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Aspects of Noncommutative Differential Geometry

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Resumo

Hom-conexões e formas integrais associadas foram introduzidas e estudadas por T. Brzeziński como uma versão adjunta da noção usual de uma conexão em geometria não-comutativa. Dada uma Hom-conexão plana num Cálculo Diferencial (Ω, d) sobre uma álgebra A, obtém-se o complexo integral, que para diversas álgebras prova-se ser isomorfo ao complexo de de Rham (que também é denominado cálculo diferencial no contexto de Grupos Quânticos). Para uma álgebra A com uma Hom-conexão plana, clarificamos quando é que os complexos de de Rham e integral são isomorfos. Especializamos o nosso estudo ao caso em que um cálculo diferencial de dimensão n possa ser construído numa álgebra exterior quântica sobre um A-bimódulo. Alguns critérios são fornecidos para bimódulos livres com estrutura de bimódulo diagonal ou triangular superior. Ilustramos os resultados para cálculos diferenciais numa álgebra polinomial quântica multivariada e num n-espaço quântico de Manin.

Hom-bimódulos covariantes são introduzidos e, nesse "Hom-cenário", onde (co)álgebras associadas com um certo endomorfismo satisfazem umas condições de (co)associatividade e de (co)unidade torcidas, a sua teoria de estrutura é estudada em detalhe. Esses resultados estruturais sobre Hom-bimódulos bicovariantes e covariantes à esquerda são também representados em forma de coordenadas. Prova-se que a categoria dos Hom-bimódulos bicovariantes é uma categoria monoidal (pré-)trançada. A noção de Hom-módulos de Yetter-Drinfel'd é apresentada e, em seguida, é provado que a categoria dos Hom-módulos de Yetter-Drinfel'd é também uma categoria monoidal (pré-)trançada. Por fim, sob certas condições, é provado que estas categorias monoidais são equivalentes no sentido monoidal trançado.

As noções de Hom-coanel, estrutura Hom-entrelaçada e Hom-módulo entrelaçado associado são introduzidas. Um teorema de extensão do anel de base de um Hom-coanel é provado e, em seguida, é usado para adquirir uma versão "Hom" do coanel de Sweedler. Motivado por um resultado de Brzeziński, associado a uma estrutura Hom-entrelaçada, é construído um Hom-coanel e uma identificação dos Hom-módulos entrelaçados com os Hom-comódulos desse Hom-coanel é demonstrada. É provado, então, que a álgebra dual desse Hom-coanel é uma álgebra de convolução ψ-torcida. Por construção, mostra-se que um Hom-Doi-Koppinen datum é obtido a partir de uma estrutura Hom-entrelaçada e que os Doi-Koppinen Hom-Hopf módulos são os mesmos que os Hom-módulos entrelaçados associados. Uma construção semelhante, com respeito ao Hom-Doi-Koppinen datum, é também fornecida. Uma coleção de Hom-Hopf módulos são apresentadas como exemplos especiais de estruturas Hom-entrelaçadas e Hom-módulos entrelaçados correspondentes. E também são consideradas estruturas de todos os Hom-coanéis relevantes.

As definições de Cálculo Diferencial de Primeira Ordem (FODC) numa Hom-álgebra monoidal e FODC à esquerda covariante sobre um espaço Hom-quântico à esquerda, com respeito a uma Hom-Hopf álgebra monoidal, são dadas. Em seguida, a covariância à esquerda de um Hom-FODC é caracterizada. Também é descrita a extensão de um FODC sobre uma Hom-álgebra monoidal para um cálculo Hom-diferencial universal. Introduz-se os conceitos de FODC covariante à esquerda e FODC bicovariante sobre uma Hom-Hopf álgebra monoidal e, após isso, os Hom-ideais e espaços quânticos Hom-tangentes associados são estudados. A noção de Hom-Lie álgebra quântica (ou generalizada) de um FODC bicovariante sobre uma Hom-Hopf álgebra monoidal, em que versões generalizadas de relações de anti-simetria e identidades de Hom-Jacobi são satisfeitas, é obtida.

Abstract

Hom-connections or noncommutative connections of the second type and associated integral forms have been introduced and studied by T.Brzeziński as an adjoint version of the usual notion of a noncommutative connection in a right module over an associative algebra. Given a flat hom-connection on a differential calculus (Ω, d) over an algebra A yields the integral complex which for various algebras has been shown to be isomorphic to the noncommutative de Rham complex (which is also termed the differential calculus in the context of quantum groups). We shed further light on the question when the integral and the de Rham complex are isomorphic for an algebra A with a flat hom-connection. We specialize our study to the case where an n-dimensional differential calculus can be constructed on a quantum exterior algebra over an A-bimodule. Criteria are given for free bimodules with diagonal or upper triangular bimodule structure. Our results are illustrated for a differential calculus on a multivariate quantum polynomial algebra and for a differential calculus on Manin's quantum n-space.

Covariant Hom-bimodules, as a generalization of Woronowicz' covariant bimodules, are introduced and the structure theory of them in the Hom-setting, where (co)algebras have twisted (co)associativity and (co)unity conditions along with an associated endomorphism, is studied in a detailed way. These structural results about left-covariant and bicovariant Hom-bimodules were also restated in coordinate form. The category of bicovariant Hom-bimodules is proved to be a (pre-)braided monoidal category. The notion of Yetter-Drinfel'd Hom-module is presented and it is shown that the category of Yetter-Drinfel'd Hom-modules is a (pre-)braided tensor category as well. Finally, it is verified that these tensor categories are braided monoidal equivalent under certain conditions.

The notions of Hom-coring, Hom-entwining structure and associated entwined Hom-module are introduced. A theorem regarding base ring extension of a Hom-coring is proven and then is used to acquire the Hom-version of Sweedler's coring. Motivated by a result of Brzeziński, a Hom-coring associated to an Hom-entwining structure is constructed and an identification of entwined Hom-modules with Hom-comodules of this Hom-coring is shown. The dual algebra of this Hom-coring is proven to be a ψ-twisted convolution algebra. By a construction, it is shown that a Hom-Doi-Koppinen datum comes from a Hom-entwining structure and that the Doi-Koppinen Hom-Hopf modules are the same as the associated entwined Hom-modules. A similar construction regarding an alternative Hom-Doi-Koppinen datum is also given. A collection of Hom-Hopf-type modules are gathered as special examples of Hom-entwining structures and corresponding entwined Hom-modules, and structures of all relevant Hom-corings are also considered.

The definitions of first order differential calculus (FODC) on a monoidal Hom-algebra and left-covariant FODC over a left Hom-quantum space with respect to a monoidal Hom-Hopf algebra are given, and the left-covariance of a Hom-FODC is characterized. The extension of a FODC over a monoidal Hom-algebra to a universal Hom-differential calculus is described. The concepts of left-covariant and bicovariant FODC over monoidal Hom-Hopf algebras are introduced, and their associated right Hom-ideals and quantum Hom-tangent spaces are studied. The notion of quantum (or generalized) Hom-Lie algebra of a bicovariant FODC over a monoidal Hom-Hopf algebra is obtained, in which generalized versions of antisymmetry relation and Hom-Jacobi identity are satisfied .

Contents

Introduction

Hom-connections and associated integral forms have been introduced and studied by T.Brzeziński, in [15], as an adjoint version of the usual notion of a connection in noncommutative geometry. Given a flat hom-connection on a differential calculus (Ω, d) over an algebra A yields the integral complex which for various algebras has been shown to be isomorphic to the noncommutative de Rham complex (which is also termed the differential calculus in the context of quantum groups). The purpose of Chapter 2 is to provide further examples of algebras which contribute to the general study of algebras with this property. Hereby, necessary and sufficient conditions to extend the associated first order differential calculus (abbreviated, FODC) (Ω^1, d) of a right twisted multi-derivation (∂, σ) on an algebra A to a full differential calculus (Ω, d) on the quantum exterior algebra Ω of Ω^1 is presented. A chain map between the de Rham complex and the integral complex is defined and a criterion is given to assure an isomorphism between the de Rham and the integral complexes for free right upper-triangular twisted multi-derivations whose associated FODC can be extended to a full differential calculus on the quantum exterior algebra. Easier criteria for FODCs with a diagonal bimodule structure are established and are applied to show that a multivariate quantum polynomial algebra satisfies the strong Poincaré duality in the sense of T.Brzeziński with respect to some canonical FODC. Lastly, it is shown that for a certain two-parameter n-dimensional (upper-triangular) calculus over Manin's quantum n-space the de Rham and integral complexes are isomorphic.

The first examples of Hom-type algebras arose in connection with quasi-deformations of Lie algebras of vector fields, particularly q-deformations of Witt and Virasoro algebras (see [1, 22, 23, 24, 25, 33, 34, 48, 43, 57]), which have a crucial role in conformal field theory. These deformed algebras are obtained by replacing the derivation with a twisted derivation (σ-derivation), and are no longer Lie algebras due to the fact that they satisfy a twisted Jacobi identity. Motivated by these examples and their generalization, the notions of quasi-Lie algebras, quasi-Hom-Lie algebras and Hom-Lie algebras were introduced by Hartwig, Larsson and Silvestrov in [42, 54, 55, 56] to deal with Lie algebras, Lie superalgebras and color Lie algebras within the same framework. The Hom-associative algebras generalizing associative algebras by introducing twisted associativity law along a linear endomorphism were suggested by Makhlouf and Silvestrov in [62] to give rise to Hom-Lie algebras by means of commutator bracket defined using the multiplication in Hom-associative algebras. For other features of Hom-associative algebras regarding the unitality and twist property one should also see [37, 38]. The construction of the free Hom-associative algebra and the enveloping algebra of a Hom-Lie algebra was given [83], and the so-called *twisting principle* was introduced in [85] to construct examples of Hom-type objects and related algebraic structures from classical structures. The concepts of Hom-coassociative coalgebras, Hom-bialgebras and Hom-Hopf algebras and their properties were considered in [63, 64, 84]. Hence, representation theory, cohomology and deformation theory of Hom-associative and Hom-Lie algebras were studied, and Hom-analogues of many classical structures such as n-ary Nambu algebras, alternative, Jordan, Malcev, Novikov, Rota-Baxter algebras were considered in [2, 3, 4, 7, 8, 61, 65, 67, 74, 90, 91, 92]. Hom-type generalizations of (co)quasitriangular bialgebras and (quantum) Yang-Baxter equation were also studied by Yau in [86, 87, 88, 89, 93, 94].

For a given braided tensor(=monoidal) category $\mathcal{C}$, a braided monoidal category $\widetilde{\mathcal{H}}(\mathcal{C})$ with non-trivial associativity and unity constraints was constructed by Caenepeel and Goyvaerts in [21], and the counterparts of Hom-type structures are investigated in the context of monoidal categories. They obtained the symmetric monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ for the category of modules over a commutative ring k and introduced *monoidal* Hom-(co)algebras, Hom-bialgebras and Hom-Hopf algebras as (co)algebras, bialgebras and Hopf algebras in this tensor category. Besides its appropriateness to highlight the general structures systematically, the framework of monoidal categories provides a way to see what additional requirements in the definitions of Hom-structures are needed and convenient for certain kinds of applications. In the original definitions of Hom-type structures in [62, 63, 64], the deforming linear endomorphism (structure map) was not required to be either multiplicative or bijective; one should check the results in [21] and [66], respectively, to see the necessity of the multiplicativity and bijectivity assumptions on the structure map in order to have monoidal structures on the categories of modules and Yetter-Drinfeld modules over (monoidal) Hom-biagebras, respectively. Further properties of monoidal Hom-Hopf algebras and many structures on them have been lately studied [26],[27],[28], [29], [39], [58].

Covariant bimodules were introduced by Woronowicz in [82] to construct differential calculi on Hopf algebras, where bicovariant bimodules (or Hopf bimodules) are considered as Hopf algebraic analogue to the notion of vector bundles over a Lie group. In Chapter 3, the notions of left(right)-covariant Hom-bimodules and bicovariant Hom-bimodules are introduced to have twisted, generalized versions of the concepts of left(right)-covariant bimodules and bicovariant bimodules. Afterwards, the structure theory of covariant bimodules over monoidal Hom-Hopf algebras is studied in coordinate-free setting and then the main results are restated in coordinate form. Furthermore, it is shown that the categories of left(right)-covariant Hom-bimodules and bicovariant Hom-bimodules are tensor categories equipped with a monoidal structure defined by a coequalizer which is modified by a suitable insertion of a related nontrivial associator. Additionally, it is proven that the category of bicovariant bimodules over a monoidal Hom-Hopf algebra forms a (pre-)braided monoidal category (with nontrivial associators and unitors). In the meantime, (right-right) Hom-Yetter-Drinfeld modules are proposed as a deformed version of the classical ones and it is demonstrated that the category of Hom-Yetter-Drinfeld modules can be set as a (pre-)braided tensor category endowed with a tensor product over a commutative ring k described by the diagonal Hom-action and codiagonal Hom-coaction (together with nontrivial associators and unitors). As one of the main consequences of the chapter, the fundamental theorem of Hom-Hopf modules, which is provided in [21], is extended to a (pre-)braided monoidal equivalence between the category of bicovariant Hom-bimodules and the category of (right-right) Hom-Yetter-Drinfeld modules.

Motivated by the study of symmetry properties of noncommutative principal bundles, *entwining structures* (over a commutative ring k) were introduced in [11] as a triple $(A, C)_\psi$ consisting of a k-algebra A, a k-coalgebra C and a k-module map $\psi : C \otimes A \to A \otimes C$ satisfying four conditions regarding the relationships between the so-called entwining map and algebra and coalgebra structures. The main aim of Chapter 4 is to generalize the entwining structures, entwined modules and the associated corings within the framework of monoidal Hom-structures and then to study Hopf-type modules such as (relative) Hopf modules, (anti) Yetter-Drinfeld modules, Doi-Koppinen Hopf modules, Long dimodules, etc., in the Hom-setting. The idea is to replace algebra and coalgebra in a classical entwining structure with a monoidal Hom-algebra and a monoidal Hom-coalgebra to make a definition of Hom-entwining structures and associated entwined Hom-modules. Following [13], these entwined Hom-modules are identified with Hom-comodules of the associated Hom-coring. The dual algebra of this Hom-coring is proven to be the Koppinen smash. Furthermore, we give a construction regarding Hom-Doi-Kopinen datum and Doi-Koppinen Hom-Hopf modules as special cases of Hom-entwining structures and associated entwined Hom-modules. Besides, we introduce alternative Hom-Doi-Koppinen datum. By using these constructions, we get Hom-versions of the aforementioned Hopf-type modules as special cases of entwined Hom-modules, and give examples of Hom-corings in addition to trivial Hom-coring and canonical Hom-coring.

The general theory of covariant differential calculus on quantum groups was presented in [82], [80], [81]. Following the work [82] of Woronowicz, in Chapter 5, after the notions of first order differential calculus (FODC) on a monoidal Hom-algebra and left-covariant FODC over a left Hom-quantum space with respect to a monoidal Hom-Hopf algebra being introduced, the left-covariance of a Hom-FODC is characterized as well. The extension of a FODC over a monoidal Hom-algebra to a universal Hom-differential calculus is described (for the classical case, that is, for an introduction on the differential envelope of an algebra A one should refer to [32], [31]). In the rest of the chapter, the concepts of left-covariant and bicovariant FODC over a monoidal Hom-Hopf algebra (H,α) are studied in detail. A subobject of $ker\varepsilon$, which is right Hom-ideal of (H,α), and a quantum Hom-tangent space are associated to each left-covariant FODC over a monoidal Hom-Hopf algebra: It is indicated that left-covariant Hom-FODCs are in one-to one correspondence with these right Hom-ideals, and that the quantum Hom-tangent space and the left coinvariant of the monoidal Hom-Hopf algebra on Hom-FODC form a nondegenerate dual pair. The quantum Hom-tangent space associated to a bicovariant Hom-FODC is equipped with an analogue of Lie bracket (or commutator) through Woronowicz' braiding and it is proven that this commutator satisfies quantum (or generalized) versions of the antisymmetry relation and Hom-Jacobi identity, which is therefore called the quantum (or generalized) Hom-Lie algebra of that bicovariant Hom-FODC.

The content of Chapter 2 consists of the results from a paper by the author and

Christian Lomp in [47]. Much of the contents of Chapter 3 and Chapter 4 consists of results from the preprints [45] and [46], respectively, by the author.

Chapter 1

Preliminaries

This chapter contains some definitions and results regarding fundamental algebraic structures such as (co)algebras, bialgebras, Hopf algebras and their (co)modules, which are constructed in a (braided) tensor category. For a solid background on (braided) monoidal categories one should refer to [60] and [49].

1.1 Monoidal Categories

Let $\mathcal{C}$ be a category and consider a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. An *associativity constraint* for $\otimes$ is a natural isomorphism

$$a : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \to \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes).$$

This means that for any triple (U, V, W) objects of $\mathcal{C}$ there exist an isomorphism

$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W),$$

such that the following diagram commutes:

$$\begin{array}{ccc} (U \otimes V) \otimes W & \xrightarrow{a_{U,V,W}} & U \otimes (V \otimes W) \\ {\scriptstyle (f\otimes g)\otimes h}\downarrow & & \downarrow{\scriptstyle f\otimes(g\otimes h)} \\ (U' \otimes V') \otimes W' & \xrightarrow{a_{U',V',W'}} & U' \otimes (V' \otimes W') \end{array}$$

whenever $f : U \to U'$, $g : V \to V'$ and $h : W \to W'$ are morphisms in the category.

The associativity constraint a is said to satisfy the *Pentagon Axiom* if the following diagram commutes:

$$\begin{array}{ccc}
(U\otimes(V\otimes W))\otimes X & \xleftarrow{a_{U,V,W}\otimes \mathrm{id}_X} & ((U\otimes V)\otimes W)\otimes X \\
 & & \downarrow{\scriptstyle a_{U\otimes V,W,X}} \\
\downarrow{\scriptstyle a_{U,V\otimes W,X}} & & (U\otimes V)\otimes(W\otimes X) \\
 & & \downarrow{\scriptstyle a_{U,V,W\otimes X}} \\
U\otimes((V\otimes W)\otimes X) & \xrightarrow[\mathrm{id}_U\otimes a_{V,W,X}]{} & U\otimes(V\otimes(W\otimes X))
\end{array} \qquad \text{(Pentagon)}$$

for all objects U,V,W,X of $\mathcal{C}$. A *left unit constraint* (resp. a *right unit constraint*) with respect to an object I is a natural isomorphism l (resp. r) between the functors $I\otimes -$ (resp. $-\otimes I$) and the identity functor of $\mathcal{C}$. This means that there are natural isomorphisms $l_V : I\otimes V \to V$ and $r_V : V\otimes I \to V$, for all object $V\in\mathcal{C}$. The naturality means that, for any $f : V \to U$, the equations

$$f\circ l_V = l_U\circ(\mathrm{id}_I\otimes f) \qquad f\circ r_V = r_U\circ(f\otimes\mathrm{id}_I)$$

hold.

The *Triangle Axiom* holds for a given associativity constraint a and left and right unit constraints l,r with respect to an object I if the following diagram commutes:

$$\begin{array}{ccc}
(U\otimes I)\otimes V & \xrightarrow{a_{U,I,V}} & U\otimes(I\otimes V) \\
 & {\scriptstyle r_U\otimes\mathrm{id}_V}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_U\otimes l_V} & \\
 & U\otimes V &
\end{array} \qquad \text{(Triangle)}$$

Definition 1.1.1 *A monoidal category $(\mathcal{C},\otimes,I,a,l,r)$ is a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and an associativity constraints a, a left and right unit constraint l and r with respect to I such that the Pentagon and Triangle Axioms hold. The monoidal category is called strict if a, l and r are identities in $\mathcal{C}$.*

Let us denote by $\tau : \mathcal{C}\times\mathcal{C}\to\mathcal{C}\times\mathcal{C}$ the flip functor defined by $\tau(U,V) = (V,U)$ on any pair of objects. A *commutativity constraint* c is a natural isomorphism $c : \otimes \to \otimes\circ\tau$ and we say that it satisfies the Hexagon Axiom if the hexagonal diagrams commute:

$$
\begin{array}{ccc}
U\otimes(V\otimes W) & \xrightarrow{c_{U,V\otimes W}} & (V\otimes W)\otimes U \\
{\scriptstyle a_{U,V,W}}\uparrow & & \downarrow{\scriptstyle a_{V,W,U}} \\
(U\otimes V)\otimes W & & V\otimes(W\otimes U) \\
{\scriptstyle c_{U,V}\otimes id_W}\downarrow & & \uparrow{\scriptstyle id_V\otimes c_{U,V}} \\
(V\otimes U)\otimes W & \xrightarrow{a_{V,U,W}} & V\otimes(U\otimes W)
\end{array}
\qquad
\begin{array}{ccc}
(U\otimes V)\otimes W & \xrightarrow{c_{U\otimes V,W}} & W\otimes(U\otimes V) \\
{\scriptstyle a^{-1}_{U,V,W}}\uparrow & & \downarrow{\scriptstyle a^{-1}_{W,U,V}} \\
U\otimes(V\otimes W) & & (W\otimes U)\otimes V \\
{\scriptstyle id_U\otimes c_{V,W}}\downarrow & & \uparrow{\scriptstyle c_{U,W}\otimes id_V} \\
U\otimes(W\otimes V) & \xrightarrow{a^{-1}_{U,W,V}} & (U\otimes W)\otimes V
\end{array}
\qquad \text{(Hex)}
$$

Definition 1.1.2 *Let $(\mathcal{C},\otimes,I,a,l,r)$ be a monoidal category. A commutativity constraint satisfying the Hexagon Axiom is called a braiding in $\mathcal{C}$. A braided tensor category $(\mathcal{C},\otimes,I,a,l,r,c)$ is a tensor category with a braiding. A monoidal category is said to be symmetric if it is equipped with a braiding c such that $c_{V,U}\circ c_{U,V} = id_{U\otimes V}$ for all objects U,V in the category.*

Convention: In order to ease notation we will drop the subscripts from the associator $a_{U,V,W}$ and unitors l_V, r_V. Moreover we will simply write 1 for the identity morphism of an object.

1.2 Algebras and modules in monoidal categories

Definition 1.2.1 *An algebra A in a monoidal category $\mathcal{C}$ is an object A with morphisms $m: A\otimes A\to A$ and $\eta: I\to A$ in $\mathcal{C}$ such that the following diagrams commute:*

$$
\begin{array}{ccccc}
(A\otimes A)\otimes A & & \xrightarrow{\quad a \quad} & & A\otimes(A\otimes A) \\
{\scriptstyle m\otimes 1}\downarrow & & & & \downarrow{\scriptstyle 1\otimes m} \\
A\otimes A & & & & A\otimes A \\
& \searrow^{m} & & {}^{m}\swarrow & \\
& & A & &
\end{array}
\qquad
\begin{array}{ccccc}
I\otimes A & \xrightarrow{\eta\otimes 1} & A\otimes A & \xleftarrow{1\otimes\eta} & A\otimes I \\
& {\scriptstyle l}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle r} & \\
& & A & &
\end{array}
\qquad \text{(Alg)}
$$

The first diagram corresponds to the associativity axiom and the second one is the unity axiom. A homomorphism of algebras $\alpha: A\to B$ in $\mathcal{C}$ is a morphism in $\mathcal{C}$ such that $\alpha\circ m_A = m_B\circ(\alpha\otimes\alpha)$ and $\alpha\circ\eta_A = \eta_B$.

Definition 1.2.2 *Given an algebra A in a monoidal category $\mathcal{C}$, a left A-module is an object M in $\mathcal{C}$ with a morphism $\varphi: A\otimes M\to M$ such that the following diagrams com-*

mute:

$$
\begin{array}{ccc}
(A\otimes A)\otimes M & \xrightarrow{\;a\;} & A\otimes(A\otimes M) \\
\Big\downarrow{\scriptstyle m\otimes 1} & & \Big\downarrow{\scriptstyle 1\otimes\varphi} \\
A\otimes M & & A\otimes M \\
& {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\
& M &
\end{array}
\qquad
\begin{array}{ccc}
I\otimes M & \xrightarrow{\eta\otimes 1} & A\otimes M \\
& {\scriptstyle l}\searrow & \Big\downarrow{\scriptstyle\varphi} \\
& & M
\end{array}
\tag{Mod}
$$

The first of the above diagrams is the associativity condition and the second is the left unity condition.

The subcategory of left A-modules in $\mathcal{C}$ shall be denoted by ${}_A\mathcal{C}$. Analogously one defines right A-modules and the category of right A-modules shall be denoted by $\mathcal{C}_A$.

Definition 1.2.3 *Let A be an algebra in a monoidal category $\mathcal{C}$. For any object V of $\mathcal{C}$ there exists so-called canonical left A-module structure on $F(V) = A\otimes V$ defined by the composition of morphisms*

$$\varphi : A\otimes F(V) \xrightarrow{a^{-1}} (A\otimes A)\otimes V \xrightarrow{m\otimes 1} A\otimes V = F(V) \tag{1.1}$$

together with a morphism $i_V : V \to F(V)$ defined by

$$i_V : V \xrightarrow{l^{-1}} I\otimes V \xrightarrow{\eta\otimes 1} F(V)$$

The morphism $i_V : V \to F(V)$ satisfies the following property:

$$\varphi\circ(1\otimes i_V) = 1_{F(V)}. \tag{1.2}$$

To prove this we consider the diagram

$$
\begin{array}{ccccccc}
F(V) & \xrightarrow{1\otimes l^{-1}} & A\otimes(I\otimes V) & \xrightarrow{1\otimes(\eta\otimes 1)} & A\otimes F(V) & \xrightarrow{\;\varphi\;} & F(V) \\
& {\scriptstyle r^{-1}\otimes 1}\searrow & \Big\downarrow{\scriptstyle a^{-1}} & & \Big\downarrow{\scriptstyle a^{-1}} & \nearrow{\scriptstyle m\otimes 1} & \\
& & (A\otimes I)\otimes V & \xrightarrow[(1\otimes\eta)\otimes 1]{} & (A\otimes A)\otimes V & &
\end{array}
$$

The first triangle commutes since the Triangle Axiom holds, the naturality of a implies the commutativity of the square and the triangle at the end is the definition of φ. The upper line is the composition $\varphi\circ(1\otimes i_V)$ while the lower line is equal to the identity by the right unity condition of the algebra A.

Definition 1.2.4 *Let A and B be algebras in $\mathcal{C}$. A (A,B)-bimodule M in $\mathcal{C}$ is an object in $\mathcal{C}$ with a left A-module structure $\varphi : A \otimes M \to M$ and a right B-module structure $\phi : M \otimes B \to M$ such that the following diagram commutes:*

$$\begin{array}{ccc} (A \otimes M) \otimes B & \xrightarrow{\ a\ } & A \otimes (M \otimes B) \\ \downarrow{\scriptstyle \varphi\otimes 1} & & \downarrow{\scriptstyle 1\otimes\phi} \\ M \otimes B & & A \otimes M \\ & {\scriptstyle \phi}\searrow \quad \swarrow{\scriptstyle \varphi} & \\ & M & \end{array} \tag{Bimod}$$

We shall denote the category of (A,B)-bimodules by ${}_A\mathcal{C}_B$.

The following theorem has been proven by Schauenburg for strict monoidal categories (see [72], Theorem 5.2). For the reader's sake we include here a proof for an arbitrary monoidal category.

