous two sections.

(i) In the monoid algebra $K[G]$, the antipode property takes the form

$$gS(g) = S(g)g = \varepsilon(g) = 1,$$

for all $g \in G$, which means that the elements of G are invertible in $K[G]$. If there exists an antipode S, we show that these inverses are also in G. Given $g \in G$, we can write $S(g) = \sum_{h \in G} \lambda_h h$ for some finitely many nonzero $\lambda_h \in G$. Thus we have $1_G = \sum_{h \in G} \lambda_h (hg) = \sum_{h \in G} \lambda_h (gh)$ and since the elements of G form a basis of $K[G]$, this means that there exists $h \in G$ such that $hg = 1_G = gh$. Hence, we conclude that there exists an antipode in $K[G]$ if and only if G is a group, in which case $S(g) = g^{-1}$, for all $g \in G$. In particular, defining the comultiplication map in $K[x] = K[\mathbb{N}]$ by $\Delta(x) = x \otimes x$ and the counit map by $\varepsilon(x) = 1$ *does not* give a Hopf algebra structure to $K[x]$, since x is not invertible in $K[x]$.

(ii) However, still in $K[x]$, the bialgebra structure given by $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\varepsilon(x) = 0$ can be extended to a Hopf algebra structure if we define an antipode by $S(x) = -x$. Since S is an algebra endomorphism of $K[x]$ and x generates $K[x]$, it is enough to check the antipode property for x, i.e, $1S(x) + xS(1) = S(1)x + S(x)1 = 0 = \varepsilon(x)1$.

(iii) If H is a Hopf algebra, then H^{opcop}, as defined in the previous section, is also a Hopf algebras with antipode S. If S is bijective, then H^{op} and H^{cop} are also Hopf algebras, but with antipode S^{-1}.

(iv) If H and H' are Hopf algebras, then $H \otimes H'$ has a Hopf algebra structure with antipode $S_H \otimes S_{H'}$.

We now study some of the properties of the antipode in a Hopf algebra.

Proposition 2.3.2. *Let H be a Hopf algebra with antipode S. Then:*

(i) $S(gh) = S(h)S(g)$*, for any* $g, h \in H$

(ii) $S(1_H) = 1_H$.

(iii) $\Delta(S(h)) = \sum S(h_2) \otimes S(h_1)$*, for any* $h \in H$.

(iv) $\varepsilon(S(h)) = \varepsilon(h)$*, for any* $h \in H$.

Properties (i) and (ii) mean that S is an antihomomorphism of algebras, while properties (iii) and (iv) mean that S is an antihomomorphism of coalgebras.

Proof. We follow [Swe69, Proposition 4.0.1].

(i) Let $F, G\colon H \otimes H \to H$ be the linear maps $F = \mu \circ (S \otimes S) \circ \tau$ and $G = S \circ \mu$. We want to prove precisely that $F = G$. Consider the convolution product $\star$ in $\mathrm{Hom}_K(H \otimes H, H)$ and recall the coalgebra structure on $H \otimes H$, given by the comultiplication $\Delta_{H\otimes H} = (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \circ (\Delta \otimes \Delta)$ and counit $\varepsilon_{H\otimes H} = \mu_K \circ (\varepsilon \otimes \varepsilon)$. For $g, h \in H$, we compute the following two convolutions

$$\begin{aligned}
(\mu \star F)(g \otimes h) &= \sum \mu(g_1 \otimes h_1) F(g_2 \otimes h_2) && \text{(definition of } \star) \\
&= \sum g_1 h_1 S(h_2) S(g_2) \\
&= \sum g_1 \left(\varepsilon(h) 1_H\right) S(g_2) && \text{(antipode property for } h) \\
&= \varepsilon(g)\varepsilon(h) 1_H && \text{(antipode property for } g) \\
&= (\eta \circ \varepsilon_{H\otimes H})(g \otimes h)
\end{aligned}$$

and

$$\begin{aligned}
(G \star \mu)(g \otimes h) &= \sum G(g_1 \otimes h_1) \mu(g_2 \otimes h_2) \\
&= \sum S(g_1 h_1) g_2 h_2 \\
&= \sum S((gh)_1)(gh)_2 && (\Delta \text{ is an algebra hom.}) \\
&= \varepsilon(gh) 1_H && \text{(antipode property for } gh) \\
&= \varepsilon(g)\varepsilon(h) 1_H && (\varepsilon \text{ is an algebra hom.}) \\
&= (\eta \circ \varepsilon_{H\otimes H})(g \otimes h).
\end{aligned}$$

By Proposition 2.2.4, $\eta \circ \varepsilon_{H\otimes H}$ is the convolution identity of the algebra $\mathrm{Hom}_K(H \otimes H, H)$. Hence, we have just proved that μ is convolution invertible with left inverse G and right inverse F. Thus, by the uniqueness of inverses, we conclude that $F = G$ like we wanted.

(ii) Apply the antipode property to $1_H \in H$, for which we know that $\Delta(1_H) = 1_H \otimes 1_H$ and $\varepsilon(1_H) = 1$. We get that $S(1_H)1_H = \varepsilon(1_H) = 1$.

(iii) Let now $F, G\colon H \to H \otimes H$ be the linear maps $F = \Delta \circ S$ and $G = (S \otimes S) \circ \tau \circ \Delta$ and consider the convolution product in $\mathrm{Hom}_K(H, H \otimes H)$. Given $h \in H$, we compute

$$\begin{aligned}(\Delta \star F)(h) &= \sum \Delta(h_1)F(h_2) \\ &= \sum \Delta(h_1)\Delta(S(h_2)) \\ &= \Delta\left(\sum h_1 S(h_2)\right) && (\Delta \text{ is an algebra hom.}) \\ &= \Delta\left(\varepsilon(h)1_H\right) && (\text{antipode property}) \\ &= \varepsilon(h)1_H \otimes 1_H && (\Delta \text{ is linear}) \\ &= (\eta_{H\otimes H} \circ \varepsilon)(h)\end{aligned}$$

and

$$\begin{aligned}(G \star \Delta)(h) = \sum G(h_1)\Delta(h_2) &= \sum (S(h_2) \otimes S(h_1))(h_3 \otimes h_4) \\ &= \sum S(h_2)h_3 \otimes S(h_1)h_4 \\ &= \sum \varepsilon(h_2)1_H \otimes S(h_1)h_3 && (\text{antipode property}) \\ &= \sum 1_H \otimes S(h_1\varepsilon(h_2))h_3 && (S \text{ is linear}) \\ &= \sum 1_H \otimes S(h_1)h_2 && (\text{counit axiom}) \\ &= \varepsilon(h)1_H \otimes 1_H && (\text{antipode property})\end{aligned}$$

which shows that $\Delta \star F = G \star \Delta = \eta_{H\otimes H} \circ \varepsilon$. Since $\eta_{H\otimes H} \circ \varepsilon$ is the convolution identity of $\mathrm{Hom}_K(H, H \otimes H)$, by Proposition 2.2.4, it follows that Δ is convolution invertible with left inverse G and right inverse F. Hence, $F = G$.

(iv) Take the counit property, $\varepsilon(h) = \sum h_1 S(h_2)$, and apply ε. By linearity of both ε and S, we get $\varepsilon(h) = \sum \varepsilon(h_1)\varepsilon(S(h_2)) = \varepsilon\left(S\left(\sum \varepsilon(h_1)h_2\right)\right) = \varepsilon\left(S(h)\right)$ by the counit axiom. □

We introduce some terminology regarding Hopf algebras.

Definition 2.3.3. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a Hopf algebra. A **Hopf subalgebra** H' of H is a subalgebra of H that is also a subcoalgebra and such that $S(H') \subset H'$. Equivalently, H' is subset of H such that $(H', \mu|_{H'}, \eta|_{H'}, \Delta|_{H'}, \varepsilon|_{H'})$ is a Hopf algebra. . The **augmentation ideal** of H is $H^+ := \mathrm{Ker}\,\varepsilon$.

If H is a bialgebra, a **bi-ideal** of H is a subset $I \subseteq H$ that is simultaneously an ideal and a coideal. Furthermore, if H is a Hopf algebra, a **Hopf ideal** is a bi-ideal I such that $S(I) \subseteq I$.

The next result tells us that Hopf ideals behave like we expect them, that is, they induce a quotient Hopf algebra.

Proposition 2.3.4. *Let H be a Hopf algebra and I a Hopf ideal of H. Then the quotient H/I is a Hopf algebra.*

Proof. Since I is an ideal, H/I is an algebra and since I is also a coideal, H/I is a coalgebra by Proposition 2.1.7. The comultiplication and counit maps in H/I are induced by the ones in H and therefore, are also algebra homomorphisms, which makes H/I into a bialgebra with structure maps induced by the corresponding ones in H. It remains to see that the antipode S of H induces an antipode $\bar{S}$ in H/I. Since $S(I) \subset I$, the map $\bar{S}\colon H/I \to H/I$ given by $\bar{S}(h+I) = S(h) + I$ is well-defined. The antipode property on H/I follows at once from its analogue in H. □

The augmentation ideal of a Hopf algebra is an example of a Hopf ideal, as the next proposition shows.

Proposition 2.3.5. *Let H be a Hopf algebra. Then the augmentation ideal H^+ is a Hopf ideal and $H/H^+ \simeq K$.*

Proof. To see that it is a coideal, take $h \in H^+$. Write

$$\Delta(h) = \sum h_1 \otimes h_2 = \sum (h_1 - \varepsilon(h_1)) \otimes h_2 + \sum \varepsilon(h_1) \otimes h_2$$

and note that $h_1 - \varepsilon(h_1) \in H^+$ and $\sum \varepsilon(h_1) \otimes h_2 = 1 \otimes \sum \varepsilon(h_1) h_2 = 1 \otimes h \in H \otimes H^+$ by the counit axiom. Hence, $\Delta(h) \in H^+ \otimes H + H \otimes H^+$. Obviously, by definition, $\varepsilon(H^+) = 0$. From Proposition 2.3.2(iv), it follows at once that $S(H^+) \subseteq H^+$. Hence, H^+ is a Hopf ideal. We have that $\varepsilon\colon H \to K$ is an homomorphism of algebras, which is surjective because $\varepsilon(1_H) = 1$. Then ε factors through $\operatorname{Ker}\varepsilon = H^+$ to an isomorphism $H/H^+ \to K$, by the first isomorphism theorem. □

For instance, in the Hopf algebra $K[x]$ where $\Delta(x) = 1 \otimes x + x \otimes 1$, $\varepsilon(x) = 0$ and $S(x) = -x$, we can prove that $K[x]^+ = \langle x \rangle$. Indeed, by definition, we have that $x \in K[x]^+$ and since $K[x]^+$ is an ideal, then the ideal $\langle x \rangle$ is contained in $K[x]^+$. On the other hand, any element in $K[x]$ is of the form $\lambda + f$ for some $f \in \langle x \rangle$. So $\varepsilon(\lambda + f) = 0$ if and only if $\lambda = 0$ and therefore, $K[x]^+ = \langle x \rangle$. It is clear that $K[x]/\langle x \rangle \simeq K$.

The convolution product $\star$ restricts to unital algebra homomorphisms from H to K, which are also called **characters** of H. Indeed, such a map $\alpha\colon H \to K$ can be identified with the linear endomorphism $\eta \circ \alpha$ of H. Denote by $\mathrm{Alg}_K(H, K)$ the K-vector space of unital algebra homomorphisms from H to K. We have the following corollary of Proposition 2.2.4.

Corollary 2.3.6. *Let H be a bialgebra. The space $\mathrm{Alg}_K(H, K)$ is a monoid under the convolution product $\star$ with ε as the convolution identity element. Furthermore, if H is a Hopf algebra, with antipode S, then $\mathrm{Alg}_K(H, K)$ becomes a group, in which for every $\alpha \in \mathrm{Alg}_K(H, K)$, its convolution inverse is $\alpha \circ S$.*

Proof. The first assertion is simply translating Proposition 2.2.4 to $\mathrm{Alg}_K(H, K)$. Assume now that H is a Hopf algebra and S is the antipode. Given $h \in H$, we compute

$$(\alpha \star (\alpha \circ S))(h) = \sum \alpha(h_1)\alpha(S(h_2)) = \alpha\left(\sum h_1 S(h_2)\right) = \alpha(\varepsilon(h)) = \varepsilon(h), \tag{2.3.1}$$

because α is an algebra homomorphism and because of the antipode property. Analogously, one proves that $(\alpha \circ S) \star \alpha = \varepsilon$. □

We can use these maps in $\mathrm{Alg}_K(H, K)$ to define special algebra automorphisms in H.

Definition 2.3.7. Let H be a Hopf algebra. For $\alpha \in \mathrm{Alg}_K(H, K)$, the **left winding automorphisms** τ^ℓ_α is the algebra endomorphism $\mu \circ (\alpha \otimes \mathrm{Id}) \circ \Delta\colon H \to H$, i.e., in Sweedler's notation:

$$\tau^\ell_\alpha(h) = \sum \alpha(h_1)h_2, \tag{2.3.2}$$

for all $h \in H$. Likewise, the **right winding automorphism** τ^r_α is the map $\mu \circ (\mathrm{Id} \otimes \alpha) \circ \Delta\colon H \to H$, or in Sweedler's notation:

$$\tau^r_\alpha(h) = \sum h_1\alpha(h_2), \tag{2.3.3}$$

for all $h \in H$.

Left and right refer to the component of the comultiplication on which the map α acts on. One thing that is not clear from the definition is that these maps are indeed automorphisms. The next lemma adresses that point and the compositon structure of winding automorphisms.

Lemma 2.3.8. *Let H be a Hopf algebra and let $\alpha, \beta \in \mathrm{Alg}_K(H, K)$.*

(i) $\tau^\ell_\alpha \circ \tau^\ell_\beta = \tau^\ell_{\beta\star\alpha}$.

(ii) τ^ℓ_α *is bijective with* $\left(\tau^\ell_\alpha\right)^{-1} = \tau^\ell_{\alpha\circ S}$.

(iii) $\tau^r_\alpha \circ \tau^r_\beta = \tau^r_{\alpha\star\beta}$.

(iv) τ^r_α *is bijective with* $(\tau^r_\alpha)^{-1} = \tau^r_{\alpha\circ S}$.

In other words, there are an injective group homomorphism $\tau^r\colon (\mathrm{Alg}_K(H, K), \star) \to (\mathrm{Alg}_K(H, H), \circ)$*,* $\alpha \mapsto \tau^r_\alpha$ *and an injective group antihomomorphism* $\tau^\ell\colon (\mathrm{Alg}_K(H, K), \star) \to (\mathrm{Alg}(H, H), \circ)$*,* $\alpha \mapsto \tau^\ell_\alpha$*.*

Proof. (i) Given $h \in H$, we compute

$$\begin{aligned}\tau^\ell_\alpha(\tau^\ell_\beta(h)) = \tau^\ell_\alpha\left(\sum \beta(h_1)h_2\right) &= \sum \beta(h_1)\tau^\ell_\alpha(h_2)\\ &= \sum \beta(h_1)\alpha(h_2)h_3 = \sum(\beta\star\alpha)(h_1)h_2 = \tau^\ell_{\beta\star\alpha}(h).\end{aligned} \tag{2.3.4}$$

(ii) The counit axiom means that $\tau^\ell_\varepsilon = \mathrm{Id}_H$. We have $\alpha \star (\alpha \circ S) = (\alpha \circ S) \star \alpha = \varepsilon$ from Corollary 2.3.6 and therefore the result follows by (i).

(iii) & (iv) These are analogous to (i) and (ii). □

We introduce now a family of examples of Hopf algebras which are related to our results in Chapter 3. In order to do so, we first need to define some concepts from algebraic geometry, that can be found for instance in [BG02, §I.9].

Definition 2.3.9. An **affine algebraic group** over K is a group G which is also an affine variety over K, that is, a subset of K^n for some $n \in \mathbb{N}$, and such that the group multiplication and inverse operator are polynomial maps. The **coordinate algebra** $\mathcal{O}(G)$ of G is the algebra of polynomial functions from G to K.

A trivial example of an affine algebraic group is the field K itself. More substantial examples are matrix groups, such as the group of invertible $n \times n$ matrices $GL(n, K)$ or its subgroup $SL(n, K)$, of $n \times n$ matrices of determinant 1. If $G \subseteq K^n$ is an affine algebraic group, then any element $g \in G$ can be identified with a tuple $(\alpha_1, \cdots, \alpha_n)$ with $\alpha_i \in K$, $1 \le i \le n$. The maps $x_i\colon G \to K$ given by $x_i(\alpha_1, \cdots, \alpha_n) = \alpha_i$, for $1 \le i \le n$ are elements of $\mathcal{O}(G)$ called the **coordinate functions** of G. An arbitrary element in $\mathcal{O}(G)$ is a map $f\colon G \to K$ that is a polynomial in the variables $x_1, \cdots, x_n$.

There is a canonical way to define a Hopf algebra structure on $\mathcal{O}(G)$, which we present in the next proposition. Observe that there is a linear isomorphism $\varphi\colon \mathcal{O}(G) \otimes \mathcal{O}(G) \xrightarrow{\simeq} \mathcal{O}(G \times G)$ (see [Har77, Ex. 3.15, p. 22]). If $f_1 \otimes f_2 \in \mathcal{O}(G) \otimes \mathcal{O}(G)$, then $\varphi(f_1 \otimes f_2)$ is given by $\varphi(f_1 \otimes f_2)(g, h) = f_1(g)f_2(h)$, for all $g, h \in G$. Therefore we can identify $\mathcal{O}(G) \otimes \mathcal{O}(G)$ with $\mathcal{O}(G \times G)$.

Proposition 2.3.10. *Let G be an affine algebraic group. Then the coordinate algebra $\mathcal{O}(G)$ is Hopf algebra with structure maps $\Delta\colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$, $\varepsilon\colon \mathcal{O}(G) \to K$ and $S\colon \mathcal{O}(G) \to \mathcal{O}(G)$ given by*

$$\Delta(f)(g, h) = f(gh), \quad \varepsilon(f)(g) = f(1_G), \quad S(f)(g) = f(g^{-1}), \tag{2.3.5}$$

for all $f \in \mathcal{O}(G)$ and $g, h \in G$.

Proof. We write for $g, h, i \in G$,

$$(\Delta \otimes \mathrm{Id})(\Delta(f))(g, h, i) = \Delta(f)(gh, i) = f((gh)i),$$
$$(\mathrm{Id} \otimes \Delta)(\Delta(f))(g, h, i) = \Delta(f)(g, hi) = f(g(hi)).$$

It is clear that the two equations above agree because of the associativity of the group multiplication.

The counit axiom can be expressed as $(\varepsilon \otimes \mathrm{Id})(\Delta(f))(g) = \Delta(f)(1_G, g) = f(1_G g) = f(g)$ and $(\mathrm{Id} \otimes \varepsilon)(\Delta(f))(g, 1_G) = \Delta(f)(g, 1_G) = f(g1_G) = f(g)$, for all $g \in G$. Finally, the antipode property in $\mathcal{O}(G)$ can be expressed as $(S \otimes \mathrm{Id})(\Delta(f))(g) = \Delta(f)(g^{-1}, g) = f(g^{-1}g) = f(1_G) = \varepsilon(f)(g)$ and likewise, $(\mathrm{Id} \otimes S)(\Delta(f))(g) = \Delta(f)(g^{-1}, g) = f(gg^{-1}) = \varepsilon(f)(g)$, for all $g \in G$. □

The next definition is reminiscent of the first examples of coalgebras that we studied in Section 2.1.

Definition 2.3.11. Let C be a coalgebra. An element $g \in C$ is called **grouplike** if $g \neq 0$ and $\Delta(g) = g \otimes g$. The set of grouplike elements of C is denoted $G(C)$.

Let H be a bialgebra. An element $h \in H$ is called **primitive** if $\Delta(h) = 1 \otimes h + h \otimes 1$. The set of primitive elements of H is denoted $P(H)$.

If $g \in C$ is grouplike, then by the counit axiom we have $\varepsilon(g)g = g$, from where it follows that $\varepsilon(g) = 1$. Likewise, if $h \in H$ is primitive, then by the counit axiom it follows that $\varepsilon(h) = 0$.

The next lemma tells us some properties of the set of grouplikes and the set of primitive elements.

Lemma 2.3.12. *Let H be a bialgebra. The set of grouplike elements $G(H)$ under regular multiplication is a monoid and furthermore, if H is a Hopf algebra, then $G(H)$ is a group. The set of primitive elements, $P(H)$ is a Lie algebra, under the commutator bracket $[h, h'] = hh' - h'h$, for $h, h' \in P(H)$.*

Proof. It is straightforward to check that $G(H)$ is closed under the multiplication of H and that $1_H \in G(H)$, which proves that $G(H)$ is a monoid. If H is a Hopf algebra, then we have $gS(g) = S(g)g = \varepsilon(g) = 1$, for every $g \in G(H)$, by the antipode property. This means that g is invertible in H, but it turns out the its inverse $S(g)$ is also grouplike, because we have $\Delta(S(g)) = (S \otimes S)(\tau(\Delta(g))) = S(g) \otimes S(g)$, by Proposition 2.3.2. Hence if H is a Hopf algebra, then $G(H)$ is a group.

