ON THE CLASSIFICATION OF THE RIGID MOTIONS OF THE PLANE

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We present here two very well-known and elementary theorems in Euclidean Geometry. The first one is a form of the result which says that a rigid motion can be written as a composition of at most three reflections and is usually proven at the beginning of a typical course in Geometry. The second one is the Classification Theorem for rigid motions of the plane, which is presented typically at the end of the course.

In this article we show that with the tools we use to prove the first theorem a proof of the second one follows almost immediately. This shows that the Classification Theorem is of a much more elementary nature than is usually thought and so it is a relevant example of a comprehensive yet basic result. Thus, beyond a Geometry course, the approach presented here might be used in an Introduction to Proofs course or, perhaps, as a way of bringing in a geometric perspective when reflections are treated in a basic Linear Algebra course.

We do not claim here originality: indeed, the interest lies in the construction we give in the first proof, and in the way this construction is used to derive the classification theorem (in a way not dissimilar to [1], but simpler, we hope).

We recall that the classification theorem (often attributed to Michel Chasles [1]), states that every rigid motion of the plane different from the identity is either a *reflection in a line*, a *translation*, defined as the product of the reflections in two parallel lines, a *rotation*, the product of the reflections in two intersecting lines [2], or a *glide reflection*, a translation followed (or preceded) by a reflection in a line parallel to the direction of the translation. It is perhaps worth mentioning that the isometries in dimension 3 were classified much earlier by Euler. Whereas in the plane a general "rigid body displacement" is hence a translation or a rotation about a point, in space it may also be the product of a translation along a line by a rotation about that line (Mozzi-Chasles' theorem).

Our starting point is the elementary fact that, given two different points A and B, and strictly positive numbers a, b such that $a+b > |AB|$

[1] See e.g. Coolidge [2, p. 273], or Nikulin-Shafarevich [3, p. 72]. Michel Chasles (1793 –1880) was a renowned French mathematician and geometer.

[2] Note that the rotation is a half turn if and only if the two lines are perpendicular to each other.

(the distance between A and B), there exist exactly two points, X_1 and X_2, located on opposite sides of the line through A and B, $r = \overleftrightarrow{AB}$, such that $|AX_i| = a$ and $|BX_i| = b$, $i = 1, 2$. Moreover, the points X_1 and X_2 are interchanged by σ^r, the reflection in the line r, which is also the perpendicular bisector of the segment determined by them.

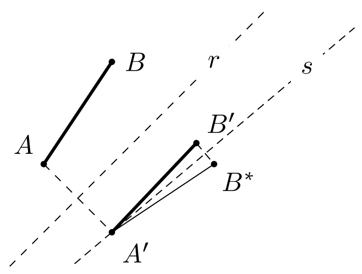
A first obvious consequence is that any point in the plane is characterised by its distances to three given noncollinear points. Therefore, the images of any three noncollinear points characterise any isometry. Another consequence is the following theorem, which can be proved simply and elegantly.

Theorem 1. *Given points A, A', B and B' in the plane such that $|AB| = |A'B'| \neq 0$, there exist exactly* [1] *two rigid motions sending A to A' and B to B'. The first of these can be written as the composition of two reflections. The second one is obtained composing the first one with the reflection in the line $\overleftrightarrow{A'B'}$.*

Our proof proceeds by defining i, the first rigid motion satisfying $i(A) = A'$ and $i(B) = B'$. Then, the second motion will be $j := \sigma^{\overleftrightarrow{A'B'}} \circ i \neq i$. In the general case, we define $r = \operatorname{bis} \overline{AA'}$ as the perpendicular bisector of the segment $\overline{AA'}$, so that A' is the reflection on r of A, $A' = \sigma^r(A)$, and we define $s = \operatorname{bis} \overline{B^* B'}$ where $B^* = \sigma^r(B)$. Finally, we define $i = \sigma^s \circ \sigma^r$. Since

$$|A'B^*| = |\sigma^r(A)\, \sigma^r(B)| = |AB| = |A'B'|,$$

$i(A) = A'$ and, of course, $i(B) = B'$. Now, if $A = A'$ then we may instead define $r = \overleftrightarrow{AB}$. Also, if $B' = B^*$ then we may define $s = \overleftrightarrow{A'B'}$. In both cases, one still has $i(A) = A'$ and $i(B) = B'$. Note that if $A = A'$ and $B = B'$ then i is the identity. Here ends the proof.