Theorem 1.2.5 *Let A and B be algebras in $\mathcal{C}$. Let V be any object in $\mathcal{C}$, $F(V) = A \otimes V$ be the canonical left A-module with module structure given by φ as in (1.1) and let $i_V : V \to F(V)$ be the canonical morphism attached to $F(V)$. Then the following statements are equivalent:*

(a) $F(V)$ is a (A,B)-bimodule with the canonical left A-module structure;

(b) There exists a morphism $f : V \otimes B \to F(V)$ such that

$$f \circ (1 \otimes m_B) \circ a = \varphi \circ (1 \otimes f) \circ a \circ (f \otimes 1) \tag{Cond1}$$

$$f \circ (1 \otimes \eta_B) = i_V \circ r_V \tag{Cond2}$$

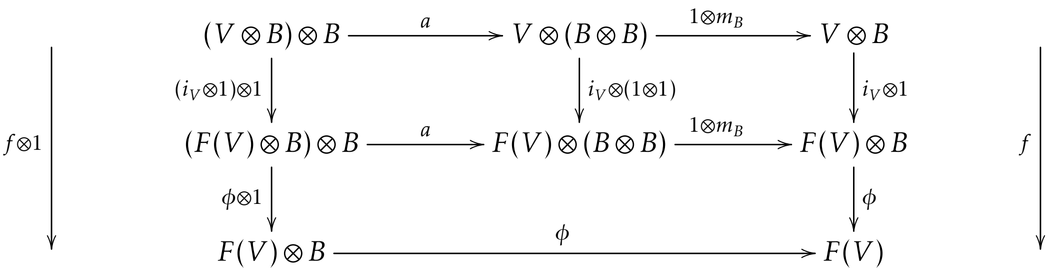
In this case, if $\phi : F(V) \otimes B \to F(V)$ denotes the right B-module structure on $F(V)$, then the morphism f is defined by the composition of morphisms:

$$f : V \otimes B \xrightarrow{i_V \otimes 1} F(V) \otimes B \xrightarrow{\ \phi\ } F(V) \tag{1.3}$$

On the other hand if f is given satisfying (Cond1) and (Cond2), then the right B-module structure on $F(V)$ is given by the composition of morphisms:

$$\phi : F(V) \otimes B \xrightarrow{\ a\ } A \otimes (V \otimes B) \xrightarrow{1 \otimes f} A \otimes F(V) \xrightarrow{\ \varphi\ } F(V) \tag{1.4}$$

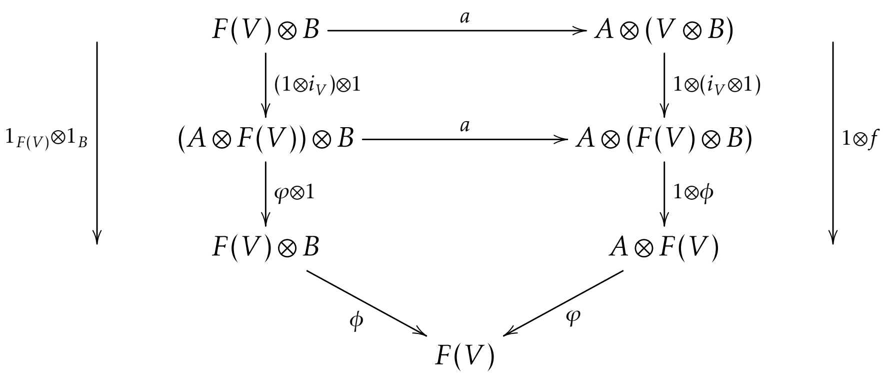
Proof: $(a) \Rightarrow (b)$: Suppose $F(V)$ is an (A,B)-bimodule with right B-module structure ϕ. Let f as in (1.3), i.e. $f = \phi \circ (i_V \otimes 1)$ and consider the following diagram:



The upper left square commutes because of the naturality of a, the upper right square commutes for the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves the composition of the morphisms, while the commutativity of the lower diagram corresponds to the associativity condition for the right B-module structure on $F(V)$. In particular we get

$$f \circ (1 \otimes m_B) \circ a = \phi \circ (f \otimes 1) \tag{1.5}$$

and we will show that $\phi = \varphi \circ (1 \otimes f) \circ a$ which is a consequence of the compatibility condition (Bimod) and the identity (1.2). We have namely the following diagram:

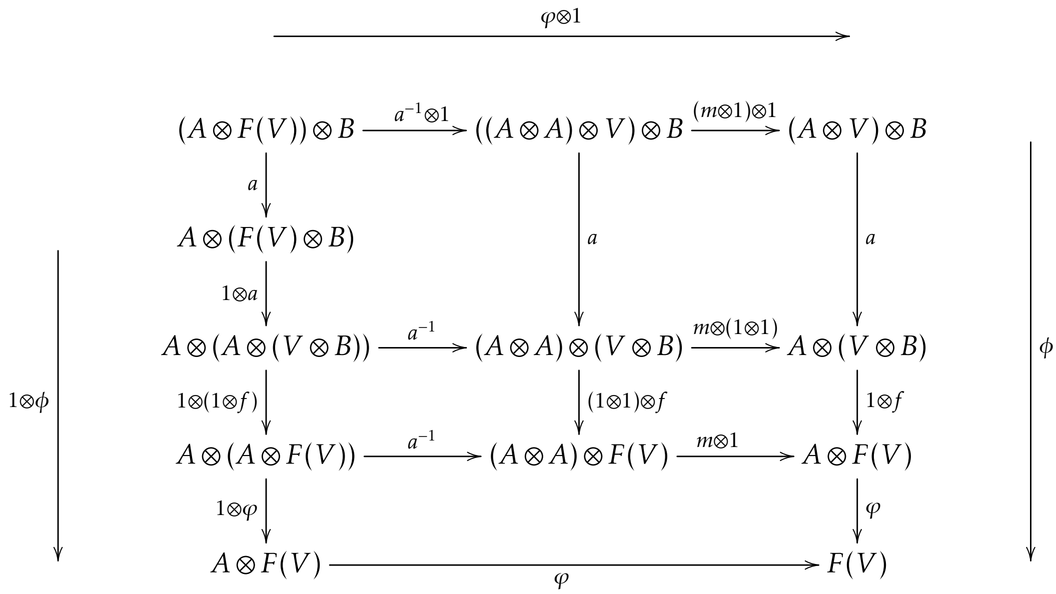


The commutativity of the square follows from the naturality of a and the pentagon is the bimodule compatibility condition (Bimod). We then conclude that $\varphi \circ (1 \otimes f) \circ a = \phi$ holds, which, when substituted in (1.5) yields (Cond1). (Cond2) indeed holds:

$$f \circ (1 \otimes \eta_B) = \phi \circ (i_V \otimes 1) \circ (1 \otimes \eta_B) = \phi \circ (1 \otimes \eta_B) \circ (i_V \otimes 1) = r_{F(V)} \circ (i_V \otimes 1) = i_V \circ r_V,$$

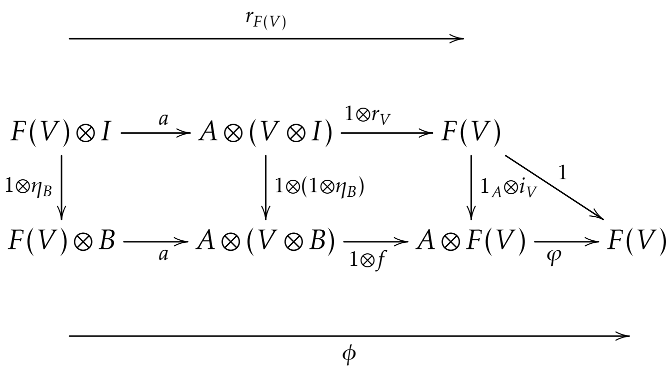
where the penultimate equality results from the right unity condition for ϕ, which is $\phi \circ (1 \otimes \eta_B) = r_{F(V)}$, the last equality is induced by the naturality of r.

$(b) \Rightarrow (a)$ Let $f : V \otimes B \to F(V)$ be given satisfying (Cond1) and (Cond2). There are three diagrams to check, namely the two diagrams in (Mod) and the compatibility diagram (Bimod): Let us first check the compatibility condition by considering the following diagram:

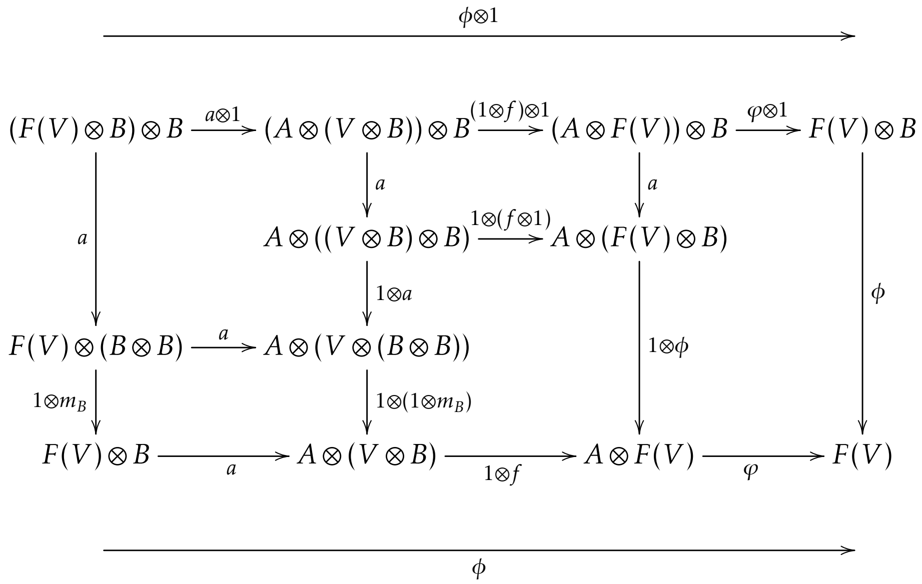


The upper left pentagon commutes by the Pentagon Axiom. The commutativity of the upper right and the middle left squares results from the naturality of a. The middle right square commute since the functor $\otimes$ preserves the composition of the morphisms. The associativity condition for the canonical left A-module structure φ on $F(V)$ implies the commutativity of the lower pentagon. Hence the compatibility condition (Bimod) follows.

Now we check the unity condition of (Mod) for $\phi : F(V) \otimes B \to F(V)$ by considering the following diagram:



The triangle is the property (1.2), the square in the middle follows from the condition (Cond2) and the left square is induced by the naturality of a. Lastly, we prove the associativity condition of (Mod) for ϕ by the following diagram:

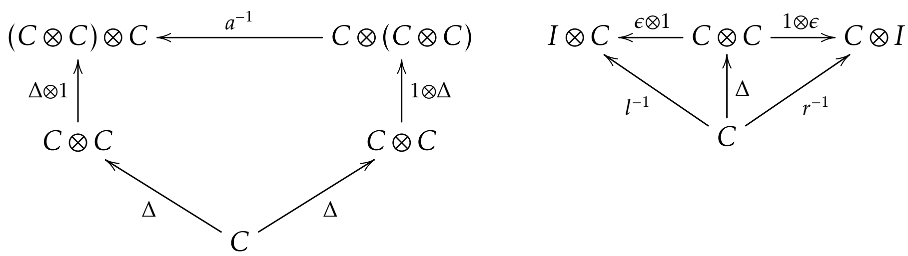


where the lower middle pentagon is precisely the condition (Cond1) and the right pentagon is the compatibility condition (Bimod), and the commutativity of the rest comes from the Pentagon Axiom and the naturality of the associativity constraint a. □

1.3 Coalgebras and comodules in monoidal categories

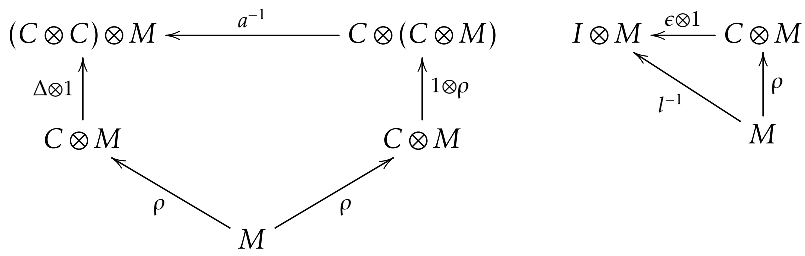
Inverting the direction of arrows in the diagram that defines algebras and modules in a monoidal category $\mathcal{C}$ we define colagebras and comodules in $\mathcal{C}$. Hence

Definition 1.3.1 *A coalgebra C over a monoidal category $\mathcal{C}$ is an object C with morphisms $\Delta : C \to C \otimes C$ and $\epsilon : C \to I$ in $\mathcal{C}$ such that the following diagrams commute:*



(CoAlg)

Definition 1.3.2 *Given a coalgebra C in a monoidal category $\mathcal{C}$, a left C-comodule is an object M in $\mathcal{C}$ with a morphism $\rho : M \to C \otimes M$ such that the following diagrams commute:*



(CoMod)

The subcategory of left C-comodules in $\mathcal{C}$ shall be denoted by ${}^{C}\mathcal{C}$. Analogously one defines right C-comodules and the category of right C-comodules shall be denoted by $\mathcal{C}^{C}$.

Definition 1.3.3 *Let C be a coalgebra in a monoidal category $\mathcal{C}$. For any object V of $\mathcal{C}$ there exists a canonical left C-comodule structure on $F(V) = C \otimes V$ defined by the composition of morphisms*

$$\rho : F(V) \xrightarrow{\Delta\otimes 1} (C \otimes C) \otimes V \xrightarrow{a} C \otimes F(V) \tag{1.6}$$

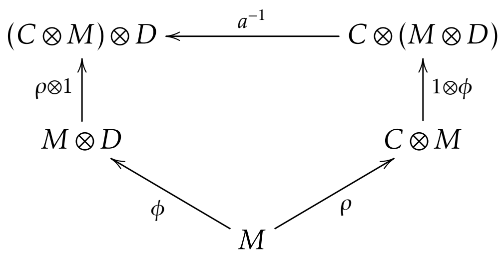
together with a morphism $j_V : F(V) \to V$ defined by

$$j_V : F(V) \xrightarrow{\epsilon\otimes 1} I \otimes V \xrightarrow{l} V$$

The morphism $j_V : F(V) \to V$ satisfies the following property:

$$(1 \otimes j_V) \circ \rho = 1_{F(V)}. \tag{1.7}$$

Definition 1.3.4 *Let C and D be coalgebras in $\mathcal{C}$. A (C,D)-bicomodule M in $\mathcal{C}$ is an object in $\mathcal{C}$ with a left C-comodule structure $\rho : M \to C \otimes M$ and a right B-comodule structure $\phi : M \to M \otimes D$ such that the following diagram commutes:*



(BiComod)

We shall denote the category of (C,D)-bicomodules by ${}^{C}\mathcal{C}^{D}$.

1.4 Bialgebras and Hopf modules in braided monoidal categories

Let $\mathcal{C}$ be a braided monoidal category with commutativity constraint $c_{U,V} : U \otimes V \to V \otimes U$, for all objects U, V in $\mathcal{C}$. To simplify notation we define the following isomorphism τ_{23} for objects U, V, W, X in $\mathcal{C}$:

$$\begin{array}{ccc} (U \otimes V) \otimes (W \otimes X) & \xrightarrow{b} & U \otimes ((V \otimes W) \otimes X) \\ \downarrow{\scriptstyle \tau_{23}} & & \downarrow{\scriptstyle 1 \otimes c_{V,W} \otimes 1} \\ (U \otimes W) \otimes (V \otimes X) & \xleftarrow{b^{-1}} & U \otimes ((W \otimes V) \otimes X) \end{array} \tag{flip}$$

where

$$b : (U \otimes V) \otimes (W \otimes X) \xrightarrow{a} U \otimes (V \otimes (W \otimes X)) \xrightarrow{1 \otimes a^{-1}} U \otimes ((V \otimes W) \otimes X)$$

Definition 1.4.1 *Given two algebras (A, m_A, η_A) and (B, m_B, η_B) in a symmetric monoidal category, their tensor product $A \otimes B$ carries a canonical algebra structure in $\mathcal{C}$ with the product:*

$$m_{A \otimes B} : \ (A \otimes B) \otimes (A \otimes B) \xrightarrow{\tau_{23}} (A \otimes A) \otimes (B \otimes B) \xrightarrow{m_A \otimes m_B} A \otimes B$$

and unit:

$$\eta_{A \otimes B} : \ I \xrightarrow{l^{-1}} I \otimes I \xrightarrow{\eta_A \otimes \eta_B} A \otimes B$$

Analogously for two coalgebras $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ in $\mathcal{C}$, their tensor product $C \otimes D$ carries a canonical coalgebra structure in $\mathcal{C}$ with the coproduct:

$$\Delta_{C \otimes D} : \ C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} (C \otimes C) \otimes (D \otimes D) \xrightarrow{\tau_{23}^{-1}} (C \otimes D) \otimes (C \otimes D)$$

and the counit:

$$\epsilon_{C \otimes D} : \ C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} I \otimes I \xrightarrow{l} I$$

With these canonical algebra and coalgebra structures on the tensor product of two algebras resp. coalgebras we can define bialgebras in a braided monoidal category.

Definition 1.4.2 *A bialgebra H in $\mathcal{C}$ is an object H that has an algebra structure (H, m, η) in $\mathcal{C}$ and a coalgebra structure (H, Δ, ϵ) in $\mathcal{C}$, such that Δ and ϵ are algebra homomorphisms with respect to the canonical algebra structure on $H \otimes H$ respectively on I.*

Let H be a bialgebra in a braided monoidal category $\mathcal{C}$. Then the categories of left (resp. right) H-modules ${}_H\mathcal{C}$ (resp. $\mathcal{C}_H$) as well as the categories of left (resp. right) H-comodules ${}^H\mathcal{C}$ (resp. $\mathcal{C}^H$) are braided monoidal subcategories of $\mathcal{C}$. For example for $U, V \in {}_H\mathcal{C}$ the tensor product $U \otimes V$ belongs to ${}_H\mathcal{C}$ by the *left diagonal action* defined as

$$\phi_{U\otimes V}:\ H\otimes(U\otimes V)\xrightarrow{\Delta\otimes 1\otimes 1}(H\otimes H)\otimes(U\otimes V)\xrightarrow{\tau_{23}}(H\otimes U)\otimes(H\otimes V)\xrightarrow{\phi_U\otimes\phi_V}U\otimes V\ .$$

Analogously the right diagonal action of a bialgebra H on tensor products is defined. For any object V in $\mathcal{C}$ the *trivial left H-action* on V is defined as follows:

$$\phi_V:\ H\otimes V\xrightarrow{\epsilon\otimes 1}I\otimes V\xrightarrow{l}V\ .$$

In particular the unit object I becomes a left H-module.

Definition 1.4.3 *Let H be a bialgebra in a braided monoidal category $\mathcal{C}$. Then H is itself a coalgebra in ${}_H\mathcal{C}$, where tensor products carry the diagonal left H-module structure. Hence it makes sense to consider the category ${}^H({}_H\mathcal{C})$ of left H-comodules in this category of left H-modules. Objects of this category are left H-modules and right H-comodules M in $\mathcal{C}$ such that the coaction $\rho_M : M \to H\otimes M$ is left H-linear. This category is denoted by ${}^H_H\mathcal{C}$ and objects are termed left H-Hopf modules. Analogously the categories $\mathcal{C}^H_H$, ${}_H\mathcal{C}^H$ and ${}^H\mathcal{C}_H$ of right, left-right and right-left H-Hopf modules are defined respectively.*

The tensor product of objects $U, V \in {}^H\mathcal{C}$ carries a *left diagonal coaction* defined as

$$\rho_{U\otimes V}:\ U\otimes V\xrightarrow{\rho_U\otimes\rho_V}(H\otimes U)\otimes(H\otimes V)\xrightarrow{\tau_{23}}(H\otimes H)\otimes(U\otimes V)\xrightarrow{m\otimes 1\otimes 1}H\otimes(U\otimes V)\ .$$

Analogously the right diagonal coaction of a bialgebra H on tensor products is defined. For any object V in $\mathcal{C}$ the trivial left H-coaction on an object V is defined

$$\rho_V:\ V\xrightarrow{l^{-1}}I\otimes V\xrightarrow{\eta\otimes 1}H\otimes V\ .$$

In particular the identity object I becomes a left H-comodule. Note that if H is a bialgebra in $\mathcal{C}$. Then the assignment $F(V) = H\otimes V$ for any object $V \in \mathcal{C}$ is a functor $F : \mathcal{C} \to {}^H_H\mathcal{C}$. Moreover if V is considered a trivial left H-module (resp. trivial left H-comodule), then the map $i_V : V \to F(V)$ is a morphism in ${}_H\mathcal{C}$ (resp. in ${}^H\mathcal{C}$).

Definition 1.4.4 *Let H be a bialgebra in a braided monoidal category $\mathcal{C}$. Then H is itself an algebra in ${}^H\mathcal{C}$, where tensor products carry the diagonal left H-comodule structure. Hence it makes sense to consider the category ${}_H({}^H\mathcal{C})_H$ of H-bimodules in this category of left H-comodules. Objects of this category are left and right H-modules and left H-comodules M in $\mathcal{C}$ such that the left and right H-module actions are H-colinear and the bimodule condition is satisfied. This category is denoted by ${}_H^H\mathcal{C}_H$ and objects are termed left covariant H-bimodules. Analogously the category ${}_H\mathcal{C}_H^H$ of right covariant H-bimodules is defined.*

Definition 1.4.5 *A left-covariant (resp. right-covariant) H-bicomodule is a left-covariant (resp. right-covariant) H-bimodule in the dual category $\mathcal{C}^{op}$.*

We denote by ${}_H^H\mathcal{C}^H$ (resp. ${}^H\mathcal{C}_H^H$) the category of left-covariant (resp. right-covariant) H-bicomodules together with those morphisms in $\mathcal{C}$ that are left and right H-colinear and left (resp. right) H-linear.

We now make use of the Theorem 1.2.5, which has been given in the general tensor category framework, to prove the undermentioned theorem (see [72], Theorem 5.1):

Theorem 1.4.6 *Let $V \in \mathcal{C}$ and let $H \otimes V \in {}_H^H\mathcal{C}$ with the canonical H-module and H-comodule structures. Then there is a bijection between right H-module structures making $H \otimes V$ a left-covariant H-bimodule and right H-module structures on V.*

Proof: By performing the previous theorem to the left H-module $H \otimes V$ in the category of left H-comodules, we obtain a bijection between right H-module structures making $H \otimes V$ an H-bimodule and left H-colinear morphisms $f : V \otimes H \to H \otimes V$ fulfilling

1. $f \circ (1_V \otimes m) \circ a = (m \otimes 1_V) \circ a^{-1} \circ (1_H \otimes f) \circ a \circ (f \otimes 1_H)$,
2. $f \circ (1_V \otimes \eta) = (\eta \otimes 1_V) \circ l_V^{-1} \circ r_V$.

For any left H-comodule X with the coaction $\rho : X \to H \otimes X$, there is the bijective mapping

$$F_X : {}^H Hom(X, H \otimes V) \to Hom(X, V),\ f \mapsto l_V \circ (\varepsilon \otimes 1_V) \circ f$$

with the inverse given by $g \mapsto (1_H \otimes g) \circ \rho$. Let us take $f : V \otimes H \to H \otimes V$ and put

$$\psi = l_V \circ (\varepsilon \otimes 1_V) \circ f : V \otimes H \to V.$$

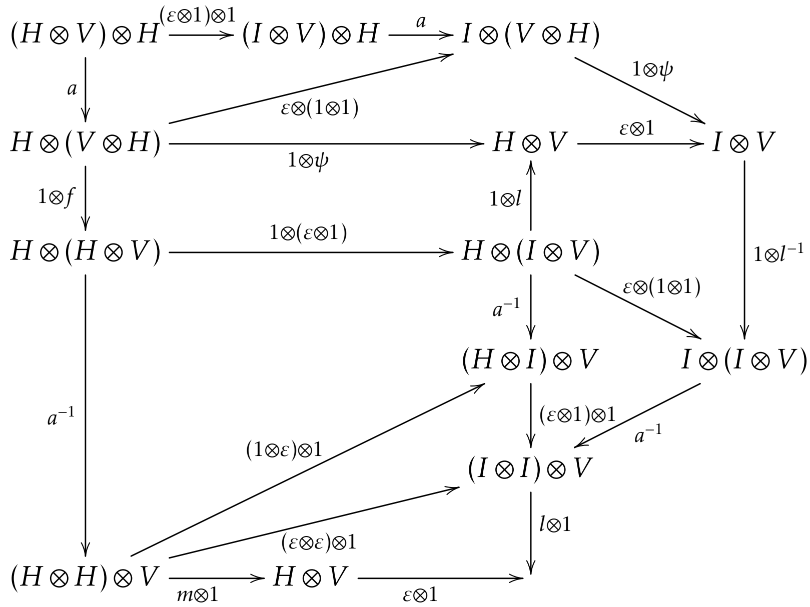
Then we prove that f satisfies the above equations if and only if ψ defines a right H-module structure on V.

$$
\begin{aligned}
F_{(V\otimes H)\otimes H}&((m\otimes 1_V)\circ a^{-1}\circ(1_H\otimes f)\circ a\circ(f\otimes 1_H))\\
&= l_V\circ(\varepsilon\otimes 1_V)\circ(m\otimes 1_V)\circ a^{-1}\circ(1_H\otimes f)\circ a\circ(f\otimes 1_H)\\
&= \psi\circ(\psi\otimes 1).
\end{aligned}
$$

Indeed, we first show that the equality

$$(\varepsilon\otimes 1_V)\circ(m\otimes 1_V)\circ a^{-1}\circ(1_H\otimes f)\circ a=(1\otimes\psi)\circ a\circ((\varepsilon\otimes 1_V)\otimes 1_H)$$

holds by the diagram



where the fact of ε being an algebra map implies the commutativity of the lower square, the middle left square is the definition of ψ and the commutativity of the rest follows from the naturality of a and the fact that $\otimes$ preserves the composition of the morphisms. To get the above equality we also used the the Triangle Axiom, which implies $a^{-1}\circ(1\otimes l_V^{-1})=$

$r_I^{-1} \otimes 1_V$, and the fact that $r_I = l_I$. Thus

$$\begin{aligned} & l_V \circ (\varepsilon \otimes 1_V) \circ (m \otimes 1_V) \circ a^{-1} \circ (1_H \otimes f) \circ a \circ (f \otimes 1_H) \\ &= l_V(1 \otimes \psi) \circ a \circ ((\varepsilon \otimes 1_V) \otimes 1_H) \circ (f \otimes 1_H) \\ &= \psi \circ l_{V \otimes H} \circ a \circ ((\varepsilon \otimes 1_V) \circ f \otimes 1_H) \\ &= \psi \circ (l_V \otimes 1) \circ (l_V^{-1} \circ l_V \circ (\varepsilon \otimes 1_V) \circ f \otimes 1_H) \\ &= \psi \circ (l_V \otimes 1) \circ (l_V^{-1} \otimes 1)(\psi \otimes 1_H) \\ &= \psi \circ (\psi \otimes 1) \end{aligned}$$

where the second equality results from the naturality of l and the third one is obtained by the Lemma XI.2.2 in ([49]). We also have

$$\begin{aligned} F_{(V \otimes H) \otimes H}(f \circ (1_V \otimes m) \circ a) &= l_V \circ (\varepsilon \otimes 1_V) \circ f \circ (1_V \otimes m) \circ a \\ &= \psi \circ (1_V \otimes m) \circ a. \end{aligned}$$

Therefore, the associativity of ψ holds if and only if $F_{(V \otimes H) \otimes H}(f \circ (1_V \otimes m) \circ a) = F_{(V \otimes H) \otimes H}((m \otimes 1_V) \circ a^{-1} \circ (1_H \otimes f) \circ a \circ (f \otimes 1_H))$, which is equivalent to the relation (1) due to the fact that $F_{(V \otimes H) \otimes H}$ is a bijective map. By a similar argument, we get the equivalence between the unity condition of ψ and the relation (2) since $F_{V \otimes I}(f \circ (1_V \otimes \eta)) = \psi \circ (1_V \otimes \eta)$ and $F_{V \otimes I}((\eta \otimes 1_V) \circ l_V^{-1} \circ r_V) = r_V$ and $F_{V \otimes I}$ is a bijection. □

Definition 1.4.7 *$M \in \mathcal{C}$ is called bicovariant H-bimodule if it is an H-bimodule and an H-bicomodule such that $M \in \mathcal{C}_H^H$, ${}_H\mathcal{C}^H$, ${}^H\mathcal{C}_H$, $\mathcal{C}_H^H$.*

We denote by ${}_H^H\mathcal{C}_H^H$ the category of bicovariant H-bimodules together with those morphisms in C that are H-linear and H-colinear on both sides. By applying the Theorem (1.4.6) in the opposite category we get

Corollary 1.4.8 *Let $V \in \mathcal{C}$) and let $H \otimes V \in {}_H^H\mathcal{C}$ with the canonical H-module and H-comodule structures. There is a one-to-one correspondence between right H-comodule structures on $H \otimes V$ making it a left-covariant H-bicomodule and the right H-comodule structures on V.*

Definition 1.4.9 *A right-right Yetter-Drinfel'd module V in $\mathcal{C}$ is a right H-module with an action $\psi : V \otimes H \to V$ and a right H-comodule with a coaction $\rho : V \to V \otimes H$ such that the following condition*

$$v_{(0)} \triangleleft h_1 \otimes v_{(1)} h_2 = (v \triangleleft h_2)_{(0)} \otimes h_1 (v \triangleleft h_2)_{(1)} \tag{1.8}$$

holds, for $h \in H$ and $v \in V$, if we write $\psi(v \otimes h) = v \triangleleft h$ and $\rho(v) = v_{(0)} \otimes v_{(1)}$.