It is also straightforward to see that $P(C)$ is closed under addition and scalar multiplication and hence, it is a vector subspace of C. Finally, given $x, y \in P(C)$ we compute

$$\begin{aligned}\Delta(xy - yx) &= (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) - (1 \otimes y + y \otimes 1)(1 \otimes x + x \otimes 1) \\ &= 1 \otimes (xy - yx) + (xy - yx) \otimes 1, \end{aligned} \tag{2.3.6}$$

which shows that $P(C)$ is closed under the commutator bracket. □

A corollary of this result is that if H is a Hopf algebra, then the group algebra $K[G(H)]$ is always a Hopf subalgebra of H. This follows at once from Lemma 2.3.12 together with the fact that $G(H)$ is linearly independent over K (of which one can find a proof in [HGK10, Proposition 3.6.12], for instance).

Lemma 2.3.12 also allows us to introduce another interesting family of Hopf algebras, which contains in particular the Hopf algebra structure on the polynomial algebra $K[x]$ introduced previously (and polynomial algebras in n variables as well, for any $n \in N$). These are the universal enveloping algebras of Lie algebras, which we introduced in Section 0.1. If L is any Lie algebra of finite dimension, then setting the elements of L to be primitive defines a coalgebra structure in L. We recall that the universal enveloping algebra $U(L)$ of L is the unique associative algebra (up to isomorphism) that contains L as its Lie subalgebra, when we consider the Lie algebra $L(U(L))$ determined by the commutator bracket on $U(L)$, and that satisfies the universal property for enveloping algebras (described in the end of Section 0.1).

Proposition 2.3.13. *Let L be a Lie algebra of finite dimension. The universal enveloping algebra $U(L)$ has a Hopf algebra structure in which*

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \varepsilon(x) = 0, \quad S(x) = -x, \tag{2.3.7}$$

for all $x \in L \subset U(L)$.

Proof. Define the linear maps $\Delta\colon L \to U(L) \otimes U(L)$, $\varepsilon\colon L \to K$ and $S\colon L \to U(L)$ given by (2.3.7). We are going to prove that they extend respectively to a comultiplication map, a counit map and an antipode on $U(L)$. We can rewrite the computation in (2.3.6) as

$$[\Delta(x), \Delta(y)]_{U(L)\otimes U(L)} = 1 \otimes [x,y]_{U(L)} + [x,y]_{U(L)} \otimes 1 = \Delta([x,y]_L)$$

for any $x, y \in L$ and therefore, we have that Δ is actually a Lie algebra homomorphism $L \to L(U(L) \otimes U(L))$. Hence, by the universal property of the universal enveloping algebra, there exists a unique algebra homomorphism $\Delta\colon U(L) \to U(L) \otimes U(L)$ extending the original one. Analogously, we can extend the original map $\varepsilon \equiv 0$ to a unique algebra homomorphism $\varepsilon\colon U(L) \to K$, because any map constantly zero is always trivially a Lie algebra homomorphism.

It is a straightforward computation (the same we already did for the polynomial algebra $K[x]$) to show that the elements of L satisfy the coassociativity axiom using (2.3.7). Hence, we have that the map

$$\varphi_{\text{Coass}} := [(\Delta \otimes \mathrm{Id}) \circ \Delta - (\mathrm{Id} \otimes \Delta) \circ \Delta] : L \to U(L) \otimes U(L) \otimes U(L)$$

is constantly zero and in particular, it is trivially a Lie algebra homomorphism from L to $L(U(L) \otimes U(L) \otimes U(L))$. Therefore, by the universal property, we can extend φ_{Coass} to an algebra homomorphism $\varphi_{\text{Coass}}\colon U(L) \to U(L) \otimes U(L) \otimes U(L)$. Being an algebra homomorphism, its image is completely determined by the images of the elements of L, which are a set of generators of $U(L)$. Hence, since φ_{Coass} is zero for the elements of L, it must be zero for every element in $U(L)$, i.e., $U(L)$ satisfies the coassociativity axiom for Δ. We can analogously prove that it suffices to check the counit axiom for elements of L, which is straightforward to do using (2.3.7).

Now consider the map $S\colon L \to U(L)$ such that $S(x) = -x$, for all $x \in L$. Since $U(L)$ is anti-isomorphic to $U(L)^{\mathrm{op}}$, we can view S as a map $S\colon L \to U(L)^{\mathrm{op}}$ which satisfies, for any $x, y \in L$,

$$S([x,y]_L) = -[x,y]_{U(L)} = yx - xy = x * y - y * x = [x,y]_{U(L)^{\mathrm{op}}},$$

where $*$ is the multiplication on $U(L)^{\mathrm{op}}$. The last term above is of course equal to $[S(x), S(y)]_{U(L)^{\mathrm{op}}}$ which means that $S\colon L \to L\,(U(L)^{\mathrm{op}})$ is actually a Lie algebra homomorphism. By the universal property of the universal enveloping algebra, there exists an algebra homomorphism $S\colon U(L) \to U(L)^{\mathrm{op}}$ extending the original one. Hence, we can view S as an anti-homomorphism $S\colon U(L) \to U(L)$. It is enough to show the antipode property for the elements of L because L generates $U(L)$ as an algebra and the antipode property is preserved by the algebra operations (which we can show using an argument analogous to the one given for the coassociativity). Finally, the computation to show that the antipode property holds for $x \in L$ is the same as the one presented for the polynomial algebra previously. □

Examples of universal enveloping algebras include the polynomial algebra in n variables, which correspond to the abelian Lie algebra of dimension n (for any $n \in \mathbb{N}$). There is a noncommutative universal enveloping algebra coming from the (unique) nonabelian Lie algebra of dimension 2, which turns out to be a differential operator ring, a type of Ore extension, later discussed in Section 4.4. More interesting examples arise in higher dimension, for example the special linear Lie algebra $\mathfrak{sl}(2)$, which we briefly discuss in Section 4.2. With Proposition 2.3.13, we have proved that all of these are cocommutative Hopf algebras (because they are generated by primitive elements, whose comultiplication is symmetric).

We now address an important invariant of a coalgebra, its coradical. A nonzero subcoalgebra of a coalgebra is called simple it does not have any nontrivial proper subcoalgebras.

Definition 2.3.14. Let C be a coalgebra. The **coradical** C_0 of C is the sum of the simple subcoalgebras of C. The coalgebra C is called **connected** if C_0 is trivial, i.e., $C_0 = K$.

Note that a grouplike element $g \in C$ spans a one-dimensional subcoalgebra Kg of C because $\Delta(g) = g \otimes g$. Since it is one-dimensional, it is necessarily a simple subcoalgebra of C. Hence, the set of $G(C)$ is contained in C_0. In particular, if H is a connected bialgebra, then there is only one grouplike element in C, its identity 1_H.

Proposition 2.3.15. *Let C be a coalgebra. Define inductively $C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C)$, for $n \geq 1$. Then $\{C_n\}_{n\in\mathbb{N}}$ is a family of subcoalgebras of C, called* ***the coradical filtration****, that satisfies*

(i) $C = \cup_{n\in\mathbb{N}} C_n$

(ii) $C_n \subseteq C_{n+1}$

(iii) $\Delta(C_n) \subseteq \sum_{i=0}^{n} C_i \otimes C_{n-i}$.

Proof. See [Mon93, Theorem 5.2.2]. □

A family $\{A_i\}_{i\in\mathbb{N}}$ of subcoalgebras of a coalgebra that satisfies (i) to (iii) above is called a **coalgebra filtration**. Thus Proposition 2.3.15 tells that the coradical filtration is a coalgebra filtration. It turns out that the coradical of a coalgebra is the smallest possible initial subset of any coalgebra filtration.

Lemma 2.3.16. *Let C be a coalgebra and $\{A_i\}_{i\in\mathbb{N}}$ be a coalgebra filtration of C. Then $C_0 \subseteq A_0$.*

Proof. See [Mon93, Lemma 5.3.4]. □

Chapter 3

Hopf algebras on Ore extensions

In this chapter, we start combining elements from the previous two chapters. Consider an Ore extension over a Hopf algebra. We want to study the problem of giving it a Hopf algebra structure extending the original one. In general, for which automorphisms and twisted-derivations is it possible?

We will first focus on an article by Panov [Pan03], which was precisely the first article studied in the preparation of the present thesis. Then, we will proceed to an article by Brown, O'Hagan, Zhang and Zhuang, [BOZZ15], that improves Panov's result and hence became the main reference in this particular topic. Throughout the chapter, R will be a Hopf algebra and $T = R[y; \sigma, \delta]$ will be an Ore extension of R. We recall that T is the algebra generated by R and by y subject to the relations

$$yr = \sigma(r)y + \delta(r), \tag{3.0.1}$$

for all $r \in R$, where σ is an automorphism of R and δ is a $\sigma-$derivation. Every element in T can be written uniquely as $\sum_{i \in \mathbb{N}} r_i y^i$, for finitely many nonzero $r_i \in R$.

3.1 Panov's theorem

In [Pan03], Panov proved necessary and sufficient conditions for $T = R[y; \sigma, \delta]$ to be a Hopf algebra, having R as its Hopf subalgebra, under the hypothesis that the comultiplication has a specific form:

$$\Delta(y) = a \otimes y + y \otimes b \tag{3.1.1}$$

for some $a, b \in R$. The hypothesis that $\Delta(y)$ has the form (3.1.1) is somewhat natural, since it can be seen as a generalization of the notion of primitive element (which corresponds to $a = b = 1$). This is in conformity with how the classical polynomial algebra $K[x]$, where x is primitive, is generalized by the concept of Ore extension (also called skew polynomial algebra, in a direct nod to this motivation). Accordingly, if y satisfies (3.1.1) then it can be proved that a and b are grouplike elements and hence, we call y a **skew primitive** element.

When all the conditions above are satisfied, Panov called such an extension T a Hopf Ore extension of R. We will, however, reserve that term for later. In the next section, we give a broader definition of Hopf Ore extension and use it to prove a stronger result, Theorem 3.3.1, which yields Panov's theorem as a corollary. But to give an idea of what comes in the next section and to duly acknowledge Panov's contribution, we present in this section the statement of Panov's theorem without proof.

In a lemma before his theorem, Panov proves that a and b in (3.1.1) are invertible in R and that we can assume without loss of generality that $b = 1$. Since we will also prove these facts in a stronger setting, let us just assume them for now. We could also assume instead that $a = 1$ but not both simultaneously. We recall Sweedler's notation, defined in Chapter 2, in which we write $\Delta(r) = \sum r_1 \otimes r_2$ for the comultiplication of a general element $r \in R$.

Theorem 3.1.1 ([Pan03, Theorem 1.3])**.** *The Ore extension $T = R[y; \sigma, \delta]$ is a Hopf algebra with $\Delta(y) = a \otimes y + y \otimes 1$, having R as its Hopf subalgebra, if and only if*

(i) there is an algebra homomorphism $\chi \colon R \to K$ such that $\sigma(r) = \sum \chi(r_1) r_2$, for any $r \in R$;
(ii) the relations $\sum \chi(r_1) r_2 = \sum \left(a r_1 a^{-1}\right) \chi(r_2)$ hold for all $r \in R$;
(iii) the σ-derivation δ satisfies the relation $\Delta\left(\delta(r)\right) = \sum \delta(r_1) \otimes r_2 + \sum a r_1 \otimes \delta(r_2)$ for any $r \in R$.

Condition (i) above means that σ is a left winding automorphism τ^{ℓ}_{χ} of R, as introduced in Definition 2.3.7) and condition (ii) means that this left winding automorphism is also equal to a right winding automorphism composed with the map of conjugation by a, i.e., $\tau^{\ell}_{\chi} = \mathbf{ad}_a \circ \tau^{r}_{\chi}$. In light of Lemma 2.3.8, this condition can be rewritten as $\tau^{\ell}_{\chi} \circ \tau^{r}_{\chi \circ S} = \mathbf{ad}_a$ which means that conjugation with respect to χ (in the sense of winding automorphisms) is the same as actual conjugation by the element a.

For the sake of completeness, we end this section by mentioning that the counit and the antipode of T are defined, in Theorem 3.1.1, by $\varepsilon(y) = 0$ and $S(y) = -a^{-1}y$, respectively. We also note that condition (iii) can be written as $\Delta \circ \delta = (\delta \otimes 1) \circ \Delta + (a \otimes \delta) \circ \Delta$. We call a map that satisfies it, a **twisted coderivation**, or more precisely, an a-**coderivation**. It is the dual concept of a derivation, see [Dup03].

3.2 Preparation for generalizing Panov's theorem

The main result in this chapter expands Panov's theorem. Like its precursor, it establishes a criterion to assess when we can extend a Hopf algebra structure on R through an Ore extension $T = R[y; \sigma, \delta]$, that is, define on T a Hopf algebra structure compatible with the given structure on R. However, instead of requiring that y is skew primitive, we will require that $\Delta(y)$ satisfies a more general hypothesis. In the next section, we will obtain a complete characterization of these Ore extensions which are also Hopf algebras. Understandably, we will call them Hopf Ore extensions.

Definition 3.2.1. A **Hopf Ore extension** (**HOE**) of R is an algebra T such that:

(i) T is a Hopf algebra with R as its Hopf subalgebra;

(ii) There exist a $y \in T$, an algebra automorphism σ of R and a σ-derivation δ such that $T = R[y; \sigma, \delta]$;

(iii) There are $a, b \in R$ and $v, w \in R \otimes R$ such that

$$\Delta(y) = a \otimes y + y \otimes b + v(y \otimes y) + w. \tag{3.2.1}$$

The condition given by (3.2.1) can be seen as imposing that y is not too "far" from being skew primitive. We observe that the "Hopf Ore extensions" originally defined by Panov are the Hopf Ore extensions (as defined above) in which $v = w = 0$. It is natural to ask just how restrictive this condition (3.2.1) is. We will see later in Section 3.5 that we can almost dispense it (but not quite), if we assume that $R \otimes R$ is a domain. Furthermore, we will also see that if we assume that R is a connected Hopf algebra, then (3.2.1) is automatically true.

Before we start to study Hopf Ore extensions, we establish two observations that will be useful in the sequel.

Lemma 3.2.2. *If $c \in R$ is such that $\Delta(c) = c_1 \otimes 1$ or $\Delta(c) = 1 \otimes c_2$ for some $c_1, c_2 \in R$, then $c \in K$. Moreover, if $a \in R \otimes R$ and there are $a_1, a_2 \in R$ such that $a = a_1 \otimes 1 = 1 \otimes a_2$, then $a \in K$, in the sense that $a = \lambda 1 \otimes 1$, for some $\lambda \in K$.*

Proof. If $\Delta(c) = c_1 \otimes 1$, apply $\varepsilon \otimes \mathrm{Id}$. By the counit axiom, we have $c = \varepsilon(c_1) \in K$. For the other case, apply $\mathrm{Id} \otimes \varepsilon$. As for the second part, we have $a_1 \otimes 1 - 1 \otimes a_2 = 0$. If $a_1 \in K$, then we are done. Otherwise, 1 and a_1 form a linearly independent set and by Lemma 0.1.3, we conclude that $1 = 0$, which is absurd. □

We proceed by studying what conditions does a Hopf Ore extension necessarily satisfy. With this in mind, we look to the elements $a, b \in R$, $v, w \in R \otimes R$ in (3.2.1). We recall that $G(R)$ is the set of grouplike elements of R.

Lemma 3.2.3. *Let $T = R[y; \sigma, \delta]$ be a HOE.*

(i) If $v = 0$ or $w = 0$, then $a, b \in G(R)$.
(ii) If $R \otimes R$ is a domain, then $v \in K$.
(iii) The relation $a \otimes w + (\mathrm{Id} \otimes \Delta)(w) = w \otimes b + (\Delta \otimes \mathrm{Id})(w)$ holds.

Proof. This result is Lemma 2 in §2.2 of [BOZZ15] and we follow its proof. The idea is to apply the axioms of coassociativity and counit to (3.2.1).

(i) We start by writing $(\Delta \otimes \mathrm{Id})\Delta(y)$ and $(\mathrm{Id} \otimes \Delta)\Delta(y)$:

$$\begin{aligned}(\Delta \otimes \mathrm{Id})\Delta(y) &= (\Delta \otimes \mathrm{Id})(a \otimes y + y \otimes b + v(y \otimes y) + w) \\ &= \Delta(a) \otimes y + (a \otimes y + y \otimes b + v(y \otimes y) + w) \otimes b \\ &\quad + (\Delta \otimes \mathrm{Id})(v)[(a \otimes y + y \otimes b + v(y \otimes y) + w) \otimes y] + (\Delta \otimes \mathrm{Id})(w).\end{aligned} \tag{3.2.2}$$

and

$$\begin{aligned}(\mathrm{Id} \otimes \Delta)\Delta(y) &= (\mathrm{Id} \otimes \Delta)(a \otimes y + y \otimes b + v(y \otimes y) + w) \\ &= a \otimes (a \otimes y + y \otimes b + v(y \otimes y) + w) + y \otimes \Delta(b) \\ &\quad + (\mathrm{Id} \otimes \Delta)(v)[y \otimes (a \otimes y + y \otimes b + v(y \otimes y) + w)] + (\mathrm{Id} \otimes \Delta)(w).\end{aligned} \tag{3.2.3}$$

Regarding $T \otimes T \otimes T$ as a free module over $R \otimes R \otimes R$, we can compare the coefficients of $1 \otimes 1 \otimes y$ in (3.2.2) and (3.2.3):

$$\Delta(a) \otimes 1 + ((\Delta \otimes \mathrm{Id})v)(w \otimes 1) = a \otimes a \otimes 1. \tag{3.2.4}$$

Likewise, we can compare the coefficients of $y \otimes 1 \otimes 1$, yielding

$$1 \otimes b \otimes b = 1 \otimes \Delta(b) + ((\mathrm{Id} \otimes \Delta)v)(1 \otimes w). \tag{3.2.5}$$

If either $v = 0$ or $w = 0$, then (3.2.4) and (3.2.5) imply that $\Delta(a) = a \otimes a$ and $\Delta(b) = b \otimes b$, respectively. It remains to prove that both a and b are nonzero. Write v and w as sums of tensors $v = \sum v_1 \otimes v_2$ and $w = \sum w_1 \otimes w_2$, in a notation akin to Sweedler's notation (but v and w are not necessarily comultiplications of elements!). By the counit axiom applied to y,

$$y = \varepsilon(a)y + \varepsilon(y)b + \sum \varepsilon(v_1)\varepsilon(y)v_2y + \sum \varepsilon(w_1)w_2 \tag{3.2.6}$$

and

$$y = a\varepsilon(y) + y\varepsilon(b) + \sum v_1y\varepsilon(y)\varepsilon(v_2) + \sum w_1\varepsilon(w_2). \tag{3.2.7}$$

If $v = 0$, then we have $\varepsilon(a) = \varepsilon(b) = 1$ by comparing the coefficients of y and we are done. Otherwise, if $w = 0$, then we have $\varepsilon(y)a = b\varepsilon(y) = 0$. If $\varepsilon(y) \neq 0$, we must have $a = b = 0$. In this case, $\Delta(y) = v(y \otimes y)$. By the antipode property, $\varepsilon(y) = \sum S(v_1y)v_2y$, which means that y is left invertible which is a contradiction, by Lemma 1.2.2. Therefore, $\varepsilon(y) = 0$ and after comparing the coefficients of y on both sides from (3.2.6) and from (3.2.7), we finally get that $\varepsilon(a) = \varepsilon(b) = 1$.

(ii) We divide this proof into two cases: $w \neq 0$ and $w = 0$. Write explicitly $v = \sum_i v_{1i} \otimes v_{2i}$ and assume without loss of generality that $\{v_{1i}\}_i$ and $\{v_{2i}\}_i$ are linearly independent sets. Just require that the number of summands $v_{1i} \otimes v_{2i}$ is minimal, any linear dependence relation contradicts this minimality. In the case $w \neq 0$, we write (3.2.4) as $(\Delta(a) - a \otimes a) \otimes 1 + \sum_i \Delta(v_{1i})w \otimes v_{2i} = 0$. For a fixed i, if v_{2i} which is not in K (i.e, it is linearly independent of 1), then we have by Lemma 0.1.3 that $\Delta(v_{1i})w = 0$. Because $R \otimes R$ is a domain and $w \neq 0$, it follows that $\Delta(v_{1i}) = 0$ and since Δ is injective, $v_{1i} = 0$ for every i such that $v_{2i} \notin K$. Hence, $v = v_1 \otimes 1$. Analogously, comparing the leftmost tensorands in (3.2.5) yields $v = 1 \otimes v_2$. The result then follows from Lemma 3.2.2.