Now, the classification of the rigid motions of the plane is accomplished by the following corollary. The basic idea is that the first rigid motion i from Theorem 1 can be identified by the three points A, $i(A)$ and $i^2(A)$ for a generic point A.

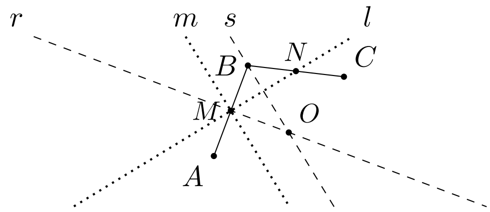
[1]The fact that there exist no third motion with the same properties follows easily from the fact mentioned above that the images of any three noncollinear points characterise any isometry.

Corollary 2. *Let i be a rigid motion of the plane.*

(1) *If there exists A such that $B = i(A)$ is different from A and such that $i(B) = A$, then i is either the reflection in a line or a half turn.*
(2) *If there exists A such that A, $B = i(A)$ and $C = i(B)$ are all three distinct and* collinear*, then i is either a translation or a glide reflection*
(3) *If there exists A such that A, $B = i(A)$ and $C = i(B)$ are all three distinct and* not collinear*, then i is either a rotation or a glide reflection.*

Proof. We define r and s as in the proof of Theorem 1. Since in all three cases $A' = B \neq A$, $r = \operatorname{bis}\overline{AB}$ and $B^* = A$.

(1) In this case, $B^* = A = B'$, and hence $s = \overleftrightarrow{AB}$ and $r \perp s$. Then $i = \sigma^s \circ \sigma^r$ is the half turn about the midpoint of $\overline{AB}$. The second motion is
$$j = \sigma^{\overleftrightarrow{A'B'}} \circ \sigma^{\overleftrightarrow{AB}} \circ \sigma^{\operatorname{bis}\overline{AB}} = \sigma^{\operatorname{bis}\overline{AB}}.$$
(2) Here, $B^* = A \neq C = B'$, $s = \operatorname{bis}\overline{AC}$ and $B \in s$. Hence $r \perp \overleftrightarrow{AB} = \overleftrightarrow{AC} \perp s$, thus i is a translation and $j = \sigma^{\overleftrightarrow{AB}} \circ i$ is a glide reflection.
(3) Now, again $B^* = A \neq C = B'$, $s = \operatorname{bis}\overline{AC}$ and $B \in s$. But now $r \not\perp \overleftrightarrow{AC} \perp s$, so r and s meet [1] at a point O and i is a rotation about O. It remains to see that $j = \sigma^{\overleftrightarrow{A'B'}} \circ i$ is a glide reflection.



For this, let M and N be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively, and let m be the perpendicular to $l = \overleftrightarrow{MN}$, through M. Then $s \perp l$, so $k = \sigma^l \circ \sigma^s \circ \sigma^m$ is a glide reflection. Moreover, $B = k(A)$ and $C = k(B)$ so, in view of Theorem 1, we have $k = i$ or $k = j$. In order to exclude the case $k = i$, we recall that the composition of reflections in two perpendicular lines is a half turn about their point of intersection and, therefore, is independent of the order in which the reflections are taken.

[1] This is the only place we use the Euclidean parallel axiom. In the hyperbolic plane, alternatively, i could be a limit rotation or a translation, depending on the relative position of r and s. Thus the argument presented here also gives the classification of isometries in the hyperbolic plane (cf. [1]).

Thus $k = i$ would mean that $\sigma^l \circ \sigma^s \circ \sigma^m = \sigma^s \circ \sigma^l \circ \sigma^m = \sigma^s \circ \sigma^r$ and hence $\sigma^l \circ \sigma^m = \sigma^r$, which is impossible.

$\square$

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