The category of right-right Yetter-Drinfel'd modules together with those morphisms in $\mathcal{C}$ that are both H-linear and H-colinear is indicated by $\mathcal{YD}_H^H$.

In what follows we prove ([72], Theorem 5.4) in an arbitrary category, where the notion of generalized elements of objects in a category $\mathcal{C}$ is used, refer to ([70]). We use the notations, $\Delta(c) = c_1 \otimes c_2$ for a generalized element c of a coalgebra C and $\rho^M(m) = m_{(0)} \otimes m_{(1)}$ for a generalized element m of a right C-comodule M with the structure morphism $\rho^M : M \to M \otimes C$; and for the left comodules we use ${}^M\rho(m) = m_{(-1)} \otimes m_{(0)}$, which is the Sweedler's notation where the summation is dropped, and that notation is used throughout the thesis.

Theorem 1.4.10 *Let $V \in \mathcal{C}$ and let $H \otimes V \in {}^H_H\mathcal{C}$ with the canonical H-module and H-comodule structures. Then there is a one-to-one correspondence between*

1. *right H-module structures and right H-comodule structures making $H \otimes V$ bicovariant H-bimodule,*

2. *right-right Yetter-Drinfel'd module structures on V.*

Proof: The right H-module structure $v \otimes h \mapsto v \triangleleft h$ and the right H-comodule structure $v \mapsto v_{(0)} \otimes v_{(1)}$ on V are induced by the correspondences in (1.4.6) and (1.4.8). What is left to finish the proof is to show the equivalence of the right H-Hopf module condition on $H \otimes V$ to the compatibility condition (1.8) on V. Let's write $\phi' : (H \otimes V) \otimes H \to H \otimes V$ for the diagonal right action on $H \otimes V$, $\phi'' : ((H \otimes V) \otimes H) \otimes H \to (H \otimes V) \otimes H$ for the diagonal right action on $(H \otimes V) \otimes H$ and $\sigma' : H \otimes V \to (H \otimes V) \otimes H$ for the codiagonal right coaction on $H \otimes V$. So we have, for $g, h \in H$ and $v \in V$,

$$\begin{aligned}
\sigma'(\phi'((g \otimes v) \otimes h)) &= \sigma'(gh_1 \otimes v \triangleleft h_2) \\
&= ((gh_1)_1 \otimes (v \triangleleft h_2)_{(0)}) \otimes (gh_1)_2 (v \triangleleft h_2)_{(1)} \\
&= (g_1 h_{11} \otimes (v \triangleleft h_2)_{(0)}) \otimes (g_2 h_{12})(v \triangleleft h_2)_{(1)} \\
&= (g_1 h_1 \otimes (v \triangleleft h_{22})_{(0)}) \otimes (g_2 h_{21})(v \triangleleft h_{22})_{(1)} \\
&= (g_1 h_1 \otimes (v \triangleleft h_{22})_{(0)}) \otimes g_2 (h_{21} (v \triangleleft h_{22})_{(1)}),
\end{aligned} \tag{1.9}$$

$$
\begin{aligned}
&\phi''((\sigma' \otimes id_H)((g \otimes v) \otimes h)) \\
&\quad = \phi''(((g_1 \otimes v_{(0)}) \otimes g_2 v_{(1)}) \otimes h) \\
&\quad = (g_1 \otimes v_{(0)}) \cdot h_1 \otimes (g_2 v_{(1)}) h_2 \\
&\quad = (g_1 h_{11} \otimes v_{(0)} \triangleleft h_{12}) \otimes g_2 (v_{(1)} h_2) \\
&\quad = (g_1 h_1 \otimes v_{(0)} \triangleleft h_{21}) \otimes g_2 (v_{(1)} h_{22}). \qquad (1.10)
\end{aligned}
$$

Thereby if the condition (1.8) on V holds then the right hand sides of (1.9) and (1.10) are equal, and thus the left hand sides of (1.9) and (1.10) are equal, that is, the requirement that $H \otimes V$ be a right Hopf module is fulfilled. Conversely, if we assume that $H \otimes V$ is a right $\tilde{H}$-Hopf module, then by applying $(\varepsilon \otimes (1_V \otimes 1_H)) \circ a$ to the equation $(\sigma' \circ \phi')((1 \otimes v) \otimes h) = (\phi'' \circ (\sigma' \otimes id_H))((1 \otimes v) \otimes h)$ we obtain the condition (1.8) on V. □

Chapter 2

Integral Calculus On Quantum Exterior Algebras

2.1 Introduction

Let A be an algebra over a field K. A derivation $d : A \to \Omega^1$ of a K-algebra A into an A-bimodule is a K-linear map satisfying the Leibniz rule $d(ab) = ad(b) + d(a)b$ for all $a, b \in A$. The pair (Ω^1, d) is called a *first order differential calculus* (FODC) on A. More generally a differential graded algebra $\Omega = \bigoplus_{n\geq 0} \Omega^n$ is an $\mathbb{N}$-graded algebra with a linear mapping $d : \Omega \to \Omega$ of degree 1 that satisfies $d^2 = 0$ and the graded Leibniz rule. This means that $d(\Omega^n) \subseteq \Omega^{n+1}$, $d^2 = 0$ and for all homogeneous elements $a, b \in \Omega$ the graded Leibniz rule:

$$d(ab) = d(a)b + (-1)^{|a|} a d(b) \tag{2.1}$$

holds, where $|a|$ denotes the degree of a, i.e. $a \in \Omega|a|$ (see for example [30]). We shall call (Ω, d) an n-dimensional differential calculus on A if $\Omega^m = 0$ for all $m \geq n$. The zero component $A = \Omega^0$ is a subring of Ω and hence Ω^n are A-bimodule for all $n > 0$. In particular $d : A \to \Omega^1$ is a bimodule derivation and (Ω^1, d) is a FODC over A. The elements of Ω^n are then called n-forms and the product of Ω is denoted by $\wedge$. Given an FODC (Ω^1, d) over A, a connection in a right A-module M is a K-linear map $\nabla^0 : M \to M \otimes_A \Omega^1$ satisfying

$$\nabla^0(ma) = \nabla^0(m)a + m \otimes_A d(a) \qquad \forall a \in A, m \in M. \tag{2.2}$$

In [15] T.Brzezinski introduced an adjoint version of a connection by defining the notion of a right hom-connection as a pair (M, ∇_0), where M is a right A-module and $\nabla_0 : \mathrm{Hom}_A(\Omega^1, M) \to M$ is a K-linear map such that

$$\nabla_0(fa) = \nabla_0(f)a + f(d(a)) \qquad \forall a \in A, f \in \mathrm{Hom}_A(\Omega^1, M) \tag{2.3}$$

Here the multiplication $(fa)(\omega) := f(a\omega)$, for all $\omega \in \Omega^1$, makes $\mathrm{Hom}_A(\Omega^1, M)$ a right A-module. In case the FODC stems from a differential calculus (Ω, d), then a hom-connection ∇_0 on M can be extended to maps $\nabla_m : \mathrm{Hom}_A(\Omega^{m+1}, M) \longrightarrow \mathrm{Hom}_A(\Omega^m, M)$ with

$$\nabla_m(f)(v) = \nabla(fv) + (-1)^{m+1} f(dv), \qquad \forall f \in \mathrm{Hom}_A(\Omega^{m+1}, M), v \in \Omega^m. \tag{2.4}$$

If $\nabla_0\nabla_1 = 0$, the hom-connection ∇_0 is called flat. In this paper we will be mostly interested in the case $M = A$. Set $\Omega^*_m := \mathrm{Hom}_A(\Omega^m, A)$ as well as $\Omega^* = \bigoplus_m \Omega^*_m$ and define $\nabla : \Omega^* \to \Omega^*$ by $\nabla(f) = \nabla_m(f)$ for all $f \in \Omega^*_{m+1}$.

If ∇_0 is flat, then (Ω^*, ∇) builds up the *integral complex*:

$$\cdots \xrightarrow{\nabla_3} \Omega^*_3 \xrightarrow{\nabla_2} \Omega^*_2 \xrightarrow{\nabla_1} \Omega^*_1 \xrightarrow{\nabla_0} A$$

It had been shown in [16, 18] that for some finite dimensional differential calculi the integral complex is isomorphic to the *de Rham complex* given by (Ω, d):

$$A \xrightarrow{d} \Omega_1 \xrightarrow{d} \Omega_2 \xrightarrow{d} \Omega_3 \xrightarrow{d} \cdots$$

i.e. for certain algebras A and n-dimensional differential calculi $\Omega = \bigoplus_{m=0}^{n} \Omega^m$ it had been proven that there is a commutative diagram

$$\begin{array}{ccccccccc}
\Omega^*_n & \xrightarrow{\nabla_{n-1}} & \Omega^*_{n-1} & \xrightarrow{\nabla_{n-2}} & \cdots & \xrightarrow{\nabla_1} & \Omega^*_1 & \xrightarrow{\nabla_0} & A \\
\Theta_0 \uparrow & & \Theta_1 \uparrow & & & & \Theta_{n-1} \uparrow & & \Theta \uparrow \\
A & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n-1} & \xrightarrow{d} & \Omega^n
\end{array}$$

in which vertical maps are right A-module isomorphisms: In this case, we say that A satisfies the *strong Poincaré duality* with respect to (Ω, d) and ∇, following T.Brzezinski [16].

The purpose of this chapter is to provide further examples of algebras whose corresponding de Rham and integral complexes are isomorphic with respect to some differential calculi which contributes to the general study of algebras with this property. It should be noticed that the Poincaré duality in the sense of M.Van den Bergh [79] (see also the work of U.Krähmer [53]) is different.

2.2 Twisted multi-derivations and hom-connections

From Woronowicz' paper [80] it follows that any covariant differential calculus on a quantum group is determined by a certain family of maps which had been termed *twisted multi-derivations* in [18].

We recall from [18] that by a *right twisted multi-derivation* in an algebra A we mean a pair (∂,σ), where $\sigma : A \to M_n(A)$ is an algebra homomorphism ($M_n(A)$ is the algebra of $n \times n$ matrices with entries from A) and $\partial : A \to A^n$ is a k-linear map such that, for all $a \in A$, $b \in B$,

$$\partial(ab) = \partial(a)\sigma(b) + a\partial(b). \tag{2.5}$$

Here A^n is understood as an $(A\text{-}M_n(A))$-bimodule. We write $\sigma(a) = (\sigma_{ij}(a))_{i,j=1}^n$ and $\partial(a) = (\partial_i(a))_{i=1}^n$ for an element $a \in A$. Then (2.5) is equivalent to the following n equations

$$\partial_i(ab) = \sum_j \partial_j(a)\sigma_{ji}(b) + a\partial_i(b), \qquad i = 1,2,\ldots,n. \tag{2.6}$$

Given a right twisted multi-derivation (∂,σ) on A we construct a FODC on the free left A-module

$$\Omega^1 = A^n = \bigoplus_{i=1}^n A\omega_i \tag{2.7}$$

with basis $\omega_1,\ldots,\omega_n$ which becomes an A-bimodule by $\omega_i a = \sum_{j=1}^n \sigma_{ij}(a)\omega_j$ for all $1 \le i \le n$. The map

$$d : A \to \Omega^1, \qquad a \mapsto \sum_{i=1}^n \partial_i(a)\omega_i \tag{2.8}$$

is a derivation and makes (Ω^1, d) a first order differential calculus on A.

A map $\sigma : A \to M_n(A)$ can be equivalently understood as an element of $M_n(\mathrm{End}_k(A))$. Write $\bullet$ for the product in $M_n(\mathrm{End}_k(A))$, $\mathbb{I}$ for the unit in $M_n(\mathrm{End}_k(A))$ and σ^T for the transpose of σ.

Definition 2.2.1 *Let (∂,σ) be a right twisted multi-derivation. We say that (∂,σ) is* free, *provided there exist algebra maps $\bar\sigma : A \to M_n(A)$ and $\hat\sigma : A \to M_n(A)$ such that*

$$\bar\sigma \bullet \sigma^T = \mathbb{I}, \qquad \sigma^T \bullet \bar\sigma = \mathbb{I}, \tag{2.9}$$

$$\hat\sigma \bullet \bar\sigma^T = \mathbb{I}, \qquad \bar\sigma^T \bullet \hat\sigma = \mathbb{I}. \tag{2.10}$$

Theorem [18, Theorem 3.4] showed that for any free right twisted multi-derivation $(\partial,\sigma;\bar{\sigma},\hat{\sigma})$ on A, and associated first order differential calculus (Ω^1,d) with generators ω_i, the map

$$\nabla : \mathrm{Hom}_A(\Omega^1,A) \to A, \qquad f \mapsto \sum_i \partial_i^{\sigma}\left(f\left(\omega_i\right)\right). \tag{2.11}$$

is a hom-conection, where $\partial_i^{\sigma} := \sum_{j,k} \bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}$, for each $i = 1,2,\ldots,n$. Moreover ∇ had been shown to be unique with respect to the property that $\nabla(\xi_i) = 0$, for all $i = 1,2,\ldots,n$, where $\xi_i : \Omega^1 \to A$ are right A-linear maps defined by $\xi_i(\omega_j) = \delta_{ij}$, $i,j = 1,2,\ldots,n$.

We shall be mostly interested in right twisted multi-derivation (∂,σ) that are upper triangular, for which $\sigma_{ij} = 0$ for all $i > j$ holds. It had been shown in [18, Proposition 3.3] that an upper triangular right twisted multi-derivation is free if and only if $\sigma_{11},\ldots,\sigma_{nn}$ are automorphisms of A.

2.3 Differential calculi on quantum exterior algebras

Let A be a unital associative algebra over a field K. Given an A-bimodule M which is free as left and right A-module with basis $\{\omega_1,\ldots,\omega_n\}$ one defines the tensor algebra of M over A as

$$T_A(M) = A \oplus M \oplus (M \otimes M) \oplus M^{\otimes 3} \oplus \cdots = \bigoplus_{n=0}^{\infty} M^{\otimes n} \tag{2.12}$$

which is a graded algebra whose product is the concatenation of tensors and whose zero component is A. Following [10, I.2.1] we call an $n \times n$-matrix $Q = (q_{ij})$ over K a *multiplicatively antisymmetric matrix* if $q_{ij}q_{ji} = q_{ii} = 1$ for all i,j. The *quantum exterior algebra* of M over A with respect to a multiplicatively antisymmetric matrix Q is defined as

$$\bigwedge\nolimits^Q(M) := T_A(M)/\langle \omega_i \otimes \omega_j + q_{ij}\omega_j \otimes \omega_i, \omega_i \otimes \omega_i \mid i,j = 1,\ldots,n\rangle.$$

This construction for a vector space $M = V$ and a field $A = K$ appears in [68, 78]. The product of $\bigwedge^Q(M)$ is written as $\wedge$. The quantum exterior algebra is a free left and right A-module of rank 2^n with basis

$$\{1\} \cup \{\omega_{i_1} \wedge \omega_{i_2} \cdots \wedge \omega_{i_k} \mid i_1 < i_2 < \cdots < i_k,\ 1 \le k \le n\}.$$

Write $\sup(\omega_{i_1} \wedge \omega_{i_2} \cdots \wedge \omega_{i_k}) = \{i_1, i_2 \cdots, i_k\}$ for any basis element. Given a bimodule derivation $d : A \to M$, we will examine when d can be extended to an exterior derivation of $\bigwedge^Q(M)$, i.e. to a graded map $d : \bigwedge^Q(M) \to \bigwedge^Q(M)$ of degree 1 such that $d^2 = 0$ and such that the graded Leibniz rule is satisfied.

$$a\omega_i = \sum_{j=1}^{n} \omega_j \bar{\sigma}_{ji}(a) \qquad \forall a \in A, i = 1, \ldots, n. \tag{2.13}$$

Proposition 2.3.1 *Let (∂, σ) be a right twisted multi-derivation of rank n on a K-algebra A with associated FODC (Ω^1, d). Let Q be an $n \times n$ multiplicatively antisymmetric matrix over k. Then $d : A \to \Omega^1$ can be extended to make $\Omega = \bigwedge^Q(\Omega^1)$ an n-dimensional differential calculus on A with $d(\omega_i) = 0$ for all $i = 1, \ldots, n$ if and only if*

$$\partial_i \partial_j = q_{ji} \partial_j \partial_i, \qquad \text{and} \qquad \partial_i \sigma_{kj} - q_{ji} \partial_j \sigma_{ki} = q_{ji} \sigma_{kj} \partial_i - \sigma_{ki} \partial_j \qquad \forall i < j, \forall k. \tag{2.14}$$

Proof: Suppose d extends to make Ω a differential calculus on A with $d(\omega_i) = 0$. Then for all $a \in A$ and $k = 1, \ldots, n$ the following equations hold:

$$d(\omega_k a) = d(\omega_k)a - \omega_k \wedge d(a) = \sum_{j=1}^{n} -\omega_k \wedge \partial_j(a)\omega_j = \sum_{i,j=1}^{n} -\sigma_{ki}(\partial_j(a))\omega_i \wedge \omega_j \tag{2.15}$$

$$d\left(\sum_{j=1}^{n} \sigma_{kj}(a)\omega_j\right) = \sum_{i,j=1}^{n} \partial_i(\sigma_{kj}(a))\omega_i \wedge \omega_j + \sum_{j=1}^{n} \sigma_{kj}(a) d(\omega_j) = \sum_{i,j=1}^{n} \partial_i(\sigma_{kj}(a))\omega_i \wedge \omega_j \tag{2.16}$$

Hence, as $\omega_k a = \sum_{j=1}^{n} \sigma_{kj}(a)\omega_j$ and $\omega_j \wedge \omega_i = -q_{ji}\omega_i \wedge \omega_j$ for $i < j$, we have

$$-\sigma_{ki}\partial_j + q_{ji}\sigma_{kj}\partial_i = \partial_i \sigma_{kj} - q_{ji}\partial_j \sigma_{ki} \qquad \forall i < j \tag{2.17}$$

Furthermore $d^2 = 0$ implies for all $a \in A$:

$$0 = d^2(a) = \sum_{i,j=1}^{n} \partial_i \partial_j(a)\omega_i \wedge \omega_j = \sum_{i<j} (\partial_i \partial_j - q_{ji}\partial_j \partial_i)(a)\omega_i \wedge \omega_j, \tag{2.18}$$

which shows $\partial_i \partial_j = q_{ji}\partial_j \partial_i$, for $i < j$.

On the other hand if (2.14) holds, then set for any homogeneous element $a\omega \in \Omega^m$ with $a \in A$ and $\omega = \omega_{j_1} \wedge \omega_{j_2} \wedge \cdots \wedge \omega_{j_m}$, with $j_1 < j_2 < \cdots < j_m$, a basis element of Ω^m:

$$d(a\omega) := d(a) \wedge \omega = \sum_{i=1}^{n} \partial_i(a)\omega_i \wedge \omega_{j_1} \wedge \omega_{j_2} \wedge \cdots \wedge \omega_{j_m}. \tag{2.19}$$

We will show that $d : \Omega \to \Omega$ in that way, will satisfy $d^2 = 0$ and the graded Leibniz rule. For any $a\omega \in \Omega^m$ as above:

$$d^2(a\omega) = \sum_{i,j=1}^{n} \partial_i \partial_j(a)\omega_i \wedge \omega_j \wedge \omega = \sum_{i<j} (\partial_i \partial_j - q_{ji}\partial_j \partial_i)(a)\omega_i \wedge \omega_j \wedge \omega = 0 \tag{2.20}$$

Since (2.6) implies that $\partial_i(1) = \sum_j \partial_j(1)\sigma_{ji}(1) + \partial_i(1) = 2\partial_i(1)$, as $\sigma_{ji}(1) = 0$ if $i \neq j$, we have $\partial_i(1) = 0$ and hence$d(\omega_i) = d(1) \wedge \omega_i = 0$ for all i.

We prove the graded Leibniz rule

$$d(a\omega \wedge b\nu) = d(a\omega) \wedge b\nu + (-1)^m a\omega \wedge d(b\nu) \tag{2.21}$$

inductively on the grade of ω, where $\omega = \omega_{j_1} \wedge \cdots \wedge \omega_{j_m}$ and $\nu = \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$ are basis elements of Ω and $a, b \in A$. For a $m = 0$, ie. $a\omega = a$, equation (2.21) follows from the definition and $d(\nu) = 0$. Let $m > 0$ and suppose that (2.21) has been proven for all basis elements ω of grade $|\omega| \leq m-1$. Let ω be a basis element with $|\omega| = m$ and write $\omega = \omega' \wedge \omega_k$.

$$\begin{aligned}
d(a\omega \wedge b\nu) &= d(a\omega' \wedge \omega_k \wedge b\nu) \\
&= \sum_{j=1}^{n} d\left(a\omega' \wedge \sigma_{kj}(b)\omega_j \wedge \nu\right) \\
&= \sum_{j=1}^{n} d(a\omega') \wedge \sigma_{kj}(b)\omega_j \wedge \nu + (-1)^{m-1} \sum_{j=1}^{n} a\omega' \wedge d\left(\sigma_{kj}(b)\omega_j \wedge \nu\right) \\
&= d(a\omega') \wedge \omega_k \wedge b\nu + (-1)^{m-1} a\omega' \wedge \sum_{i,j=1}^{n} \partial_i(\sigma_{kj}(b))\omega_i \wedge \omega_j \wedge \nu \\
&= d(a\omega) \wedge b\nu - (-1)^m a\omega' \wedge \sum_{i<j} \left[\partial_i(\sigma_{kj}(b)) - q_{ji}\partial_j(\sigma_{ki}(b))\right] \omega_i \wedge \omega_j \wedge \nu \\
&= d(a\omega) \wedge b\nu + (-1)^m a\omega' \wedge \sum_{i<j} \left[\sigma_{ki}(\partial_j(b)) - q_{ji}\sigma_{kj}(\partial_i(b))\right] \omega_i \wedge \omega_j \wedge \nu \\
&= d(a\omega) \wedge b\nu + (-1)^m a\omega' \wedge \sum_{i,j=1}^{n} \sigma_{ki}(\partial_j(b))\omega_i \wedge \omega_j \wedge \nu \\
&= d(a\omega) \wedge b\nu + (-1)^m a\omega' \wedge \omega_k \wedge \sum_{j=1}^{n} \partial_j(b)\omega_j \wedge \nu \\
&= d(a\omega) \wedge b\nu + (-1)^m a\omega \wedge d(b\nu)
\end{aligned}$$

which shows the graded Leibniz rule, where the induction hypothesis has been used in the third line and where (2.14) has been used in the sixth line . □

Suppose that (∂, σ) is a free right twisted multi-derivation satisfying the equations (2.14) and that (Ω, d) is the associated n-dimensional differential calculus over A for some $n \times n$ matrix Q. Then, as mentioned above, $\nabla : \mathrm{Hom}_A(\Omega^1, A) \to A$ with $\nabla(f) =$

$\sum_{i=1}^n \partial_i^\sigma(f(\omega_i))$ for all $f \in \mathrm{Hom}_A(\Omega^1, A)$ is hom-connection. For each $1 \leq m < n$ one defines also $\nabla_m : \mathrm{Hom}_A(\Omega^{m+1}, A) \longrightarrow \mathrm{Hom}_A(\Omega^m, A)$ with

$$\nabla_m(f)(u) = \nabla(fu) + (-1)^{m+1} f(d(u)), \qquad \forall f \in \mathrm{Hom}_A(\Omega^{m+1}, A), u \in \Omega^m, \tag{2.22}$$

where $fu \in \mathrm{Hom}_A(\Omega^1, A)$ is defined by $fu(v) = f(u \wedge v)$ for all $v \in \Omega^1$. As every element $u \in \Omega^m$ can be uniquely written as a right A-linear combination of basis elements $\omega = \omega_{i_1} \wedge \cdots \wedge \omega_{i_m}$ and since $\nabla_m(f)$ is right A-linear and furthermore by Proposition 2.3.1 $d(\omega) = 0$ is satisfied, we conclude that for $u = \omega a$:

$$\nabla_m(f)(\omega a) = \nabla_m(f)(\omega)a = \nabla(f\omega)a + (-1)^{m+1} f(d(\omega))a = \nabla(f\omega)a \tag{2.23}$$

holds. If $\partial_i^\sigma(1) = 0$ for all i, the hom-connection is flat, because for any dual basis element $f = \beta_{s,t} \in \mathrm{Hom}_A(\Omega^2, A)$ with $s < t$, i.e. $\beta_{s,t}(\omega_i \wedge \omega_j) = \delta_{s,i}\delta_{t,j}$ one has

$$\begin{aligned} \nabla(\nabla_1(f)) &= \sum_{i=1}^n \partial_i^\sigma(\nabla_1(f)(\omega_i)) = \sum_{i=1}^n \partial_i^\sigma(\nabla(f\omega_i)) \\ &= \sum_{i=1}^n \sum_{j=1}^n \partial_i^\sigma(\partial_j^\sigma(f(\omega_i \wedge \omega_j))) = \partial_s^\sigma(\partial_t^\sigma(1)) = 0. \end{aligned}$$

Set $\Omega^* = \mathrm{Hom}_A(\Omega, A) = \bigoplus_{m=0}^n \mathrm{Hom}_A(\Omega^m, A)$ and note that ∇ induces a map of degree -1 on Ω^*. We want to establish an isomorphism between the de Rham complex given by $d : \Omega \to \Omega$ and the integral complex given by $\nabla : \Omega^* \to \Omega^*$. More precisely we are looking for a bijective chain map $\Theta : (\Omega, d) \to (\Omega^*, \nabla)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} A & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} \cdots \xrightarrow{d} & \Omega^{n-1} & \xrightarrow{d} & \Omega^n \\ \downarrow{\scriptstyle\Theta_0} & & \downarrow{\scriptstyle\Theta_1} & & \downarrow{\scriptstyle\Theta_{n-1}} & & \downarrow{\scriptstyle\Theta_n} \\ \mathrm{Hom}_A(\Omega^n, A) & \xrightarrow[\nabla_{n-1}]{} & \mathrm{Hom}_A(\Omega^{n-1}, A) & \xrightarrow[\nabla_{n-2}]{} \cdots \xrightarrow[\nabla_1]{} & \mathrm{Hom}_A(\Omega^1, A) & \xrightarrow[\nabla]{} & A \end{array}$$

One attempt is to define the maps Θ_m via the dual basis element of Ω^n. Define

$$\overline{\omega} = \omega_1 \wedge \cdots \wedge \omega_n \in \Omega^n$$

for the base element of Ω^n. Let $\beta \in \Omega^{n*}$ be the dual basis of Ω^n as a right A-module, i.e. $\beta(\overline{\omega}a) = a$ for all $a \in A$. For any $0 \leq m < n$ define $\Theta_m : \Omega^m \longrightarrow \mathrm{Hom}_A(\Omega^{n-m}, A)$ through

$\Theta_m(v) = (-1)^{m(n-1)}\beta v$ for all $v \in \Omega^m$. Note that $\Theta_n = \beta$. Moreover the maps Θ_m are right A-linear taking into account the right A-module structure of $\mathrm{Hom}_A(\Omega^{n-m}, A)$, namely for $a \in A, v \in \Omega^m$ and $w \in \Omega^{n-m}$:

$$\Theta_m(va)(w) = (-1)^{m(n-1)}\beta(va \wedge w) = (-1)^{m(n-1)}\beta(v \wedge aw) = \Theta_m(v)(aw) = (\Theta_m(v)a)(w).$$

Hence $\Theta_m(va) = \Theta_m(v)a$.

For a certain class of twisted multi-derivations, extended to a quantum exterior algebra, we will show that the maps Θ_m are always isomorphisms. We say that a twisted multi-derivation (∂, σ) on an algebra A is upper triangular if $\sigma_{ij} = 0$ for all $i > j$. By [18, Proposition 3.3] any upper triangular twisted multi-derivation is free if and only if σ_{ii} are automorphisms of A for all i. The corresponding maps $\bar{\sigma}$ and $\hat{\sigma}$ are defined inductively by $\bar{\sigma}_{ii} = \sigma_{ii}^{-1}$ for all i, $\bar{\sigma}_{ij} = -\sum_{k=j}^{i-1} \sigma_{ii}^{-1}\sigma_{ki}\bar{\sigma}_{kj}$ for all $i > j$ and $\bar{\sigma}_{ij} = 0$ for $i < j$. The map $\hat{\sigma}$ is defined analogously using $\bar{\sigma}$.

