

Seminar on Semigroups, Automata and Languages

Factoring the determinant of a finite semigroup

Benjamin Steinberg
City College of New York

Abstract: In 1875, Smith computed the determinant of the $n \times n$ matrix whose i, j -entry is $\gcd(i, j)$. This matrix is nothing other than the multiplication table of the semigroup $\{1, \dots, n\}$ under the binary operation of $\gcd$. Dedekind introduced in 1896, in correspondence with Frobenius, the group determinant of a finite group, which is the formal determinant of the multiplication table of the group. His motivation came from trying to compute the discriminant of a finite Galois extension of $\mathbb{Q}$. Frobenius famously invented the representation theory of finite groups precisely to factor Dedekind's group determinant. Studying a generalization of the group determinant also led Frobenius to characterizing the algebras nowadays called Frobenius algebras.

Of course, there is no reason to only compute the determinant of a group multiplication table. People have looked at both semigroups and Latin squares. Generalizing the work of Smith mentioned earlier, Lindstrom and Wilf independently published papers in 1967 computing the semigroup determinant of a meet semilattice. For Wilf, the primary motivation was to compute determinants of various matrices arising from combinatorial objects and his paper, entitled Hadamard determinants, Möbius functions, and the chromatic number of a graph, was published in the Bulletin of the AMS.

In 1998, Jay Wood factored the determinant of the multiplicative semigroup of a finite commutative chain ring; these are rings whose ideals form a chain like $\mathbb{Z}/p^n\mathbb{Z}$ with p a prime. His motivation was to prove a generalization of the MacWilliams extension theorem for codes over a finite field to codes over a finite ring. This theorem says that a partial isometry between codes can be extended to a global isometry with respect to the Hamming metric. Eventually, it was shown using nonsemigroup theoretic methods that the MacWilliams extension theorem holds precisely for finite Frobenius rings.

In this talk, I'll survey some results I've obtained in a more systematic attempt to factor the semigroup determinant. The semigroup determinant is a special case of Frobenius's paratrophic determinant of an algebra and so the semigroup determinant is nonzero if and only if the semigroup algebra is Frobenius. A fair amount of work on Frobenius semigroup algebras was done by Ponizovsky, Wenger, Okninski and others.

Our main results are a factorization of the determinant of an inverse semigroup (generalizing simultaneously Frobenius and the Lindstrom-Wilf theorem) and a factorization of the semigroup determinant of a commutative semigroup. The latter result is more difficult and clarifies and refines the earlier work of Ponizovskii and Wenger on Frobenius commutative semigroup algebras, as well as generalizing Wood's computation for chain rings. Our final result says that the semigroup algebra of a finite Frobenius ring is a Frobenius algebra. This implies as a special case the celebrated result of Okninski and Putcha on the semisimplicity of the algebra of $n \times n$ matrices over a finite field. It also suggests that the MacWilliams extension theorem for finite Frobenius rings should be provable using the semigroup determinant method.

Date: Friday, 12 November 2021, 14:30

Place: Online Zoom meeting



Com o apoio do projeto UIDB/00144/2020