In the case $w = 0$, comparing the coefficients of $1 \otimes y \otimes y$ in (3.2.2) and (3.2.3) gives the following equality in $R \otimes R \otimes R$:

$$(\Delta \otimes \mathrm{Id})\,(v)(a \otimes 1 \otimes 1) = a \otimes v. \tag{3.2.8}$$

This is equivalent to $\sum_i [\Delta(v_{1i})(a \otimes 1) - a \otimes v_{1i}] \otimes v_{2i} = 0$. By the linear independence of $\{v_{2i}\}_i$, we get $\Delta(v_{1i})(a \otimes 1) = a \otimes v_{1i}$. As $R \otimes R$ is a domain and $a \neq 0$ by (i), we can cancel $a \otimes 1$ from the right side, which gives $\Delta(v_{1i}) = 1 \otimes v_{1i}$. By Lemma 3.2.2, we conclude that $v_1 \in K$. Likewise, comparing the coefficients of $y \otimes y \otimes 1$ in (3.2.2) and (3.2.3) gives

$$v \otimes b = (\mathrm{Id} \otimes \Delta)\,(v)(1 \otimes 1 \otimes b) \tag{3.2.9}$$

and analogously, we conclude that $v_2 \in K$ and thus, $v \in K$.

(iii) It follows directly from comparing the coefficients of $1 \otimes 1 \otimes 1$ in (3.2.2) and (3.2.3). □

Lemma 3.2.3 corrects some minor details of [BOZZ15, Section 2.2, Lemma 2]. In point (ii), the hypothesis that $R \otimes R$ is a domain is needed and in point (iii), the condition $v = 0$ was unnecessary.

Regarding now the properties of the antipode of the element y, we have the next lemma:

Lemma 3.2.4. *Let $T = R[y; \sigma, \delta]$ be a HOE. Write $w = \sum w_1 \otimes w_2$ as before.*

(i) If S is bijective and R is a domain, then $S(y) = \alpha y + \beta$ for $\alpha \in R^\times$ and $\beta \in R$.
Assume that $S(y)$ has the form in (i) and $v \in K$. Then:

(ii) $v = 0$.

(iii) $a, b \in G(R)$. In particular, they are invertible.

(iv) $\alpha = -a^{-1}\sigma(b^{-1})$.

(v) $\beta = a^{-1}(\varepsilon(y) - \delta(b^{-1}) - \sum w_1 S(w_2))$.

Proof. (i) The antipode S is an anti-homomorphism of T by Proposition 2.3.2. Being bijective, S is an anti-automorphism and since R is a Hopf subalgebra of T, S restricts to $S|_R\colon R \to R$. Hence, it falls into the conditions of Proposition 1.2.5 and the result follows.

(ii) Applying the antipode property to y, together with (i) and $v \in K$, yields

$$\begin{aligned} \varepsilon(y) &= aS(y) + yS(b) + vyS(y) + \sum w_1 S(w_2) \\ &= a\alpha y + a\beta + yS(b) + vy\alpha y + vy\beta + \sum w_1 S(w_2). \end{aligned} \tag{3.2.10}$$

The only term above with y^2 is $v\sigma(\alpha)y^2$ which comes from $vy\alpha y$. Since the powers of y form a basis of T over R, we must have $v\sigma(\alpha) = 0$. But α is invertible and σ is an automorphism, therefore, $v = 0$.

(iii) It follows immediately by (ii) and by Lemma 3.2.3(i). In a Hopf algebra, grouplike elements are invertible by the antipode property.

(iv) & (v) Given that $v = 0$ and that a, b are grouplike, we can rewrite (3.2.10) as

$$\begin{aligned} \varepsilon(y) &= a\alpha y + a\beta + yb^{-1} + \sum w_1 S(w_2) \\ &= \left(a\alpha + \sigma(b^{-1})\right) y + a\beta + \delta(b^{-1}) + \sum w_1 S(w_2). \end{aligned}$$

By comparing the coefficients of y and of the independent term on both sides, we get exactly (iv) and (v). □

We will in general want to assume that the antipode S is bijective. This assumption is reasonable because it is automatic if R is a noetherian domain. In this case, then so is T by Propositions 1.2.1 and 1.2.4. In [Skr06], it is proved that noetherian Hopf algebra domains have a bijective antipode and some counterexamples are mentioned, namely a construction by Takeuchi of a Hopf algebra with non bijective antipode. The examples we will see will almost always be noetherian domains.

We have explored so far the "Hopf algebra" side of a Hopf Ore extension and have not yet dwelt much upon its properties as an Ore extension. In general, an Ore extension $T = R[y; \sigma, \delta]$ is far from being uniquely determined by the automorphism σ, the $\sigma-$derivation δ or even the element y that generates it with R. In the next lemma, we explore this to our advantage, seeking to simplify the Hopf algebra structure on $T = R[y; \sigma, \delta]$.

Lemma 3.2.5. *Let $T = R[y; \sigma, \delta]$ be an Ore extension.*

(i) Given $\lambda \in K$, we have that $\delta_\lambda = \delta + \lambda(\mathrm{Id} - \sigma)$ is a σ-derivation and $T = R[y + \lambda; \sigma, \delta_\lambda]$. Hence, we can replace y by $y + \lambda$ without affecting T as an Ore extension.

(ii) Given $s \in R^{\times}$, denote by ad_s the map of conjugation by s, i.e., $\mathrm{ad}_s(r) = srs^{-1}$, for all $r \in R$. Then, $T = R[sy; \mathrm{ad}_s \circ \sigma, s\delta]$. Hence, we can replace y by sy without affecting T as an Ore extension.

Proof. (i) Take $u, v \in R$. Then, using the properties of σ and δ, we have

$$\begin{aligned} \delta_\lambda(uv) &= \delta(uv) + \lambda uv - \lambda\sigma(u)\sigma(v) \\ &= \delta(u)v + \sigma(u)\delta(v) + \lambda uv - \lambda\sigma(u)\sigma(v) \\ &= (\delta(u) + \lambda u)\, v + \sigma(u)\,(\delta(v) - \lambda\sigma(v)) \,. \end{aligned} \tag{3.2.11}$$

Adding and subtracting $\lambda\sigma(u)v$ yields $\delta_\lambda(uv) = \delta_\lambda(u)v + \sigma(u)\delta_\lambda(v)$. Hence, δ_λ is a σ-derivation. Furthermore, we have

$$(y + \lambda)r = yr + \lambda r = \sigma(r)y + \delta(r) + \lambda r = \sigma(r)(y + \lambda) + \delta_\lambda(r), \tag{3.2.12}$$

for all $r \in R$ and with this, we prove that $R[y + \lambda; \sigma, \delta_\lambda]$ is an Ore extension. Since T can be generated by R and $y + \lambda$, it follows that $T = R[y + \lambda; \sigma, \delta_\lambda]$. We emphasize that we mean equality and not only isomorphism.

(ii) The map ad_s is an automorphism (with inverse $\mathrm{ad}_{s^{-1}}$). Thus, $\mathrm{ad}_s \circ \sigma$ is an automorphism. Given $u, v \in R$, we check that

$$s\delta(uv) = s\delta(u)v + s\sigma(u)\delta(v) = s\delta(u)v + \mathrm{ad}_s(\sigma(u))s\delta(v) \tag{3.2.13}$$

and hence, $s\delta$ is a $(\mathrm{ad}_s \circ \sigma)$-derivation. We also check that

$$syr = s[\sigma(r)y + \delta(r)] = \mathrm{ad}_s(\sigma(r))sy + s\delta(r), \tag{3.2.14}$$

for any $r \in R$, from where it follows that $R[sy; \mathrm{ad}_s \circ \sigma, s\delta]$ is an Ore extension. Since s is invertible, we have that T is generated by R and sy and thus, $T = R[sy; \mathrm{ad}_s \circ \sigma, s\delta]$. □

3.3 Generalization of Panov's theorem

We recall that the augmentation ideal Ker_ε of T (resp. R) is denoted by T^+ (resp. R^+). The small typo corrected in Lemma 3.2.3 also affects the hypotheses of [BOZZ15, Theorem §2.4], which accounts for our additional assumption that $v \in K$ in the following result.

Theorem 3.3.1 ([BOZZ15]). *Let R be a Hopf algebra.*

(i) Let $T = R[y;\sigma,\delta]$ be a Hopf Ore extension of R, in which $\Delta(y) = a\otimes y + y\otimes b + v(y\otimes y) + w$ for $a, b \in R$ and $v, w \in R\otimes R$. Suppose that $v \in K$ and $S(y) = \alpha y + \beta$, for $\alpha, \beta \in R$ with $\alpha \in R^{\times}$. Write $w = \sum w_1 \otimes w_2 \in R\otimes R$. Then, we have the following properties:

(a) a and b are grouplike (in particular, invertible) and $v = 0$.

(b) After suitable changes of variable and the corresponding changes in σ, δ, a and w, we can assume that

$$\Delta(y) = a\otimes y + y\otimes 1 + w, \tag{3.3.1}$$

and that $\varepsilon(y) = 0$.

(c) $S(y) = -a^{-1}\left(y + \sum w_1 S(w_2)\right)$.

(d) There exists an algebra homomorphism $\chi\colon R \to K$ such that

$$\sigma(r) = \sum \chi(r_1) r_2 = \sum a r_1 \chi(r_2) a^{-1}, \tag{3.3.2}$$

for all $r \in R$. Hence, σ is both a left winding automorphism τ^{ℓ}_{χ} and a right winding automorphism τ^{r}_{χ} composed with conjugation by a, ad_a.

(e) The $\sigma-$derivation δ satisfies

$$\Delta(\delta(r)) - \sum \delta(r_1)\otimes r_2 - \sum a r_1 \otimes \delta(r_2) = w\Delta(r) - \Delta(\sigma(r))w, \tag{3.3.3}$$

for all $r \in R$.

(f) We have $w \in R^+ \otimes R^+$ and w satisfies

$$\sum S(w_1) w_2 = a^{-1} \sum w_1 S(w_2), \tag{3.3.4}$$

and

$$w\otimes 1 + (\Delta\otimes \mathrm{Id})(w) = a\otimes w + (\mathrm{Id}\otimes\Delta)(w). \tag{3.3.5}$$

(ii) Conversely, suppose that $a \in G(R)$, $w \in R\otimes R$, an algebra automorphism σ of R and a σ-derivation δ are given, satisfying conditions (d), (e) and (f) above. Then, the Ore extension $T = R[y;\sigma,\delta]$ admits a Hopf algebra structure having R as a Hopf subalgebra and with the comultiplication, counit and antipode of R being extended to T as in (b) and (c). As a consequence, T is a Hopf-Ore extension of R.

While condition (c) above may seem to privilege a specific way of writing $S(y)$, this is offset by (3.3.4) in (f). Combining the two, we can write alternatively $S(y) = -a^{-1}y - \sum S(w_1)w_2$. Likewise, in condition (d), we also see the symmetry given by the relation $\tau_\chi^\ell = \mathrm{ad}_a \circ \tau_\chi^r$. Of course, if in condition (b), we had set $a = 1$ instead of b, these statements would all be symmetric.

Comparing Panov's theorem with Theorem 3.3.1, we see that condition (d) in the latter is the equivalent to conditions (i) and (ii) in the former, condition (e) is equivalent to condition (iii) and (e) is void (because $w = 0$ in Panov's case).

Proof of Theorem 3.3.1. We follow the proof in [BOZZ15, Theorem in §2.4].

(i) (a) It is just (ii) and (iii) of Lemma 3.2.4.

(b) By Lemma 3.2.5, we can replace y with $y - \varepsilon(y)$ and we observe that $y - \varepsilon(y) \in T^+$. With the corresponding change in w, $\Delta(y - \varepsilon(y))$ still satisfies (3.2.1). Since b is grouplike (and thus so is b^{-1}) and $v = 0$, we can write

$$\Delta(b^{-1}y) = (b^{-1} \otimes b^{-1})(a \otimes y + y \otimes b + w) = b^{-1}a \otimes b^{-1}y + b^{-1}y \otimes 1 + (b^{-1} \otimes b^{-1})w. \quad (3.3.6)$$

Replacing y by $b^{-1}y$, a by $b^{-1}a$ and w by $(b^{-1} \otimes b^{-1})w$, the result follows.

(c) With the changes in the previous point, we now have $\varepsilon(y) = 0$ and $\delta(b^{-1}) = 0$, because $b = 1$ and δ is a derivation. Thus, the result follows directly from (iv) and (v) of Lemma 3.2.4.

(d) & (e) Start by applying Δ to the defining relations (3.0.1). Since Δ is an algebra homomorphism, the defining relations in the Ore extension $R[y; \sigma, \delta]$ must be preserved, that is

$$\Delta(y)\Delta(r) = \Delta(\sigma(r))\Delta(y) + \Delta(\delta(r)), \quad (3.3.7)$$

for all $r \in R$. We compute

$$\begin{aligned}
\Delta(y)\Delta(r) &= (a \otimes y + y \otimes 1 + w)\left(\sum r_1 \otimes r_2\right) \\
&= \sum ar_1 \otimes [\sigma(r_2)y + \delta(r_2)] + \sum [\sigma(r_1)y + \delta(r_1)] \otimes r_2 + w\left(\sum r_1 \otimes r_2\right) \\
&= \left(\sum ar_1a^{-1} \otimes \sigma(r_2)\right)(a \otimes y) + \left(\sum \sigma(r_1) \otimes r_2\right)(y \otimes 1) \\
&\quad + \sum\left(ar_1 \otimes \delta(r_2) + \delta(r_1) \otimes r_2\right) + w\left(\sum r_1 \otimes r_2\right)
\end{aligned}$$

and

$$\Delta(\sigma(r))\Delta(y) + \Delta(\delta(r)) = \Delta(\sigma(r))(a \otimes y) + \Delta(\sigma(r))(y \otimes 1) + \Delta(\sigma(r))w + \Delta(\delta(r)).$$

By comparing coefficients above we get the following identities in $R \otimes R$.

$$\begin{cases} \Delta(\sigma(r)) = \sum \mathrm{ad}_a(r_1) \otimes \sigma(r_2), & \text{(3.3.8a)} \\ \Delta(\sigma(r)) = \sum \sigma(r_1) \otimes r_2, & \text{(3.3.8b)} \\ \Delta(\sigma(r))w + \Delta(\delta(r)) = \sum \Big[ar_1 \otimes \delta(r_2) + \delta(r_1) \otimes r_2\Big] + w\Delta(r). & \text{(3.3.8c)} \end{cases}$$

It is clear that (3.3.7) holds if and only if equations (3.3.8a) to (3.3.8c) do. Define $\chi\colon R \to R$ as $\chi = \mu \circ (\sigma \otimes S) \circ \Delta$, where $\mu\colon R \otimes R \to R$ is the multiplication map. That is, χ maps an element $r \in R$ to $\chi(r) = \sum \sigma(r_1)S(r_2)$. Applying the antipode property to $\sigma(r)$ and using (3.3.8b) yields

$$\varepsilon(\sigma(r)) = \mu(1 \otimes S)\Delta(\sigma(r)) = \mu(1 \otimes S)\left(\sum \sigma(r_1) \otimes r_2\right) = \sum \sigma(r_1)S(r_2) = \chi(r)$$

which shows that χ is actually a map $R \to K$ and it also shows that χ is an algebra homomorphism, being the composition of two such maps. If we now apply the counit axiom to $\sigma(r)$, together with (3.3.8b), we get

$$\sigma(r) = \mu(\varepsilon \otimes 1)\Delta(\sigma(r)) = \mu(\varepsilon \otimes 1)\left(\sum \sigma(r_1) \otimes r_2\right) = \sum \varepsilon(\sigma(r_1))r_2 = \sum \chi(r_1)r_2$$

which shows that $\sigma = \tau^{\ell}_{\chi}$. The same argument using (3.3.8a) proves that $\sigma(r) = \sum \mathrm{ad}_a(r_1)\chi(r_2)$. Since $\chi(r_2) \in K$ and ad_a is linear, we can write $\sigma = \mathrm{ad}_a \circ \tau^{r}_{\chi}$. Finally, observe that (3.3.3) is simply (3.3.8c) rearranged.

(f) Write explicitly $w = \sum_i w_{1i} \otimes w_{2i}$. We can assume without loss of generality that $\{w_{1i}\}_i$ and $\{w_{2i}\}_i$ are linearly independent sets. Just assume that the number of summands $w_{1i} \otimes w_{2i}$ is minimal, any linear dependence relation contradicts the minimality. Applying the counit axiom together with $\Delta(y) = a \otimes y + y \otimes 1 + w$ yields

$$y + \sum_i w_{1i}\varepsilon(w_{2i}) = y = y + \sum_i \varepsilon(w_{1i})w_{2i}$$

because $\varepsilon(y) = 0$ and $\varepsilon(a) = 1$. This implies that $\varepsilon(w_{1i}) = \varepsilon(w_{2i}) = 0$ for all i because of the linear independence. Thus, $w \in R^+ \otimes R^+$. Applying now the antipode property to $\Delta(y) = a \otimes y + y \otimes 1 + w$

yields

$$0 = \varepsilon(y) = S(a)y + S(y) + \sum S(w_1)w_2 = -a^{-1}\sum w_1 S(w_2) + \sum S(w_1)w_2$$

and (3.3.4) follows. Finally, (3.3.5) is (iii) of Lemma 3.2.3.

(ii) To extend the maps Δ, ε and S from R to $T = R[y;\sigma,\delta]$, we need to check that the extended maps preserve the defining relations (3.0.1), with the images $\Delta(y)$, $\varepsilon(y)$ and $S(y)$ given by conditions (b) and (c). Since R and y generate T, the image of an element $\sum r_i y^i \in T$ is entirely determined by the images of r_i and y and by the maps being homomorphisms of algebras (or in the case of S, an anti-homomorphism). After proving the well-definedness of these maps, we need to check that they define a Hopf algebra structure on T. The fact that R is a Hopf subalgebra of T is then automatic, because the restrictions to R of these extended maps are obviously the original ones. We break down this proof into several claims.

Claim 1. *The extension of Δ from R to T given by $\Delta(y) = a\otimes y + y\otimes 1 + w$ is well defined.* As observed above, Δ preserves the defining relations if and only if equations (3.3.8a) to (3.3.8c) hold.

If we compose Δ with $\sigma = \tau^{\ell}_{\chi}$, we get

$$\Delta(\sigma(r)) = \sum \Delta(\chi(r_1)r_2) = \sum \chi(r_1)\Delta(r_2) = \sum (\chi(r_1)r_2)\otimes r_3 = \sum \sigma(r_1)\otimes r_2$$

for any $r\in R$ because $\chi(r_1)\in K$. This yields (3.3.8b). We have implicitly used the coassociativity axiom, in its Sweedler notation form, on the third equality: $\sum \chi(r_1)(r_2\otimes r_3) = \sum(\chi(r_1)r_2)\otimes r_3$. We will use this argument throughout this section, avoiding writing explicitly one step for coassociativity, as that would make the computations too cumbersome. Analogously, we prove (3.3.8a) using instead $\sigma = \mathrm{ad}_a\circ\tau^{r}_{\chi}$ and noting that $\Delta\circ\mathrm{ad}_a = (\mathrm{ad}_a\otimes\mathrm{ad}_a)\circ\Delta$ since a is grouplike. Finally, condition (3.3.8c) is simply (3.3.3) in (e) rewritten.

Claim 2. *The extension of ε from R to T given by $\varepsilon(y) = 0$ is well defined.* Applying ε to (3.0.1) yields $\varepsilon(y)\varepsilon(r) = \varepsilon(\sigma(r))\varepsilon(y) + \varepsilon(\delta(r))$. Since $\varepsilon(y) = 0$, this relation holds if and only if $\varepsilon(\delta(r)) = 0$, for all $r\in R$. We can prove it by applying the counit axiom with $(\mathrm{Id}\otimes\varepsilon)$ to (3.3.3) in (e). Since $w\in R^+\otimes R^+$, we get

$$\delta(r) - \sum\delta(r_1)\varepsilon(r_2) - \sum a r_1\varepsilon\delta(r_2) = 0.$$

Because δ is linear, we have $\sum\delta(r_1)\varepsilon(r_2) = \delta\left(\sum r_1\varepsilon(r_2)\right) = \delta(r)$, again by the counit axiom.

Hence, the expression above is equivalent to $\sum r_1\varepsilon(\delta(r_2)) = 0$. If we apply ε, we finally get

$$0 = \varepsilon\left(\sum r_1\varepsilon(\delta(r_2))\right) = \sum \varepsilon(r_1)\varepsilon(\delta(r_2)) = \varepsilon\left(\delta\left(\sum \varepsilon(r_1)r_2\right)\right) = \varepsilon(\delta(r)),$$

using once more the counit axiom.

Claim 3. *The extended maps Δ and ε give T a bialgebra structure.* As argued above, it is sufficient to check the coassociativity and counit axioms for y, because y and R generate T and by hypothesis, Δ and ε are a comultiplication and counit in R, respectively. We compute $(\Delta \otimes \mathrm{Id})\Delta(y)$ and $(\Delta \otimes \mathrm{Id})\Delta(y)$, essentially just writing equations (3.2.2) and (3.2.3) with $a \in G(R)$, $v = 0$ and $b = 1$:

$$(\Delta \otimes \mathrm{Id})(\Delta(y)) = a \otimes a \otimes y + \Big(a \otimes y + y \otimes 1 + w\Big) \otimes 1 + (\Delta \otimes \mathrm{Id})\,(w),$$
$$(\mathrm{Id} \otimes \Delta)(\Delta(y)) = a \otimes \Big(a \otimes y + y \otimes 1 + w\Big) + y \otimes 1 \otimes 1 + (\mathrm{Id} \otimes \Delta)(w).$$

It is clear that the two expressions above are equal by (3.3.5). This proves coassociativity. The counit axiom is also clear:

$$\varepsilon(a)y + \varepsilon(y)1 + \sum \varepsilon(w_1)w_2 = y = a\varepsilon(y) + y\varepsilon(1) + \sum w_1\varepsilon(w_2),$$

since $a \in G(R)$, $y \in T^+$ and $w \in R^+ \otimes R^+$, by hypothesis.