Theorem 2.3.2 *Let (∂, σ) be a free upper triangular twisted multi-derivation on A with associated FODC (Ω^1, d). Suppose that $d : A \to \Omega^1$ can be extended to an n-dimensional differential calculus (Ω, d) where $\Omega = \bigwedge^Q(\Omega^1)$ is the quantum exterior algebra of Ω^1 for some matrix Q. Then the following hold:*

1. *$\overline{\omega}a = \det(\sigma)(a)\,\overline{\omega}$, for all $a \in A$, where $\det(\sigma) = \sigma_{11} \circ \cdots \circ \sigma_{nn}$.*
2. *The maps $\Theta_m : \Omega^m \to \mathrm{Hom}_A(\Omega^{n-m}, A)$ given by $\Theta_m(v) = (-1)^{m(n-1)}\beta v$ for all $v \in \Omega^m$ are isomorphisms of right A-modules.*
3. *Moreover if*
$$\partial_i^{\sigma} = \left(\prod_j q_{ij}\right) \det(\sigma)^{-1}\partial_i \det(\sigma) \qquad \forall i = 1, \ldots, n \tag{2.24}$$
holds, then $\Theta = (\Theta_m)_{m=0}^n$ is a chain map, that is, A satisfies the strong Poincaré duality with respect to (Ω, d) in the sense of T.Brzezinski.

Proof: (1) By the definition of the bimodule structure of $\bigwedge^Q(\Omega^1)$ and by the fact that $\bar{\sigma}$ is lower triangular we have

$$a\overline{\omega} = \sum_{j_n \geq n} \cdots \sum_{j_1 \geq 1} \omega_{j_1} \wedge \cdots \wedge \omega_{j_n} \bar{\sigma}_{nj_n} \circ \cdots \circ \bar{\sigma}_{1j_1}(a).$$

By the definition of the quantum exterior algebra the non-zero terms $\omega_{j_1} \wedge \cdots \wedge \omega_{j_n}$ must have distinct indices, i.e. $j_k \neq j_l$ for all $k \neq l$. In particular $j_n = n$ and hence inductively we can conclude that $j_i = i$ for all i. This shows that $a\overline{\omega} = \overline{\omega}\det(\sigma)^{-1}(a)$.

(2) For every basis element of $\omega = \omega_{i_1} \wedge \cdots \wedge \omega_{i_{n-m}}$ of Ω^{n-m}, there exists a unique complement basis element $\omega' = \omega'_{j_1} \wedge \cdots \wedge \omega'_{j_m}$ of Ω^m such that $\omega' \wedge \omega \neq 0$. Let C_ω be the non-zero scalar such that $\omega' \wedge \omega = C_\omega \overline{\omega}$ Let $f \in \mathrm{Hom}_A(\Omega^{n-m}, A)$ be any non-zero element and set

$$a_\omega = (-1)^{m(n-1)} C_\omega^{-1} \det(\sigma)(f(\omega))$$

for any basis element $\omega \in \Omega^{n-m}$. Set $v = \sum a_\omega \omega'$. Then

$$\Theta_m(v)(\omega) = (-1)^{m(n-1)}\beta(a_\omega \omega' \wedge \omega) = (-1)^{m(n-1)}\beta(a_\omega C_\omega \overline{\omega}) = \det(\sigma)^{-1}(\det(\sigma)(f(\omega)) = f(\omega).$$

Hence $\Theta_m(v) = f$, which shows that Θ_m is surjective. To prove injectivity, assume that $v = \sum a_\omega \omega' \in \Omega^m$ is an element such that $\Theta_m(v)$ is the zero function. Then for any basis element $\omega \in \Omega^{n-m}$, one has

$$\Theta_m(v)(\omega) = (-1)^{m(n-1)}\beta(a_\omega \omega' \wedge \omega) = (-1)^{m(n-1)} C_\omega \det(\sigma)^{-1}(a_\omega) = 0$$

which implies a_ω to be zero. Thus $v = 0$ and Θ_m is an isomorphism.

(3) We will show that $(\Theta_m)_m$ is a chain map, i.e. that $\Theta_{m+1} \circ d = \nabla_{n-m-1} \circ \Theta_m$. Let $\omega = \omega_{j_1} \wedge \cdots \wedge \omega_{j_m}$ be a basis element of Ω^m and let $a \in A$. For any basis element $\nu = \omega_{k_1} \wedge \cdots \wedge \omega_{k_{n-m-1}} \in \Omega^{n-m-1}$ we have

$$\Theta_{m+1}(d(a\omega))(\nu) = (-1)^{(m+1)(n-1)} \sum_{i=1}^{n} \beta(\partial_i(a)\omega_i \wedge \omega \wedge \nu).$$

On the other hand

$$\nabla_{n-m-1}(\Theta_m(a\omega))(\nu) = (-1)^{m(n-1)} \nabla(\beta(a\omega \wedge \nu) = (-1)^{m(n-1)} \sum_{i=1}^{n} \partial_i(\beta(a\omega \wedge \nu \wedge \omega_i)),$$

as $d(\nu) = 0$. Note that $\Theta_{m+1}(d(a\omega))(\nu) = 0$ and $\nabla_{n-m-1}(\Theta_m(a\omega))(\nu) = 0$ if $\sup(\omega) \cap \sup(\nu) \neq \emptyset$. Hence suppose that ω and ν have disjoint support. Then there exists a unique index i that does not belong to $\sup(\omega) \cup \sup(\nu)$. Let C be the constant such that

$$\omega \wedge \nu \wedge \omega_i = C\overline{\omega}.$$

Recall also that by the definition of the quantum exterior algebra we have:

$$\omega_i \wedge \omega \wedge \nu = \left(\prod_{j \neq i} -q_{ij}\right) \omega \wedge \nu \wedge \omega_i = (-1)^{n-1} C \left(\prod_j q_{ij}\right) \overline{\omega}.$$

Note that hypothesis (2.24) is moreover equivalent to

$$\partial_i^\sigma \circ \det(\sigma)^{-1} = \left(\prod_j q_{ij}\right) \det(\sigma)^{-1} \circ \partial_i \tag{2.25}$$

These equations yield now the following:

$$\begin{aligned}
\Theta_{m+1}(d(a\omega))(\nu) &= (-1)^{(m+1)(n-1)} \beta(\partial_i(a)\omega_i \wedge \omega \wedge \nu) \\
&= (-1)^{m(n-1)} C \left(\prod_j q_{ij}\right) \beta(\partial_i(a)\overline{\omega}) \\
&= (-1)^{m(n-1)} C \left(\prod_j q_{ij}\right) \det(\sigma)^{-1}(\partial_i(a)) \\
&= (-1)^{m(n-1)} C\, \partial_i^\sigma \left(\det(\sigma)^{-1}(a)\right) \\
&= \partial_i^\sigma \left((-1)^{m(n-1)} C\, \beta(a\overline{\omega})\right) \\
&= \partial_i^\sigma \left((-1)^{m(n-1)} \beta(a\omega \wedge \nu \wedge \omega_i)\right) = \nabla_{n-m-1}\left(\Theta_m(a\omega)(\nu)\right)
\end{aligned}$$

Thus $\Theta_{m+1} \circ d = \nabla_{n-m-1} \circ \Theta_m$. Hence Θ is a chain map between the de Rham and the integral complexes of right A-modules.

□

Remark 1 *Let (∂, σ) be an upper-triangular twisted multi-derivation of rank n on A and let Q be an $n \times n$ matrix with $q_{ij}q_{ji} = q_{ii} = 1$. The conditions to extend the multi-derivations to the quantum exterior algebra $\Omega = \bigwedge^Q(\Omega^1)$ such that the complex of integral forms on A and the de Rham complex are isomorphic with respect to (Ω, d) are:*

1. *σ_{ii} is an automorphism of A for all i;*
2. *$\partial_i \partial_j = q_{ji} \partial_j \partial_i$ for all $i < j$;*
3. *$\partial_i \sigma_{kj} - q_{ji} \sigma_{kj} \partial_i = q_{ji} \partial_j \sigma_{ki} - \sigma_{ki} \partial_j$ for all $i < j$ and all k;*
4. *$\partial_i^\sigma = \left(\prod_j q_{ij}\right) \det(\sigma)^{-1} \partial_i \det(\sigma)$ for all i.*

2.4 Differential calculi from skew derivations

The simplest bimodule structure on $\Omega^1 = A^n$ is a *diagonal* one, i.e. if $\sigma_{ij} = \delta_{ij}\sigma_i$ for all i, j where $\sigma_1, \ldots, \sigma_n$ are endomorphisms of A. Moreover if σ is diagonal and (∂, σ) is a right twisted multi-derivation on A, then the maps ∂_i are right σ_i-derivations, i.e. for all $a, b \in A$ and i:

$$\partial_i(ab) = \partial_i(a)\sigma_i(b) + a\partial_i(b). \tag{2.26}$$

Conversely, given any right σ_i-derivations ∂_i on A, for $i = 1, \ldots, n$ one can form a corresponding diagonal twisted multi-derivation (∂, σ) on A. Such *diagonal* twisted multi-derivation (∂, σ) is free if and only if the maps $\sigma_1, \ldots, \sigma_n$ are automorphisms. The associated A-bimodule structure on $\Omega^1 = A^n$ with left A-basis $\omega_1, \ldots, \omega_n$ is given by $\omega_i a = \sigma_i(a)\omega_i$ for all i and $a \in A$. From Proposition 2.3.1 we obtain the following corollary for diagonal bimodule structures.

Corollary 2.4.1 *Let A be an algebra over a field K, σ_i automorphisms and ∂_i right σ_i-skew derivations on A, for $i = 1, \ldots, n$ and let (Ω^1, d) be the associated first order differential calculus on A.*

1. *The derivation $d : A \to \Omega^1$ extends to an n-dimensional differential calculus (Ω, d) where $\Omega = \bigwedge^Q(\Omega^1)$ is the quantum exterior algebra with respect to some Q such that $d(\omega_i) = 0$ for all $i = 1, \ldots, n$ if and only if*

$$\partial_i\sigma_j = q_{ji}\sigma_j\partial_i \qquad \text{and} \qquad \partial_i\partial_j = q_{ji}\partial_j\partial_i \qquad \forall i < j \tag{2.27}$$

2. *If $\partial_i\sigma_j = q_{ji}\sigma_j\partial_i$ for all i, j and $\partial_i\partial_j = q_{ji}\partial_j\partial_i$ for all $i < j$, then the de Rham and the integral complexes on A are isomorphic relative to (Ω, d).*

Proof: (1) Since $\sigma_{ki} = 0$ for all $k \neq i$, equation (2.14) reduces to equation (2.27).

(2) Note that $\partial_i^\sigma = \sigma_i^{-1}\partial_i\sigma_i = \partial_i$. On the other hand by hypothesis $\partial_i \det(\sigma) = \left(\prod_j q_{ji}\right) \det(\sigma)\partial_i$. Hence

$$\left(\prod_j q_{ij}\right) \det(\sigma)^{-1}\partial_i \det(\sigma) = \partial_i = \partial_i^\sigma.$$

Thus by Theorem 2.3.2, A satisfies the strong Poincaré duality with respect to (Ω, d) in the sense of T.Brzeziński. □

2.5 Multivariate quantum polynomials

Let K be a field, $n > 1$, and $Q = (q_{ij})$ a $n \times n$ multiplicatively antisymmetric matrix over K. The multivariate quantum polynomial algebra with respect to Q is defined as:

$$A = \mathcal{O}_Q(K^n) := K\langle x_1, \dots, x_n\rangle / \langle x_i x_j - q_{ij} x_j x_i \mid 1 \le i, j \le n\rangle.$$

This means that x_i and x_j commute up to the scalar q_{ij} in A. Moreover every element is a linear combination of ordered monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. The set of n-tuples $\mathbb{N}^n$ is a submonoid of $\mathbb{Z}^n$ by componentwise addition. For any $\alpha \in \mathbb{Z}$ we set $x^\alpha = 0$ if there exists $i = 1, \dots, n$ such that $\alpha_i < 0$. Furthermore $\mathbb{N}^n$ is partially ordered as follows: $\alpha \le \beta$ if and only if $\alpha_i \le \beta_i, i = 1, \cdots, n$ for $\alpha, \beta \in \mathbb{N}^n$. If $\alpha \le \beta$, then $\beta - \alpha \in \mathbb{N}^n$ and $x^{\beta-\alpha} \neq 0$.

For two generic monomials x^α and x^β with $\alpha, \beta \in \mathbb{N}^n$ one has

$$x^\alpha x^\beta = \left(\prod_{1 \le j < i \le n} q_{ij}^{\alpha_i \beta_j} \right) x^{\alpha+\beta} = \mu(\alpha, \beta) x^{\alpha+\beta}, \tag{2.28}$$

where $\mu(\alpha, \beta) = \prod_{1 \le j < i \le n} q_{ij}^{\alpha_i \beta_j}$. The algebra A has been well-studied by Artamonov [5, 6] as well by Goodearl and Brown [10] and others. The Manin's quantum n-space is obtained in case there exists $q \in K$ with $q_{ij} = q$ for all $i < j$. In particular for $n = 2$ one obtains the quantum plane.

We define automorphisms $\sigma_1, \dots, \sigma_n$ and right σ_i-derivations of A as follows: For a generic monomial x^α with $\alpha \in \mathbb{N}^n$ one sets

$$\sigma_i(x^\alpha) := \lambda_i(\alpha) x^\alpha \qquad \text{and} \qquad \partial_i(x^\alpha) := \alpha_i \delta_i(\alpha) x^{\alpha - \epsilon^i} \tag{2.29}$$

where $\lambda_i(\alpha) = \prod_{j=1}^n q_{ij}^{\alpha_j}$, $\delta_i(\alpha) = \prod_{i<j} q_{ij}^{\alpha_j}$ and $\epsilon^i \in \mathbb{N}^n$ such that $\epsilon^i_j = \delta_{ij}$. Let $\overline{\delta_i}(\alpha) = \prod_{i>j} q_{ij}^{\alpha_j}$ and note that $\lambda_i(\alpha) = \delta_i(\alpha)\overline{\delta_i}(\alpha)$. Since $\mu(\alpha, \beta) = \mu(\alpha - \epsilon^i, \beta)\overline{\delta_i}(\beta)$ if $\alpha_i \neq 0$ and $\mu(\alpha, \beta) = \mu(\alpha, \beta - \epsilon^i)\delta_i(\alpha)^{-1}$ if $\beta_i \neq 0$, we have:

$$\begin{aligned}
\partial_i(x^\alpha x^\beta) &= (\alpha_i + \beta_i)\mu(\alpha, \beta)\delta_i(\alpha + \beta) x^{\alpha+\beta-\epsilon^i} \\
&= \alpha_i \mu(\alpha - \epsilon^i, \beta)\overline{\delta_i}(\beta)\delta_i(\alpha)\delta_i(\beta) x^{\alpha-\epsilon^i+\beta} + \beta_i \mu(\alpha, \beta - \epsilon^i)\delta_i(\alpha)^{-1}\delta_i(\alpha)\delta_i(\beta) x^{\alpha+\beta-\epsilon^i} \\
&= \alpha_i \delta_i(\alpha) x^{\alpha-\epsilon^i} \lambda_i(\beta) x^\beta + x^\alpha \beta_i \delta_i(\beta) x^{\beta-\epsilon^i} \\
&= \partial_i(x^\alpha)\sigma_i(x^\beta) + x^\alpha \partial_i(x^\beta)
\end{aligned}$$

Let $i < j$ and $\alpha \in \mathbb{N}^n$. Then $\delta_j(\alpha - \epsilon^i) = \delta_j(\alpha)$, while $\delta_i(\alpha - \epsilon^j) = \delta_i(\alpha) q_{ji}$. Hence

$$\partial_j(\partial_i(x^\alpha)) = \alpha_i\alpha_j\delta_i(\alpha)\delta_j(\alpha-\epsilon^i)x^{\alpha-\epsilon^i-\epsilon^j} = \alpha_i\alpha_j q_{ij}\delta_i(\alpha-\epsilon^j)\delta_j(\alpha)x^{\alpha-\epsilon^i-\epsilon^j} = q_{ij}\partial_i(\partial_j(x^\alpha)) \tag{2.30}$$

Thus $\partial_j\partial_i = q_{ij}\partial_i\partial_j$ for all $i < j$.

Let $i \le j$ and $\alpha \in \mathbb{N}^n$. Then

$$\sigma_i(\partial_j(x^\alpha)) = \alpha_j\delta_j(\alpha)\lambda_i(\alpha-\epsilon^j)x^{\alpha-\epsilon^j} = \alpha_j\delta_j(\alpha)\lambda_i(\alpha)q_{ji}x^{\alpha-\epsilon^j} = q_{ji}\lambda_i(\alpha)\partial_j(x^\alpha) = q_{ji}\partial_j(\sigma_i(x^\alpha)). \tag{2.31}$$

Hence $\sigma_i\partial_j = q_{ji}\partial_j\sigma_i$ for all $i \le j$. By Corollary 2.4.1 we can conclude:

Corollary 2.5.1 *Let $A = \mathcal{O}_Q(K^n)$ be the multivariate quantum polynomial algebra and let $\Omega = \bigwedge^Q(\Omega^1)$ be the associated quantum exterior algebra. Then the derivation $d : A \to \Omega^1$ with $d(x^\alpha) = \sum_{i=1}^n \partial_i(x^\alpha)\omega_i$ makes Ω into a differential calculus such that the de Rham complex and the integral complex are isomorphic.*

2.6 Manin's quantum n-space

In this section we will show that for a special case of the multivariate quantum polynomial algebra there exists a differential calculus whose bimodule structure is not diagonal, but upper triangular and nevertheless the de Rham complex and the integral complex are isomorphic.

Let $q \in K \setminus \{0\}$. For the matrix $Q = (q_{ij})$ with $q_{ij} = q$ and $q_{ji} = q^{-1}$ for all $i < j$ and $q_{ii} = 1$, the algebra $\mathcal{O}_Q(K^n)$ is called the *coordinate ring of quantum n-space* or *Manin's quantum n-space* and will be denoted by $A = K_q[x_1, \dots, x_n]$. We have the following defining relations of the algebra A

$$x_i x_j = q x_j x_i, \qquad i < j. \tag{2.32}$$

Note that for $\alpha \in \mathbb{N}^n$ and $1 \le i \le n$ we have:

$$\lambda_i(\alpha)x^\alpha x_i = x^{\alpha+\epsilon^i} = \overline{\lambda}_i(\alpha)x_i x^\alpha,$$

where

$$\lambda_i(\alpha) = \prod_{i<j} q^{\alpha_j} \qquad \text{and} \qquad \overline{\lambda}_i(\alpha) = \prod_{j<i} q^{-\alpha_j}.$$

More generally

$$x^{\alpha+\beta} = \left(\prod_{j=1}^{n-1} \lambda_j(\alpha)^{\beta_j}\right) x^\alpha x^\beta = \prod_{1\le s<j\le n} q^{\alpha_s\beta_j} x^\alpha x^\beta$$

Let $\mu(\alpha,\beta)$ be the scalar such that $x^\alpha x^\beta = \mu(\alpha,\beta)x^{\alpha+\beta}$.

We take the following two-parameter first order differential calculus Ω^1 (see [51, p.468] for the case $p = q^2$ and [18, Example 3.9] for the case $n = 2$), which is freely generated by $\{\omega_1, \dots \omega_n\}$ over A subject to the relations

$$\omega_i x_j = q x_j \omega_i + (p-1) x_i \omega_j, \qquad i < j, \tag{2.33}$$

$$\omega_i x_i = p x_i \omega_i, \tag{2.34}$$

$$\omega_j x_i = p q^{-1} x_i \omega_j, \qquad i < j, \tag{2.35}$$

There exists an algebra map $\sigma : A \to M_n(A)$ whose associated matrix of endomorphisms $\sigma = (\sigma_{ij})$ is upper triangular and such that $\omega_i x^\alpha = \sum_{i\le j} \sigma_{ij}(x^\alpha)\omega_j$. The next lemma will characterize the algebra map σ. For any $\alpha \in \mathbb{N}^n$ and $i = 1, \dots, n$ set $\pi_i(\alpha) = \prod_{s<i} p^{\alpha_s}$.

Lemma 2.6.1 *For $\alpha \in \mathbb{N}^n$ the entries of the matrix $\sigma(x^\alpha)$ are as follows $\sigma_{ij}(x^\alpha) = 0$ for $i > j$ and*

$$\sigma_{ij}(x^\alpha) = \eta_{ij}(\alpha) x^{\alpha+\epsilon^i-\epsilon^j} \qquad \text{where} \qquad \eta_{ij}(\alpha) = \begin{cases} \pi_j(\alpha)\overline{\lambda}_i(\alpha)\lambda_j(\alpha)(p^{\alpha_j}-1) & \text{for} \quad i<j, \\ \pi_i(\alpha)\overline{\lambda}_i(\alpha)\lambda_i(\alpha)p^{\alpha_i} & \text{for} \quad i=j \end{cases}$$

Proof: Fix a number i between 1 and n. We prove the relations for σ_{ij} by induction on the length of α, which by length we mean $|\alpha| = \alpha_1 + \dots + \alpha_n$. For $|\alpha| = 0$ the relation is clear, because $\alpha_j = 0$ for all j, i.e. $x^\alpha = 1$. Hence $\omega_i x^\alpha = \omega_i$, i.e. $\sigma_{ij}(x^\alpha) = \delta_{ij}$. Since $p^{\alpha_j} - 1 = 0$ for all j and $p^{\alpha_i} = 1$ the relation holds.

Now suppose that $m \ge 0$ and that the relations (2.6.1) hold for all $\alpha \in \mathbb{N}^n$ of length m. Let $\beta \in \mathbb{N}^n$ be an element of length $m+1$ and let k be the largest index j such that $\beta_j \ne 0$. Set $\alpha = \beta - \epsilon^k$, i.e. $\beta = \alpha + \epsilon^k$. We have to discuss the three cases $k < i$, $k = i$ and $k > i$.

If $k < i$, then for all $i < j$, $\alpha_j = 0$, i.e. $\sigma_{ij}(x^\alpha) = 0$. Hence

$$\omega_i x^\beta = \omega_i x^\alpha x_k = \sigma_{ii}(x^\alpha)\omega_i x_k = pq^{-1}\sigma_{ii}(x^\alpha)x_k\omega_i = p\pi_i(\alpha)q^{-1}\overline{\lambda}_i(\alpha)x^\alpha x_k \omega_i = \pi_i(\beta)\overline{\lambda}_i(\beta)x^\beta\omega_i,$$

since $\lambda_i(\alpha) = p^{\alpha_i} = 1$, $\pi_i(\alpha+\epsilon^k) = p\pi_i(\alpha)$ and $\overline{\lambda}_i(\alpha+\epsilon^k) = q^{-1}\overline{\lambda}_i(\alpha)$ for any $k < i$ and $\alpha \in \mathbb{N}^n$. Thus $\sigma_{ii}(x^\beta) = \pi_i(\beta)\overline{\lambda}_i(\beta)\lambda_i(\beta)p^{\beta_i}x^\beta$.

If $k = i$, then again $\sigma_{ij}(x^\alpha) = 0$ for all $j > i$. Moreover $\lambda_j(\alpha) = 1$ for all $j > i$. Thus

$$\omega_i x^\beta = \sigma_{ii}(x^\alpha)\omega_i x_i = \sigma_{ii}(x^\alpha)px_i\omega_i = \pi_i(\alpha)\overline{\lambda}_i(\alpha)p^{\alpha_i+1}x^\alpha x_i\omega_i = \pi_i(\beta)\overline{\lambda}_i(\beta)p^{\beta_i}x^\beta\omega_i,$$

since $\alpha_s = \beta_s$ for all $s < i$, i.e. $\pi_i(\beta) = \pi_i(\alpha)$ and $\overline{\lambda}_i(\beta) = \overline{\lambda}_i(\alpha)$.

If $i < k$, then note that $\sigma_{ij}(x^\alpha) = 0$ for all $k < j$, because $p^{\alpha_j} = 1$. Thus

$$\begin{aligned}
\omega_i x^\beta &= \sigma_{ii}(x^\alpha)\omega_i x_k + \sum_{i<j<k}\sigma_{ij}(x^\alpha)\omega_j x_k + \sigma_{ik}(x^\alpha)\omega_k x_k \\
&= \sigma_{ii}(x^\alpha)[qx_k\omega_i + (p-1)x_i\omega_k] + \sum_{i<j<k}\sigma_{ij}(x^\alpha)[qx_k\omega_j + (p-1)x_j\omega_k] + \sigma_{ik}(x^\alpha)px_k\omega_k \\
&= q\sigma_{ii}(x^\alpha)x_k\omega_i + \sum_{i<j<k}q\sigma_{ij}(x^\alpha)x_k\omega_j \\
&+ \underbrace{\left[(p-1)\sigma_{ii}(x^\alpha)x_i + \sum_{i<j<k}(p-1)\sigma_{ij}(x^\alpha)x_j + p\sigma_{ik}(x^\alpha)x_k\right]}_{(*)}\omega_k
\end{aligned}$$

Note that for any $j < k$ we have $q\lambda_j(\alpha) = \lambda_j(\beta)$. Hence $q\sigma_{ij}(x^\alpha)x_k = \sigma_{ij}(x^\beta)$ for all $j < k$. It is left to show that the expression $(*)$ equals $\sigma_{ik}(x^\beta)$. Recall that $\lambda_l(\alpha)x^\alpha x_l = x^{\alpha+\epsilon^l}$. Hence $\lambda_j(\alpha)x^{\alpha+\epsilon^i-\epsilon^j}x_j = x^{\alpha+\epsilon^i}$. Note also that $p^{\alpha_j}\pi_j(\alpha) = \pi_{j+1}(\alpha)$.

$$\begin{aligned}
(*) &= (p-1)\overline{\lambda}_i(\alpha)\left[\pi_i(\alpha)\lambda_i(\alpha)p^{\alpha_i}x^\alpha x_i + \sum_{i<j<k}\pi_j(\alpha)\lambda_j(\alpha)(p^{\alpha_j}-1)x^{\alpha+\epsilon^i-\epsilon^j}x_j\right] + p\sigma_{ik}(x^\alpha)x_k \\
&= (p-1)\overline{\lambda}_i(\alpha)\left[p^{\alpha_i}\pi_i(\alpha) + \sum_{i<j<k}\pi_j(\alpha)(p^{\alpha_j}-1)\right]x^{\alpha+\epsilon^i} + p\sigma_{ik}(x^\alpha)x_k \\
&= (p-1)\overline{\lambda}_i(\alpha)\left[\pi_{i+1}(\alpha) + \sum_{i<j<k}(\pi_{j+1}(\alpha)-\pi_j(\alpha))\right]x^{\alpha+\epsilon^i} + p\pi_k(\alpha)\overline{\lambda}_i(\alpha)(p^{\alpha_k}-1)x^{\alpha+\epsilon^i} \\
&= (p-1)\overline{\lambda}_i(\alpha)[\pi_{i+1}(\alpha) + \pi_k(\alpha) - \pi_{i+1}(\alpha)]x^{\alpha+\epsilon^i} + p\pi_k(\alpha)\overline{\lambda}_i(\alpha)(p^{\alpha_k}-1)x^{\alpha+\epsilon^i} \\
&= \overline{\lambda}_i(\alpha)[(p-1)\pi_k(\alpha) + p\pi_k(\alpha)(p^{\alpha_k}-1)]x^{\alpha+\epsilon^i} \\
&= \overline{\lambda}_i(\alpha)(p^{\alpha_k+1}-1)\pi_k(\alpha)x^{\alpha+\epsilon^i} \\
&= \pi_k(\beta)\overline{\lambda}_i(\beta)\lambda_k(\beta)(p^{\beta_k}-1)x^{\beta+\epsilon^i-\epsilon^k} = \sigma_{ik}(x^\beta),
\end{aligned}$$

since $\lambda_k(\beta) = 1 = \lambda_k(\alpha)$ and $\pi_k(\alpha) = \pi_k(\beta)$ as α and β differ only in the kth position. □

We will define a derivation $d : K_q[x_1,\dots,x_n] \to \Omega^1$ such that $d(x_i) = \omega_i$ for all i. For any $\alpha \in \mathbb{N}^n$ we set $d(x^\alpha) = \sum_{i=1}^n \partial_i(x^\alpha)\omega_i$ where

$$\partial_i(x^\alpha) = \delta_i(\alpha)x^{\alpha-\epsilon^i} \qquad \text{and} \qquad \delta_i(\alpha) = \pi_i(\alpha)\lambda_i(\alpha)\frac{p^{\alpha_i}-1}{p-1}. \tag{2.36}$$

for all $i = 1,\dots,n$. Note that for i,k we have:

$$\delta_i(\alpha) = q^{\mp 1}\delta_i(\alpha \pm \epsilon^k), \text{ if } i < k \qquad \text{and} \qquad \delta_i(\alpha) = p^{\mp 1}\delta_i(\alpha \pm \epsilon^k), \text{ if } i > k.$$

Lemma 2.6.2 *The pair (∂,σ) is a right twisted multi-derivation of $K_q[x_1,\dots,x_n]$ satisfying the equations* (2.14) *with respect to the multiplicatively antisymmetric matrix Q' whose entries are $Q'_{ij} = p^{-1}q$ for $i < j$. In particular*

$$\partial_i\partial_j = pq^{-1}\partial_j\partial_i, \qquad \forall i < j \tag{2.37}$$

holds as well as for all i,k,j:

$$\begin{aligned}
\partial_i\sigma_{kj} &= pq^{-1}\sigma_{kj}\partial_i, & i < k \le j\\
\partial_i\sigma_{kj} &= pq^{-1}\partial_j\sigma_{ki}, & k < i < j\\
\sigma_{ki}\partial_j &= pq^{-1}\sigma_{kj}\partial_i, & k < i < j\\
\partial_i\sigma_{ij} - pq^{-1}\partial_j\sigma_{ii} &= pq^{-1}\sigma_{ij}\partial_i - \sigma_{ii}\partial_j, & i < j
\end{aligned}$$

Proof: Let $\alpha,\beta \in \mathbb{N}^n$. To prove that the pair (∂,σ) is a right twisted multi-derivation, we show the following n equations hold

$$\partial_l(x^\alpha x^\beta) = \sum_k \partial_k(x^\alpha)\sigma_{kl}(x^\beta) + x^\alpha\partial_l(x^\beta), \qquad l = 1,\dots,n. \tag{2.38}$$

Since $x_ix_j = q^{-1}x_jx_i$ for $i > j$, we have $x_i^{\alpha_i}x_j^{\beta_j} = q^{-\alpha_i\beta_j}x_j^{\beta_j}x_i^{\alpha_i}$ for $i > j$, and hence $x^\alpha x^\beta = \mu(\alpha,\beta)x^{\alpha+\beta}$, where $\mu(\alpha,\beta) = \prod_{1\le r<s\le n} q^{-\alpha_s\beta_r}$. We then obtain

$$\partial_l(x^\alpha x^\beta) = \mu(\alpha,\beta)\delta_l(\alpha+\beta)x^{\alpha+\beta-\epsilon^l} = \pi_l(\alpha+\beta)\lambda_l(\alpha+\beta)\frac{p^{\alpha_l+\beta_l}-1}{p-1}\mu(\alpha,\beta)x^{\alpha+\beta-\epsilon^l}.$$

On the other hand, we compute

$$\begin{aligned}
&\sum_{k=1}^{n}\partial_k(x^\alpha)\sigma_{kl}(x^\beta)\\
=\ &\sum_{k=1}^{l-1}\partial_k(x^\alpha)\sigma_{kl}(x^\beta)+\partial_l(x^\alpha)\sigma_{ll}(x^\beta)\\
=\ &\sum_{k=1}^{l-1}\delta_k(\alpha)\pi_l(\beta)\overline{\lambda}_k(\beta)\lambda_l(\beta)(p^{\beta_l}-1)x^{\alpha-\epsilon^k}x^{\beta+\epsilon^k-\epsilon^l}+p^{\beta_l}\delta_l(\alpha)\pi_l(\beta)\overline{\lambda}_l(\beta)\lambda_l(\beta)x^{\alpha-\epsilon^l}x^\beta\\
=\ &\left[\pi_l(\beta)\frac{p^{\beta_l}-1}{p-1}\sum_{k=1}^{l-1}\pi_k(\alpha)(p^{\alpha_k}-1)+p^{\beta_l}\pi_l(\alpha+\beta)\frac{p^{\alpha_l}-1}{p-1}\right]\lambda_l(\alpha+\beta)\mu(\alpha,\beta)x^{\alpha+\beta-\epsilon^l}\\
=\ &\left[\pi_l(\beta)\frac{p^{\beta_l}-1}{p-1}(\pi_l(\alpha)-1)+p^{\beta_l}\pi_l(\alpha+\beta)\frac{p^{\alpha_l}-1}{p-1}\right]\lambda_l(\alpha+\beta)\mu(\alpha,\beta)x^{\alpha+\beta-\epsilon^l}\\
=\ &\left[\pi_l(\alpha+\beta)\frac{p^{\alpha_l+\beta_l}-1}{p-1}-\pi_l(\beta)\frac{p^{\beta_l}-1}{p-1}\right]\lambda_l(\alpha+\beta)\mu(\alpha,\beta)x^{\alpha+\beta-\epsilon^l}
\end{aligned}\tag{2.39}$$

where the third equality holds because

$$\lambda_k(\alpha)\overline{\lambda}_k(\beta)x^{\alpha-\epsilon^k}x^{\beta+\epsilon^k-\epsilon^l}=\lambda_l(\alpha)\mu(\alpha,\beta)x^{\alpha+\beta-\epsilon^l}\qquad\text{and}\qquad x^{\alpha-\epsilon^l}x^\beta=\lambda_l(\beta)\mu(\alpha,\beta)x^{\alpha+\beta-\epsilon^l}.$$

The fourth equation follows since $\pi_k(\alpha)p^{\alpha_k}=\pi_{k+1}(\alpha)$. As we also have

$$x^\alpha\partial_l(x^\beta)=\delta_l(\beta)x^\alpha x^{\beta-\epsilon^l}=\pi_l(\beta)\lambda_l(\alpha+\beta)\frac{p^{\beta_l}-1}{p-1}\mu(\alpha,\beta)x^{\alpha+\beta-\epsilon^l}.\tag{2.40}$$

We can conclude, combining (2.39) and (2.40) that (2.38) holds:

$$\sum_{k=1}^{n}\partial_k(x^\alpha)\sigma_{kl}(x^\beta)+x^\alpha\partial_l(x^\beta)=\pi_l(\alpha+\beta)\lambda_l(\alpha+\beta)\frac{p^{\alpha_l+\beta_l}-1}{p-1}\mu(\alpha,\beta)x^{\alpha+\beta-\epsilon^l}=\partial_l(x^\alpha x^\beta).\tag{2.41}$$

For any $i<j$ we have:

$$\partial_i\partial_j(x^\alpha)=\delta_i(\alpha-\epsilon^j)\delta_j(\alpha)x^{\alpha-\epsilon^i-\epsilon^j}=q^{-1}\delta_i(\alpha)p\delta_j(\alpha-\epsilon^i)x^{\alpha-\epsilon^i-\epsilon^j}=pq^{-1}\partial_j\partial_i(x^\alpha)\tag{2.42}$$

For $i<k<j$, we have $\eta_{kj}(\alpha)=pq^{-1}\eta_{kj}(\alpha-\epsilon^i)$. Hence

$$\sigma_{kj}\partial_i(x^\alpha)=\delta_i(\alpha)\eta_{kj}(\alpha-\epsilon^i)x^{\alpha-\epsilon^i+\epsilon^k-\epsilon^j}=p^{-1}q\eta_{kj}(\alpha)\delta_i(\alpha)x^{\alpha-\epsilon^i+\epsilon^k-\epsilon^j}=p^{-1}q\partial_i(\sigma_{kj}(x^\alpha))\tag{2.43}$$

which shows that $\partial_i\sigma_{kj}=pq^{-1}\sigma_{kj}\partial_i$ for all $i<k<j$.

For $i < k = j$, we have $\eta_{jj}(\alpha) = pq^{-1}\eta_{jj}(\alpha - \epsilon^i)$. Thus

$$\partial_i\sigma_{jj}(x^\alpha) = \eta_{jj}(\alpha)\delta_i(\alpha)x^{\alpha-\epsilon^i} = pq^{-1}\delta_i(\alpha)\eta_{jj}(\alpha-\epsilon^i)x^{\alpha-\epsilon^i} = pq^{-1}\sigma_{jj}(\partial_i(x^\alpha)), \tag{2.44}$$

showing $\partial_i\sigma_{jj} = pq^{-1}\sigma_{jj}\partial_i$ for $i < j$.

For $k < i < j$ using $\eta_{kj}(\alpha)\delta_i(\alpha) = \eta_{ki}(\alpha)\delta_j(\alpha)$ we get:

$$\begin{aligned}
\partial_i\sigma_{kj}(x^\alpha) &= \eta_{kj}(\alpha)\delta_i(\alpha+\epsilon^k-\epsilon^j)x^{\alpha-\epsilon^i+\epsilon^k-\epsilon^j} \\
&= pq^{-1}\eta_{kj}(\alpha)\delta_i(\alpha)x^{\alpha-\epsilon^i+\epsilon^k-\epsilon^j} \\
&= pq^{-1}\eta_{ki}(\alpha)\delta_j(\alpha)x^{\alpha-\epsilon^i+\epsilon^k-\epsilon^j} \\
&= pq^{-1}\eta_{ki}(\alpha)\delta_j(\alpha+\epsilon^k-\epsilon^i)x^{\alpha-\epsilon^i+\epsilon^k-\epsilon^j} = pq^{-1}\partial_j\sigma_{ki}(x^\alpha)
\end{aligned} \tag{2.45}$$

showing $\partial_i\sigma_{kj}(x^\alpha) - pq^{-1}\partial_j\sigma_{ki}(x^\alpha) = 0$. In a similar way, the relation

$$pq^{-1}\sigma_{kj}\partial_i(x^\alpha) - \sigma_{ki}\partial_j(x^\alpha) = 0$$

holds for $k < i < j$. Lastly, we show that the equations

$$\partial_i\sigma_{ij}(x^\alpha) - pq^{-1}\partial_j\sigma_{ii}(x^\alpha) = pq^{-1}\sigma_{ij}\partial_i(x^\alpha) - \sigma_{ii}\partial_j(x^\alpha), \qquad i < j$$

are satisfied, because of the following equations for $i < j$

$$\sigma_{ii}\partial_j(x^\alpha) = \frac{q^{-1}p^{\alpha_i}}{p-1}\eta_{ij}(\alpha)\pi_i(\alpha)\lambda_i(\alpha)x^{\alpha-\epsilon^j} = q^{-1}\partial_j\sigma_{ii}(x^\alpha)$$

$$\partial_i\sigma_{ij}(x^\alpha) = \frac{q^{-1}}{p-1}\eta_{ij}(\alpha)\pi_i(\alpha)\lambda_i(\alpha)(p^{\alpha_i+1}-1)x^{\alpha-\epsilon^j},$$

$$\sigma_{ij}\partial_i(x^\alpha) = \frac{p^{-1}(p^{\alpha_i}-1)}{p-1}\eta_{ij}(\alpha)\pi_i(\alpha)\lambda_i(\alpha)x^{\alpha-\epsilon^j},$$

By using these equations we attain the equation:

$$\partial_i\sigma_{ij}(x^\alpha) - pq^{-1}\partial_j\sigma_{ii}(x^\alpha) = -\frac{q^{-1}}{p-1}\eta_{ij}(\alpha)\pi_i(\alpha)\lambda_i(\alpha)x^{\alpha-\epsilon^j}$$

and

$$pq^{-1}\sigma_{ij}\partial_i(x^\alpha) - \sigma_{ii}\partial_j(x^\alpha) = -\frac{q^{-1}}{p-1}\eta_{ij}(\alpha)\pi_i(\alpha)\lambda_i(\alpha)x^{\alpha-\epsilon^j},$$

which completes the proof the lemma. □

Denote by $\Omega = \bigwedge^{p^{-1}q}(\Omega^1)$ the quantum exterior algebra of Ω^1 over $K_q[x_1,\ldots,x_n]$ with respect to the matrix Q'.

Theorem 2.6.3 *The derivation $d : K_q[x_1,\dots,x_n] \to \Omega^1$ extends to a differential calculus $\bigwedge^{p^{-1}q}(\Omega^1)$ on $K_q[x_1,\dots,x_n]$. Furthermore the de Rham and the integral complex associated to the differential calculus $(\bigwedge^{p^{-1}q}(\Omega^1), d)$ are isomorphic.*

Proof: The first statement follows from Proposition 2.3.1 and Lemma 2.6.2. We have an upper-triangular $\sigma = (\sigma_{ij})$ matrix by Lemma 2.6.1, of which the diagonal entries $\sigma_{ii}, i = 1,\dots,n$ are automorphisms. Hence we construct the corresponding lower-triangular matrix $\bar{\sigma}$ according to [18, Proposition 3.3]. The entries of $\bar{\sigma}$ are $\bar{\sigma}_{ij} = 0$ for $i < j$ and $\bar{\sigma}_{ii} = \sigma_{ii}^{-1}$ while

$$\bar{\sigma}_{ij}(x^\alpha) = q\pi_i(\alpha)^{-1}\overline{\lambda}_j(\alpha)^{-1}\lambda_i(\alpha)^{-1}(p^{-\alpha_i} - 1)q^{\alpha_j-\alpha_i}x^{\alpha+\epsilon^j-\epsilon^i}, \tag{2.46}$$

for $\alpha \in \mathbb{N}^n$ and $i > j$. Applying [18, Proposition 3.3] again yields the map $\hat{\sigma}$. The entries of $\hat{\sigma}$ are $\hat{\sigma}_{ij} = 0$ for $i > j$ and $\hat{\sigma}_{ii} = \sigma_{ii}$ while $\hat{\sigma}_{ij} = p^{j-i}\sigma_{ij}$ for $i < j$.

By using these formulas for the entries of the matrices $\bar{\sigma}(x^\alpha)$ and $\hat{\sigma}(x^\alpha)$, we obtain an explicit expression for

$$\partial_i^\sigma(x^\alpha) \;=\; \sum_{1\le j\le k\le i} \bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}(x^\alpha).$$

for any fixed $i = 1,\dots,n$. For $j < k < i$ we get:

$$\bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}(x^\alpha) = -p^{i-k}\pi_j(\alpha)\pi_k(\alpha)^{-1}(p - p^{-\alpha_k})(p^{\alpha_j} - 1)\partial_i(x^\alpha)$$

while for $j = k < i$ we have:

$$\bar{\sigma}_{kk} \circ \partial_k \circ \hat{\sigma}_{ki}(x^\alpha) = p^{i-k}(p - p^{-\alpha_k})\partial_i(x^\alpha)$$

Thus for any $k < i$ we get the partial sum:

$$\begin{aligned}
\Lambda_k &= \sum_{j=1}^{k} \bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}(x^\alpha) \\
&= \sum_{j=1}^{k-1} -p^{i-k}\pi_j(\alpha)\pi_k(\alpha)^{-1}(p - p^{-\alpha_k})(p^{\alpha_j} - 1)\partial_i(x^\alpha) + p^{i-k}(p - p^{-\alpha_k})\partial_i(x^\alpha) \\
&= \left[1 - \sum_{j=1}^{k-1} \pi_j(\alpha)(p^{\alpha_j} - 1)\pi_k(\alpha)^{-1}\right] p^{i-k}(p - p^{-\alpha_k})\partial_i(x^\alpha) \\
&= [\pi_k(\alpha) - \pi_k(\alpha) + 1]\,\pi_k(\alpha)^{-1}p^{i-k}(p - p^{-\alpha_k})\partial_i(x^\alpha) = \pi_k(\alpha)^{-1}p^{i-k}(p - p^{-\alpha_k})\partial_i(x^\alpha)
\end{aligned}$$

Similarly, for $k=i$ we have for $j<k=i$: $\bar{\sigma}_{ij}\circ\partial_j\circ\hat{\sigma}_{ji}(x^\alpha)=-p\pi_j(\alpha)(p^{\alpha_j}-1)\pi_i(\alpha)^{-1}\partial_i(x^\alpha)$ and for $j=k=i$ we have $\bar{\sigma}_{ii}\circ\partial_i\circ\hat{\sigma}_{ii}(x^\alpha)=p\partial_i(x^\alpha)$. This gives

$$\Lambda_i=\sum_{j=1}^{i}\bar{\sigma}_{ij}\circ\partial_j\circ\hat{\sigma}_{ii}(x^\alpha)=p\pi_i(\alpha)^{-1}\partial_i(x^\alpha).$$

The sum of these partial sums Λ_k yields:

$$\begin{aligned}
\partial_i^\sigma(x^\alpha) &= \sum_{k=1}^{i}\Lambda_k=\sum_{k=1}^{i-1}\pi_k(\alpha)^{-1}p^{i-k}(p-p^{-\alpha_k})\partial_i(x^\alpha)+p\pi_i(\alpha)^{-1}\partial_i(x^\alpha)\\
&= \left[\sum_{k=1}^{i-1}\pi_k(\alpha)^{-1}p^{i-k}(p-p^{-\alpha_k})+p\pi_i(\alpha)^{-1}\right]\partial_i(x^\alpha)\\
&= p\lambda_i(\alpha)\frac{p^{\alpha_i}-1}{p-1}\left[1+p^{i-1}\sum_{k=1}^{i-1}p^{-k}(p^{\alpha_k+1}-1)\left(\prod_{k<s<i}p^{\alpha_s}\right)\right]x^{\alpha-\epsilon^i}\\
&= p\lambda_i(\alpha)\frac{p^{\alpha_i}-1}{p-1}\left[1+p^{i-1}\sum_{k=1}^{i-1}\left((p^{-(k-1)}\left(\prod_{k-1<s<i}p^{\alpha_s}\right)-p^{-k}\left(\prod_{k<s<i}p^{\alpha_s}\right)\right)\right]x^{\alpha-\epsilon^i}\\
&= p\lambda_i(\alpha)\frac{p^{\alpha_i}-1}{p-1}\left[1+p^{i-1}\left(\pi_i(\alpha)-p^{-(i-1)}\right)\right]x^{\alpha-\epsilon^i}\\
&= p^i\partial_i(x^\alpha)
\end{aligned}$$

In order to apply Theorem 2.3.2, we need to calculate $\det(\sigma)$ as well as $\prod_j q'_{ij}$ where $Q'=(q'_{ij})$ is the corresponding multiplicatively antisymmetric matrix with $q'_{ij}=p^{-1}q$ for $i<j$. Let $\alpha\in\mathbb{N}^n$. By Theorem 2.3.2 it is enough to show that $\partial_i^\sigma(x^\alpha)=\left(\prod_j q'_{ij}\right)\det(\sigma)^{-1}(\partial_i(\det(\sigma)(x^\alpha)))$ holds, i.e.

$$p^i\partial_i(x^\alpha)=\left(\prod_j q'_{ij}\eta_{jj}(\alpha)\eta_{jj}(\alpha-\epsilon^i)^{-1}\right)\partial_i(x^\alpha).$$

By the definition of η_{ij} we obtain $p^{-1}q\eta_{jj}(\alpha)\eta_{jj}(\alpha-\epsilon^i)^{-1}=1$ for $i<j$ and $pq^{-1}\eta_{jj}(\alpha)\eta_{jj}(\alpha-\epsilon^i)^{-1}=p$ for $i>j$, while $\eta_{ii}(\alpha)\eta_{ii}(\alpha-\epsilon^i)^{-1}=p$. Hence the product of the $q'_{ij}\eta_{jj}(\alpha)\eta_{jj}(\alpha-\epsilon^i)^{-1}$ equals p^i and by Theorem 2.3.2 $K_q[x_1,\dots,x_n]$ satisfies the strong Poincaré duality with respect to the differential calculus $(\bigwedge^{p^{-1}q}(\Omega^1),d)$.

□

Chapter 3

Covariant Bimodules Over Monoidal Hom-Hopf Algebras

3.1 Introduction

Covariant bimodules have been studied in [82] to construct differential calculi on Hopf algebras over a field k. The concept of bicovariant bimodule (or Hopf bimodule) in [82] is considered as Hopf algebraic analogue to the notion of vector bundle over a Lie group equipped with the left and right actions of the group, that is, it replaces the the module of differential 1-forms of a Lie group, which is a H-bimodule and a H-bicomodule satisfying Hopf module compatibility condition between each of the H-actions and each of H-coactions. The structure theory of covariant bimodules in a coordinate-free setting was introduced in [72], where bicovariant bimodules are termed two-sided two-cosided Hopf modules; see also [51] for a detailed discussion of the theory both in coordinate-free setting and in coordinate form. With regard to knot theory and solutions of the quantum Yang-Baxter equation, the notion of a Yetter Drinfeld module over a bialgebra H has been investigated profoundly in [95, 71], where it is defined as an H-module and an H-comodule with a compatibility condition different than the one describing a Hopf module. One of the most essential features in [95, 71] is the fact that Yetter-Drinfel'd modules over a bialgebra H constitute a prebraided monoidal category which is braided monoidal one if H is a Hopf algebra with an invertible antipode. For a symmetric tensor category admitting (co-)equalizers the main result (Thm. 5.7) in [72] expresses that the structure theorem of Hopf modules extends to an equivalence between the category of bicovariant bimodules and the category of Yetter-Drinfeld modules over an Hopf algebra H. If the category of Hopf bimodules is equipped with a monoidal structure over H and the category of Yetter-Drinfeld modules is endowed with a tensor product over k with the diagonal action and codiagonal coaction, then the aforementioned equivalence is braided monoidal as well, in case H has a bijective antipode.

In the present chapter, we introduce the notions of left(right)-covariant Hom-bimodules and bicovariant Hom-bimodules to have twisted, generalized versions of the concepts of left(right)-covariant bimodules and bicovariant bimodules. Afterwards, we study the structure theory of covariant bimodules over monoidal Hom-Hopf algebras in coordinate-free setting and then we summarize the main results in coordinate form. Moreover, we show that the categories of left(right)-covariant Hom-bimodules and bicovariant Hom-bimodules are tensor categories equipped with a monoidal structure defined by a co-equalizer which is modified by a suitable insertion of a related nontrivial associator. In addition, we prove that the category of bicovariant bimodules over a monoidal Hom-Hopf algebra forms a (pre-)braided monoidal category (with nontrivial associators and unitors). Meanwhile, we propose (right-right) Hom-Yetter-Drinfeld modules as a deformed version of the classical ones and we attest that the category of Hom-Yetter-Drinfeld modules can be set as a (pre-)braided tensor category endowed with a tensor product over a commutative ring k described by the diagonal Hom-action and codiagonal Hom-coaction (together with nontrivial associators and unitors). As one of the main consequences of the chapter, we prove that the fundamental theorem of Hom-Hopf modules, which is provided in [21], can be extended to a (pre-)braided monoidal equivalence between the category of bicovariant Hom-bimodules and the category of (right-right) Hom-Yetter-Drinfeld modules.

3.2 Monoidal Hom-structures

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the monoidal category of k-modules, where k is a commutative ring throughout the chapter. We associate to $\mathcal{M}_k$ a new monoidal category $\mathcal{H}(\mathcal{M}_k)$ whose objects are ordered pairs (M, μ), with $M \in \mathcal{M}_k$ and $\mu \in Aut_k(M)$, and morphisms $f : (M, \mu) \to (N, \nu)$ are morphisms $f : A \to B$ in $\mathcal{M}_k$ satisfying $\nu \circ f = f \circ \mu$. The monoidal structure is given by $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$ and $(k, 1)$. If we speak briefly, all monoidal Hom-structures are objects in the tensor category $\widetilde{\mathcal{H}}(\mathcal{M}_k) =$

$(\mathcal{H}(\mathcal{M}_k), \otimes, (k, id), \tilde{a}, \tilde{l}, \tilde{r})$ introduced in ([21]), with the associativity constraint $\tilde{a}$ defined by

$$\tilde{a}_{A,B,C} = a_{A,B,C} \circ ((\alpha \otimes id) \otimes \gamma^{-1}) = (\alpha \otimes (id \otimes \gamma^{-1})) \circ a_{A,B,C}, \tag{3.1}$$

for (A, α), (B, β), $(C, \gamma) \in \mathcal{H}(\mathcal{M}_k)$, and the right and left unit constraints $\tilde{r}$, $\tilde{l}$ given by

$$\tilde{r}_A = \alpha \circ r_A = r_A \circ (\alpha \otimes id); \ \tilde{l}_A = \alpha \circ l_A = l_A \circ (id \otimes \alpha), \tag{3.2}$$

which we write elementwise:

$$\tilde{a}_{A,B,C}((a \otimes b) \otimes c) = \alpha(a) \otimes (b \otimes \gamma^{-1}(c)),$$

$$\tilde{l}_A(x \otimes a) = x\alpha(a) = \tilde{r}_A(a \otimes x).$$

The category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is termed *Hom-category* associated to $\mathcal{M}_k$, where a k-submodule $N \subset M$ is called a subobject of (M, μ) if $(N, \mu_{|_N}) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$, that is μ restricts to an automorphism of N. We now recall some definitions of monoidal Hom-structures.

Definition 3.2.1 *[21] An algebra in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called a monoidal Hom-algebra and a coalgebra in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is termed a monoidal Hom-coalgebra, that is, respectively,*

1. *A monoidal Hom-algebra is an object $(A, \alpha) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $m : A \otimes A \to A$, $a \otimes b \mapsto ab$ and an element $1_A \in A$ such that*

$$\alpha(a)(bc) = (ab)\alpha(c) \ ; \ a1_A = \alpha(a) = 1_A a \ ; \ \alpha(1_A) = 1_A \tag{3.3}$$

for all $a, b, c \in A$.

Remark 2 *The so-called multiplicativity , that is, the equality, for $a, b \in A$,*

$$\alpha(ab) = \alpha(a)\alpha(b) \tag{3.4}$$

follows from the equations in (3.3):

$$\alpha(a)\alpha(b) = (1_A a)\alpha(b) = \alpha(1_A)(ab) = 1_A(ab) = \alpha(ab),$$

which is in fact the requirement for $m : A \otimes A \to A$ to be a morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

2. A monoidal Hom-coalgebra is an object $(C,\gamma) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with k-linear maps $\Delta : C \to C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon : C \to k$ such that

$$\gamma^{-1}(c_1)\otimes c_{21}\otimes c_{22} = c_{11}\otimes c_{12}\otimes\gamma^{-1}(c_2); c_1\varepsilon(c_2) = \gamma^{-1}(c) = \varepsilon(c_1)c_2; \varepsilon(\gamma(c)) = \varepsilon(c) \quad (3.5)$$

for all $c \in C$.

Remark 3 *The so-called comultiplicativity, that is, the equality, for $c \in C$,*

$$\Delta(\gamma(c)) = \gamma(c_1)\otimes\gamma(c_2) \quad (3.6)$$

is a consequence of the equalities in (3.5):

$$\begin{aligned}\Delta(\gamma^{-1}(c)) &= \Delta(c_1\varepsilon(c_2)) = c_{11}\otimes c_{12}\varepsilon(c_2)\\ &= \gamma^{-1}(c_1)\otimes c_{21}\varepsilon(c_{22}) = \gamma^{-1}(c_1)\otimes\gamma^{-1}(c_2),\end{aligned}$$

which is actually the condition for $\Delta : C \to C\otimes C$ to be a morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

Definition 3.2.2 *[21] Now we consider modules and comodules over a Hom-algebra and a Hom-coalgebra, respectively.*

1. A right (A,α)-Hom-module consists of an object $(M,\mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $\psi : M\otimes A \to M$, $\psi(m\otimes a) = ma$ satisfying the following

$$\mu(m)(ab) = (ma)\alpha(b)\,;\, m1_A = \mu(m), \quad (3.7)$$

for all $m \in M$ and $a,b \in A$. The equation , for $a \in A$ and $m \in M$,

$$\mu(ma) = \mu(m)\alpha(a), \quad (3.8)$$

follows from (3.7) and (3.3) as in the Remark (2). ψ is termed a right Hom-action of (A,α) on (M,μ). Let (M,μ) and (N,ν) be two right (A,α)-Hom-modules. We call a morphism $f : M \to N$ right (A,α)-linear if it preserves Hom-action, that is, $f(ma) = f(m)a$ for all $m \in M$ and $a \in A$. Since we have, for any $m \in M$, $f(\mu(m)) = f(m1_A) = f(m)1_A = \nu(f(m))$, the equality $f\circ\mu = \nu\circ f$ holds.