Claim 4. *The extension of S from R to T given by $S(y) = -a^{-1}(y + \sum w_1 S(w_2))$ is well defined.* We recall that the antipode S is an anti-homomorphism of algebras. Hence, when we apply it to the defining relations (3.0.1), we get $S(r)S(y) = S(y)S(\sigma(r)) + S(\delta(r))$. By replacing $S(y)$ by its defining expression and multiplying on both sides by $-a$, we compute

$$\begin{aligned} -aS(r)S(y) &= \mathrm{ad}_a(S(r))\left(y + \sum w_1 S(w_2)\right) \\ &= \mathrm{ad}_a(S(r))y + \mathrm{ad}_a(S(r))\left(\sum w_1 S(w_2)\right), \end{aligned}$$

and

$$\begin{aligned} -aS(y)S(\sigma(r)) - aS(\delta(r)) &= \left(y + \sum w_1 S(w_2)\right) S(\sigma(r)) - aS(\delta(r)) \\ &= \sigma(S(\sigma(r)))y + \delta(S(\sigma(r))) + \left(\sum w_1 S(w_2)\right) S(\sigma(r)) - aS(\delta(r)). \end{aligned}$$

Comparing the coefficients of y and the constant terms, we get

$$\begin{cases} \mathrm{ad}_a(S(r)) = \sigma(S(\sigma(r))), & \text{(3.3.9a)} \\ \mathrm{ad}_a(S(r))\left(\sum w_1 S(w_2)\right) = \delta(S(\sigma(r))) + \left(\sum w_1 S(w_2)\right) S(\sigma(r)) - aS(\delta(r)). & \text{(3.3.9b)} \end{cases}$$

We recall that S is also an anti-homomorphism of coalgebras, which means that we have $\Delta(S(r)) = \sum S(r_2) \otimes S(r_1)$, for all $r \in R$. Hence, using $\sigma = \tau_\chi^r$, we compute

$$\sigma(S(r)) = \sum \mathrm{ad}_a(S(r_2))\chi S(r_1).$$

Using now $\sigma = \tau_\chi^\ell$, we compute

$$\sigma(S(\sigma(r))) = \sigma\left(S\left(\sum \chi(r_1)r_2\right)\right) = \sum \chi(r_1)\sigma(S(r_2)).$$

Combining both expressions, we get

$$\begin{aligned} \sigma(S(\sigma(r))) &= \sum \chi(r_1)\mathrm{ad}_a(S(r_3))\chi(S(r_2)) \\ &= \sum \chi\left(\sum r_1 S(r_2)\right)\mathrm{ad}_2(S(r_3)) \qquad \text{(because } \chi\colon R \to K \text{ is a homomorphism)} \\ &= \sum \chi(\varepsilon(r_1))\mathrm{ad}_a(S(r_2)) \qquad \text{(by antipode property)} \\ &= \mathrm{ad}_a\left(S\left(\sum \varepsilon(r_1)r_2\right)\right) \qquad \text{(because } \chi|_K = \mathrm{Id}) \\ &= \mathrm{ad}_a(S(r)) \qquad \text{(by counit axiom)}, \end{aligned}$$

hence, proving (3.3.9a). In order to prove (3.3.9b), we start by rewriting it as

$$aS(\delta(r)) - \delta(S(\sigma(r))) = w_S S(\sigma(r)) - \mathrm{ad}_a(S(r))w_S \tag{3.3.10}$$

where $w_S = \sum w_1 S(w_2)$. We then compute

$$aS(\delta(r)) = aS\left(\delta\left(\sum \varepsilon(r_1)r_2\right)\right) = \sum a\varepsilon(r_1)S(\delta(r_2)) = \sum aS(r_1)r_2S(\delta(r_3)), \tag{3.3.11}$$

using the counit axiom and the antipode property in succession. Using $\sigma = \tau_\chi^\ell$, we write $\delta(S(\sigma(r))) = \sum \chi(r_1)\delta(S(r_2))$ and we observe that by applying the σ-derivation δ to the antipode property $\varepsilon(r) = \sum r_1 S(r_2)$, we obtain

$$0 = \delta(\varepsilon(r)) = \sum \delta(r_1)S(r_2) + \sum \sigma(r_1)\delta(S(r_2)),$$

because $\delta|_K \equiv 0$. Hence, $\sum \sigma(r_1)\delta(S(r_2)) = -\sum \delta(r_1)S(r_2)$. We will use this equality in the following long series of calculations

$$
\begin{aligned}
\sum \chi(r_1)\delta(S(r_2)) &= \sum \chi(r_1)\varepsilon(r_2)\delta(S(r_3)), && \text{(by the counit axiom)}\\
&= \sum \chi(r_1)\sigma(\varepsilon(r_2))\delta(S(r_3)), && \text{(because } \sigma|_K = \mathrm{Id})\\
&= \sum \chi(r_1)\sigma(S(r_2))\sigma(r_3)\delta(S(r_4)), && \text{(by the antipode property)}\\
&= -\sum \sigma(S(\chi(r_1)r_2))\delta(r_3)S(r_4), && \text{(by the identity above)}\\
&= -\sum \sigma(S(\sigma(r_1)))\delta(r_2)S(r_3), && \text{(because } \sigma = \tau^{\ell}_{\chi})\\
&= -\sum \mathrm{ad}_a(S(r_1))\delta(r_2)S(r_3), && (3.3.12)
\end{aligned}
$$

using $\sigma S \sigma = \mathrm{ad}_a S$, which was proved above. We combine (3.3.11) and (3.3.12) while rewriting $aS(r_1) = \mathrm{ad}_a(S(r_1))a$ in the first term, for convenience. This yields

$$aS(\delta(r)) - \delta(S(\sigma(r))) = \sum \mathrm{ad}_a(S(r_1))\Big[ar_2S(\delta(r_3)) + \delta(r_2)S(r_3)\Big]. \tag{3.3.13}$$

We now apply the antipode property with $(\mathrm{Id} \otimes S)$ to (3.3.3) in (e), which states

$$\Delta(\delta(r)) - \sum \delta(r_1) \otimes r_2 - \sum ar_1 \otimes \delta(r_2) = w\Delta(r) - \Delta(\sigma(r))w.$$

On the left hand side, we obtain $\varepsilon(\delta(r)) - \sum \delta(r_1)S(r_2) - \sum ar_1S(\delta(r_2))$. On the right hand side, using (3.3.8b), we get

$$\sum w_1r_1S(r_2)S(w_2) - \sum \sigma(r_1)w_1S(w_2)S(r_2) = \varepsilon(r)w_S - \sum \sigma(r_1)w_SS(r_2),$$

because S is an anti-homomorphism of algebras. As seen in claim 2, $\varepsilon(\delta(r)) = 0$. Thus, we have

$$\sum ar_1S(\delta(r_2)) + \sum \delta(r_1)S(r_2) = -\varepsilon(r)w_S + \sum \sigma(r_1)w_SS(r_2) \tag{3.3.14}$$

Inputting (3.3.14) into equation (3.3.13) yields

$$
\begin{aligned}
aS(\delta(r)) - \delta(S(\sigma(r))) &= \sum \mathrm{ad}_a(S(r_1))\Big[- \varepsilon(r_2)w_S + \sum \sigma(r_2)w_SS(r_3)\Big]\\
&= -\Big(\sum \mathrm{ad}_a(S(r_1))\varepsilon(r_2)\Big)\, w_S + \Big(\sum \mathrm{ad}_a(S(r_1))\sigma(r_2)w_SS(r_3)\Big)\,.
\end{aligned} \tag{3.3.15}
$$

The first sum equals $-\mathrm{ad}_a S(r) w_S$, by the counit axiom. To evaluate the second sum, use $\mathrm{ad}_a S = \sigma S \sigma$ to write

$$\sum \mathrm{ad}_a(S(r_1))\sigma(r_2) = \sigma\left(\sum S(\sigma(r_1))r_2\right) = \sigma(\varepsilon(\sigma(r))) = \varepsilon(\sigma(r)),$$

because $\Delta(\sigma(r)) = \sum \sigma(r_1) \otimes r_2$ and σ acts as the identity on scalars. Finally, we check that

$$\varepsilon(\sigma(r)) = \varepsilon\left(\sum \chi(r_1)r_2\right) = \sum \chi(r_1)\varepsilon(r_2) = \chi\left(\sum r_1\varepsilon(r_2)\right) = \chi(r).$$

Therefore, the second sum in (3.3.15) is equal to $\sum \chi(r_1) w_S S(r_2) = w_S \sum S(\chi(r_1)r_2) = w_S S(\sigma(r))$ and we proved

$$aS(\delta(r)) - \delta(S(\sigma(r))) = -\mathrm{ad}_a(S(r))w_S + w_S S(\sigma(r)), \tag{3.3.16}$$

like we wanted.

Claim 5. *The extended map S is the antipode in T with Δ and ε.* Like argued above for the comultiplication and counit, it suffices to check the antipode property for y. We compute

$$S(a)y + S(y)1 + \sum S(w_1)w_2 = a^{-1}y - a^{-1}(y + \sum w_1 S(w_2)) + \sum S(w_1)w_2 = 0 = \varepsilon(y),$$
$$aS(y) + yS(y) + \sum w_1 S(w_2) = -y - \sum w_1 S(w_2) + y + \sum w_1 S(w_2) = 0,$$

using the definition of $S(y)$ and equation (3.3.4). □

3.4 Examples of Hopf Ore extensions

Let L is the non-trivial Lie algebra of dimension two, i.e., there is a basis $\{x, y\}$ of L with $[x, y] = x$. Its universal enveloping algebra $U(L)$ is an Ore extension $K[x][y; \mathrm{Id}, \delta]$ of $K[x]$ with δ the derivation determined by $\delta(x) = -x$. The polynomial algebra $K[x]$ has a standard Hopf algebra structure with x primitive, i.e., $\Delta(x) = 1 \otimes x + x \otimes 1$, $\varepsilon(x) = 0$ and $S(x) = -x$. It is straightforward to check that $U(L)$ satisfies the conditions of Theorem 3.3.1(ii) with $a = 1$ and $w = 0$. Condition (d) for $\sigma = \mathrm{Id}$ is simply the counit axiom using $\chi = \varepsilon$. Condition (e) follows from the primitivity of x. Finally, condition (f) is void because $w = 0$. Therefore, $U(L)$ is a Hopf Ore extension.

The quantum plane $K_q[x, y]$ (with $q \neq 1$) is an Ore extension $K[x][y; \sigma, 0]$ where σ is determined by $\sigma(x) = qx$. If $K_q[x, y]$ was a Hopf Ore extension, then by Theorem 3.3.1 there would exist a character $\chi\colon K[x] \to K$ such that

$$qx = \sigma(x) = \chi(x) + x,$$

because x is primitive. This equation implies that $q = 1$ because $\{1, x\}$ form a linearly independent set and this is a contradiction.

The next example shows an Ore extension over a Hopf algebra on which the generalization of Panov's theorem applies, but Panov's original result does not. It serves as a motivation for a broader definition of the comultiplication (3.2.1), rather than Panov's condition of skew primitiveness. The **Heisenberg group G** of dimension 3 is the set of upper triangular 3×3 matrices with 1 in the diagonal, i.e.,

$$G = \left\{ \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} : \alpha, \beta, \gamma \in K \right\},$$

with the usual matrix multiplication as group operation. This group is related to the equivalence of different formulations of quantum mechanics and it is named after the German physicist Werner Heisenberg, one of the pioneers of quantum mechanics.The Heisenberg group is what we call an affine variety of dimension 3, although we will not enter in too many details about what that means. In short, an element of *G* can be identified with a triple (α, β, γ) and in light of this identification, multiplication is a polynomial function. As a matter of fact, if we identify two elements $g, g' \in G$ with triples, say $g = (\alpha_1, \beta_1, \gamma_1)$ and $g' = (\alpha_2, \beta_2, \gamma_2)$, then its multiplication is given by

$$g \cdot g' = (\alpha_1 + \alpha_2, \beta_1 + \beta_2 + \alpha_1\gamma_2, \gamma_1 + \gamma_2) \tag{3.4.1}$$

and each coordinate above is a polynomial in the variables α_i, β_i and γ_i, for $i = 1, 2$. The identity is identified with the triple $(0, 0, 0)$ and inverses are given by $(\alpha, \beta, \gamma)^{-1} = (-\alpha, \alpha\gamma - \beta, -\gamma)$.

Denote by H the coordinate algebra $\mathcal{O}(G)$ of G (see Definition 2.3.9). It consists of the polynomial maps $G \to K$, which we can identify with polynomials in the variables x, y and z. These three variables are the coordinate functions, which map (α, β, γ) to α, β and γ, respectively. Hence, we see that $H = K[x, y, z]$ and therefore we can regard H as the Ore extension $R[y]$, where $R = K[x, z]$. The element $1 \in H$ is the constant map $G \to K$ that maps every element in G to 1. The tensor product $H \otimes H$ can be seen as consisting of the polynomial maps $G \otimes G \to K$. We know that H has a Hopf algebra structure in which the comultiplication is induced by the multiplication in G, as seen in Proposition 2.3.10. This means, for instance, that $\Delta(x)$ is the map $G \times G \to K$ such that

$$\Delta(x)(g, g') = x(g \cdot g') = \alpha_1 + \alpha_2.$$

where $g = (\alpha_1, \beta_1, \gamma_1)$ and $g' = (\alpha_2, \beta_2, \gamma_2)$. On the other hand, we have

$$\begin{aligned} \alpha_1 &= (x \otimes 1)(g \otimes g'), & \alpha_2 &= (1 \otimes x)(g \otimes g'), \\ \beta_1 &= (y \otimes 1)(g \otimes g'), & \beta_2 &= (1 \otimes y)(g \otimes g'), \\ \gamma_1 &= (z \otimes 1)(g \otimes g'), & \gamma_2 &= (1 \otimes z)(g \otimes g'). \end{aligned}$$

via $K \simeq K \otimes K$, meaning for instance that $\alpha_1 = \alpha_1(1 \otimes 1)$. Hence, we conclude that x is primtive, i.e., $\Delta(x) = x \otimes 1 + 1 \otimes x$. Likewise, z is also primitive. As for y, the fact that

$$\Delta(y)(g \otimes g') = y(g \cdot g') = \beta_1 + \beta_2 + \alpha_1 \gamma_2$$

means that $\Delta(y) = 1 \otimes y + y \otimes 1 + (x \otimes 1)(1 \otimes z)$. Summarizing, we have

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \Delta(y) = 1 \otimes y + y \otimes 1 + x \otimes z, \qquad \Delta(z) = 1 \otimes z + z \otimes 1.$$

Since $(0, 0, 0)$ is the identity in G, the counit map is given by $\varepsilon(x) = x(0, 0, 0) = 0$ and likewise, $\varepsilon(y) = \varepsilon(z) = 0$. Thus, the Hopf algebra structure on $R = K[x, z]$ is the usual one and it is a Hopf subalgebra of H. This shows that $H = K[x, z][y]$ is a Hopf Ore extension in which $\Delta(y)$ satisfies (3.2.1) with $w = x \otimes z \neq 0$.

3.5 Additional results

Note that in Theorem 3.3.1(ii), the conditions (a) to (f) are sufficient for the existence of a Hopf Ore extension of the Hopf algebra R holds for any Hopf algebra R. However, for the converse statement Theorem 3.3.1(i), in which we prove the necessity of the same conditions (a) to (f), we had to make two extra assumptions: that the parameter v is a scalar and that the antipode S a specific form. We present the following corollary as a situation in which (i) and (ii) are equivalent.

Corollary 3.5.1. *Let R be a Hopf algebra. Suppose that R is a noetherian and that $R \otimes R$ is a domain. Then, the Hopf Ore extensions $T = R[y; \sigma, \delta]$ over R are exactly those given by Theorem 3.3.1.*

Proof. Observe that if $R \otimes R$ is a domain, then so is R because the map $[r \mapsto r \otimes 1]$ gives an embedding $R \hookrightarrow R \otimes R$. Then, the hypothese of Lemmas 3.2.3 and 3.2.4 hold and these two lemmas give us the necessary conditions to apply Theorem 3.3.1(i), which gives necessity. Theorem 3.3.1(ii) gives sufficiency. □

As we mentioned at the beginning of Section 3.2, condition (3.2.1) in the definition of a Hopf Ore extension can almost be dispensed with. I.e., if $T = R[y; \sigma, \delta]$ is a Hopf algebra with R as its Hopf subalgebra, then $\Delta(y)$ is necessarily of the form $a \otimes y + y \otimes b + v(y \otimes y) + w$. We explore this idea for the remaining of this section, under two settings: first, when $R \otimes R$ is a domain and second, when R is connected as a Hopf algebra.

Proposition 3.5.2. *Let $T = R[y; \sigma, \delta]$ be a Hopf algebra with R as a Hopf subalgebra. If $R \otimes R$ is a domain, then $\Delta(y) = s(1 \otimes y) + t(y \otimes 1) + v(y \otimes y) + w$, for some $s, t, v, w \in R \otimes R$.*

Proof. See [BOZZ15, Lemma 1 of §2.2]. □

The difference between the comultiplication given above and (3.2.1) is in the coefficients $s, t \in R \otimes R$. In the latter, we simply have $s = a \otimes 1$ and $t = 1 \otimes b$, for some $a, b \in R$. However, this apparently small detail makes a great difference since we build all the results in Section 3.2 around it. Since $a, b \in R$, we can use the Ore extension structure on $T = R[y; \sigma, \delta]$, while there is not *a priori* an Ore extension strucuture on $T \otimes T$ over $R \otimes R$. In this result, we assumed that $R \otimes R$ is a domain, but whether this hypothesis can be replaced with the weaker hypothesis of R itself being a domain, we do not know yet.

We recall that, in a Hopf algebra R, a coalgebra filtration is a family of vector subspaces $\{F_n R\}_{n \in \mathbb{N}}$ such that $R = \bigcup_{n \in \mathbb{N}} F_n R$ and

$$\Delta(F_n R) \subseteq \sum_{i=0}^{n} F_i R \otimes F_{n-i} R$$

. One particular coalgebra filtration is the coradical filtration $\{R_n\}_{n \in \mathbb{N}}$, defined in Proposition 2.3.15. The coradical of R is R_0 is the sum of the simple subcoalgebras of R.

The next proposition relates the coradical of R and that of $T = R[y; \sigma, \delta]$.

Proposition 3.5.3. *Let $T = R[y;\sigma,\delta]$ be a HOE such that $v = 0$. Then, $T_0 = R_0$, i.e., the coradical of T is the coradical of R.*

Proof. See [BOZZ15, Proposition 2.5]. □

Proposition 3.5.3 tells us that every simple subcoalgebra of T is contained in R. In particular, every grouplike element of T lies in R. Recall from Proposition 2.3.15 that R admits a coalgebra filtration $\{R_i\}_{i\in\mathbb{N}}$ called the coradical filtration, where R_0 is the coradical of R and $R_i = \Delta^{-1}(R \otimes R_{i-1} + R_0 \otimes R)$ for $i \geq 1$. Then it is straightforward to check that $A_n := \sum_{i=0}^{n} R_i \otimes R_{n-i}$ defines a coalgebra filtation of $R \otimes R$ and $B_n := \sum_{i+j+k=n} R_i \otimes R_j \otimes R_k$ defines a coalgebra filtration of $R \otimes R \otimes R$. Note that $A_0 = R_0 \otimes R_0$ and $B_0 = R_0 \otimes R_0 \otimes R_0$.

In Chapter 2, we called a Hopf algebra R connected if its coradical was trivial, that is, $R_0 = K$. In particular, the only grouplike element in R is 1. Keeping the notation from the previous paragraph, it is clear that $A_0 = K \otimes K \simeq K$ and likewise $B_0 \simeq K$. By Lemma 2.3.16, the coradical of $R \otimes R$ is contained in $A_0 = K$ and hence, $R \otimes R$ is also connected. Similarly, $R \otimes R \otimes R$ is connected too. In [Zhu13], we have the following result.

Proposition 3.5.4. *If H be a connected Hopf algebra over a field of characteristic 0, then H is a domain.*

Proof. See [Zhu13, Theorem 6.6]. □

Therefore it follows from this result and the observations that preceded it that R, $R \otimes R$ and $R \otimes R \otimes R$ are all connected Hopf algebra domains. In this situation, several of the hypotheses we have made in Sections 3.2 and 3.3 on R and its Ore extension $T = R[y;\sigma,\delta]$ become automatically true. Namely, if T is a Hopf algebra with R as its Hopf subalgebra, then $\Delta(y)$ must necessarily have form (3.2.1). The next proposition summarizes these claims. We note that our assumption of the characteristic of K being 0 is crucial here, because of Proposition 3.5.4.