2. *A right (C,γ)-Hom-comodule consists of an object $(M,\mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $\rho : M \to M \otimes C$, $\rho(m) = m_{[0]} \otimes m_{[1]}$ such that*

$$\mu^{-1}(m_{[0]}) \otimes m_{[1]1} \otimes m_{[1]2} = m_{[0][0]} \otimes m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})\,;\ m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m) \quad (3.9)$$

for all $m \in M$. The equality, for $m \in M$,

$$\mu(m)_{[0]} \otimes \mu(m)_{[1]} = \mu(m_{[0]}) \otimes \gamma(m_{[1]}) \quad (3.10)$$

is a consequence of (3.9) and (3.5) in a similar manner as in Remark (3). ρ is called a right Hom-coaction of (C,γ) on (M,μ). Let (M,μ) and (N,ν) be two right (C,γ)-Hom-comodules, then we call a morphism $f : M \to N$ right (C,γ)-colinear if it preserves Hom-coaction, i.e., $f(m_{[0]}) \otimes m_{[1]} = f(m)_{[0]} \otimes f(m)_{[1]}$ for all $m \in M$. The equation $f \circ \mu = \nu \circ f$ follows from (3.9) and (C,γ)-colinearity: For $m \in M$,

$$f(\mu^{-1}(m)) = f(m_{[0]})\varepsilon(m_{[1]}) = f(m)_{[0]}\varepsilon(f(m)_{[1]}) = \nu^{-1}(f(m)).$$

Definition 3.2.3 *[21] A bialgebra in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called a monoidal Hom-bialgebra and a Hopf algebra in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called a monoidal Hom-Hopf algebra, in other words*

1. *A monoidal Hom-bialgebra (H,α) is a sextuple $(H,\alpha,m,\eta,\Delta,\varepsilon)$ where (H,α,m,η) is a monoidal Hom-algebra and $(H,\alpha,\Delta,\varepsilon)$ is a monoidal Hom-coalgebra such that*

$$\Delta(hh') = \Delta(h)\Delta(h')\,;\ \Delta(1_H) = 1_H \otimes 1_H, \quad (3.11)$$

$$\varepsilon(hh') = \varepsilon(h)\varepsilon(h')\,;\ \varepsilon(1_H) = 1, \quad (3.12)$$

for any $h,h' \in H$.

2. *A monoidal Hom-Hopf algebra (H,α) is a septuple $(H,\alpha,m,\eta,\Delta,\varepsilon,S)$ where $(H,\alpha,m,\eta,\Delta,\varepsilon)$ is a monoidal Hom-bialgebra and $S : H \to H$ is a morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that $S * id_H = id_H * S = \eta \circ \varepsilon$.*

S is called antipode and it has the following properties

$$S(gh) = S(h)S(g)\,;\ S(1_H) = 1_H\,;$$

$$\Delta(S(h)) = S(h_2) \otimes S(h_1)\,;\ \varepsilon \circ S = \varepsilon,$$

for any elements g,h of the monoidal Hom-Hopf algebra H.

Definition 3.2.4 *([21]) Let* (H,α) *be a monoidal Hom-Hopf algebra. Then an object* (M,μ) *in* $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ *is called a left* (H,α)*-Hom-Hopf module if* (M,μ) *is both a left* (H,α)*-Hom-module and a left* (H,α)*-Hom-comodule such that the compatibility relation*

$$\rho(hm) = h_1 m_{(-1)} \otimes h_2 m_{(0)} \tag{3.13}$$

holds for $h \in H$ *and* $m \in M$*, where* $\rho : M \to H \otimes M$ *is a left* H*-coaction on* M*. A morphism of two* (H,α)*-Hom-Hopf modules is a* k*-linear map which is both left* (H,α)*-linear and left* (H,α)*-colinear. The category of left* (H,α)*-Hom-Hopf modules and the morphisms between them is denoted by* ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)$*.*

We also have the fundamental theorem of Hopf modules in the Hom-setting as follows.

Theorem 3.2.5 *([21])* (F,G) *is a pair of inverse equivalences, where the functors* F *and* G *are defined by*

$$F = (H \otimes -, \alpha \otimes -) : \widetilde{\mathcal{H}}(\mathcal{M}_k) \to {}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k), \tag{3.14}$$

$$G = {}^{coH}(-) : {}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k) \to \widetilde{\mathcal{H}}(\mathcal{M}_k). \tag{3.15}$$

Above, we get ${}^{coH}M = \{m \in M | \rho(m) = 1_H \otimes \mu^{-1}(m)\}$ for a left (H,α)-Hom-Hopf module (M,μ), which is called the *left coinvariant* of (H,α) on (M,μ), and $({}^{coH}M, \mu|_{{}^{coH}M})$ is in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

In the following, we introduce the concepts of Hom-bimodules, Hom-(co)module algebras and left (right) adjoint Hom-actions of a monoidal Hom-Hopf algebra on itself.

Definition 3.2.6 *Let* (A,α) *and* (B,β) *be two monoidal Hom-algebras. A left* (A,α)*, right* (B,β) *Hom-bimodule consists of an object* $(M,\mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ *together with a left* (A,α)*-Hom-action* $\phi : A \otimes M \to M$*,* $\phi(a \otimes m) = am$ *and a right* (B,β)*-Hom-action* $\varphi : M \otimes B \to M$*,* $\varphi(m \otimes b) = mb$ *fulfilling the compatibility condition, for all* $a \in A$*,* $b \in B$ *and* $m \in M$*,*

$$(am)\beta(b) = \alpha(a)(mb). \tag{3.16}$$

We call a left (A,α)*, right* (B,β) *Hom-bimodule a* $[(A,\alpha),(B,\beta)]$*-Hom-bimodule. Let* (M,μ) *and* (N,ν) *be two* $[(A,\alpha),(B,\beta)]$*-Hom-bimodules. A morphism* $f : M \to N$ *is called a morphism of* $[(A,\alpha),(B,\beta)]$*-Hom-bimodules if it is both left* (A,α)*-linear and right* (B,β)*-linear.* f *satisfies the following, for all* $a \in A$*,* $b \in B$ *and* $m \in M$*,*

$$(af(m))\beta(b) = \alpha(a)(f(m)b), \tag{3.17}$$

directly from (3.16).

Lemma 3.2.7 *Let (H,α) be a monoidal Hom-Hopf algebra and (M,μ) a (H,α)-Hom-bimodule. For $h \in H$ and $m \in M$,*

1. *the linear map*

$$M \otimes H \to M,\ m \otimes h \mapsto \widetilde{ad}_R(h)(m) = (S(h_1)\mu^{-1}(m))\alpha(h_2)$$

   defines a right (H,α)-Hom-module structure on (M,μ), and

2. *the linear mapping*

$$H \otimes M \to M,\ h \otimes m \mapsto \widetilde{ad}_L(h)(m) = \alpha(h_1)(\mu^{-1}(m)S(h_2))$$

   gives (M,μ) a left (H,α)-Hom-module structure.

Proof:

1. We first set $m \triangleleft h = \widetilde{ad}_R(h)(m)$ for $h \in H$ and $m \in M$. Let g also be in H, then

$$\begin{aligned}
\mu(m) \triangleleft (hg) &= (S((hg)_1)\mu^{-1}(\mu(m)))\alpha((hg)_2) \\
&= (S(g_1)S(h_1)m)(\alpha(h_2)\alpha(g_2)) \\
&= (\alpha(S(g_1))(S(h_1)\mu^{-1}(m)))(\alpha(h_2)\alpha(g_2)) \\
&= \alpha^2(S(g_1))((S(h_1)\mu^{-1}(m))(h_2g_2)) \\
&= \alpha^2(S(g_1))(((\alpha^{-1}(S(h_1))\mu^{-2}(m))h_2)\alpha(g_2)) \\
&= (\alpha(S(g_1))((\alpha^{-1}(S(h_1))\mu^{-2}(m))h_2))\alpha^2(g_2)) \\
&= (S(\alpha(g_1))\alpha^{-1}((S(h_1)\mu^{-1}(m))\alpha(h_2)))\alpha^2(g_2)) \\
&= (S(\alpha(g_1))\alpha^{-1}(m \triangleleft h))\alpha(\alpha(g_2)) \\
&= (m \triangleleft h) \triangleleft \alpha(g).
\end{aligned}$$

   $m \triangleleft 1_H = (S(1_H)\mu^{-1}(m))\alpha(1_H) = (1_H\mu^{-1}(m))1_H = m1_H = \mu(m)$, which finishes the proof.

2. The proof is carried out as in (1).

□

Remark 4 *Since a monoidal Hom-Hopf algebra (H,α) is a (H,α)-Hom bimodule, by taking (M,μ) as (H,α) in the above lemma, the mappings $\widetilde{ad}_R$ and $\widetilde{ad}_L$ give us the so-called right and left adjoint Hom-action of (H,α) on itself, respectively.*

Definition 3.2.8 *Let (B,β) be a monoidal Hom-bialgebra. A right (B,β)-Hom-comodule algebra (or Hom-quantum space) (A,α) is a monoidal Hom-algebra and a right (B,β)-Hom-comodule with a Hom-coaction $\rho^A : A \to A \otimes B$, $a \mapsto a_{(0)} \otimes a_{(1)}$ such that ρ^A is a Hom-algebra morphism, i.e., for any $a, a' \in A$*

$$(aa')_{(0)} \otimes (aa')_{(1)} = a_{(0)}a'_{(0)} \otimes a_{(1)}a'_{(1)}, \ \rho^A(1_A) = 1_A \otimes 1_B. \tag{3.18}$$

By using the properties of (A,α) and (B,β) as monoidal Hom-algebras and the equalities in (3.18), we get

$$\rho^A \circ \alpha = (\alpha \otimes \beta) \circ \rho^A.$$

Definition 3.2.9 *Let (B,β) be a monoidal Hom-bialgebra. A right (B,β)-Hom-module algebra (A,α) is a monoidal Hom-algebra and a right (B,β)-Hom-module with a Hom-action $\rho_A : A \otimes B \to A$, $a \otimes b \mapsto a \cdot b$ such that, for any $a, a' \in A$ and $b \in B$*

$$(aa') \cdot b = (a \cdot b_1)(a' \cdot b_2), \ 1_A \cdot b = \varepsilon(b)1_A. \tag{3.19}$$

The equation

$$\rho_A \circ (\alpha \otimes \beta) = \alpha \circ \rho_A$$

follows from the defining relations of Hom-module algebra in (3.19), Hom-counity of (B,β) and Hom-unity of (A,α).

Proposition 3.2.10 *The right adjoint Hom-action $\widetilde{ad}_R$ (resp. the left adjoint Hom-action $\widetilde{ad}_L$) turns the monoidal Hom-Hopf algebra (H,α) into a right (H,α)-Hom-module algebra (resp. a left (H,α)-Hom-module algebra).*

Proof: We prove only the case of $\widetilde{ad}_R$. Since we have already verified in the Lemma (3.2.7) that $\widetilde{ad}_R$ determines a right (H,α)-Hom-module structure on itself, we are left to prove that the conditions in (3.19) are accomplished: In fact,

$$
\begin{aligned}
(g \triangleleft k_1)(h \triangleleft k_2) &= ((S(k_{11})\alpha^{-1}(g))\alpha(k_{12}))((S(k_{21})\alpha^{-1}(h))\alpha(k_{22}))\\
&= ((S(k_{11})\alpha^{-1}(g))\alpha(k_{12}))(\alpha(S(k_{21}))(\alpha^{-1}(h)k_{22}))\\
&= (S(\alpha(k_{11}))g)(\alpha(k_{12})(S(k_{21})(\alpha^{-2}(h)\alpha^{-1}(k_{22}))))\\
&= (S(\alpha(k_{11}))g)((k_{12}S(k_{21}))(\alpha^{-1}(h)k_{22}))\\
&= (S(k_1)g)((\alpha(k_{211})S(\alpha(k_{212})))(\alpha^{-1}(h)k_{22}))\\
&= (S(k_1)g)(\alpha(\varepsilon(k_{21})1_H)(\alpha^{-1}(h)k_{22}))\\
&= (S(k_1)g)(1_H(\alpha^{-1}(h)\alpha^{-1}(k_2)))\\
&= (S(k_1)g)(hk_2)\\
&= \alpha(S(k_1))((\alpha^{-1}(g)\alpha^{-1}(h))k_2)\\
&= (S(k_1)\alpha^{-1}(gh))\alpha(k_2)\\
&= (gh) \triangleleft k,
\end{aligned}
$$

where the fifth line is a consequence of the equality

$$
h_1 \otimes h_{211} \otimes h_{212} \otimes h_{22} = \alpha(h_{11}) \otimes \alpha^{-1}(h_{12}) \otimes \alpha^{-1}(h_{21}) \otimes h_{22}, \tag{3.20}
$$

which follows from the relation

$$
(id \otimes (\Delta \otimes id)) \circ (id \otimes \Delta) \circ \Delta = (id \otimes \tilde{a}^{-1}_{H,H,H}) \circ \tilde{a}_{H,H,H\otimes H} \circ (id_{H\otimes H} \otimes \Delta) \circ (\Delta \otimes id) \circ \Delta, \tag{3.21}
$$

and

$$
\begin{aligned}
1_H \triangleleft h &= (S(h_1)\alpha^{-1}(1_H))\alpha(h_2) = \alpha(S(h_1))\alpha(h_2)\\
&= \alpha(\varepsilon(h)1) = \varepsilon(h)1_H.
\end{aligned}
$$

In the case of $\widetilde{ad}_L$, similar computations are performed. □

Definition 3.2.11 *Let (M,μ) be a right (A,α)-Hom-module and (N,ν) be a left (A,α)-Hom-module. The tensor product $(M \otimes_A N, \mu \otimes \nu)$ of (M,μ) and (N,ν) over (A,α) is the coequalizer of $\rho \otimes id_N$, $(id_M \otimes \bar{\rho}) \circ \tilde{a}_{M,A,N} : (M \otimes A) \otimes N \to M \otimes N$, where $\rho : M \otimes A \to M$, $m \otimes a \mapsto ma$ and $\bar{\rho} : A \otimes N \to N$, $a \otimes n \mapsto an$, for $a \in A$, $m \in M$ and $n \in N$, are the right and left Hom-actions of (A,α) on (M,μ) and (N,ν) respectively. That is,*

$$
m \otimes_A n = \{m \otimes n \in M \otimes N \mid ma \otimes n = \mu(m) \otimes a\nu^{-1}(n), \forall a \in A\}. \tag{3.22}
$$

3.3 Left-Covariant Hom-Bimodules

Definition 3.3.1 *A left-covariant (H,α)-Hom-bimodule is an (H,α)-Hom-bimodule $(M,\mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ which is a left (H,α)-Hom-comodule, with Hom-coaction $\rho : M \to H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that*

$$\rho((hm)\alpha(g)) = \Delta(\alpha(h))(\rho(m)\Delta(g)). \tag{3.23}$$

We here recall the left coinvariant of (H,α) on (M,μ) for a left (H,α)-Hom-Hopf module (M,μ), ${}^{coH}M = \{m \in M | \rho(m) = 1_H \otimes \mu^{-1}(m)\}$, which is in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

Lemma 3.3.2 *Let (M,μ) be a left-covariant (H,α)-Hom-bimodule. There exists a unique k-linear projection $P_L : M \longrightarrow {}^{coH}M$, $m \mapsto S(m_{(-1)})m_{(0)}$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, such that, for all $h \in H$ and $m \in M$,*

$$P_L(hm) = \varepsilon(h)\mu(P_L(m)). \tag{3.24}$$

We also have the following relations

$$m = m_{(-1)}P_L(m_{(0)}), \tag{3.25}$$

$$P_L(mh) = \widetilde{ad}_R(h)(P_L(m)). \tag{3.26}$$

Proof: We show that $P_L(m)$ is in ${}^{coH}M$: Indeed,

$$\begin{aligned}
\rho(P_L(m)) &= \rho(S(m_{(-1)})m_{(0)}) = (S(m_{(-1)})m_{(0)})_{(-1)} \otimes (S(m_{(-1)})m_{(0)})_{(0)} \\
&= S(m_{(-1)})_1 m_{(0)(-1)} \otimes S(m_{(-1)})_2 m_{(0)(0)} \\
&= S(m_{(-1)2})m_{(0)(-1)} \otimes S(m_{(-1)1})m_{(0)(0)} \\
&= S(\alpha(m_{(0)(-1)1}))\alpha(m_{(0)(-1)2}) \otimes S(\alpha^{-1}(m_{(-1)}))m_{(0)(0)} \\
&= \alpha(S(m_{(0)(-1)1})m_{(0)(-1)2}) \otimes \alpha^{-1}(S(m_{(-1)}))m_{(0)(0)} \\
&= \alpha(\varepsilon(m_{(0)(-1)})1_H) \otimes \alpha^{-1}(S(m_{(-1)}))m_{(0)(0)} \\
&= 1_H \otimes \alpha^{-1}(S(m_{(-1)}))\varepsilon(m_{(0)(-1)})m_{(0)(0)} \\
&= 1_H \otimes \alpha^{-1}(S(m_{(-1)}))\mu^{-1}(m_{(0)}) \\
&= 1_H \otimes \mu^{-1}(S(m_{(-1)}m_{(0)})) = 1_H \otimes \mu^{-1}(P_L(m)),
\end{aligned}$$

where in the fifth equality we have used

$$m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)(-1)} \otimes m_{(0)(0)} = \alpha^{-1}(m_{(-1)}) \otimes \alpha(m_{(0)(-1)1}) \otimes \alpha(m_{(0)(-1)2}) \otimes m_{(0)(0)}, \tag{3.27}$$

which results from the fact that the following relation holds:

$$(\Delta\otimes id)\circ(id\otimes\rho)\circ\rho=\tilde{a}^{-1}_{H,H,H\otimes M}\circ(id\otimes\tilde{a}_{H,H,M})\circ(id\otimes(\Delta\otimes id))\circ(id\otimes\rho)\circ\rho. \quad (3.28)$$

Now we prove that $M=H\cdot{}^{coH}M$

$$\begin{aligned}
m_{(-1)}P_L(m_{(0)}) &= m_{(-1)}(S(m_{(0)(-1)})m_{(0)(0)})\\
&= (\alpha^{-1}(m_{(-1)})S(m_{(0)(-1)}))\mu(m_{(0)(0)})\\
&= (m_{(-1)1}S(m_{(-1)2}))m_{(0)}\\
&= \varepsilon(m_{(-1)})1_Hm_{(0)}\\
&= \mu(\varepsilon(m_{(-1)})m_{(0)})\\
&= \mu(\mu^{-1}(m))=m,
\end{aligned}$$

where we have used the Hom-coassociativity condition for the left Hom-comodules in the third equation.

$$\begin{aligned}
P_L(hm) &= S(h_1m_{(-1)})(h_2m_{(0)})\\
&= (S(m_{(-1)})S(h_1))(h_2m_{(0)})\\
&= \alpha(S(m_{(-1)}))(S(h_1)(\alpha^{-1}(h_2)\mu^{-1}(m_{(0)})))\\
&= \alpha(S(m_{(-1)}))((\alpha^{-1}(S(h_1))\alpha^{-1}(h_2))m_{(0)})\\
&= \alpha(S(m_{(-1)}))(\alpha^{-1}(S(h_1)h_2)m_{(0)})\\
&= \varepsilon(h)\alpha(S(m_{(-1)}))\mu(m_0)\\
&= \varepsilon(h)\mu(S(m_{(-1)}m_0))=\varepsilon(h)\mu(P_L(m)).
\end{aligned}$$

$$\begin{aligned}
P_L(mh) &= S(m_{(-1)}h_1)(m_{(0)}h_2)\\
&= (S(h_1)S(m_{(-1)}))(m_{(0)}h_2)\\
&= [(\alpha^{-1}(S(h_1))\alpha^{-1}(S(m_{(-1)})))m_{(0)}]\alpha(h_2)\\
&= [S(h_1)(\alpha^{-1}(S(m_{(-1)}))\mu^{-1}(m_{(0)}))]\alpha(h_2)\\
&= (S(h_1)\mu^{-1}(S(m_{(-1)})m_{(0)}))\alpha(h_2)\\
&= \widetilde{ad}_R(h)(P_L(m)).
\end{aligned}$$

If m belongs to ${}^{coH}M$, then

$$P_L(m)=S(1_H)\mu^{-1}(m)=m$$

proving that P_L is a k-projection of M onto ${}^{coH}M$. Let $P'_L : M \longrightarrow {}^{coH} M$ be another k-projection, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, such that $P'_L(hm) = \varepsilon(h)\mu(P'_L(m))$, then, by the fact that P'_L is a morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, we have for all $m \in M$

$$\begin{aligned} P'_L(m) &= P'_L(m_{(-1)}P_L(m_{(0)})) = \varepsilon(m_{(-1)})\mu(P'_L(P_L(m_{(0)})) \\ &= \varepsilon(m_{(-1)})\mu(P_L(m_{(0)})) = P_L(\mu(\varepsilon(m_{(-1)})m_{(0)})) \\ &= P_L(\mu(\mu^{-1}(m))) = P_L(m), \end{aligned}$$

which shows the uniqueness of P_L. □

Proposition 3.3.3 *Let $(N,\nu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ be a right (H,α)-Hom-module by the Hom-action $N \otimes H \to N$, $n \otimes h \mapsto n \triangleleft h$. The following morphisms*

$$H \otimes (H \otimes N) \to H \otimes N,\ h \otimes (g \otimes n) \mapsto \alpha^{-1}(h)g \otimes \nu(n), \tag{3.29}$$

$$(H \otimes N) \otimes H \to H \otimes N,\ (h \otimes n) \otimes g \mapsto hg_1 \otimes n \triangleleft g_2, \tag{3.30}$$

$$\rho : H \otimes N \to H \otimes (H \otimes N),\ h \otimes n \mapsto \alpha(h_1) \otimes (h_2 \otimes \nu^{-1}(n)), \tag{3.31}$$

in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, define a left-covariant (H,α)-Hom-bimodule structure on $(H \otimes N, \alpha \otimes \nu)$.

Proof: We verify the Hom-associativity and Hom-unity conditions for the left and the right Hom-multiplications of (H,α) on $(H \otimes N, \alpha \otimes \nu)$, respectively: For all $h,k,g \in H$ and $n \in N$, we get

$$\begin{aligned} \alpha(k)(h(g \otimes n)) &= \alpha(k)(\alpha^{-1}(h)g \otimes \nu(n)) = k((\alpha^{-1}(h)g) \otimes \nu^2(n) \\ &= \alpha^{-1}(kh)\alpha(g) \otimes \nu^2(n) = (kh)((\alpha \otimes \nu)(g \otimes n)), \end{aligned}$$

$$1_H(g \otimes n) = \alpha^{-1}(1_H)g \otimes \nu(n) = \alpha(g) \otimes \nu(n) = (\alpha \otimes \nu)(g \otimes n),$$

$$\begin{aligned} ((\alpha \otimes \nu)(h \otimes n))(gk) &= \alpha(h)(g_1k_1) \otimes \nu(n) \triangleleft (g_2k_2) = (hg_1)\alpha(k_1) \otimes (n \triangleleft g_2) \triangleleft \alpha(k_2) \\ &= (hg_1 \otimes n \triangleleft g_2)\alpha(k) = ((h \otimes n)g)\alpha(k), \end{aligned}$$

$$(h \otimes n)1_H = h1_H \otimes n \triangleleft 1_H = (\alpha \otimes \nu)(h \otimes n).$$

We now show that the compatibility condition is satisfied:

$$\begin{aligned} (g(h \otimes n))\alpha(k) &= (\alpha^{-1}(g)h \otimes \nu(n))\alpha(k) = (\alpha^{-1}(g)h)\alpha(k_1) \otimes \nu(n) \triangleleft \alpha(k_2) \\ &= g(hk_1) \otimes \nu(n) \triangleleft \alpha(k_2) = \alpha^{-1}(\alpha(g))(hk_1) \otimes \nu(n \triangleleft k_2) \\ &= \alpha(g)(hk_1 \otimes n \triangleleft k_2) = \alpha(g)((h \otimes n)k). \end{aligned}$$

ρ satisfies the Hom-coassociativity and Hom-counity condition: Indeed, on one hand we have

$$\begin{aligned}\Delta((h\otimes n)_{(-1)})\otimes(\alpha^{-1}\otimes\nu^{-1})((h\otimes n)_{(0)}) &= \Delta(\alpha(h_1))\otimes(\alpha^{-1}\otimes\nu^{-1})(h_2\otimes\nu^{-1}(n))\\ &= (\alpha(h_{11})\otimes\alpha(h_{12}))\otimes(\alpha^{-1}(h_2)\otimes\nu^{-2}(n))\\ &= (h_1\otimes\alpha(h_{21}))\otimes(h_{22}\otimes\nu^{-2}(n))\\ &= (\alpha^{-1}((h\otimes n)_{(-1)})\otimes(h\otimes n)_{(0)(-1)})\otimes(h\otimes n)_{(0)(0)},\end{aligned}$$

where in the first equality we have used $\rho(h\otimes n)=(h\otimes n)_{(-1)}\otimes(h\otimes n)_{(0)}=\alpha(h_1)\otimes(h_2\otimes\nu^{-1}(n))$, the third equality has resulted from the relation

$$\alpha(h_{11})\otimes\alpha(h_{12})\otimes h_2\otimes\nu^{-1}(n)=h_1\otimes\alpha(h_{21})\otimes\alpha(h_{22})\otimes\nu^{-1}(n), \tag{3.32}$$

which follows from

$$(\Delta\otimes id)\circ\rho=\tilde{a}^{-1}_{H,H,H\otimes N}\circ(id\otimes\tilde{a}_{H,H,N})\circ(id\otimes(\Delta\otimes id))\circ\rho,$$

and in the last line we have used $\rho((h\otimes n)_{(0)})=(h\otimes n)_{(0)(-1)}\otimes(h\otimes n)_{(0)(0)}=\alpha(h_{21})\otimes(h_{22}\otimes\nu^{-2}(n))$. On the other hand,

$$\begin{aligned}\varepsilon((h\otimes n)_{(-1)})(h\otimes n)_{(0)} &= \varepsilon(\alpha(h_1))(h_2\otimes\nu^{-1}(n))\\ &= \varepsilon(h_1)h_2\otimes\nu^{-1}(n)=(\alpha^{-1}\otimes\nu^{-1})(h\otimes n).\end{aligned}$$

To finish the proof of the fact that the above Hom-actions and Hom-coaction of (H,α) on $H\otimes N$ define a left-covariant (H,α)-Hom-bimodule structure on $(H\otimes N,\alpha\otimes\nu)$ we show that the following relation holds:

$$
\begin{aligned}
\Delta(\alpha(g))(\rho(h\otimes n)\Delta(k)) &= (\alpha(g_1)\otimes\alpha(g_2))((\alpha(h_1)\otimes(h_2\otimes\nu^{-1}(n)))(k_1\otimes k_2))\\
&= (\alpha(g_1)\otimes\alpha(g_2))(\alpha(h_1)k_1\otimes(h_2\otimes\nu^{-1}(n))k_2)\\
&= \alpha(g_1)(\alpha(h_1)k_1)\otimes\alpha(g_2)((h_2\otimes\nu^{-1}(n))k_2)\\
&= \alpha(g_1)(\alpha(h_1)k_1)\otimes(\alpha^{-1}(\alpha(g_2))(h_2k_{21})\otimes\nu(\nu^{-1}(n)\lhd k_{22}))\\
&= \alpha(g_1)(\alpha(h_1)k_1)\otimes(g_2(h_2k_{21})\otimes n\lhd\alpha(k_{22}))\\
&= \alpha(g_1)(\alpha(h_1)\alpha(k_{11}))\otimes(g_2(h_2k_{12})\otimes n\lhd k_2)\\
&= (g_1\alpha(h_1))\alpha^2(k_{11})\otimes((\alpha^{-1}(g_2)h_2)\alpha(k_{12})\otimes\nu^{-1}(\nu(n)\lhd\alpha(k_2)))\\
&= \alpha(((\alpha^{-1}(g)h)\alpha(k_1))_1)\otimes(((\alpha^{-1}(g)h)\alpha(k_1))_2\otimes\nu^{-1}(\nu(n)\lhd\alpha(k_2)))\\
&= \rho((\alpha^{-1}(g)h)\alpha(k_1)\otimes\nu(n)\lhd\alpha(k_2))\\
&= \rho((\alpha^{-1}(g)h\otimes\nu(n))\alpha(k))\\
&= \rho((g(h\otimes n))\alpha(k)),
\end{aligned}
$$

where the sixth equality has resulted from Hom-coassociativity of Δ for $k\in H$. □

Proposition 3.3.4 *If $(M,\mu)\in\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is a left-covariant (H,α)-Hom-bimodule, the k-linear map*

$$\theta: H\otimes {}^{coH}M\longrightarrow M,\ h\otimes m\mapsto hm, \tag{3.33}$$

in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, is an isomorphism of left-covariant (H,α)-Hom-bimodules, where the right (H,α)-Hom-module structure on $({}^{coH}M,\mu|_{{}^{coH}M})$ is defined, by using (3.26), as follows

$$m\lhd h := P_L(mh)=\widetilde{ad}_R(h)(m), \tag{3.34}$$

for $h\in H$ and $m\in {}^{coH}M$.

Proof: Define $\vartheta: M\to H\otimes {}^{coH}M$ as follows: For any $m\in M$

$$\vartheta(m)=m_{(-1)}\otimes P_L(m_{(0)}),$$

which is shown that ϑ is the inverse of θ:

$$
\begin{aligned}
\theta(\vartheta(m)) &= \theta(m_{(-1)}\otimes P_L(m_{(0)}))\\
&= = m_{(-1)}P_L(m_{(0)})=m,
\end{aligned}
$$

where in the last equality the equation (3.25) has been used. On the other hand, for $m \in {}^{coH}M$ and $h \in H$ we obtain

$$\begin{aligned}
\vartheta(\theta(h\otimes m)) &= \vartheta(hm)\\
&= (hm)_{(-1)}\otimes P_L((hm)_{(0)})\\
&= h_1 m_{(-1)}\otimes P_L(h_2 m_{(0)})\\
&= h_1 1_H \otimes P_L(h_2\mu^{-1}(m))\\
&= \alpha(h_1)\otimes \varepsilon(h_2)\mu(P_L(\mu^{-1}(m)))\\
&= \alpha(h_1\varepsilon(h_2))\otimes P_L(m)\\
&= \alpha(\alpha^{-1}(h))\otimes m = h\otimes m,
\end{aligned}$$

where in the fourth equality the fact that the Hom-coaction of (H,α) on (M,μ) is a morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ has been used. Now we show that θ is both (H,α)-bilinear and left (H,α)-colinear:

$$\theta(g(h\otimes m)) = \theta(\alpha^{-1}(g)h\otimes \mu(m)) = (\alpha^{-1}(g)h)\mu(m)) = g(hm) = g\theta(h\otimes m),$$

$$\begin{aligned}
\theta((h\otimes m)k) &= \theta(hk_1\otimes m\triangleleft k_2) = (hk_1)(\widetilde{ad}_R(k_2)m)\\
&= (hk_1)((S(k_{21})\mu^{-1}(m))\alpha(k_{22}))\\
&= (hk_1)(\alpha(S(k_{21}))(\mu^{-1}(m)k_{22}))\\
&= ((\alpha^{-1}(h)\alpha^{-1}(k_1))\alpha(S(k_{21})))(m\alpha(k_{22}))\\
&= (h(\alpha^{-1}(k_1)S(k_{21})))(m\alpha(k_{22}))\\
&= (h(k_{11}S(k_{12})))(mk_2)\\
&= \alpha(h)(m\alpha^{-1}(k)) = \theta(h\otimes m)k,
\end{aligned}$$

where the penultimate line follows from the first relation of (3.5). Lastly, put ${}^{M}\rho : M \to H\otimes M$ and ${}^{H\otimes {}^{coH}M}\rho : H\otimes {}^{coH}M \to H\otimes (H\otimes {}^{coH}M)$ for the left Hom-coaction of (H,α) on (M,μ) and $(H\otimes {}^{coH}M, \alpha\otimes \mu|_{{}^{coH}M})$, resp., thus

$$\begin{aligned}
{}^{M}\rho(\theta(h\otimes m)) &= {}^{M}\rho(hm)\\
&= h_1 1_H\otimes h_2\mu^{-1}(m)\\
&= \alpha(h_1)\otimes h_2\mu^{-1}(m)\\
&= (id\otimes\theta)(\alpha(h_1)\otimes(h_2\otimes\mu^{-1}(m)))\\
&= (id\otimes\theta)({}^{H\otimes {}^{coH}M}\rho(h\otimes m)).
\end{aligned}$$

□

By Propositions (3.3.3) and (3.3.4), we have the following

Theorem 3.3.5 *There is a bijection, given by (3.29)-(3.31) and (3.34), between left-covariant* (H,α)*-Hom-bimodules* (M,μ) *and the right* (H,α)*-Hom-module structures on* $({}^{coH}M, \mu|_{{}^{coH}M})$.

If the antipode S of the monoidal Hom-Hopf algebra (H,α) is invertible, we have, for $m \in {}^{coH}M$ and $h \in H$

$$hm = (\mu^{-1}(m) \lhd S^{-1}(h_2))\alpha(h_1). \tag{3.35}$$

Indeed;

$$\begin{aligned}
(m \lhd S^{-1}(h_2))\alpha(h_1) &= (1_H\mu^{-1}(m \lhd S^{-1}(h_2)))\alpha(h_1) \\
&= (1_H(\mu^{-1}(m) \lhd \alpha^{-1}(S^{-1}(h_2))))\alpha(h_1) \\
&= (1_H\alpha(h_{11}))((\mu^{-1}(m) \lhd \alpha^{-1}(S^{-1}(h_2))) \lhd \alpha(h_{12}) \\
&= \alpha^2(h_{11})(\mu(\mu^{-1}(m)) \lhd (\alpha^{-1}(S^{-1}(h_2))\alpha^{-1}(\alpha(h_{12})))) \\
&= \alpha(h_1)(m \lhd (\alpha^{-1}(S^{-1}(\alpha(h_{22})))h_{21}) \\
&= \alpha(h_1)(m \lhd (S^{-1}(h_{22})h_{21}) \\
&= \alpha(h_1)(m \lhd \varepsilon(h_2)1_H) = h\mu(m),
\end{aligned}$$

which implies that $M = {}^{coH}M \cdot H$.

We indicate by ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)_H$ the category of left-covariant (H,α)-Hom-bimodules; the objects are the left-covariant Hom-bimodules and the morphisms are the ones in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ that are (H,α)-linear on both sides and left (H,α)-colinear.

We next show that the category ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)_H$ of left-covariant (H,α)-Hom-bimodules forms a monoidal category.

Proposition 3.3.6 *Let* (H,α) *be a monoidal Hom-Hopf algebra and* (M,μ), (N,ν) *be two left-covariant* (H,α)*-Hom-bimodules. Define the* k*-linear maps*

$$H \otimes (M \otimes_H N) \to M \otimes_H N,\ h \otimes (m \otimes_H n) = \alpha^{-1}(h)m \otimes_H \nu(n), \tag{3.36}$$

$$(M \otimes_H N) \otimes H \to M \otimes_H N,\ (m \otimes_H n) \otimes h = \mu(m) \otimes_H n\alpha^{-1}(h), \tag{3.37}$$

$$\rho : M \otimes_H N \to H \otimes (M \otimes_H N),\ m \otimes_H n = m_{(-1)}n_{(-1)} \otimes (m_{(0)} \otimes_H n_{(0)}), \tag{3.38}$$

Then $(M \otimes_H N, \mu \otimes_H \nu)$ *becomes a left-covariant* (H,α)*-Hom-bimodule with these structures.*

Proof: We first prove that the map (3.36) gives $M \otimes_H N$ a left (H,α)-Hom-module structure:

$$\begin{aligned}
\alpha(g)(h(m\otimes_H n)) &= \alpha(g)(\alpha^{-1}(h)m\otimes_H \nu(n)) = g(\alpha^{-1}(h)m)\otimes_H \nu^2(n)\\
&= \alpha^{-1}(gh)\mu(m)\otimes_H \nu(\nu(n)) = (gh)(\mu\otimes_H \nu)(m\otimes_H n),
\end{aligned}$$

$$1_H(m\otimes_H n) = \alpha^{-1}(1_H)m\otimes_H \nu(n) = \mu(m)\otimes_H \nu(n).$$

Similarly, one can also show that the map (3.37) makes $M \otimes_H N$ a right Hom-module. We now prove that the compatibility condition is satisfied:

$$\begin{aligned}
(g(m\otimes_H n))\alpha(h) &= (\alpha^{-1}(g)m\otimes_H \nu(n))\alpha(h) = \mu(\alpha^{-1}(g)m)\otimes_H \nu(n)h\\
&= g\mu(m)\otimes_H \nu(n)h = \alpha^{-1}(\alpha(g))\mu(m)\otimes_H \nu(n\alpha^{-1}(h))\\
&= \alpha(g)(\mu(m)\otimes_H n\alpha^{-1}(h)) = \alpha(g)((m\otimes_H n)h).
\end{aligned}$$

We next demonstrate that $M\otimes_H N$ possesses a left (H,α)-Hom-comodule structure with ρ which is given by $\rho(m\otimes_H n) = m_{(-1)}n_{(-1)}\otimes(m_{(0)}\otimes_H n_{(0)})$.

$$\begin{aligned}
&\Delta((m\otimes_H n)_{(-1)})\otimes(\mu^{-1}\otimes_H \nu^{-1})((m\otimes_H n)_{(0)})\\
&= \Delta(m_{(-1)})\Delta(n_{(-1)})\otimes(\mu^{-1}(m_{(0)})\otimes_H \nu^{-1}(n_{(0)}))\\
&= (\alpha^{-1}(m_{(-1)})\alpha^{-1}(n_{(-1)})\otimes m_{(0)(-1)}n_{(0)(-1)})\otimes(m_{(0)(0)}\otimes_H n_{(0)(0)})\\
&= (\alpha^{-1}((m\otimes_H n)_{(-1)})\otimes(m\otimes_H n)_{(0)(-1)})\otimes(m\otimes_H n)_{(0)(0)},
\end{aligned}$$

$$\begin{aligned}
\varepsilon((m\otimes_H n)_{(-1)})(m\otimes_H n)_{(0)} &= \varepsilon(m_{(-1)}n_{(-1)})m_{(0)}\otimes_H n_{(0)}\\
&= \varepsilon(m_{(-1)})m_{(0)}\otimes_H \varepsilon(n_{(-1)})n_{(0)}\\
&= \mu^{-1}(m)\otimes_H \nu^{-1}(n),
\end{aligned}$$

which prove the Hom-coassociativity and Hom-counity of ρ, respectively. We then finish the proof by the below computation:

$$
\begin{aligned}
\Delta(\alpha(g))(\rho(m\otimes_H n)\Delta(h)) &= (\alpha(g_1)\otimes\alpha(g_2))((m_{(-1)}n_{(-1)}\otimes(m_{(0)}\otimes_H n_{(0)}))(h_1\otimes h_2))\\
&= \alpha(g_1)((m_{(-1)}n_{(-1)})h_1)\otimes\alpha(g_2)((m_{(0)}\otimes_H n_{(0)})h_2)\\
&= \alpha(g_1)(\alpha(m_{(-1)})(n_{(-1)}\alpha^{-1}(h_1)))\otimes\alpha(g_2)(\mu(m_{(0)})\otimes_H n_{(0)}\alpha^{-1}(h_2))\\
&= (g_1\alpha(m_{(-1)}))(\alpha(n_{(-1)})h_1)\otimes(g_2\mu(m_{(0)})\otimes_H \nu(n_{(0)}\alpha^{-1}(h_2)))\\
&= (g_1\mu(m)_{(-1)})(\nu(n)_{(-1)}h_1)\otimes(g_2\mu(m)_{(0)}\otimes_H \nu(n)_{(0)}h_2)\\
&= (g\mu(m))_{(-1)}(\nu(n)h)_{(-1)}\otimes((g\mu(m))_{(0)}\otimes_H (\nu(n)h)_{(0)})\\
&= \rho(g\mu(m)\otimes_H \nu(n)h)\\
&= \rho((\alpha^{-1}(g)m\otimes_H \nu(n))\alpha(h))\\
&= \rho((g(m\otimes_H n))\alpha(h)).
\end{aligned}
$$

□

Proposition 3.3.7 *Let* (H,α) *be a monoidal Hom-Hopf algebra and* (M,μ), (N,ν), (P,π) *be left-covariant* (H,α)*-Hom-bimodules. Then the linear map*

$$
\tilde{a}_{M,N,P}:(M\otimes_H N)\otimes_H P\to M\otimes_H(N\otimes_H P),\ \tilde{a}_{M,N,P}((m\otimes_H n)\otimes_H p)=\mu(m)\otimes_H(n\otimes_H \pi^{-1}(p)),\tag{3.39}
$$

is an isomorphism of (H,α)*-Hom-bimodules and left* (H,α)*-Hom-comodules.*

Proof: It is clear that $\tilde{a}_{M,N,P}$ is bijective and fulfills the relation $\tilde{a}_{M,N,P}\circ(\mu\otimes\nu\otimes\pi)=(\mu\otimes\nu\otimes\pi)\circ\tilde{a}_{M,N,P}$. In what follows we prove the left and right (H,α)-linearity, and left (H,α)-colinearity of $\tilde{a}_{M,N,P}$: The calculation

$$
\begin{aligned}
\tilde{a}_{M,N,P}(h((m\otimes_H n)\otimes_H p)) &= \tilde{a}_{M,N,P}(\alpha^{-1}(h)(m\otimes_H n)\otimes_H \pi(p))\\
&= \tilde{a}_{M,N,P}((\alpha^{-2}(h)m\otimes_H \nu(n))\otimes_H \pi(p))\\
&= \mu(\alpha^{-2}(h)m)\otimes_H(\nu(n)\otimes_H p)\\
&= \alpha^{-1}(h)\mu(m)\otimes_H((\nu\otimes_H\pi)(n\otimes_H\pi^{-1}(p)))\\
&= h(\mu(m)\otimes_H(n\otimes_H\pi^{-1}(p)))\\
&= h\tilde{a}_{M,N,P}((m\otimes_H n)\otimes_H p)
\end{aligned}
$$

shows that $\tilde{a}_{M,N,P}$ is left (H,α)-linear. By performing a similar computation, one can also affirm that $\tilde{a}_{M,N,P}(((m\otimes_H n)\otimes_H p)h) = \tilde{a}_{M,N,P}((m\otimes_H n)\otimes_H p)h$, i.e., $\tilde{a}_{M,N,P}$ is right (H,α)-linear too.

Now we verify the left (H,α)-colinearity of $\tilde{a}_{M,N,P}$:

$$\begin{aligned}
&{}^{Q}\rho(\tilde{a}_{M,N,P}((m\otimes_H n)\otimes_H p)) \\
&= {}^{Q}\rho(\mu(m)\otimes_H (n\otimes_H \pi^{-1}(p))) \\
&= \mu(m)_{(-1)}(n\otimes_H \pi^{-1}(p))_{(-1)}\otimes(\mu(m)_{(0)}\otimes_H (n\otimes_H \pi^{-1}(p))_{(0)}) \\
&= \mu(m)_{(-1)}(n_{(-1)}\otimes_H \pi^{-1}(p)_{(-1)})\otimes(\mu(m)_{(0)}\otimes_H (n_{(0)}\otimes_H \pi^{-1}(p)_{(0)}) \\
&= \alpha(m_{(-1)})(n_{(-1)}\alpha^{-1}(p_{(-1)}))\otimes(\mu(m_{(0)})\otimes_H (n_{(0)}\otimes_H \pi^{-1}(p_{(0)})) \\
&= (m_{(-1)}n_{(-1)})p_{(-1)}\otimes\tilde{a}_{M,N,P}((m_{(0)}\otimes_H n_{(0)})\otimes_H p_{(0)}) \\
&= (m\otimes_H n)_{(-1)}p_{(-1)}\otimes\tilde{a}_{M,N,P}((m_{(0)}\otimes_H n_{(0)})\otimes_H p_{(0)}) \\
&= (id\otimes\tilde{a}_{M,N,P})((m\otimes_H n)_{(-1)}p_{(-1)}\otimes((m\otimes_H n)_{(0)}\otimes_H p_{(0)})) \\
&= (id\otimes\tilde{a}_{M,N,P})({}^{Q'}\rho((m\otimes_H n)\otimes_H p)),
\end{aligned}$$

where ${}^{Q}\rho$ and ${}^{Q'}\rho$ are the left codiagonal Hom-coactions of (H,α) on the objects $Q = M\otimes_H (N\otimes_H P)$ and $Q' = (M\otimes_H N)\otimes_H P$ resp. □

Proposition 3.3.8 *Let (H,α) be a monoidal Hom-Hopf algebra and (M,μ) be a left-covariant (H,α)-Hom-bimodule. Then the following linear maps*

$$\tilde{l}_M : H\otimes_H M \to M,\ h\otimes_H m \mapsto hm, \tag{3.40}$$

$$\tilde{r}_M : M\otimes_H H \to M,\ m\otimes_H h \mapsto mh. \tag{3.41}$$

are isomorphisms of (H,α)-Hom-bimodules and left (H,α)-Hom-comodules.

Proof: With the left and right (H,α)-Hom-module structures given by Hom-multiplication $H\otimes H\to H$, $h\otimes g\mapsto m_H(h\otimes g) = hg$ and the left (H,α)-Hom-comodule structure given by Hom-multiplication $H\to H\otimes H$, $h\mapsto h_1\otimes h_2$, (H,α) is a left-covariant (H,α)-Hom-bimodule. We show only that $\tilde{l}_M$ is (H,α)-linear on both sides and left (H,α)-colinear. For $\tilde{r}_M$ the argument is analogous. Obviously, $\tilde{l}_M$ is a k-isomorphism with the inverse $\tilde{l}_M^{-1} : M\to H\otimes_H M$, $m\mapsto 1\otimes\mu^{-1}(m)$ and the relation $\mu\circ\tilde{l}_M = \tilde{l}_M\circ(id_H\otimes\mu)$ is satisfied. We show now left and right (H,α)-linearity, and (H,α)-colinearity of $\tilde{l}_M$, respectively: For any $h,g\in H$ and $m\in M$,

$$\begin{aligned}\tilde{l}_M(h(g\otimes_H m)) &= \tilde{l}_M(\alpha^{-1}(h)g\otimes_H \mu(m)) = (\alpha^{-1}(h)g)\mu(m)\\ &= h(gm) = h\tilde{l}_M(g\otimes_H m),\end{aligned}$$

$$\begin{aligned}\tilde{l}_M((g\otimes_H m)h) &= \tilde{l}_M(\alpha(g)\otimes_H m\alpha^{-1}(h))\\ &= \alpha(g)(m\alpha^{-1}(h)) = (gm)h = \tilde{l}_M(g\otimes_H m)h,\end{aligned}$$

$$\begin{aligned}(id_H\otimes\tilde{l}_M)({}^{H\otimes_H M}\rho(h\otimes_H m)) &= (id_H\otimes\tilde{l}_M)(h_{(-1)}m_{(-1)}\otimes(h_{(0)}\otimes_H m_{(0)}))\\ &= (id_H\otimes\tilde{l}_M)(h_1 m_{(-1)}\otimes(h_2\otimes_H m_{(0)}))\\ &= h_1 m_{(-1)}\otimes h_2 m_{(0)} = h\,{}^M\rho(m)\\ &= {}^M\rho(hm) = {}^M\rho(\tilde{l}_M(h\otimes_H m)),\end{aligned}$$

where ${}^{H\otimes_H M}\rho$ and ${}^M\rho$ are the left Hom-coactions of (H,α) on the objects $H\otimes_H M$ and M, respectively. □

Theorem 3.3.9 *Let (H,α) be a monoidal Hom-Hopf algebra. Then the category ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)_H$ of left-covariant (H,α)-Hom-bimodules forms a monoidal category, with tensor product $\otimes_H$, associativity constraints $\tilde{a}$, and left and right unity constraints $\tilde{l}$ and $\tilde{r}$, defined in Propositions 3.3.6, 3.3.7 and 3.3.8, respectively.*

Proof: The naturality of $\tilde{a}$ and the fact that $\tilde{a}$ satisfies the Pentagon Axiom follow from Proposition 1.1 in [21]. Let $f: M\to M'$ be a morphism in ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)_H$ and let (M,μ) be a left-covariant (H,α)-Hom-bimodule. Then, for $m\in M$ and $h\in H$, we have

$$(f\circ\tilde{l}_M)(h\otimes_H m) = f(hm) = hf(m) = \tilde{l}_{M'}(h\otimes_H f(m)),$$

showing that $\tilde{l}$ is natural. The naturality of $\tilde{r}$ can be proven similarly. We finally verify that the Triangle Axiom is satisfied: For $h\in H$, $m\in M$ and $n\in N$,

$$\begin{aligned}((id_M\otimes_H\tilde{l}_N)\circ\tilde{a}_{M,H,N})((m\otimes_H h)\otimes_H n) &= (id_M\otimes_H\tilde{l}_N)(\mu(m)\otimes_H(h\otimes_H\nu^{-1}(n)))\\ &= \mu(m)\otimes_H h\nu^{-1}(n) = mh\otimes_H n\\ &= \tilde{r}_M(m\otimes_H h)\otimes_H n = (\tilde{r}_M\otimes id_N)((m\otimes_H h)\otimes_H n).\end{aligned}$$

□

In the rest of the section, we study the structure theory of left-covariant Hom-bimodules.

Let (H,α) be a monoidal Hom-coalgebra with Hom-comultiplication $\Delta : H \to H \otimes H,\ h \mapsto h_1 \otimes h_2$ and Hom-counit $\varepsilon : H \to k$. Then the dual $(H' = Hom(H,k), \bar{\alpha})$ is a monoidal Hom-algebra with the convolution product $(ff')(h) = f(h_1)f'(h_2)$ for functionals $f, f' \in H'$ and $h \in H$, as Hom-multiplication, and ε as Hom-unit, where $\bar{\alpha}(f) = f \circ \alpha^{-1}$ for any $f \in H'$: For $f, g, k \in H'$ and $h \in H$,

$$
\begin{aligned}
(\bar{\alpha}(f)(gk))(h) &= \bar{\alpha}(f)(h_1)(gk)(h_2) = f(\alpha^{-1}(h_1))g(a_{21})k(a_{22}) \\
&= f(h_{11}))g(a_{12})k(\alpha^{-1}(a_2)) = (fg)(h_1)\bar{\alpha}(k)(h_2) \\
&= ((fg)\bar{\alpha}(k))(h),
\end{aligned}
$$

which is the Hom-coassociativity, and

$$
(\varepsilon f)(h) = \varepsilon(h_1)f(h_2) = f(\alpha^{-1}(h)) = \bar{\alpha}(f)(h) = (f\varepsilon)(h),
$$

which is the Hom-unity. Then we have the following

Lemma 3.3.10 *1. The linear map $H' \otimes H \to H, f \otimes h \mapsto f \bullet h := \alpha^2(h_1)f(\alpha(h_2))$ defines a left Hom-action of $(H', \bar{\alpha})$ on (H, α).*

2. The linear map $H \otimes H' \to H,\ h \otimes f \mapsto h \bullet f := f(\alpha(h_1))\alpha^2(h_2)$ defines a right Hom-action of $(H', \bar{\alpha})$ on (H, α).

Proof: We prove only the item (1). Let $f, f' \in H'$ and $h \in H$. Then,

$$
\begin{aligned}
\bar{\alpha}(f) \bullet (f' \bullet h) &= (f \circ \alpha^{-1}) \bullet (\alpha^2(h_1)f'(\alpha(h_2))) \\
&= \alpha^2(\alpha^2(h_1)_1)(f \circ \alpha^{-1})(\alpha(\alpha^2(h_1)_2))f'(\alpha(h_2)) \\
&= \alpha^4(h_{11})f(\alpha^2(h_{12}))f'(\alpha(h_2)) = \alpha^4(\alpha^{-1}(h_1))f(\alpha^2(h_{21}))f'(\alpha(\alpha(h_{22}))) \\
&= \alpha^3(h_1)f(\alpha^2(h_{21}))f'(\alpha^2(h_{22})) = \alpha^3(h_1)f(\alpha^2(h_2)_1)f'(\alpha^2(h_2)_2) \\
&= \alpha^3(h_1)(ff')(\alpha^2(h_2)) = \alpha^2(\alpha(h)_1)(ff')(\alpha(\alpha(h)_2)) \\
&= (ff') \bullet \alpha(h),
\end{aligned}
$$

$$
\varepsilon \bullet h = \alpha^2(h_1)\varepsilon(\alpha(h_2)) = \alpha^2(h_1)\varepsilon(h_2) = \alpha^2(\alpha^{-1}(h)) = \alpha(h),
$$

which are the Hom-associativity and Hom-unity properties, respectively. We also have $\bar{\alpha}(f) \bullet \alpha(h) = (f \circ \alpha^{-1}) \bullet \alpha(h) = \alpha^3(h_1) f(\alpha(h_2)) = \alpha(f \bullet h)$, which finishes the proof that (H, α) is a left $(H', \bar{\alpha})$-Hom-module with the given map. □

For the discussion below we assume k as a field. Suppose that I is an index set. The matrix $(v^i_j)_{i,j \in I}$ with entries $v^i_j \in H$ is said to be *pointwise finite* if for any $i \in I$, only a finite number of terms v^i_j do not vanish. The matrix $(f^i_j)_{i,j \in I}$ of functionals $f^i_j \in H'$ is called pointwise finite if for arbitrary $i \in I$ and $h \in H$, all but finitely many terms $f^i_j(h)$ vanish. If (M, μ) be a left-covariant (H, α)-Hom-bimodule and $\{m_i\}_{i \in I}$ is a linear basis of ${}^{coH}M$, then there exist uniquely determined coefficients μ^i_j, $\bar{\mu}^i_j \in k$, which are the entries of pointwise fine matrices $(\mu^i_j)_{i,j \in I}$ and $(\bar{\mu}^i_j)_{i,j \in I}$, such that $\mu|_{{}^{coH}M}(m_i) = \mu^i_j m_j$, $(\mu|_{{}^{coH}M})^{-1}(m_i) = \bar{\mu}^i_j m_j$ (Einstein summation convention is used, i.e., there is a summation over repeating indices) satisfying $\mu^i_j \bar{\mu}^j_k = \delta_{ik} = \bar{\mu}^i_j \mu^j_k$. Thus, by using the above lemma, we express some of the results obtained about left-covariant Hom-bimodules in coordinate form as follows

Theorem 3.3.11 *Let (M, μ) be a left-covariant (H, α)-Hom-bimodule and $\{m_i\}_{i \in I}$ be a linear basis of ${}^{coH}M$. Then $\{m_i\}_{i \in I}$ is a free left (H, α)-Hom-module basis of M and there exists a pointwise finite matrix $(f^i_j)_{i,j \in I}$ of linear functionals $f^i_j \in H'$ satisfying, for any $h, g \in H$ and $i, j \in I$,*

$$\mu^i_j f^j_k(hg) = f^i_j(h) f^j_k(\alpha(g)),\ f^i_j(1) = \mu^i_j, \tag{3.42}$$

$$m_i h = (\bar{\mu}^i_j f^j_k \bullet \alpha^{-1}(h)) m_k. \tag{3.43}$$

Moreover, $\{m_i\}_{i \in I}$ is a free right (H, α)-Hom-module basis of M and we have

$$h m_i = m_j((\bar{\mu}^i_k f^k_j \circ S^{-1}) \bullet \alpha^{-1}(h)). \tag{3.44}$$

Proof: By the equation (3.25) and the fact that $P_L(m) \in {}^{coH}M$ for any $m \in M$, we write any element $m \in M$ in the form $m = \sum_i h_i m_i$, where $h_i \in H$, $i \in I$. Then, applying the left Hom-coaction to the both sides of $m = \sum_i h_i m_i$, we get $\rho(m) = \sum_i \Delta(h_i)(1 \otimes \mu^{-1}(m_i))$, and hence by the equations (3.24) and $P_L(m_i) = m_i$, $i \in I$, we have

$$\begin{aligned}(id \otimes P_L)(\rho(m)) &= \sum_i h_{i,1} 1 \otimes P_L(h_{i,2} \mu^{-1}(m_i)) = \sum_i \alpha(h_{i,1}) \otimes \varepsilon(h_{i,2}) \mu(P_L(\mu^{-1}(m_i))) \\ &= \sum_i \alpha(h_{i,1} \varepsilon(h_{i,2})) \otimes P_L(m_i) = \sum_i h_i \otimes m_i,\end{aligned}$$

where we put $\Delta(h_i) = h_{i,1} \otimes h_{i,2}$. By the linear independence of $\{m_i\}_{i\in I}$, we conclude that $h_i \in H$ are uniquely determined.