Proposition 3.5.5. *Let R be a connected Hopf algebra with bijective antipode and let $T = R[y;\sigma,\delta]$ be a Hopf algebra with R as its Hopf subalgebra. Then we have*

$$\Delta(y) = 1 \otimes y + y \otimes 1 + w$$

for some $w \in R \otimes R$. As a consequence, T is a HOE and is a connected Hopf algebra. Furthermore, (i) of Theorem 3.3.1 holds.

Proof. See [BOZZ15, Proposition in §2.8]. □

The next result concerns the Gelfand-Kirillov dimension of a Hopf Ore extension. See Section 1.3 for the definition.

Proposition 3.5.6. *Let $T = R[y;\sigma,\delta]$ be a Hopf Ore extension and assume that R is finitely generated as an algebra. Then*

$$\operatorname{GKdim} T = \operatorname{GKdim} R + 1.$$

Proof. By Corollary 1.3.3, it is enough to prove that σ is locally algebraic, i.e., for every $r \in R$, there exists a finite dimensional vector subspace V of R such that $\{\sigma^n(r)\}_{n\in\mathbb{N}} \subseteq V$. Before we do so, we prove the following claim: if $\chi\colon H \to K$ is the algebra homomorphism such that $\sigma = \tau^{\ell}_{\chi}$, then in Sweedler's notation, we have

$$\sigma^n(r) = \sum \chi(r_1 r_2 \cdots r_n) r_{n+1} \tag{3.5.1}$$

for all $r \in R$. We proceed by induction on $n \geq 1$. For $n = 1$, this is precisely (d) of Theorem 3.3.1. Suppose that (3.5.1) holds for $n \geq 1$. Then, for $r \in R$, we have

$$\sigma^{n+1}(r) = \sigma^n\left(\sum \chi(r_1)r_2\right) = \sum \chi(r_1)\chi(r_{2_1}\cdots r_{2_n})r_{2_{n+1}} = \sum \chi(r_1 r_2 \cdots r_{n+1})r_{n+2}$$

by the coassociativity axiom, finishing the induction step. Fix $r \in R$ and write $\Delta(r) = \sum r_1 \otimes r_2$ in Sweedler's notation. Let $V = \sum K r_2$ be the linear span of the right tensorands $\{r_2\}$ in $\Delta(r)$. We will prove that $\sigma^n(r) \in V$ for all $n \in \mathbb{N}$. For $n = 0$, we have $r = \sum \varepsilon(r_1)r_2 \in V$ by the counit axiom. Note that (3.5.1) can be written as $\sigma^n(r) = \sum \chi(r_{1_1}\cdots r_{1_n})r_2$ by the coassociavity axiom, for $n \geq 1$. Hence, it follows immediately that $\sigma^n(r) \in V$ for $n \geq 1$. □

3.6 Iterated Hopf Ore extensions

As Proposition 3.5.5 shows, connectedness is a property passed through Hopf Ore extensions. This motivates us to consider a chain of Hopf Ore extensions, all of them connected and all of them satisfying (i) of Theorem 3.3.1. With this in mind, we introduce the following definition.

Definition 3.6.1. An **iterated Hopf Ore extensions** of K (IHOE) of order n is a Hopf algebra

$$H = K[y_1][y_2;\sigma_2,\delta_2]\cdots[y_n;\sigma_n,\delta_n] \tag{3.6.1}$$

in which

(i) The subalgebra $H_{(i)}$ of H generated by $y_1,\cdots,y_i$ is a Hopf subalgebra of H, for $1 \le i \le n$,
(ii) σ_i is an algebra automorphism of $H_{(i-1)}$ and δ_i is a σ_i-derivation, for $2 \le i \le n$.

By definition, $H_{(0)} = K$.

In [BOZZ15], the ultimate goal was to classify the affine (finitely generated) Hopf algebras of low GK dimension as Iterated Hopf Ore extensions. For us, they will be of interest in the next chapter, where we introduce the concept of a double Ore extension and study Hopf algebra structures on it. We finish this chapter with some properties of IHOEs, featuring part of a result in [BOZZ15].

Proposition 3.6.2. *Let H be an IHOE with defining chain* (3.6.1).

(i) H is a connected Hopf algebra with

$$\Delta(y_i) = 1\otimes y_i + y_i\otimes 1 + w^{i-1}, \tag{3.6.2}$$

where $w^{i-1} \in H^+_{(i-1)} \otimes H^+_{(i-1)}$, for $1 \le i \le n$ and with $w^0 = w^1 = 0$. After changes of the variables y_i and the corresponding changes in the data $\{\sigma_i,\delta_i,w^{i-1}\}_{2\le i\le n}$ but not of the chain (3.6.1)*, we have that $H_{(i)} = H_{(i-1)}[y_i;\sigma_i,\delta_i]$ is a Hopf Ore extension satisfying the conditions in (i) of Theorem 3.3.1, with $a = 1$, for $1 \le i \le n$.*

(ii) H is a noetherian domain of GK dimension n.

(iii) For $1 \le i \le n$ and $j \le i$, we have $\sigma_i(y_j) = y_j + a_{ij}$, for some $a_{ij} \in H_{(j-1)}$.

Proof. (i) We can apply Proposition 3.5.5 for $H_(i) = H_{(i-1)}[y_i;\sigma_i,\delta_i]$, for every $1 \le i \le n$ and the result follows by induction. We also have that H_2 is a connected Hopf algebra of GK dimension two (see the next item) and [Zhu13, Proposition 7.4] states that these are universal enveloping Lie algebras. Both the polynomial algebra $K[y_1,y_2]$ and the universal enveloping algebra of the non-trivial Lie algebra of dimension two are Hopf algebras that satisfy (3.6.2) with $w^0 = w^1 = 0$

(ii) It follows by induction in $H_{(i)}$, starting with $H_{(0)} = K$ which is a noetherian domain and using Proposition 1.2.1 in the induction step.

(iii) Fix $1 \leq i \leq n$ and $j \in \{1, \cdots, i\}$. There exists a character $\chi_i \colon H_{(i-1)} \to K$ such that $\sigma_i = \tau^{\ell}_{\chi_i}$. Hence, by (3.6.2), we have $\sigma_i(y_j) = y_j + \chi_i(y_j) + \sum \chi_i(w_1^{j-1})w_2^{j-1}$, where $w^{j-1} = \sum w_1^{j-1} \otimes w_2^{j-1} \in H_{(j-1)} \otimes H_{(j-1)}$. □

Chapter 4

Hopf algebra structures on double Ore extensions

The ultimate goal of this chapter is to study Hopf algebra structures on double Ore extensions $B = A_P[y, z; \sigma, \delta]$ such that A is a Hopf subalgebra of B. The definition of a double Ore extension is introduced in the Section 4.1, along with notation. In Section 4.2, we reduce the problem to the case when $A = K$ by taking the quotient of B by a suitable Hopf ideal. Afterwards, we focus on this particular setting and its ramifications. Like with Hopf Ore extensions in Chapter 3, we denote by (Δ, ε, S) the structure maps on both A and B , attaching a subscript when we need to distinguish them.

4.1 Double Ore extensions

A double Ore extension is a generalization of an Ore extension, in which we add two generators at the same time and a relation between them. The resulting object can be an iterated Ore extension of order two, but that is not necessarily the case. As a matter of fact, neither of these two classes of algebras is contained in the other. The intersection of these classes, that is, double Ore extensions which are also iterated Ore extensions of order two, has been studied in [CLM11].

Let A be an algebra. Like in Chapter 3, we start by introducing an appropriate notion of twisted derivation in A, since the one from the previous chapter is not quite what we need. If y_1 and y_2 are the new indeterminates introduced, then we now look at three defining relations instead of only one: the relation between y_1 and an arbitrary $a \in A$, the relation between y_2 and an arbitrary element $a \in A$ and finally, the relation between y_1 and y_2.

Let $\mathcal{M}_A = M_{2\times 2}(\mathrm{End}_K(A))$ be the algebra of 2×2 matrices of linear endomorphisms of A, with matrix product as multiplication. The operation in $\mathrm{End}_K(A)$ is the composition of maps. Hence, an element $\sigma \in \mathcal{M}_A$ is of the form $\sigma = \left[\begin{smallmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{smallmatrix}\right]$, for some linear endomorphisms σ_{ij} of A. Equivalently, we can identify σ with the linear map $A \to M_{2\times 2}(A)$ by

$$\sigma(a) = \begin{bmatrix} \sigma_{11}(a) & \sigma_{12}(a) \\ \sigma_{21}(a) & \sigma_{22}(a) \end{bmatrix}. \tag{4.1.1}$$

If we denote the matrix product in $\mathcal{M}_A$ by $\bullet$ and the map composition in $\mathrm{End}_K(A)$ by $\circ$, then we have

$$\sigma \bullet \psi = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \bullet \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{11} \circ \psi_{11} + \sigma_{12} \circ \psi_{21} & \sigma_{11} \circ \psi_{12} + \sigma_{12} \circ \psi_{22} \\ \sigma_{21} \circ \psi_{11} + \sigma_{22} \circ \psi_{21} & \sigma_{21} \circ \psi_{12} + \sigma_{22} \circ \psi_{22} \end{bmatrix},$$

for all $\sigma, \psi \in \mathcal{M}_A$. The identity element in $\mathcal{M}_A$ is of course $\mathrm{Id}_{\mathcal{M}_A} = \left[\begin{smallmatrix} \mathrm{Id}_A & 0 \\ 0 & \mathrm{Id}_A \end{smallmatrix}\right]$. We say that $\sigma \in \mathcal{M}_A$ is $\bullet-$**invertible** if there exists a map $\psi \in \mathcal{M}_A$, such that, $\sigma \bullet \psi = \psi \bullet \sigma = \mathrm{Id}_{\mathcal{M}_A}$.

The introduction of two indeterminates at once in a double Ore extension brings us to a matricial setting, which motivates our next definition.

Definition 4.1.1. Let A be an algebra and $\sigma \in \mathcal{M}_A = M_{2\times 2}(\mathrm{End}_K(A))$. A **twisted multiderivation**, more precisely, a $\sigma-$**multiderivation**, is a linear map $\delta\colon A \to M_{2\times 1}(A)$ such that

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b, \tag{4.1.2}$$

for all $a, b \in A$, under the identification of $\sigma\colon A \to M_{2\times 2}(A)$.

This definition is quite similar to the original definition of twisted derivation (Definition 1.1.1), but with the appropriate differences on where the images of the involved maps are. We also write $\delta = \left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right]$, where δ_1, δ_2 are linear endomorphisms of A.

The maps σ and δ address the commutation relation between the pair (y_1, y_2) and the elements of A. The relation between y_1 and y_2 themselves is a new feature of double Ore extensions. We are now ready to give such a definition.

Definition 4.1.2. Let A be a subalgebra of an algebra B and $\mathcal{M}_A = M_{2\times 2}(\mathrm{End}_K(A))$. We say that B is a **double Ore extension** of A if

(i) B is generated as an algebra by A and by two elements $y_1, y_2 \in B$;

(ii) y_1 and y_2 satisfy the relation

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1 y_1 + \tau_2 y_2 + \tau_0 \tag{4.1.3}$$

for some $p_{12}, p_{11} \in K$ with $p_{12} \neq 0$ and $\tau_1, \tau_2, \tau_0 \in A$;

(iii) there exists an algebra homomorphism $\sigma = \left[\begin{smallmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{smallmatrix}\right]: A \to M_{2\times 2}(A)$, which we can identify with an element of $\mathcal{M}_A$ and there exists a σ-multiderivation $\delta = \left[\begin{smallmatrix}\delta_1\\ \delta_2\end{smallmatrix}\right]: A \to M_{2\times 1}(A)$, such that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} a = \sigma(a) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \delta(a), \quad \forall a \in A; \tag{4.1.4}$$

(iv) The transpose of σ in $\mathcal{M}_A$, σ^T, is $\bullet-$invertible, i.e., there is an algebra homomorphism $\hat{\sigma} = \left[\begin{smallmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{smallmatrix}\right]: A \to M_{2\times 2}(A)$ such that

$$\sigma^T \bullet \hat{\sigma} = \hat{\sigma} \bullet \sigma^T = \mathrm{Id}_{\mathcal{M}_A}. \tag{4.1.5}$$

We say that σ is $\bullet^T-$**invertible** and denote the $\bullet^T-$inverse by $\hat{\sigma}$.

(v) B is a free left A-module with basis $\{y_1^i y_2^j\}_{i,j\geq 0}$ and also a free right A-module with basis $\{y_2^i y_1^j\}_{i,j\geq 0}$.

Under these conditions, we write $B = A_P[y_1, y_2; \sigma, \delta, \tau]$, where $P = (p_{12}, p_{11}) \in K^2$ is called the **parameter pair** and $\tau = (\tau_0, \tau_1, \tau_2) \in A^3$ is called the **tail** of the double Ore extension B.

The existence of σ and δ satisfying condition (iii) is equivalent to $y_1A+y_2A+A \subseteq Ay_1+Ay_2+A$. As proved in [ZZ08, Lemma 1.9], the existence of the map $\hat{\sigma}$ is equivalent to the condition $y_1A + y_2A + A = Ay_1 + Ay_2 + A$ and this A-module being free with basis $\{1, y_1, y_2\}$ on both sides. In particular, this means that if we assume (v), the inverse $\hat{\sigma}$ exists if and only if $Ay_1 + Ay_2 + A \subseteq y_1A + y_2A + A$. In other words, it is this condition that allows us to write the right-sided versions of (4.1.4).

Just an in the case of Ore extensions in Chapter 1, it is natural to ask if given an algebra A together with data P, τ , σ and δ satisfying (ii), (iii) and (iv), there exists the double Ore extension $A_P[y_1, y_2; \sigma, \delta, \tau]$. The answer is yes if and only if some additional conditions on these data is met. One can find the precise characterization in [ZZ08, Proposition 1.11] or in [CLM11, Proposition 1.5], the latter correcting some small mistakes in the relations originally published in the former.

One of the simplest examples of double Ore extensions are those taken over a field K, i.e., when $A = K$. Such double Ore extensions turn out to be iterated Ore extensions.

Proposition 4.1.3. *Let $B = K_P[y_1, y_2; \sigma, \delta, \tau]$ be a double Ore extension of a field K. Then $B \simeq K[x_1][x_2; \theta, d]$ is an iterated Ore extension, where θ is the algebra automorphism of the polynomial algebra $K[x_1]$ defined by $\theta(x_1) = p_{12}x_1 + \tau_2$ and d is the θ-derivation of $K[x_1]$ given by $d(x_1) = p_{11}x_1^2 + \tau_1 x_1 + \tau_0$.*

Proof. See [CLM11, Proposition 1.2]. □

We now give another, more substantial, example that first appeared in [ZZ08, Example 4.1]. Let $A = K[x]$ and fix $a, b, c \in K$ with $b \neq 0$. Define an algebra homomorphism $\sigma\colon A \to M_{2\times 2}(A)$ given by $\sigma(x) = \begin{bmatrix} 0 & b^{-1}x \\ bx & 0 \end{bmatrix}$ and a linear map $\delta\colon A \to M_{2\times 1}(A)$ given by $\delta(x) = \begin{bmatrix} cx^2 \\ -bcx^2 \end{bmatrix}$. Then we can consider the double Ore extension $B^2(a, b, c) = A_{(-1,0)}[y, z; \sigma, \delta, (ax^2, 0, 0)]$. It is the algebra over K generated by x, y, z subject to the relations

$$\begin{aligned} zy &= -yz + ax^2, \\ yx &= b^{-1}xz + cx^2, \\ zx &= bxy - bcx^2. \end{aligned} \tag{4.1.6}$$

The $\bullet$−inverse of σ^T is the the map $\hat{\sigma}\colon K[x] \to M_{2\times 2}(K[x])$ given by $\hat{\sigma}(x) = \begin{bmatrix} 0 & bx \\ b^{-1}x & 0 \end{bmatrix}$, i.e., $\hat{\sigma} = \sigma^T$. We would like to point to the second and third equations in (4.1.6), which could not happen in an iterated Ore extension of order two over $K[x]$. It is stated in [ZZ08] that $B^2(a, b, c)$ cannot be written as an iterated Ore extension $K[x][y; \sigma_1, d_1][z; \sigma_2, d_2]$ for some algebra automorphism σ_1 of $K[x]$, σ_1−derivation d_1, algebra automorphism σ_2 of $K[x][y; \sigma_1, d_1]$ and σ_2−derivation d_2.

Of course, examples of iterated Ore extensions $A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ which are not double Ore extensions can be constructed simply by requiring $\sigma_2(y_1)$ to be a polynomial in y of degree greater or equal than 2.

A characterization of the double Ore extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ which are also an iterated Ore extension $A[y_1; \sigma_1, d_1][y_2; \sigma, \delta]$ is proved in [CLM11].

Theorem 4.1.4. *Let A and B be algebras with $A \subseteq B$. Let P, τ, σ and δ be as in Definition 4.1.2. The following conditions are equivalent:*

(i) $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ *is a double Ore extension of A which can be presented as an iterated Ore extension $A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$;*

(ii) $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ *is a double Ore extension of A with $\sigma_{12} = 0$;*

(iii) $B = A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ *is an iterated Ore extension such that*

$$\sigma_2(A) \subseteq A, \quad \sigma_2(y_1) = p_{12}y + \tau_2,$$
$$d_2(A) \subseteq Ay_1 + A, \quad d_2(y_1) = p_{11}y_1^2 + \tau_1 + \tau_0,$$

for some $p_{12}, p_{11} \in K$ with $p_{12} \neq 0$ and such that $\sigma_2|_A$ is an automorphisms of A.

If any of these conditions holds, then the maps σ and δ are related to the maps $\sigma_1, \sigma_2, d_1, d_2$ by

$$\sigma = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_{21} & \sigma_2|_A \end{bmatrix}, \quad \delta(a) = \begin{bmatrix} d_1(a) \\ d_2(a) - \sigma_{21}(a)y_1 \end{bmatrix}, \quad \textit{for all } a \in A.$$

Proof. See [CLM11, Theorem 2.2]. □

There a similar theorem in [CLM11] concerning instead iterated Ore extensions of the form $A[y_2; \sigma_2, d_2][y_1; \sigma_1, d_1]$.

We finish this section with another result from [CLM11], which tells us what are the classes of isomorphism of double Ore extensions with respect to the parameter pair.

Proposition 4.1.5. *Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a double Ore extensions.*

(i) If $p_{11} \neq 0$ and $p_{12} = 1$, then

$$B \simeq A_{)} \left[y_1', y_2'; \left[\begin{smallmatrix} \sigma_{11} & p_{11}\sigma_{12} \\ p_{11}^{-1}\sigma 21 & \sigma_{22} \end{smallmatrix}\right], \left[\begin{smallmatrix} p_{11}\delta_1 \\ \delta_2 \end{smallmatrix}\right], \tau'\right]$$

where $\tau' = (p_{11}\tau_0, \tau_1, p_{11}\tau_2)$, $y_1' = p_{11}y_1$ and $y_2' = y_2$.

(ii) If $p_{12} \neq 1$, then

$$B \simeq A_{(p_{12},0)} \left[y_1', y_2'; \left[\begin{smallmatrix} \sigma_{11} - q\sigma_{12} & \sigma_{12} \\ \sigma_{21} + q(\sigma_{11} - \sigma_{22} - q\sigma_{12}) & \sigma_{22} + q\sigma_{12} \end{smallmatrix}\right], \left[\begin{smallmatrix} \delta_1 \\ \delta_2 + q\delta_1 \end{smallmatrix}\right], \tau'\right]$$

where $q = \frac{p_{11}}{p_{12}-1}$, $\tau' = (\tau_0, \tau_1 - q\tau_2, \tau_2)$, $y_1' = y_1$ *and* $y_2' = y_2 + qy_1$.

Proof. See [CLM11, Lemma 1.7]. □

Only one case is not mentioned in the proposition above: $p_{12} = 1$ and $p_{11} = 0$. In that case, there is no simplification to be done. In short, we can always assume that $p_{11} = 0$ except when $p_{12} = 1$, in which case we can assume that p_{11} is either 0 or 1. In other words, there are, up to isomorphism, two cases of the parameter pair: $P = (p_{12}, 0)$ (including $p_{12} = 0$) and $P = (1, 1)$.

4.2 Reduction to double Ore extensions over a field

Assume throughout this section that B is both a double Ore extension of A and a Hopf algebra such that A is its Hopf subalgebra. Write $B = A_P[y, z; \sigma, \delta, \tau]$, with the same notation from the previous section, except that we replace y_1 by y and y_2 by z to alleviate the notation. The idea is to find an appropriate Hopf algebra quotient of B that becomes a double Ore extension of the field K. By Proposition 4.1.3, this quotient will be an iterated Ore extension of the field K. The augmentation ideal B^+ would be a first candidate because it is always a Hopf ideal but in a sense, it is too large since $B/B^+ \simeq K$ by Proposition 2.3.5. The next lemma which establishes the correct Hopf ideal to consider.

Lemma 4.2.1. *The ideal* $I = BA^+B$ *is a Hopf ideal of* B.

Proof. The result follows immediately from A^+ being a Hopf ideal of A, by Proposition 2.3.5, as well as Δ and ε being algebra homomorphisms and S being an algebra anti-homomorphism. □

Let $\pi\colon B \to B/I$ denote the canonical projection. While Lemma 4.2.1 ensures that the quotient B/I is a Hopf algebra, we still need to ensure that the double Ore extension B over A projects to a double Ore extension of K. The next lemma tells us the we are proceeding in the right direction.

Lemma 4.2.2. *Let* $I = BA^+B$. *The subalgebra* $\pi(A) = A/(I \cap A)$ *of* B/I *is isomorphic to* K.

Proof. Let us prove that $I \cap A = A^+$. First, it is clear that $I \subseteq B^+$ because ε is an algebra homomorphism. Also, the augmentation ideal A^+ of A is equal to $B^+ \cap A$ and hence $I \cap A \subseteq A^+$.

The other inclusion is clear because B is unital and thus $A^+ \subseteq I$. Therefore, it follows that $A/(I \cap A) \simeq A/A^+$, simply by mapping $a + (I \cap A)$ to $a + A^+$. By Proposition 2.3.5, we have that $A/A^+ \simeq K$, which completes the proof. $\square$

From now on, we identify $A/(I \cap A)$ with K, making each class $a + I \in A/(I \cap A)$ correspond to the scalar $\varepsilon(a)$. In particular, $\bar{\tau}_i := \pi(\tau_i)$ can be seen as the scalar $\varepsilon(\tau_i)$, for $i = 0, 1, 2$, which means that the tail τ of the double Ore extension B projects to a subset of K. The identification between $A/(I \cap A)$ and K is at the heart of the next proposition. Before we present it, we introduce a definition which plays a role in its statement.

Definition 4.2.3. We say that A^+ is **stable** under σ if $\sigma(A^+) \subseteq M_{2\times 2}(A^+)$, or equivalently, $\sigma_{ij}(A^+) \subseteq A^+$, for all $i, j = 1, 2$. Analogously, we say that A^+ is **stable** under $\hat{\sigma}$ and under δ if $\hat{\sigma}(A^+) \subseteq M_{2\times 2}(A^+)$ and $\delta(A^+) \subseteq M_{2\times 1}(A^+)$, respectively.

In the example $B = B^2(a, b, c)$ introduced in the previous section (with $a, b, c \in K$, $b \neq 0$), we have $A = K[x]$, which has a Hopf algebra structure defined by

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \varepsilon(x) = 0, \quad S(x) = -x.$$

We have that $A^+ = \langle x \rangle$ as seen in the example after Proposition 2.3.5. The ideal $I = BA^+B$ is thus the ideal of B generated by x. The condition that A^+ is stable under σ, $\hat{\sigma}$ and δ is satisfied because, by definition,

$$\sigma(x) = \begin{bmatrix} 0 & b^{-1}x \\ bx & 0 \end{bmatrix}, \quad \hat{\sigma}(x) = \begin{bmatrix} 0 & bx \\ b^{-1}x & 0 \end{bmatrix}, \quad \delta(x) = \begin{bmatrix} cx^2 \\ -bcx^2 \end{bmatrix},$$

so we see that each entry in the matrices above is in A^+. It is enough to check the stability condition for x because it generates A^+ as an ideal and σ and $\hat{\sigma}$ are algebra homomorphisms and δ is a σ-multiderivation.

The example $B^2(a, b, c)$ was our motivation for considering the stability of A^+. There are however examples of double Ore extension Hopf algebra B over a Hopf subalgebra A, for which the stability condition of A^+ *does not* hold. We thank Ken Brown for pointing us one such example, the universal enveloping algebra $U(\mathfrak{sl}(2))$ of the special linear Lie algebra of order two.

The Lie algebra $\mathfrak{sl}(2)$ has dimension three, its is spanned by three elements typically denoted e, f, h satisfying: $[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$. These Lie brackets translate into the following relations in $U(\mathfrak{sl}(2))$

$$\begin{cases} ef = fe + h, \\ he = eh + 2e, \\ hf = fh - 2f. \end{cases} \tag{4.2.1}$$

By the Poincaré-Birkhoff-Witt theorem, the elements in $U(\mathfrak{sl}(2))$ are polynomials in the indeterminates e, f, h. This fact together with the above relations mean that we can view $U(\mathfrak{sl}(2))$ as a double Ore extension of $K[h]$. More precisely, in the double Ore extension notation, we can write $U(\mathfrak{sl}(2)) = K[h]_P\,[f, e; \sigma, \delta, \tau]$, where $P = (1, 0)$, $\tau = (h, 0, 0)$, $\delta \equiv 0$ and σ is determined by

$$\sigma(h) = \begin{bmatrix} h+2 & 0 \\ 0 & h-2 \end{bmatrix}.$$

The augmentation ideal $K[h]^+$ is the ideal $\langle h \rangle$ of $K[h]$. Therefore, we see for instance that $h + 2 = \sigma_{11}(h) \notin \langle h \rangle$, which means that $\langle h \rangle$ is not stable under σ. In this case, defining I as the ideal of $U(\mathfrak{sl}(2))$ generated by h, the relations in (4.2.1) show that both $e, f \in I$. Since the elements of $U(\mathfrak{sl}(2))$ are polynomials in e, f, h, it follows that the only elements of $U(\mathfrak{sl}(2))$ possibly not in I are scalars. But scalars cannot be in I because $I \subseteq U(\mathfrak{sl}(2))^+$ and they cannot be in $U(\mathfrak{sl}(2))^+$ because the counit map acts as the identity on scalars. As a consequence, we have that $I = U(\mathfrak{sl}(2))^+$ and therefore, the quotient $U(\mathfrak{sl}(2))/I$ is isomorphic to K and it cannot be a double Ore extension.

This is precisely the situation we wish to avoid in the next proposition.

Proposition 4.2.4. *Suppose that A^+ is stable under the maps σ, $\hat{\sigma}$ and δ. Then the Hopf quotient B/I is a double Ore extension of K with defining relation*

$$\bar{z}\,\bar{y} = p_{12}\bar{y}\,\bar{z} + p_{11}\bar{y}^2 + \bar{\tau}_1\,\bar{y} + \bar{\tau}_2\,\bar{z} + \bar{\tau}_0, \tag{4.2.2}$$

i.e., B/I is a double Ore extension of the form $K_P\left[\bar{y}, \bar{z}; \left[\begin{smallmatrix} \mathrm{Id}_K & 0 \\ 0 & \mathrm{Id}_K \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \bar{\tau}\right]$, where $\bar{\tau} = (\bar{\tau}_0, \bar{\tau}_1, \bar{\tau}_2)$ is a subset of K.

Proof. By the previous lemma, $\pi(A) \simeq K$ and hence, K is a Hopf subalgebra of B/I, with the latter being generated by K, $\bar{y} := \pi(y)$ and $\bar{z} := \pi(z)$. Applying π to the defining relation (4.1.3) yields the defining relation (4.2.2) with $p_{12}, p_{11}, \bar{\tau}_0, \bar{\tau}_1, \bar{\tau}_2 \in K$, in light of the preceding identification.

The quotient B/I is a well defined $A/(I \cap A)$-module, which means, via our identification, it is a K-vector space. In particular, B/I satisfies condition (v) in Definition 4.1.2. However, we still need to check that $\{\bar{y}^i \bar{z}^j : i, j \in \mathbb{N}\}$ and $\{\bar{z}^i \bar{y}^j : i, j \in \mathbb{N}\}$ are the respective bases. We prove only the first, since the second is analogous. Clearly, the fact that $\{y^i z^j : i, j \in \mathbb{N}\}$ spans B implies that $\{\bar{y}^i \bar{z}^j : i, j \in \mathbb{N}\}$ spans B/I.

The conditions of stability of A^+ imply that $I := BA^+B = A^+B = BA^+$. To prove, for instance, that $BA^+B \subseteq A^+B$, it is enough to check that $yA^+ \subseteq A^+B$ and $zA^+ \subseteq A^+B$, because y and z generate B as an algebra. Indeed, we have

$$\begin{bmatrix} y \\ z \end{bmatrix} A^+ \subseteq \underbrace{\sigma(A^+)}_{\subseteq M_{2\times 2}(A^+)} \begin{bmatrix} y \\ z \end{bmatrix} + \underbrace{\delta(A^+)}_{\subseteq M_{2\times 1}(A^+)} \subseteq M_{2\times 1}(A^+B).$$

Analogously, we prove that $BA^+B \subseteq BA^+$ using $\hat{\sigma}$. The converse inclusions are trivial

Let us now prove that $\{\bar{y}^i \bar{z}^j\}_{i,j\in\mathbb{N}}$ is linearly independent over K. Let $\lambda_{ij} \in K$, finitely many nonzero, such that $\sum \lambda_{ij} \bar{y}^i \bar{z}^j = 0$. Via our identification of K and $A/(I \cap A)$ there exist $a_{ij} \in A$ such that $\varepsilon(a_{ij}) = \lambda_{ij}$ and $\pi\left(\sum a_{ij} y^i z^j\right) = \sum \lambda_{ij} \bar{y}^i \bar{z}^j = 0$. Hence, $h := \sum a_{ij} y^i z^j \in I$. Since $I = A^+B$ and A^+ is an ideal, we can also write $h = \sum b_{ij} y^i z^j$, for some $b_{ij} \in A^+$. By the freeness of B as a left A-module we conclude that $a_{ij} = b_{ij} \in A^+$ and hence, $\lambda_{ij} = \varepsilon(a_{ij}) = 0$, for all $i, j \in \mathbb{N}$. Therefore, $\{\bar{y}^i \bar{z}^j : i, j \in \mathbb{N}\}$ is a basis of B/I over K.

The conditions of stability of A^+ allow us to factor σ, $\hat{\sigma}$ and δ through A^+. Therefore, we can define $\bar{\sigma}$ by

$$\bar{\sigma}(a + A^+) = \begin{bmatrix} \sigma_{11}(a) + A^+ & \sigma_{12}(a) + A^+ \\ \sigma_{21}(a) + A^+ & \sigma_{22}(a) + A^+ \end{bmatrix} \in M_{2\times 2}(A/A^+) \tag{4.2.3}$$

and we can define $\bar{\hat{\sigma}}$ and $\bar{\delta}$ analogously. By identifying A/A^+ with K via the counit map ε, we get maps which make the following diagrams commute

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & M_{2\times 2}(A) \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow \\ K & \xrightarrow{\bar{\sigma}} & M_{2\times 2}(K) \end{array}, \qquad \begin{array}{ccc} A & \xrightarrow{\hat{\sigma}} & M_{2\times 2}(A) \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow \\ K & \xrightarrow{\bar{\hat{\sigma}}} & M_{2\times 2}(K) \end{array}, \qquad \begin{array}{ccc} A & \xrightarrow{\delta} & M_{2\times 1}(A) \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow \\ K & \xrightarrow{\bar{\delta}} & M_{2\times 1}(K) \end{array}.$$

Using these diagrams, we can explicitly determine $\bar{\sigma}$, $\bar{\hat{\sigma}}$ and $\bar{\delta}$. We start by rewriting equations

(4.1.4) as

$$\begin{cases} ya = \sigma_{11}(a)y + \sigma_{12}(a)z + \delta_1(a), \\ za = \sigma_{21}(a)y + \sigma_{22}(a)z + \delta_2(a), \end{cases}$$

for all $a \in A$. Passing this relation to the quotient B/I and identifying $A/(I \cap A)$ with K via ε, we get

$$\begin{cases} \bar{y}\varepsilon(a) = \varepsilon(\sigma_{11}(a))\bar{y} + \varepsilon(\sigma_{12}(a))\bar{z} + \varepsilon(\delta_1(a)), \\ \bar{z}\varepsilon(a) = \varepsilon(\sigma_{21}(a))\bar{y} + \varepsilon(\sigma_{22}(a))\bar{z} + \varepsilon(\delta_2(a)). \end{cases}$$

The commutation of the diagrams means that the equations above can yet be rewritten as

$$\begin{cases} \bar{y}\varepsilon(a) = \bar{\sigma}_{11}(\varepsilon(a))\bar{y} + \bar{\sigma}_{12}(\varepsilon(a))\bar{z} + \bar{\delta}_1(\varepsilon(a)), \\ \bar{z}\varepsilon(a) = \bar{\sigma}_{21}(\varepsilon(a))\bar{y} + \bar{\sigma}_{22}(\varepsilon(a))\bar{z} + \bar{\delta}_2(\varepsilon(a)). \end{cases}$$

But now, of course, $\varepsilon(a)$ is a scalar and thus it commutes with $\bar{y}$ and with $\bar{z}$. Hence,

$$\begin{cases} \bar{\sigma}_{11}(\varepsilon(a)) = \bar{\sigma}_{22}(\varepsilon(a)) = \varepsilon(a), \\ \bar{\sigma}_{12}(\varepsilon(a)) = \bar{\sigma}_{21}(\varepsilon(a)) = 0, \\ \bar{\delta}_1(\varepsilon(a)) = \bar{\delta}_2(\varepsilon(a)) = 0, \end{cases}$$

because $\{1, \bar{y}, \bar{z}\}$ are linearly independent. This means that $\bar{\sigma} = \begin{bmatrix} \mathrm{Id}_K & 0 \\ 0 & \mathrm{Id}_K \end{bmatrix}$ and $\bar{\delta} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Analogously, we prove that $\bar{\hat{\sigma}} = \begin{bmatrix} \mathrm{Id}_K & 0 \\ 0 & \mathrm{Id}_K \end{bmatrix}$, from where the condition of invertibility between $\bar{\sigma}$ and $\bar{\hat{\sigma}}$ is evident. □

As mentioned before Proposition 4.2.4, in the example $B^2(a, b, c)$, which is a double Ore extension of $K[x]$, we have that $K[x]^+ = \langle x \rangle$ is stable under σ, $\hat{\sigma}$ and δ. Hence, we can apply Proposition 4.2.4 to B. Since $x+I = 0$ in B/I, we have that $B/I = K_{(-1,0)}[y, z; \begin{bmatrix} \mathrm{Id}_K & 0 \\ 0 & \mathrm{Id}_K \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, (0, 0, 0)]$. The relations in (4.1.6) become simply

$$zy = -yz, \tag{4.2.4}$$

which means that B/I is actually a quantum plane $K_{-1}[y, z]$.

In the setting of Proposition 4.2.4, we recall that if $p_{12} \neq 1$, then B/I is isomorphic to a double

Ore extension with parameter pair $P = (p_{12}, 0)$. Else if $p_{12} = 1$ and $p_{11} \neq 0$, then the double Ore extension B/I is isomorphic to a double Ore extension with parameter pair $P = (1, 1)$. Therefore, without loss of generality, we will assume hereafter that the parameter pair P is either $P = (p_{12}, 0)$ or $P = (1, 1)$.

From Proposition 4.1.3, we have that B/I is also an iterated Ore extension $K[\bar{y}][\bar{z}; \theta, d]$, where the automorphism θ of $K[y]$ is defined by

$$\theta(\bar{y}) = p_{12}\bar{y} + \tau_2 \tag{4.2.5}$$

and d is the θ-derivation of $K[y]$ given by

$$d(\bar{y}) = p_{11}\bar{y}^2 + \tau_1\bar{y} + \tau_0. \tag{4.2.6}$$

We will use the notation $K[y][z; \theta, d]$ onwards for B/I, dropping the bars to alleviate the notation. In particular, $K[y][z; \theta, d]$ is an affine algebra, meaning finitely generated, with Gelfand-Kirillov dimension two by Proposition 1.3.2. This is because $\theta(y) = p_{12}y + \tau_2$ and hence, the subspace V spanned by 1 and y generates $K[y]$ and is stable under θ.

If $K[y][z; \theta, d]$ is what we called an iterated Hopf Ore extension in Section 3.6, then we have a full characterization by Theorem 3.3.1, the generalization of Panov's theorem. We address this case in the next section.

However, we could also have a Hopf algebra structure on $K[y][z; \theta, d]$ that *does not* have $K[y]$ as its Hopf subalgebra. In theory, we could have for instance the comultiplication of y depending on both y and z, which cannot not happen the former case. In order to study these alternative Hopf algebra structures, we need first to classify the algebras with Gelfand-Kirillov dimension two into families. We will then check if and when we can define Hopf algebra structures in each of these families.

Let us pause for a moment to put things in perspective. Our starting point was a double Ore extension B over a Hopf algebra A together with a Hopf algebra structure on B compatible with that of A. We also assumed that this Hopf algebra structure is such that the augmentation ideal A^+ is stable under the maps σ, $\hat{\sigma}$ and δ. Through a Hopf quotient B/I, we reduced our study to a double Ore extension of the field K, which is easier to study, while keeping the same parameter pair and tail.

Ideally, we would like to use the insight gained about Hopf algebras on double Ore extensions over K to improve our understanding about Hopf algebras on double Ore extensions in general. For instance, obtain a classification of the possible Hopf algebra structures or derive restrictions on the data associated to the double Ore extension. However, going back from B/I to B is where the subjects becomes extremely complicated. Nonetheless, we can observe that if for some fixed parameter pair P and tail τ we cannot have a Hopf algebra structure on B/I, then neither can we have one on B. This gives some sort of negative answer in such cases, allowing one to focus on the remaining cases in which a positive answer may be possible.

4.3 Iterated Hopf Ore extensions of order two

In this section, we carry the hypotheses and notation of Section 4.2 as well as the conclusions of Proposition 4.2.4. In addition, we assume that the double Ore extension $B/I = K[y][z;\theta,d]$ is actually a Hopf Ore extension of $K[y]$ or an iterated Hopf Ore extension of order two, as in Definition 3.6.1. This means that $K[y]$ is a Hopf subalgebra of $K[y][z;\theta,d]$. Note that the Hopf algebra structure on $K[y]$ is given by:

$$\Delta(y) = 1\otimes y + y\otimes 1,\quad \varepsilon(y) = 0 \quad \text{and} \qquad S(y) = -y.$$

Both Theorem 3.3.1 and Proposition 3.6.2 apply. Thus, we can directly check which conditions on the data P and τ are necessary and sufficient for $K[y][z;\theta,d]$ to be a Hopf Ore extension, given by

$$\Delta(z) = 1\otimes z + z\otimes 1 + w,\quad \varepsilon(z) = 0 \quad \text{and} \quad S(z) = -z + \sum w_1 S(w_2),$$

for some $w \in K[y]^+ \otimes K[y]^+$. We summarize these results in the next proposition:

Proposition 4.3.1. *Let $K[y][z;\theta,d]$ be an iterated Hopf Ore extension, with θ and d given by* (4.2.5) *and* (4.2.6)*, respectively. Then, $p_{12} = 1$ and $\tau_0 = 0$. Furthermore, we have two possible cases:*

(i) If $\tau_2 \neq 0$, then $w = \frac{2p_{11}}{\tau_2} y\otimes y$ and the iterated Hopf Ore extension is completely determined.
(ii) If $\tau_2 = 0$, then $p_{11} = 0$.

Proof. From Proposition 3.6.2, it follows that p_{12} must be equal to 1. By (e) of Theorem 3.3.1, we have that

$$\Delta d(y) - \sum d(y_1) \otimes y_2 - \sum y_1 \otimes d(y_2) = w\Delta(y) - \Delta\left(\theta(y)\right) w,$$

using Sweedler's notation. Bearing in mind that $p_{11}, \tau_0, \tau_1, \tau_2 \in K$ and that $K[y]$ is commutative, we compute

$$\begin{aligned}
\Delta d(y) &= p_{11}\left(1 \otimes y + y \otimes 1\right)^2 + \Delta(\tau_1)\left(1 \otimes y + y \otimes 1\right) + \Delta\left(\tau_0\right) \\
&= p_{11}\left(1 \otimes y^2 + 2y \otimes y + y^2 \otimes 1\right) + \tau_1\left(1 \otimes y + y \otimes 1\right) + \tau_0 1 \otimes 1, \\
\sum d(y_1) \otimes y_2 &= \left(d \otimes \mathrm{Id}\right)\left(1 \otimes y + y \otimes 1\right) = d(y) \otimes 1 = p_{11} y^2 \otimes 1 + \tau_1 y \otimes 1 + \tau_0 1 \otimes 1, \\
\sum y_1 \otimes d(y_2) &= \left(\mathrm{Id} \otimes d\right)\left(1 \otimes y + y \otimes 1\right) = 1 \otimes d(y) = p_{11} 1 \otimes y^2 + \tau_1 1 \otimes y + \tau_0 1 \otimes 1, \\
w\Delta(y) - \Delta\left(\theta(y)\right) w &= w\Delta(y) - \Delta(y + \tau_2)w = -\Delta(\tau_2)w = -\tau_2 w.
\end{aligned}$$

Putting everything together yields

$$2p_{11} y \otimes y - \tau_0 1 \otimes 1 = -\tau_2 w.$$

By (f) of Theorem 3.3.1, we know that $w \in K[y]^+ \otimes K[y]^+$. In particular, we have $(\varepsilon \otimes \mathrm{Id})\,(w) = 0$. Applying $\varepsilon \otimes \mathrm{Id}$ to the equation above, together with $\varepsilon(y) = 0$, implies that $\tau_0 = 0$.

If $\tau_2 \neq 0$, then we conclude that $w = -\frac{2p_{11}}{\tau_2} y \otimes y$. Otherwise if $\tau_2 = 0$, since $K[y] \otimes K[y]$ is vector space with basis $y^i \otimes y^j$ (with $i, j \in \mathbb{N}$) and $\operatorname{char} K = 0$, it follows that $p_{11} = 0$. □

Conversely, if $\tau_2 \neq 0$, we can check that there exists indeed a Hopf Ore extension of $K[y]$ when $w = -\frac{2p_{11}}{\tau_2} y \otimes y$. We just have to apply Theorem 3.3.1(ii). To check condition (d), we define the algebra homomorphism $\chi\colon K[y] \to K$ by $\chi(y) = \tau_2$. Then, $\sigma(y) = y + \tau_2 = \chi(1)y + \chi(y)1$. Condition (e) follows from the computations in the proof of Proposition 4.3.1. Finally, we check condition (f):

$$\sum S(w_1)w_2 = -\frac{2p_{11}}{\tau_2} S(y)y = \frac{2p_{11}}{\tau_2} y^2 = -\frac{2p_{11}}{\tau_2} yS(y) = \sum w_1 S(w_2) \tag{4.3.1}$$

and

$$\begin{aligned}
\frac{2p_{11}}{\tau_2} y \otimes y \otimes 1 + \frac{2p_{11}}{\tau_2} \Delta\left(y\right) \otimes y &= \frac{2p_{11}}{\tau_2}\left(y \otimes y \otimes 1 + y \otimes 1 \otimes y + y \otimes y \otimes 1\right) \\
&= \frac{2p_{11}}{\tau_2} 1 \otimes y \otimes y + \frac{2p_{11}}{\tau_2} y \otimes \Delta\left(y\right).
\end{aligned} \tag{4.3.2}$$

As noted at the end of the previous section, since $p_{12} = 1$, we can assume that either $p_{11} = 0$ or $p_{11} = 1$. In the first case, we have $w = 0$ and hence, the Hopf algebra structure on $K[y][z; \theta, d]$ is just the same the one of the classic polynomial algebra $K[y, z]$. In conclusion, we have

(i) If $P = (1, 0)$ and $\tau = (0, \tau_1, \tau_2)$ with $\tau_2 \neq 0$, then $w = 0$.
(ii) If $P = (1, 1)$ and $\tau = (0, \tau_1, \tau_2)$ with $\tau_2 \neq 0$, then $w = -\frac{2}{\tau_2} y \otimes y$.
(iii) If $P = (1, 0)$ and $\tau = (0, \tau_1, 0)$, then $\theta = \mathrm{Id}$, $d(y) = \tau_1 y$ and the defining relation is $zy = yz + \tau_1 y$.

We should note that the Hopf algebra structures in items (i) and (ii) above are one and the same, even though the defining relations and the comultiplication of the element z are different. They coincide both with the universal enveloping algebra of the nonabelian Lie algebra, $K[y'][z'; \delta']$ where $\delta'(y') = y'$. For (i), we can see that with the change of variable $y' = z + \tau_2^{-1} \tau_1 y$ and $z' = \tau_2^{-1} y$. For (ii), we can see that with the change of variable $y' = z + \tau_2^{-1} \left(y^2 + \tau_1 y \right)$ and $z' = \tau_2^{-1} y$. In the next section, we fully address the description of the Ore extensions of order two, up to isomorphism (which includes the variable changes just mentioned).

Regarding the example $B = B^2(a, b, c)$ discussed in Sections 4.1 and 4.2, we had concluded that $B/I \simeq K_{-1}[y, z]$ is a quantum plane. In particular, it is an Ore extension of $K[y]$. By Proposition 4.3.1, $K_{-1}[y, z]$ cannot have a Hopf algebra structure that extends the Hopf algebra structure of $K[y]$.

4.4 Iterated Ore extensions of order two in general

In an article by Alev and Dumas [AD97, Proposition 3.2], there is a classification, up to isomorphism, of the Ore extensions of order two over any field. We can apply it to $B/I = K[y][z; \theta, d]$, where θ and d are defined by $\theta(y) = p_{12} y + \tau_2$ and $d(y) = p_{11} y^2 + \tau_1 y + \tau_0$. There are the following possibilities:

(i) If $p_{12} \neq 1$, assuming without loss of generality that $p_{11} = 0$, then $K[y][z; \theta, d] \simeq K[y'][z; \theta', d]$ where $y' = y + \frac{\tau_2}{p_{12}-1}$ and $\theta'(y') = p_{12} y'$. Furthermore, we have $d(y') = \tau_1 y' + \left(\tau_0 - \frac{\tau_1 \tau_2}{p_{12}-1} \right)$. Thus, setting $z' = \left(z + \frac{\tau_1}{p_{12}-1} \right)$, we get that

$$z' y' = p_{12} y' z' + \left(\tau_0 - \frac{\tau_1 \tau_2}{p_{12} - 1} \right).$$

If $\tau_0 = \frac{\tau_1\tau_2}{p_{12}-1}$, then we conclude that $K[y][z;\theta,d] \simeq K_{p_{12}}[y',z']$ - the p_{12}-**quantum plane**. The defining relation is $zy = p_{12}yz$.

Else, if $\tau_0 \neq \frac{\tau_1\tau_2}{p_{12}-1}$, then $B/I \simeq A_1^{p_{12}}(K)$ - the p_{12}-**quantum Weyl algebra**. More precisely, $A_1^{p_{12}}(K) = K[y'][z'';\theta',d'']$ where $\left(\tau_0 - \frac{\tau_1\tau_2}{p_{12}-1}\right) z'' = z'$ and $d''(y') = 1$. Thus, the defining relation is $z''y' = p_{12}y'z'' + 1$.

(ii) If $p_{12} = 1$, then we can assume that either $p_{11} = 1$ or $p_{11} = 0$. If $\tau_2 = 0$, then $\theta = \mathrm{Id}|_{K[y]}$ and hence, $B/I = K[y][z;d]$ is a **differential operator ring**. The defining relation is $zy = yz + d(y)$, where $d(y) = y^2 + \tau_1 y + \tau_0$ or $d(y) = \tau_1 y + \tau_0$.

(iii) If $p_{12} = 1$ and $\tau_2 \neq 0$, then $B/I \simeq K[y'][z';d']$ is again a **differential operator ring**, where $z' = \tau_2^{-1}y$, $y' = z + \tau_2^{-1}d(y)$ and $d'(y') = -y'$. The defining relation is $z'y' = y'z' - y'$. It is clear that this defining relation falls into one of the cases of the previous item, namely, the one with $\tau_1 = -1$ and $p_{11} = \tau_0 = 0$. Therefore, we can assume without loss of generality that $\tau_2 = 0$ when $p_{12} = 1$.

The classification above yields three different families of algebra structures on B/I: quantum plane, quantum Weyl algebra and differential operator rings. Through the next sections, we will study, for each of these families, if there are any algebraic constraints to endow them with Hopf algebra structures. For clarity, we display a summary of the contents of this section in the following table, while dropping the primes in the variables:

	Parameter pair	**Tail**	**Defining relation**	**Notation**	**Type**
(i)	$p_{12} \neq 1; p_{11} = 0$	$\tau_0 = \frac{\tau_1\tau_2}{p_{12}-1}$	$zy = p_{12}yz$	$K_{p_{12}}[y,z]$	Quantum plane
		$\tau_0 \neq \frac{\tau_1\tau_2}{p_{12}-1}$	$zy = p_{12}yz + 1$	$A_1^{p_{12}}(K)$	Quantum Weyl algebra
(ii)	$p_{12} = 1; p_{11} = 0, 1$	$\tau_2 = 0$	$zy = yz + d(y)$	$K[y][z;d]$	Differential operator ring
(iii)		$\tau_2 \neq 0$	$zy = yz - y$	$K[y][z;d']$	Differential operator ring

The *Parameter pair* and *Tail* columns refer to the original data of the double Ore extension, while the *Defining relation* column refers to the new defining relation of the algebra, simplified after variable changes.

Before we continue, we will establish an important result in Proposition 4.4.2 that allows us to further reduce B/I to a commutative Hopf quotient.

Lemma 4.4.1. *Let H be an algebra. The smallest ideal of H with a commutative quotient is*

$$\bigcap \Big\{ L : L \text{ is an ideal of } H \text{ and } H/L \text{ is commutative} \Big\} \tag{4.4.1}$$

*and coincides with the **commutator ideal** $[H, H]$, which is the ideal generated by the commutators $[f, g] = fg - gf$, for all $f, g \in H$. The algebra $H/[H, H]$ is called the **largest commutative quotient** of H.*

Proof. Let $J = \bigcap\{L : L$ is an ideal of H and H/L is commutative$\}$. If L is an ideal of H, then H/L is commutative if and only if $[f, g] \in L$, for all $f, g \in H$. In particular, $[H, H]$ is an ideal that induces a commutative quotient and therefore contains J, because J is the intersection of any such ideals. On the other hand, it is clear that all the commutators are in J and thus, J contains $[H, H]$. □

The importance of the commutator ideal as a tool for studying Hopf algebras will be made clear in the next proposition.

Proposition 4.4.2. *Let H be a Hopf algebra. The commutator ideal $[H, H]$ is a Hopf ideal, i.e., $H/[H, H]$ is a commutative Hopf algebra.*

Proof. Denote $J = [H, H]$. Let $\rho\colon H \otimes H \to H/J \otimes H/J$ be the map sending $a \otimes b$ to $(a+J)\otimes(b+J)$. As seen in the proof of Proposition 2.1.7, we have $\operatorname{Ker}\rho = J \otimes H + H \otimes J$. Consider the composition map $H \xrightarrow{\Delta} H \otimes H \xrightarrow{\rho} H/J \otimes H/J$ and let $L = \operatorname{Ker}\rho\circ\Delta$. Then $L = \Delta^{-1}(J \otimes H + H \otimes J)$. By the first isomorphism theorem (see Proposition 0.1.1), $\rho \circ \Delta$ induces an injective map $H/L \hookrightarrow H/J \otimes H/J$. Since $H/J \otimes H/J$ is commutative, it follows that H/L is commutative as well. By Lemma 4.4.1, we have that $J \subseteq L$ and therefore, $\Delta(J) \subseteq J \otimes H + H \otimes J$.

The counit ε maps commutators into commutators in K because it is a homomorphism of algebras and commutators in K are 0 because K is commutative. Hence, the augmentation ideal H^+ gives rise to a commutative quotient H/H^+. By definition, $J \subseteq H^+$, i.e., $\varepsilon(J) = 0$.

We have proved so far that J is both an ideal and a coideal, it remains to check that J is stable under the antipode S. Consider the composition map $H^{\mathrm{op}} \xrightarrow{S} H \to H/J$, mapping $h \in H$ to $S(h) + J$. It is an homomorphism of algebras by Proposition 2.3.2. Its kernel is the ideal $S^{-1}(J)$. Hence, we have an induced map $\big(H/S^{-1}(J)\big)^{\mathrm{op}} \hookrightarrow H/J$, which is injective. Since

H/J is commutative, then so is $H/S^{-1}(J)$. Thus, $J \subseteq S^{-1}(J)$, i.e., $S(J) \subseteq J$, which ends the proof. □

This result means that we can study the commutative Hopf algebra $H/[H,H]$, which is an easier task, and hopefully derive some properties of the original Hopf algebra H. This is exactly what we will do in the following sections for $H = B/I = K[y][z;\theta,d]$. We would like to thank Ken Brown for pointing out that looking to the largest commutative quotient of a Hopf algebra in generally a good idea.

The commutator ideal $[H,H]$ turns out to be a principal ideal, being generated as a two-sided ideal by

$$[z,y] = zy - yz = (p_{12}-1)\,yz + p_{11}y^2 + \tau_1 y + \tau_2 z + \tau_0.$$

Indeed, in the quotient $H/\langle[z,y]\rangle$, the variables $\bar{y}$ and $\bar{z}$ commute (by construction) and they generate $H/\langle[z,y]\rangle$ as an algebra. Hence, $H/\langle[z,y]\rangle$ is commutative and by definition, $[H,H]$ is contained in $\langle[z,y]\rangle$ (the other inclusion being trivial).

4.5 Hopf algebra structures on the quantum plane

In this section, we study the quantum plane $K_{p_{12}}[y,z]$, with $p_{12} \neq 1$. It is the Ore extension of $K[y]$ defined by the relation $zy = p_{12}yz$. Its commutator ideal, as defined in the previous section, takes the form $\langle yz\rangle$, because

$$[z,y] = zy - yz = \underbrace{(p_{12}-1)}_{\neq 0}yz$$

and K has characteristic zero. By Proposition 4.4.2, we have that $K_{p_{12}}[y,z]/\langle yz\rangle$ is a commutative Hopf algebra and it is affine, because it is still finitely generated. We observe that the quantum plane $K_{p_{12}}[y,z]$ becomes indistinguishable from the classical polynomial algebra $K[y,z]$, when we take the quotient by $\langle yz\rangle$ in both, that is, $K_{p_{12}}[y,z]/\langle yz\rangle \simeq K[y,z]/\langle yz\rangle$. This happens because, in the free algebra $K\{y,z\}$, the ideal generated by yz and $zy - p_{12}yz$ is the same as the ideal generated by yz and $yz - zy$. We identify $K_{p_{12}}[y,z]/\langle yz\rangle$ with $K[y,z]/\langle yz\rangle$ so that we can regard it as an affine variety and use some techniques of algebraic geometry to study it in the following lemma. It is a small incursion into algebraic geometry, but a self-contained one.

The only requisites are knowledge about maximal ideals and winding automorphisms, which are anyway covered in Section 0.1 and Section 2.2, respectively. The aim of the following lemma is to establish an invariant of commutative Hopf algebras.

Lemma 4.5.1. *Let H be a commutative Hopf affine algebra and M be a maximal ideal of H. Then, $\dim M/M^2 = \dim H^+/{H^+}^2$. In particular, $\dim M/M^2$ remains constant, as M varies over all maximal ideals.*

Proof. The augmentation ideal H^+ is a maximal ideal, because the quotient algebra is a field (see Proposition 2.3.5) and H is commutative. An automorphism σ of H induces an isomorphism $H/M \xrightarrow{\simeq} H/\sigma(M)$ mapping $a+M$ to $\sigma(a)+\sigma(M)$. Thus, H/M is a field if and only if $H/\sigma(M)$ is. Therefore, automorphisms of H, and in particular, winding automorphisms τ_α, preserve maximal ideals. Given an algebra homomorphism $\alpha\colon H \to K$, recall that a (left) winding automorphism τ_α is defined by $\tau_\alpha(h) = \sum \alpha(h_1)h_2$. Let $G := \mathrm{Hom}(H,K)$ be the group of algebra homomorphisms from H to K, with the convolution operation (see Corollary 2.3.6). Fix $\alpha \in G$ and $h \in H$. In the following computation, remember that α maps 1_H to 1_K and hence, acts as the identity on scalars:

$$\begin{aligned}\varepsilon(h) = \alpha(\varepsilon(h)) = \alpha\left(\sum S(h_1)h_2\right) &= \sum \alpha(S(h_1))\alpha(h_2)\\ &= \alpha\left(\sum \alpha\left(S(h_1)\right)h_2\right) = \alpha(\tau_{\alpha\circ S}(h)).\end{aligned}$$

This means that $h \in H^+$ if and only if $\tau_{\alpha\circ S}(h) \in \operatorname{Ker}\alpha$. In other words, $\tau_{\alpha\circ S}(H^+) = \operatorname{Ker}\alpha$.

If M is a maximal ideal of H, then H/M is a field on one hand and an algebra over K on the other. Thus, H/M is a field extension of K. Since H/M is affine because H itself is affine, it follows by [AM94][Ex. 18, Chapter 5] that H/M is a finite algebraic extension of K. But K is algebraically closed and therefore, $H/M \simeq K$. This means that the projection map $H \to H/M \simeq K$ can be seen as an element of G. Hence, for every maximal ideal M, there exists $\alpha \in G$ such that $\operatorname{Ker}\alpha = M$ and thus, $M = \tau_{\alpha\circ S}(H^+)$. In particular, we also have that $M^2 = \tau_{\alpha\circ S}({H^+}^2)$ and $\tau_{\alpha\circ S}$ induces an isomorphism between M/M^2 and $H^+/{H^+}^2$. This implies that $\dim M/M^2 = \dim H^+/{H^+}^2$. □

The main result in this section gives a negative solution to the problem of endowing the quantum plane with a Hopf algebra structure.

Theorem 4.5.2. *The quantum plane $K_{p_{12}}[y,z]$, with $p_{12} \neq 0,1$, cannot have a Hopf algebra structure.*

Proof. Seeking a contradiction, let us assume otherwise, which is, that exists a Hopf algebra structure on $K_{p_{12}}[y,z]$. Hence, by Proposition 4.4.2 and the preceding comments in this section, the quotient $H = K[y,z]/\langle yz\rangle$ is a affine commutative Hopf algebra. Lemma 4.5.1 tells us that $\dim M/\dim M^2$ is constant. If we find two distinct maximal ideals M and M' such that $\dim M/M^2 \neq \dim M'/M'^2$, then we are done.

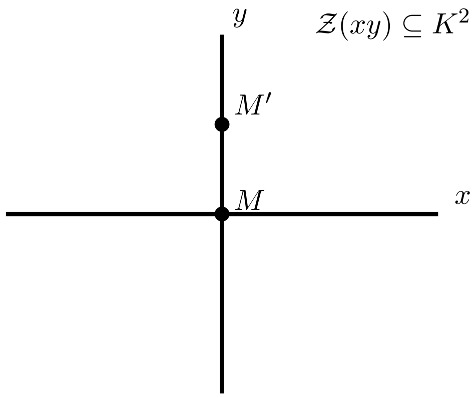
We have not said much about how maximal ideals of H look like, which is what we do next. They come from the maximal ideals of $K[y,z]$, by the second isomorphism theorem (see Proposition 0.1.1). We recall a classical result in Algebraic Geometry known as Hilbert's Nullstellensatz, whose proof can be found, for instance, in [Har77, p. 4]. It states that the maximal ideals of $K[x_1,\cdots,x_n]$ are exactly of the form $\langle x_1-\lambda_1,\cdots,x_n-\lambda_n\rangle$, for some elements $\lambda_1,\cdots,\lambda_n \in K$. Thus, we conclude that the maximal ideals of $K[y,z]/\langle yz\rangle$ are of the form $M = \langle \bar{y}-\lambda, \bar{z}-\mu\rangle$, for some $\lambda,\mu \in K$ and where $\bar{y} = y + \langle yz\rangle$ and $\bar{z} = z + \langle yz\rangle$.

We check that $M = \langle \bar{y}, \bar{z}\rangle$ has $\dim M/M^2 = 2$. Since $\bar{y}\,\bar{z} = 0$ and H is commutative, it is clear that $\{\bar{y}^i\}_{i\geq 1} \cup \{\bar{z}^i\}_{i\geq 1}$ forms a basis of M. For the same reason, it is clear that $\{\bar{y}^i\}_{i\geq 2} \cup \{\bar{z}^i\}_{i\geq 2}$ forms a basis of M^2. Hence, it follows that a basis for M/M^2 is given by $\bar{y}+M^2$ and $\bar{z}+M^2$. As such, $\dim M/M^2 = 2$.

On the other hand, we check that $M' = \langle \bar{y}, \bar{z}-1\rangle$ has $\dim M'/M'^2 = 1$. The reason for this is that M' has basis $\{\bar{y}^i\}_{i\geq 1} \cup \left\{(\bar{z}-1)^i\right\}_{i\geq 1}$, but because $\bar{y} = \bar{y}(1-\bar{z}) \in M'^2$, we have that M'^2 has basis $\{\bar{y}^i\}_{i\geq 1} \cup \{(\bar{z}-1)^i\}_{i\geq 2}$. Hence, the quotient M'/M'^2 is spanned by $(\bar{z}-1)+M'^2$, i.e., it is one-dimensional. □

Back to the example $B = B^2(a,b,c)$, we had shown that B/I is a quantum plane and in Section 4.3, we had concluded that it does not have a Hopf algebra structure extending that of $K[y]$. Now, with Theorem 4.5.2, we conclude that it does not have a Hopf algebra structure at all because $p_{12} = -1 \neq 1$.

Although we are not plunging deep into algebraic geometry and we only mention the precise results that we need, we make a small detour at the end of this section to link the algebraic argument given above with its geometric counterpart and thus, we hope to better motivate the result. What Hilbert's Nullstellensatz says is that maximal ideals in $K[x_1,\cdots,x_n]$ correspond bijectively to points $(\lambda_1, \cdots, \lambda_n) \in K^n$ (the hypothesis that K is algebraically closed is used). The ideal $\langle yz \rangle \in K[y,z]$ corresponds to the affine variety $\mathcal{Z}(yz)$ (the set of points $(y,z) \in K^2$ such that $yz = 0$), which is drawn in following figure. Given a maximal ideal M of $K[y,z]/\langle yz \rangle$ (which corresponds to a point P in the variety $\mathcal{Z}(yz)$), the $\dim M/M^2$ corresponds geometrically to the dimension of the tangent space of $\mathcal{Z}(yz)$ at P, seen as a manifold.



Intuitively, we see in the figure above that for all points in $\mathcal{Z}(yz)$ except $(0,0)$, the tangent space at $\mathcal{Z}(yz)$ in those points is a line (one-dimensional). But in $(0,0)$, which we call a singular point, a tangent line is not well defined. Not coincidently, we see that $M = \langle \bar{y}, \bar{z} \rangle$ corresponds precisely to the point $(0,0)$ and that $M' = \langle \bar{y}, \bar{z} - 1 \rangle$ corresponds to the point $(0,1)$, by Hilbert's Nullstellensatz.

4.6 Hopf algebra structures on the quantum Weyl algebra

In this section, we study the (first) quantum Weyl algebra $A_1^{p_{12}}(K)$, with $p_{12} \neq 1$ (otherwise, it would just be classical first Weyl algebra, which is studied in the next section). Denote $H = A_1^{p_{12}}(K)$. The defining relation is $zy = p_{12}yz + 1$. Thus, its commutator ideal $[H,H]$ is generated by $[z,y] = (p_{12} - 1)yz + 1$. This means that in $H/[H,H]$, which is commutative, we have $(1 - p_{12})\bar{y}\,\bar{z} = 1 = (1 - p_{12})\bar{z}\,\bar{y}$. Since $p_{12} \neq 1$, it follows that $\bar{y}$ and $\bar{z}$ are invertible and $\bar{z}^{-1} = (1 - p_{12})\bar{y}$. Therefore, $H/[H,H]$ is isomorphic to the Laurent polynomial ring $K[t^{\pm 1}]$. We write t instead of $\bar{z}$ to avoid confusion. Like in the previous section, we arrive at a negative answer.

Theorem 4.6.1. *The quantum Weyl algebra $A_1^{p_{12}}(K)$, with $p_{12} \neq 1$, cannot be a Hopf algebra.*

Proof. By [GZ10, Lemma 6.2 and Lemma 7.2], there exists a grouplike element $h \in H$ such that $\pi(h) = t$, where $\pi\colon H \to K[t^{\pm 1}]$ is the projection induced by the isomorphism $H/[H, H] \simeq K[t^{\pm 1}]$. Note that h cannot be a scalar because it projects to t. Grouplike elements are invertible in Hopf algebras, with the inverse given by the antipode, and therefore we conclude that there exists in H an invertible element which is not a scalar. This cannot happen in an Ore extension of a domain by Lemma 1.2.2, yielding a contradiction. □

4.7 Hopf algebra structures on differential operator rings

In this section, we study differential operator rings, the Ore extensions in which the automorphism θ is trivial. In Section 4.4, we saw that such cases correspond to having $p_{12} = 1$ and that we could assume without loss of generality that $\tau_2 = 0$. Then, we have $H = K[y, z; d]$, where the derivation d of $K[y]$ is determined by $d(y) = p_{11}y^2 + \tau_1 y + \tau_0$. It is a polynomial in y of degree at most two. We reduce the number of possibilities, up to isomorphism, to five:

(a) If $p_{11} = \tau_0 = \tau_1 = 0$, then $d(y) = 0$ and H is a classical **polynomial algebra**, which is a trivial differential operator ring.

(b) If $p_{11} = \tau_1 = 0$ and $\tau_0 \neq 0$, then $d(y) = \tau_0$ has degree zero. Hence, setting a variable change $z' = \frac{z}{\tau_0}$ and a new derivation given by $d'(y) = 1$ yields the classical (first) **Weyl algebra** $A_1(K)$, where the defining relation is $z'y = yz' + 1$.

(c) If $p_{11} = 0$ and $\tau_1 \neq 0$, then $d(y) = \tau_1 y + \tau_0$ has degree one. Then, setting variable changes $y' = -y - \frac{\tau_0}{\tau_1}$ and $z' = \frac{z}{\tau_1}$ yields the **universal enveloping algebra of the nontrivial Lie algebra of dimension two**, where the defining relation is $y'z' = z'y' + y'$.

(d) If $p_{11} \neq 0$, then we can assume that $p_{11} = 1$. Thus $d(y) = y^2 + \tau_1 y + \tau_0$ is a monic polynomial of degree two, which has two roots, say λ and μ, because K is algebraically closed. Factorizing $y^2 + \tau_1 y + \tau_0 = (y - \lambda)(y - \mu)$, we conclude that $\tau_1 = -\lambda - \mu$ and $\tau_0 = \lambda\mu$. Thus, if the two roots coincide, then $\tau_1 = -2\lambda$ and $\tau_0 = \lambda^2$. This implies that $\tau_1^2 = 4\lambda^2 = 4\tau_0$. Conversely, if $\tau_1^2 = 4\tau_0$, then $(\lambda + \mu)^2 = 4\lambda\mu$ which is equivalent to $(\lambda - \mu)^2 = 0$. Thus, the two roots coincide. In summary, we have that the roots of $d(y)$ **coincide** if $\tau_1^2 = 4\tau_0$ and are **distinct** otherwise.

We condense the information in the following table, dropping the primes in the variables:

Parameter pair (with $p_{12} = 1$)	**Tail** (with $\tau_2 = 0$)	**Defining relation**	**Type**
$p_{11} = 0$	$\tau_0 = \tau_1 = 0$	$zy = yz$	Polynomial algebra $K[y, z]$
	$\tau_0 \neq 0, \tau_1 = 0$	$zy = yz + 1$	Weyl Algebra $A_1(K)$
	$\tau_1 \neq 0$	$[y, z] = y$	Universal enveloping algebra
$p_{11} = 1$	$\tau_1^2 = 4\tau_0$	$\begin{cases} zy = yz + d(y) \\ \text{Roots of } d(y) \text{ coincide} \end{cases}$	Generic dif. op. ring
	$\tau_1^2 \neq 4\tau_0$	$\begin{cases} zy = yz + d(y) \\ \text{Roots of } d(y) \text{ are distinct} \end{cases}$	Generic dif. op. ring

Of these five cases, some will provide positive answers, meaning that they admit a Hopf algebra structure, and others will not. For instance, the polynomial algebra $K[y, z]$ and the universal enveloping algebra of the non-trivial Lie algebra both have classical Hopf algebra structures (see the list of examples in Section 2.3). The first Weyl algebra, which has been widely studied as one of the first examples of noncommutative algebras, does not. It is known that the first Weyl algebra is simple, which means that it has no nontrivial proper ideals (see for instance [Bre14, Example 1.13]). However, in any Hopf algebra $H \neq K$, the augmentation ideal H^+ is always nontrivial and proper. It is nontrivial because otherwise, we would have $H \simeq H/H^+ \simeq K$ (by Proposition 2.3.5) and it is proper because $1 \notin H^+$.

Suppose that $p_{11} = 1$. A theorem by Cartier states that commutative Hopf algebras over fields of characteristic zero are reduced, meaning that they do not have nilpotent elements (other than zero) (see [Wat12, Section 11.4]). In a differential operator ring H in which the two roots of $d(y)$ coincide, we have that the commutator ideal $[H, H]$ is generated by

$$[z, y] = d(y) = (y - \lambda)^2,$$

for some $\lambda \in K$ and hence, in the Hopf algebra $H/[H, H]$ the element $\bar{y} - \lambda$ is nilpotent, which is a contradiction.

It remains to study the differential operator rings H in which the two roots of $d(y)$ are distinct, say $d(y) = (y-\lambda)(y-\mu)$, for some $\lambda, \mu \in K$ with $\lambda \neq \mu$. The commutator ideal $[H, H]$ is precisely generated by $(y-\lambda)(y-\mu)$. In the next lemma, we establish the form of the commutative Hopf quotient $H/[H, H]$.

Lemma 4.7.1. *The commutative Hopf algebra $H/[H, H]$ is isomorphic to $K[t] \times K[t]$.*

Proof. We have that $H/[H, H]$ is isomorphic to $K[y, z]/\langle (y-\lambda)(y-\mu)\rangle$, very much in the spirit of what happened in the case of the quantum plane in Section 4.5. This is because in the free algebra $K\{y, z\}$, the ideal spanned by $yz - zy - d(y)$ and $d(y)$ is the same as the ideal spanned $yz - zy$ and $d(y)$. Set $I = \langle y - \lambda\rangle$ and $J = \langle y - \mu\rangle$. Define the algebra homomorphism $\varphi: K[y, z] \to K[y, z]/I \times K[y, z]/J$ mapping $h \in K[y, z]$ to $(h + I, h + J)$. We have that $\mu - \lambda = (y - \lambda) - (y - \mu) \in K\backslash\{0\} \cap (I + J)$ and thus, $I + J = K[y, z]$. This implies that φ is surjective and by Proposition 0.1.2, we have that $IJ = I \cap J$. On the other hand, it is clear that $\operatorname{Ker}\varphi = I \cap J$. Hence, φ factors through $IJ = \langle (y-\lambda)(y-\mu)\rangle$ yielding an isomorphism between $K[y, z]/\langle (y-\lambda)(y-\mu)\rangle$ and $K[y, z]/\langle y-\lambda\rangle \times K[y, z]/\langle y-\mu\rangle$. Now, since $K[y]/\langle y-\lambda\rangle \simeq K$, it follows that $K[y, z]/\langle y - \lambda\rangle \simeq K[t]$. Likewise, $K[y, z]/\langle y - \mu\rangle \simeq K[t]$. □

We stress the fact that $H/[H, H]$ is isomorphic, not to a tensor product, but to a cartesian product. In particular, it is not a domain. The next result tells us that we can have a Hopf algebra structure on the cartesian product of a Hopf algebra with itself.

Let T be a Hopf algebra. Note that $T \times T$ is a T-module via pointwise multiplication. As a consequence, $(T \times T) \otimes (T \times T)$ is a $T \otimes T$-module and hence, it makes sense to write $\Delta(h)(1, 1) \otimes (1, 1)$ meaning $\sum (h_1, h_1) \otimes (h_2, h_2)$ for $h \in T$ and in Sweedler's notation. We will denote both Hopf algebra structures on T and $T \times T$ by (Δ, ε, S), being clear which one is meant at each point by the respective argument (which lies either in T or $T \times T$).

Proposition 4.7.2. *If T is a Hopf algebra, then $T \times T$ has a Hopf algebra structure, given by*

$$\begin{aligned}
\Delta(h, h') &= \frac{\Delta(h) + \Delta(h')}{2}(1, 1) \otimes (1, 1) + \frac{\Delta(h) - \Delta(h')}{2}(1, -1) \otimes (1, -1), \\
\varepsilon(h, h') &= \varepsilon(h), \\
S(h, h') &= \left(S(h), S(h')\right).
\end{aligned}$$

for $h, h' \in T$.

Proof. While one might check the Hopf algebra axioms directly from the definition, we follow the reasoning that motivated defining these comultiplications, counit and antipode in the first place. Recall the group algebra $K[\mathbb{Z}_2]$ has a basis $\{1, g\}$, with $g^2 = 1$. We start by showing that $T \times T \simeq T \otimes K[\mathbb{Z}_2]$ as an algebra. Define $\varphi : T \times T \to T \otimes K[\mathbb{Z}_2]$ by

$$\varphi(h, h') = h \otimes \frac{1+g}{2} + h' \otimes \frac{1-g}{2}.$$

It is straightforward to check that it is a unital algebra homomorphism, because of the pointwise algebra structure on both sides and the properties of the tensor properties. Furthermore, φ is surjective because $\varphi(h, h) = h \otimes 1$ and $\varphi(h, -h) = h \otimes g$ and $\{1, g\}$ is a basis of $K[\mathbb{Z}_2]$. On the other hand, $(1 + g)/2$ and $(1 - g)/2$ also form a basis of $K[\mathbb{Z}_2]$. Hence, if $h \otimes \frac{1+g}{2} + h' \otimes \frac{1-g}{2} = 0$, then $h = h' = 0$ by Lemma 0.1.3. Thus, φ is an isomorphism.

Recall that the group algebra $K[\mathbb{Z}_2]$ has a Hopf algebra structure, with g being grouplike, that is, $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ and $S(g) = g^{-1} = g$. By Proposition 2.1.5, $T \otimes K[\mathbb{Z}_2]$ also has a Hopf algebra structure given by

$$\begin{aligned} \Delta(h \otimes 1) &= \sum h_1 \otimes 1 \otimes h_2 \otimes 1, & \Delta(h \otimes g) &= \sum h_1 \otimes g \otimes h_2 \otimes g, \\ \varepsilon(h \otimes 1) &= \varepsilon(h), & \varepsilon(h \otimes g) &= \varepsilon(h) \\ S(h \otimes 1) &= S(h) \otimes 1, & S(h \otimes g) &= S(h) \otimes g. \end{aligned}$$

which we can transport to $T \times T$ via the isomorphism just constructed.

We can actually say more about φ: it is a isomorphism of T-modules, with the module structure on $T \times T$ mentioned in the paragraph preceding the proposition and the module in $T \otimes K[\mathbb{Z}_2]$ given by left multiplication in the first tensor. In light of this isomorphism, we have $\varphi(h(1, 1)) = h \otimes 1$ and $\varphi(h(1, -1)) = h \otimes g$. Observe that we can write

$$\varphi(h, h') = h \otimes \frac{1+g}{2} + h' \otimes \frac{1-g}{2} = \frac{h+h'}{2} \otimes 1 + \frac{h-h'}{2} \otimes g.$$

At this point, we have everything to write explicitly $\Delta(h, h')$, $\varepsilon(h, h')$ and $S(h, h')$, just by chasing the isomorphisms, and the result follows. □

As a consequence of Proposition 4.7.2, we have that the algebra $H/[H, H] \simeq K[t] \times K[t]$ has a Hopf algebra structure induced by the standard Hopf algebra structure on $K[t]$. But this does not imply that H itself has a Hopf algebra structure. As a matter of fact, we can show that it does not have one with the help of [GZ10, Theorem 0.1].

In [GZ10], the noetherian Hopf algebra domains of GK dimension two which satisfy a certain homological property are classified. This property turns out to be equivalent to the Hopf algebra admitting an infinite dimensional commutative quotient. In our case, we have that $H = K[y][z; d]$ is a noetherian domain and $H/[H, H] \simeq K[t] \times K[t]$ is an infinite dimensional commutative algebra. All of the classified Hopf algebras in [GZ10, Theorem 0.1] are either commutative or have non-scalar invertible elements, except one which is the universal enveloping algebra $U(L)$ of the non-trivial Lie algebra L (with basis x, y and $[x, y] = x$). Since H is neither commutative and it does not have non-scalar invertible elements by Lemma 1.2.2, it can only be a Hopf algebra if it is isomorphic to $U(L)$. But we can prove that this is not the case as follows.

If there is an algebra isomorphism $\varphi: H \to U(L)$, then it maps bijectively the commutator ideal $[H, H]$ to the commutator ideal $[U(L), U(L)]$. Hence φ induces an algebra isomorphism $\bar{\varphi}: H/[H, H] \to U(L)/[U(L), U(L)]$. But on the other hand $H/[H, H] \simeq K[t] \times K[t]$ and $U(L)/[U(L), U(L)] \simeq K[t]$. This two algebras cannot be isomorphic because the latter is a domain but the former is not.

We summarize the results of this section in the following theorem.

Theorem 4.7.3. *Of the differential operator rings exhibited in Section 4.7, the following admit a Hopf algebra structure: the polynomial algebra $K[y, z]$ and the universal enveloping algebra of the non-trivial Lie algebra. As for the remaining ones, the first Weyl algebra and both the differential operator rings with $d(y)$ of degree 2, they cannot have a Hopf algebra structure and therefore, neither can the original double Ore extension from which they were obtained.*

4.8 Conclusions

Let A be a Hopf algebra and let $p_{12}, p_{11} \in K$, with $p_{12} \neq 0$, $\tau_0, \tau_1, \tau_2 \in A$, $\sigma: A \to M_{2\times 2}(A)$ be $\bullet^T-$invertible algebra homomorphism and $\delta: A \to M_{2\times 1}(A)$ be a $\sigma-$multiderivation. Consider the double Ore extension $B = A_{(p_{12},p_{11})}[y, z; \sigma, \delta, (\tau_0, \tau_1, \tau_2)]$ of A and let $I = BA^+B$ be the ideal of B generated by A^+. Suppose that A^+ is stable under σ, the $\bullet^T-$inverse $\hat{\sigma}$ and δ. In Section 4.2, we proved that the quotient B/I is a Hopf algebra and it is a isomorphic to a double Ore extension of K which is also an iterated Ore extension $K[y][z; \theta, d]$, where θ and d are determined by

$$\sigma(y) = p_{12}y + \varepsilon(\tau_2), \quad d(y) = p_{11}y^2 + \varepsilon(\tau_1)y + \varepsilon(\tau_0). \tag{4.8.1}$$

We have also seen that we can assume without loss of generality that either $(p_{12}, p_{11}) = (1, 1)$ or $(p_{12}, p_{11}) = (p_{12}, 0)$. In Section 4.4, we classified the algebra $K[y][z; \theta, d]$ up to isomorphism, depending on the data p_{12}, p_{11}, $\varepsilon(\tau_2)$, $\varepsilon(\tau_1)$ and $\varepsilon(\tau_2)$. Then, through Sections 4.5 to 4.7, we checked whether $K[y][z; \theta, d]$ may or may not admit a Hopf algebra structure. The caveat is that if $K[y][z; \theta, d]$ does not admit a Hopf algebra structure, then neither the algebra B. We summarize the obtained results in the following table with the appropriate references. We write $\bar{\tau}_i$ instead of $\varepsilon(\tau_i)$ for $i = 0, 1, 2$. Note that the condition $\bar{\tau}_i = 0$ translates to $\tau_i \in A^+$, when we go from B/I back to B.

p_{12}	p_{11}	$\bar{\tau}_0$	$\bar{\tau}_1$	$\bar{\tau}_2$	**Type**	**H. A. S.**	**Section**
$\neq 1$	0	$\bar{\tau}_0 = \frac{\bar{\tau}_1\bar{\tau}_2}{p_{12}-1}$			Quantum plane	No	4.5
$\neq 1$	0	$\bar{\tau}_0 \neq \frac{\bar{\tau}_1\bar{\tau}_2}{p_{12}-1}$			Quantum Weyl alg.	No	4.6
1	1	${\bar{\tau}_1}^2 = 4\bar{\tau}_0$		0	Diff. op. ring	No	4.7
1	1	${\bar{\tau}_1}^2 \neq 4\bar{\tau}_0$		0	Diff. op. ring	No	4.7
1	0	0	0	0	$K[y, z]$	Yes	4.7
1	0	$\neq 0$	0	0	Weyl algebra	No	4.7
1	0	any	$\neq 0$	0	Univ. env. algebra	Yes	4.7
1	any	any	any	$\neq 0$	Univ. env. algebra	Yes	4.7

A "No" in the "H. A. S." column in the table above means that the double Ore extension B does not admit a Hopf algebra structure for parameter pair and tail described in the five leftmost columns. Note that the condition $\bar{\tau}_i = 0$ translates to $\tau_i \in A^+$, when we go from B/I back to B. In particular, the table above tells us that there are no Hopf algebra structures on $K[y][z; \theta, d]$ (and consequently on the double Ore extension B) if $p_{12} \neq 1$. This is in agreement with Proposition 4.3.1 in Section 4.3, which concerns iterated Hopf Ore extensions of order two.

On the other direction, a "Yes" in the "H. A. S." column does not imply that B has a Hopf algebra structure, only that B/I does. We do not know how the determine the structure maps $(\Delta_B, \varepsilon_B, S_B)$ from those $(\Delta_{B/I}, \varepsilon_{B/I}, S_{B/I})$ of B/I. But we know that the Hopf algebra structure on the quotient is given by

$$\Delta_{B/I}(h+I) = \Delta_B(h) + I \otimes B + B \otimes I,$$
$$\varepsilon_{B/I}(h+I) = \varepsilon(h),$$
$$S_{B/I}(h+I) = S(h) + I$$

for all $h \in B$, so we can reconstruct $\Delta_B(h)$ up to an element of $I \otimes B + B \otimes I \subseteq B \otimes B$, we can reconstruct $\varepsilon(h)$ and we can reconstruct $S(h)$ up to an element of I. This narrows down the possibilities and helps in the task of finding a Hopf algebra structure in a double Ore extension.

It is worth mentioning that the findings presented in the table above are not new results. We proved in Section 4.2 that $K[y][z;\theta,d]$ has GK dimension two. In [Zhu13, Theorem 7.4] [BOZZ15, Theorem in §3.3], the connected Hopf algebras of GK dimension two are classified and there are only two isomorphism classes: the polynomial algebra and the universal enveloping algebra of the non-trivial Lie algebra of dimension two. These two classes comprise all the "Yes" cases in the table above. It is worth mentioning that we did not assume that B/I is connected.

In another paper, [GZ10], there is a classification of the noetherian Hopf algebras domains of GK dimension two which satisfy a certain homological property (see [GZ10, Theorem 0.1]). The algebra $B/I \simeq K[y][z;\theta,d]$ that we consider is always a noetherian domain and it satisfies that homological property, so it falls under the classification. That classification agrees with the results of table above as well. We were not aware of the article [GZ10] until the later stages of this thesis and nonetheless, our approach is a bit different except at the end of section 4.7, where we rely on a result from the mentioned article. Our ultimate goal is to focus on double Ore extensions in general, which we hope to do in the future.

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