Since, for any $h \in H$, $m_i \lhd h = \widetilde{ad}_R(h)(m_i) \in {}^{coH}M$, there exist $f^i_j(h) \in \mathbb{C}$, $i, j \in I$ such that

$$m_i \lhd h = f^i_j(h) m_j, \tag{3.45}$$

where only a finite number of $f^i_j(h)$ do not vanish. For any $h, g \in H$, we have

$$\begin{aligned} \mu^i_j f^j_k(hg) m_k &= \mu(m_i) \lhd (hg) = (m_i \lhd h) \lhd \alpha(g) \\ &= (f^i_j(h) m_j) \lhd \alpha(g) = f^i_j(h) f^j_k(\alpha(g)) m_k, \end{aligned}$$

which implies $\mu^i_j f^j_k(hg) = f^i_j(h) f^j_k(\alpha(g))$, and

$$f^i_j(1) m_j = m_i \lhd 1 = \mu(m_i) = \mu^i_j m_j$$

concludes that $f^i_j(1) = \mu^i_j$. By using the identification of hm_i with $h \otimes m_i$ and the right Hom-action of H on $H \otimes {}^{coH}M$ we obtain

$$\begin{aligned} m_i h &= (1 \otimes \mu^{-1}(m_i)) h = 1 h_1 \otimes \mu^{-1}(m_i) \lhd h_2 \\ &= \alpha(h_1) \otimes (\bar{\mu}^i_j m_j) \lhd h_2 = \alpha(h_1) \otimes \bar{\mu}^i_j f^j_k(h_2) m_k \\ &= \alpha(h_1)(\bar{\mu}^i_j f^j_k)(h_2) m_k = \alpha^2(\alpha^{-1}(h)_1)(\bar{\mu}^i_j f^j_k)(\alpha(\alpha^{-1}(h)_2)) m_k \\ &= (\bar{\mu}^i_j f^j_k \bullet \alpha^{-1}(h)) m_k. \end{aligned}$$

The equation (3.35) yields

$$\begin{aligned} hm_i &= (\mu^{-1}(m_i) \lhd S^{-1}(h_2))\alpha(h_1) = ((\bar{\mu}^i_j m_j) \lhd S^{-1}(h_2))\alpha(h_1) \\ &= (\bar{\mu}^i_j f^j_k)(S^{-1}(h_2)) m_k \alpha(h_1) = m_k \alpha(h_1)(\bar{\mu}^i_j f^j_k)(S^{-1}(h_2)) \\ &= m_k \alpha(h_1)(\bar{\mu}^i_j f^j_k \circ S^{-1})(h_2) = m_k((\bar{\mu}^i_j f^j_k \circ S^{-1}) \bullet \alpha^{-1}(h)). \end{aligned}$$

Since, for any $p, s \in I$, $f^p_s \circ \alpha = \bar{\mu}^p_q f^q_r \mu^r_s$ and $(\bar{\mu}^l_s f^p_l)(hg) = (\bar{\mu}^p_r f^r_l)(h)(\bar{\mu}^l_q f^q_s)(g)$ for $h, g \in H$, we have

$$\begin{aligned}
((\bar{\mu}^j_l f^l_i)(\bar{\mu}^k_p f^p_j \circ S^{-1}))(S(h)) &= (\bar{\mu}^j_l f^l_i)(S(h)_1)(\bar{\mu}^k_p f^p_j \circ S^{-1})(S(h)_2) \\
&= (\bar{\mu}^j_l f^l_i)(S(h_2))(\bar{\mu}^k_p f^p_j)(h_1) = (\bar{\mu}^k_p \bar{\mu}^j_l f^p_j)(h_1) f^l_i(S(h_2)) \\
&= (\bar{\mu}^k_p \bar{\mu}^p_r f^r_l \circ \alpha^{-1})(h_1)(\mu^q_i \bar{\mu}^l_s f^s_q \circ \alpha^{-1})(S(h_2)) \\
&= \bar{\mu}^k_p \mu^q_i (\bar{\mu}^p_r f^r_l)(\alpha^{-1}(h)_1)(\bar{\mu}^l_s f^s_q)(S(\alpha^{-1}(h)_2)) \\
&= \bar{\mu}^k_p \mu^q_i (\bar{\mu}^l_q f^p_l)(\alpha^{-1}(h)_1 S(\alpha^{-1}(h)_2)) \\
&= \bar{\mu}^k_p \mu^q_i \bar{\mu}^l_q f^p_l(1)\varepsilon(\alpha^{-1}(h)) = \bar{\mu}^k_p \mu^q_i \bar{\mu}^l_q \mu^p_l(1)\varepsilon(h) \\
&= \delta_{lk}\delta_{li}\varepsilon(S(h)) = \delta_{ki}\varepsilon(S(h)),
\end{aligned}$$

that is, we have shown that

$$(\bar{\mu}^j_l f^l_i)(\bar{\mu}^k_p f^p_j \circ S^{-1}) = \delta_{ik}\varepsilon. \tag{3.46}$$

In a similar way, one can also prove that

$$(\bar{\mu}^j_l f^l_k \circ S^{-1})(\bar{\mu}^i_p f^p_j) = \delta_{ki}\varepsilon. \tag{3.47}$$

Since $\{m_i\}_{i\in I}$ is a free left (H,α)-Hom-module basis of M and the equation (3.44) holds, any element $m \in M$ is also of the form $m = \sum_i m_i h_i$ for some $h_i \in H$. Let us assume that $\sum_i m_i h_i = 0$ (all but finitely many h_i vanishes, $i \in I$). So, by the equation (3.43), we get $\sum_i (\bar{\mu}^i_j f^j_k \bullet \alpha^{-1}(h_i))m_k = 0$ which implies that

$$\sum_i (\bar{\mu}^i_j f^j_k \bullet \alpha^{-1}(h_i)) = 0, \ \forall k \in I.$$

If we apply $\bar{\alpha}(\bar{\mu}^k_l f^l_p \circ S^{-1})$ from left to the both sides and use the equation (3.47), we obtain

$$\begin{aligned}
0 &= \sum_i \bar{\alpha}(\bar{\mu}^k_l f^l_p \circ S^{-1}) \bullet (\bar{\mu}^i_j f^j_k \bullet \alpha^{-1}(h_i)) \\
&= \sum_i ((\bar{\mu}^k_l f^l_p \circ S^{-1})(\bar{\mu}^i_j f^j_k)) \bullet \alpha(\alpha^{-1}(h_i)) \\
&= \sum_i \delta_{pi}\varepsilon \bullet h_i = \sum_i \delta_{pi}\alpha(h_i) = \alpha(b_p),
\end{aligned}$$

for all $p \in I$, that is $b_p = 0, \forall p \in I$. This finishes the proof that $\{m_i\}_{i\in I}$ is a free right (H,α)-Hom-module basis of M. □

3.4 Right-Covariant Hom-Bimodules

Definition 3.4.1 *A right-covariant* (H,α)*-Hom-bimodule is an* (H,α)*-Hom-bimodule* $(M,\mu)\in\widetilde{\mathcal{H}}(\mathcal{M}_k)$ *which is a right* (H,α)*-Hom-comodule, with Hom-coaction* $\sigma: M\to M\otimes H$, $m\mapsto m_{[0]}\otimes m_{[1]}$, *in* $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ *such that*

$$\sigma((hm)\alpha(g)) = \Delta(\alpha(h))(\sigma(m)\Delta(g)). \tag{3.48}$$

The set $M^{coH}=\{m\in M|\rho(m)=\mu^{-1}(m)\otimes 1_H\}$ *of* M *is called right coinvariant of* (H,α) *on* (M,μ).

Without performing details, we can develop a similar theory for the right-covariant (H,α)-Hom-bimodules as in the previous section by making the necessary changes. We define the projection by

$$P_R: M\to M^{coH},\ m\mapsto m_{[0]}S(m_{[1]}), \tag{3.49}$$

which is unique with the property

$$P_R(mh)=\varepsilon(h)\mu(P_R(m)),\ for\ all\ h\in H, m\in M. \tag{3.50}$$

Since the relation

$$(id\otimes\Delta)\circ(\sigma\otimes id)\circ\sigma = \tilde{a}_{M\otimes H,H,H}\circ(\tilde{a}^{-1}_{M,H,H}\otimes id)\circ((id\otimes\Delta)\otimes id)\circ(\sigma\otimes id)\circ\sigma \tag{3.51}$$

holds, that is, for any $m\in M$, the following equality

$$m_{[0][0]}\otimes m_{[0][1]}\otimes m_{[1]1}\otimes m_{[1]2} = m_{[0][0]}\otimes\alpha(m_{[0][1]1})\otimes\alpha(m_{[0][1]2})\otimes\alpha^{-1}(m_{[1]}) \tag{3.52}$$

is fulfilled, one can prove that

$$\sigma(P_R(m))=\mu^{-1}(P_R(m))\otimes 1_H.$$

One can also show that

$$m=P_R(m_{[0]})m_{[1]} \tag{3.53}$$

is acquired by using the Hom-coassociativity property for the right Hom-comodules, which specifies that $M=M^{coH}\cdot H$. P_R also satisfies

$$P_R(hm)=\alpha(h_1)(\mu^{-1}(P_R(m))S(h_2))\equiv\widetilde{ad}_L(h)P_R(m). \tag{3.54}$$

Since (M,μ) is an (H,α)-Hom-bimodule, M^{coH} has a left (H,α)-Hom-module structure by the formula

$$h \triangleright m := P_R(hm) = \widetilde{ad}_L(h)m. \tag{3.55}$$

$\widetilde{ad}_L$ is in fact a left Hom-action of (H,α) on M^{coH}:

$$\widetilde{ad}_L(1_H)m = 1_H \triangleright m = \alpha(1_H)(\mu^{-1}(m)S(1_H)) = 1_H m = \mu(m),$$

$$\begin{aligned}
(gh) \triangleright \mu(m) &= \alpha(g_1h_1)(\mu^{-1}(\mu(m))S(g_2h_2)) \\
&= (\alpha(g_1)\alpha(h_1))(m(S(h_2)S(g_2))) \\
&= (\alpha(g_1)\alpha(h_1))((\mu^{-1}(m)S(h_2))S(\alpha(g_2))) \\
&= ((g_1h_1)(\mu^{-1}(m)S(h_2)))\alpha(S(\alpha(g_2))) \\
&= (\alpha(g_1)(h_1\mu^{-1}(\mu^{-1}(m)S(h_2))))\alpha(S(\alpha(g_2))) \\
&= (\alpha(g_1)\mu^{-1}((\alpha(h_1)\mu^{-1}(\mu^{-1}(m)S(h_2)))))\alpha(S(\alpha(g_2))) \\
&= \alpha(\alpha(g_1))(\mu^{-1}(h \triangleright m)S(\alpha(g_2))) \\
&= \alpha(g) \triangleright (h \triangleright m),
\end{aligned}$$

for all $m \in M^{coH}$ and $g,h \in H$. Once this left Hom-module structure has been given to M^{coH}, it can be proven, in a similar way as in the proof of the Proposition (3.3.3) and the Theorem (3.3.4), that the right-covariant (H,α)-Hom-bimodule (M,μ) is isomorphic, by the morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$

$$\theta' : M^{coH} \otimes H \to M,\ m \otimes h \mapsto mh, \tag{3.56}$$

to the right-covariant (H,α)-Hom-bimodule $M^{coH} \otimes H$ with Hom-(co)module structures defined by the following maps in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$

$$(M^{coH} \otimes H) \otimes H \to M^{coH} \otimes H,\ (m \otimes h) \otimes g \mapsto \mu(m) \otimes h\alpha^{-1}(g), \tag{3.57}$$

$$H \otimes (M^{coH} \otimes H) \to M^{coH} \otimes H,\ g \otimes (m \otimes h) \mapsto g_1 \triangleright m \otimes g_2 h, \tag{3.58}$$

$$M^{coH} \otimes H \to (M^{coH} \otimes H) \otimes H,\ m \otimes h \mapsto (\mu^{-1}(m) \otimes h_1) \otimes \alpha(h_2). \tag{3.59}$$

Thus we have the following

Theorem 3.4.2 *There is a one-to-one correspondence, given by (3.55) and (3.57)-(3.59), between the right-covariant (H,α)-Hom-bimodules (M,μ) and the left (H,α)-Hom-module structures on $(M^{coH},\mu|_{M^{coH}})$.*

We denote by ${}_H\widetilde{\mathcal{H}}(\mathcal{M}_k)_H^H$ the category of right-covariant (H,α)-Hom-bimodules whose objects are the right-covariant (H,α)-Hom-bimodules with those morphisms that are left and right (H,α)-linear and right (H,α)-colinear.

Proposition 3.4.3 *Let (H,α) be a monoidal Hom-Hopf algebra and (M,μ), (N,ν) be two right-covariant (H,α)-Hom-bimodules. Along with (3.36) and (3.37), define the morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$*

$$\sigma : M\otimes_H N \to (M\otimes_H N)\otimes H,\ m\otimes_H n \mapsto (m_{[0]}\otimes_H n_{[0]})\otimes m_{[1]}n_{[1]}, \tag{3.60}$$

which is the right codiagonal Hom-coaction of (H,α) on $M\otimes_H N$. Then $(M\otimes_H N,\mu\otimes_H \nu)$ is a right-covariant (H,α)-Hom-bimodule .

Proof: It is sufficient to prove first that $M\otimes_H N$ becomes a right (H,α)-Hom-comodule with σ and then to assert that the right covariance is held.

$$\begin{aligned}
&(\mu^{-1}\otimes_H \nu^{-1})((m\otimes_H n)_{[0]})\otimes \Delta((m\otimes_H n)_{[1]})\\
&\quad= (\mu^{-1}(m_{[0]})\otimes_H \nu^{-1}(n_{[0]}))\otimes \Delta(m_{[1]})\Delta(n_{[1]})\\
&\quad= (m_{[0][0]}\otimes_H n_{[0][0]})\otimes (m_{[0][1]}n_{[0][1]}\otimes \alpha^{-1}(m_{[1]})\alpha^{-1}(n_{[1]}))\\
&\quad= (m\otimes_H n)_{[0][0]}\otimes ((m\otimes_H n)_{[0][1]}\otimes (m\otimes_H n)_{[1]}),
\end{aligned}$$

where in the second equality the Hom-coassociativity condition for right (H,α)-Hom-comodules has been used, and we also have

$$(m\otimes_H n)_{[0]}\varepsilon((m\otimes_H n)_{[1]}) = m_{[0]}\varepsilon(m_{[1]})\otimes_H n_{[0]}\varepsilon(n_{[1]}) = \mu^{-1}(m)\otimes_H \nu^{-1}(n),$$

that is, σ satisfies the Hom-coassociativity and Hom-counity, respectively.

And with the next calculation we end the proof:

$$\begin{aligned}
\sigma((g(m\otimes_H n))\alpha(h)) &= \sigma(g\mu(m)\otimes_H \nu(n)h)\\
&= ((g\mu(m))_{[0]}\otimes_H ((\nu(n)h)_{[0]})\otimes (g\mu(m))_{[1]}(\nu(n)h)_{[1]}\\
&= (g_1\mu(m_{[0]})\otimes_H \nu(n_{[0]})h_1)\otimes (g_2\alpha(m_{[1]}))(\alpha(n_{[1]})h_2)\\
&= \alpha(g_1)(\mu(m_{[0]})\otimes_H n_{[0]}\alpha^{-1}(h_1))\otimes \alpha(g_2)((m_{[1]}n_{[1]})h_2)\\
&= \Delta(\alpha(g))((m_{[0]}\otimes_H n_{[0]})h_1\otimes (m_{[1]}n_{[1]})h_2)\\
&= \Delta(\alpha(g))(\sigma(m\otimes_H n)\Delta(h)).
\end{aligned}$$

□

Theorem 3.4.4 *Let (H,α) be a monoidal Hom-Hopf algebra. Then ${}_H\widetilde{\mathcal{H}}(\mathcal{M}_k)_H^H$ is a tensor category, with tensor product $\otimes_H$ is defined in Proposition (3.4.3), and associativity constraint $\tilde{a}$, left unit constraint $\tilde{l}$ and right unit constraint $\tilde{r}$ are given by (3.39), (3.40) and (3.41), respectively.*

Proof: What is left to be proven is that the associator $\tilde{a}_{M,N,P}$, left unitor $\tilde{l}_M$ and right unitor $\tilde{r}_M$ are all right (H,α)-colinear.

$$\begin{aligned}
(\sigma^{Q'}\circ\tilde{a}_{M,N,P})((m\otimes_H n)\otimes_H p) &= \sigma^{Q'}(\mu(m)\otimes_H(n\otimes_H\pi^{-1}(p)))\\
&= (\mu(m)_{[0]}\otimes_H(n\otimes_H\pi^{-1}(p))_{[0]})\otimes\mu(m)_{[1]}(n\otimes_H\pi^{-1}(p))_{[0]}\\
&= (\mu(m_{[0]})\otimes_H(n_{[0]}\otimes_H\pi^{-1}(p_{[0]})))\otimes\alpha(m_{[1]})(n_{[1]}\alpha^{-1}(p_{[1]}))\\
&= \tilde{a}_{M,N,P}((m_{[0]}\otimes_H n_{[0]})\otimes_H p_{[0]})\otimes(m_{[1]}n_{[1]})p_{[1]}\\
&= (\tilde{a}_{M,N,P}\otimes id)(((m\otimes_H n)_{[0]}\otimes_H p_{[0]})\otimes(m\otimes_H n)_{[1]}p_{[1]})\\
&= ((\tilde{a}_{M,N,P}\otimes id)\circ\sigma^{Q})((m\otimes_H n)\otimes_H p)
\end{aligned}$$

which stands for the right (H,α)-colinearity of $\tilde{a}_{M,N,P}$, where $\sigma^{Q'}$ and σ^{Q} are the right Hom-coactions of (H,α) on $Q'=M\otimes_H(N\otimes_H P)$ and $Q=(M\otimes_H N)\otimes_H P$.

By considering the fact that (H,α) is a right-covariant (H,α)-Hom-bimodule with Hom-actions given by its Hom-multiplication and Hom-coaction by its Hom-comultiplication, we do the computation

$$\begin{aligned}
(\tilde{l}_M\otimes id_H)(\sigma^{H\otimes_H M}(h\otimes_H m)) &= (\tilde{l}_M\otimes id_H)((h_{[0]}\otimes_H m_{[0]})\otimes h_{[1]}m_{[1]})\\
&= (\tilde{l}_M\otimes id_H)((h_1\otimes_H m_{[0]})\otimes h_2 m_{[1]})\\
&= h\sigma^M(m)=\sigma^M(hm)\\
&= \sigma^M(\tilde{l}_M(h\otimes_H m)),
\end{aligned}$$

concluding $\tilde{l}_M$ is right (H,α)-colinear. By a similar argument, $\tilde{r}_M$ as well is right (H,α)-colinear . □

3.5 Bicovariant Hom-Bimodules

Definition 3.5.1 *A bicovariant (H,α)-Hom-bimodule is an (H,α)-Hom-bimodule (M,μ) together with k-linear mappings*

$$\rho : M \to H \otimes M,\ m \mapsto m_{(-1)} \otimes m_{(0)},$$

$$\sigma : M \to M \otimes H,\ m \mapsto m_{[0]} \otimes m_{[1]},$$

in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, such that

1. *(M,μ) is a left-covariant (H,α)-Hom-bimodule with left (H,α)-Hom-coaction ρ,*
2. *(M,μ) is a right-covariant (H,α)-Hom-bimodule with right (H,α)-Hom-coaction σ,*
3. *the following relation holds:*

$$\tilde{a}_{H,M,H} \circ (\rho \otimes id) \circ \sigma = (id \otimes \sigma) \circ \rho. \tag{3.61}$$

The condition (3.61) is called the Hom-commutativity of the Hom-coactions ρ and σ on M and can be expressed by Sweedler's notation as follows

$$m_{(-1)} \otimes (m_{(0)[0]} \otimes m_{(0)[1]}) = \alpha(m_{[0](-1)}) \otimes (m_{[0](0)} \otimes \alpha^{-1}(m_{[1]})),\ m \in M.$$

Proposition 3.5.2 *Let $(N,\nu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ be a right (H,α)-Hom-module by the map $N \otimes H \to N,\ n \otimes h \mapsto n \triangleleft h$ and a right (H,α)-Hom-comodule by $N \to N \otimes H,\ n \mapsto n_{(0)} \otimes n_{(1)}$ such that the compatibility condition, which is called Hom-Yetter-Drinfeld condition,*

$$n_{(0)} \triangleleft \alpha^{-1}(h_1) \otimes n_{(1)}\alpha^{-1}(h_2) = (n \triangleleft h_2)_{(0)} \otimes \alpha^{-1}(h_1(n \triangleleft h_2)_{(1)}) \tag{3.62}$$

holds for $h \in H$ and $n \in N$. The morphisms (3.29)-(3.31) and

$$\sigma : H \otimes N \to (H \otimes N) \otimes H,\ h \otimes n \mapsto (h_1 \otimes n_{(0)}) \otimes h_2 n_{(1)}, \tag{3.63}$$

in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, define a bicovariant (H,α)-Hom-bimodule structure on $(H \otimes N, \alpha \otimes \nu)$.

Proof: As has been proven in Proposition 3.3.3, the left-covariant (H,α)-Hom-bimodule structure on $(H \otimes N, \alpha \otimes \nu)$ is deduced from the right (H,α)-Hom-action on (N,ν) by the morphisms (3.29)-(3.31). The morphism (3.63) fulfills the Hom-coassociativity and Hom-counity:

$$\begin{aligned}(\alpha^{-1}\otimes\nu^{-1})((h\otimes n)_{[0]})\otimes\Delta((h\otimes n)_{[1]}) &= (\alpha^{-1}(h_1)\otimes\nu^{-1}(n_{(0)}))\otimes\Delta(h_2n_{(1)})\\ &= (\alpha^{-1}(h_1)\otimes\nu^{-1}(n_{(0)}))\otimes(h_{21}n_{(1)1}\otimes h_{22}n_{(1)2})\\ &= (h_{11}\otimes n_{(0)(0)})\otimes(h_{12}n_{(0)(1)}\otimes\alpha^{-1}(h_2)\alpha^{-1}(n_{(1)}))\\ &= (h\otimes n)_{[0][0]}\otimes((h\otimes n)_{[0][1]}\otimes\alpha^{-1}((h\otimes n)_{[1]})),\end{aligned}$$

where the fact that (N,ν) is a right (H,α)-Hom-comodule and the relation (3.32) have been used in the third equation, and we besides obtain

$$(h\otimes n)_{[0]}\varepsilon((h\otimes n)_{[1]}) = (h_1\otimes n_{(0)})\varepsilon(h_2n_{(1)}) = h_1\varepsilon(h_2)\otimes n_{(0)}\varepsilon(n_{(1)}) = (\alpha^{-1}\otimes\nu^{-1})(h\otimes n).$$

By again using the relation (3.32) and the fact that the right (H,α)-Hom-coaction on (N,ν) is a morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, we prove the Hom-commutativity condition:

$$\begin{aligned}\alpha(m_{[0](-1)})\otimes(m_{[0](0)}\otimes\alpha^{-1}(m_{[1]})) &= \alpha^2(h_{11})\otimes((h_{12}\otimes\nu^{-1}(n_{(0)}))\otimes\alpha^{-1}(h_2)\alpha^{-1}(n_{(1)}))\\ &= \alpha(h_1)\otimes((h_{21}\otimes\nu^{-1}(n_{(0)}))\otimes h_{22}\alpha^{-1}(n_{(1)}))\\ &= \alpha(h_1)\otimes((h_{21}\otimes\nu^{-1}(n)_{(0)})\otimes h_{22}\nu^{-1}(n)_{(1)})\\ &= m_{(-1)}\otimes(m_{(0)[0]}\otimes m_{(0)[1]}).\end{aligned}$$

For $g,h,k\in H$ and $n\in N$, we have

$$\begin{aligned}&\sigma((g(h\otimes n))\alpha(k))\\ &= \sigma((\alpha^{-1}(g)h\otimes\nu(n))\alpha(k))\\ &= \sigma((\alpha^{-1}(g)h)\alpha(k_1)\otimes\nu(n)\triangleleft\alpha(k_2))\\ &= (((\alpha^{-1}(g)h)\alpha(k_1))_1\otimes(\nu(n)\triangleleft\alpha(k_2))_{(0)})\otimes((\alpha^{-1}(g)h)\alpha(k_1))_2(\nu(n)\triangleleft\alpha(k_2))_{(1)}\\ &= (g_1(h_1k_{11})\otimes\nu((n\triangleleft k_2)_{(0)}))\otimes(g_2(h_2k_{12}))\alpha((n\triangleleft k_2)_{(1)})\\ &= (\alpha^{-1}(\alpha(g_1))(h_1k_{11})\otimes\nu((n\triangleleft k_2)_{(0)}))\otimes\alpha(g_2)((h_2k_{12})(n\triangleleft k_2)_{(1)})\\ &= (\alpha^{-1}(\alpha(g_1))(h_1\alpha^{-1}(k_1))\otimes\nu((n\triangleleft\alpha(k_{22}))_{(0)}))\otimes\alpha(g_2)(\alpha(h_2)(k_{21}\alpha^{-1}((n\triangleleft\alpha(k_{22})_{(1)}))\\ &= \alpha(g_1)(h_1\alpha^{-1}(k_1)\otimes(n\triangleleft\alpha(k_{22}))_{(0)})\otimes\alpha(g_2)(\alpha(h_2)(k_{21}\alpha^{-1}((n\triangleleft\alpha(k_{22})_{(1)})),\end{aligned}\tag{3.64}$$

and

$$\begin{aligned}\Delta(\alpha(g))(\sigma(h\otimes n)\Delta(k)) &= (\alpha(g_1)\otimes\alpha(g_2))(((h_1\otimes n_{(0)})\otimes h_2 n_{(1)})(k_1\otimes k_2))\\ &= (\alpha(g_1)\otimes\alpha(g_2))((h_1\otimes n_{(0)})k_1\otimes(h_2 n_{(1)})k_2)\\ &= (\alpha(g_1)\otimes\alpha(g_2))((h_1k_{11}\otimes n_{(0)}\triangleleft k_{12})\otimes\alpha(h_2)(n_{(1)}\alpha^{-1}(k_2))\\ &= \alpha(g_1)(h_1\alpha^{-1}(k_1)\otimes n_{(0)}\triangleleft k_{21})\otimes\alpha(g_2)(\alpha(h_2)(n_{(1)}k_{22})). \quad (3.65)\end{aligned}$$

The right-hand sides of (3.64) and (3.65) are equal by the compatibility condition (3.62): To see this, it is enough to set $h=\alpha(k_2)$ in (3.62) to obtain the following

$$n_{(0)}\triangleleft\alpha^{-1}(\alpha(k_2)_1)\otimes n_{(1)}\alpha^{-1}(\alpha(k_2)_2)=(n\triangleleft\alpha(k_2)_2)_{(0)}\otimes\alpha^{-1}(\alpha(k_2)_1(n\triangleleft\alpha(k_2)_2)_{(1)})$$

$$\Rightarrow\ n_{(0)}\triangleleft\alpha^{-1}(\alpha(k_{21}))\otimes n_{(1)}\alpha^{-1}(\alpha(k_{22}))=(n\triangleleft\alpha(k_{22}))_{(0)}\otimes\alpha^{-1}(\alpha(k_{21}))\alpha^{-1}((n\triangleleft\alpha(k_{22}))_{(1)})$$

$$\Rightarrow\ n_{(0)}\triangleleft k_{21}\otimes n_{(1)}k_{22}=(n\triangleleft\alpha(k_{22}))_{(0)}\otimes k_{21}\alpha^{-1}((n\triangleleft\alpha(k_{22}))_{(1)}).$$

Thus we proved that $(H\otimes N,\alpha\otimes\nu)$ is a bicovariant Hom-bimodule over (H,α). □

Proposition 3.5.3 *If $(M,\mu)\in\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is a bicovariant (H,α)-Hom-bimodule, the k-linear map (3.33) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is an isomorphism of bicovariant (H,α)-Hom-bimodules, where the right (H,α)-Hom-module structure on $({}^{coH}M,\mu|_{{}^{coH}M})$ is defined by $m\triangleleft h:=P_L(mh)=\widetilde{ad}_R(h)(m)$, for $h\in H$ and $m\in{}^{coH}M$ and the right (H,α)-Hom-comodule structure is obtained by the restriction of right (H,α)-Hom-coaction on (M,μ) fulfilling the condition (3.62).*

Proof: Let (M,μ) be a bicovariant (H,α)-Hom-bimodule with left (H,α)-Hom-coaction $\rho: M\to h\otimes M$, $m\mapsto m_{(-1)}\otimes m_{(0)}$ and right (H,α)-Hom-coaction $\sigma: M\to M\otimes H$, $m\mapsto m_{[0]}\otimes m_{[1]}$. By Hom-commutativity condition (3.61) we get,

$$\sigma({}^{coH}M)\subseteq{}^{coH}M\otimes H,$$

which implies that the restriction of φ to ${}^{coH}M$ can be taken as the right Hom-coaction of (H,α) on $({}^{coH}M,\mu|_{{}^{coH}M})$: In fact, for $m\in{}^{coH}M$,

$$\begin{aligned}(id\otimes\sigma)(\rho(m)) &= 1\otimes(\mu^{-1}(m_{[0]})\otimes\alpha^{-1}(m_{[1]}))\\ &= \tilde{a}_{H,M,H}((\rho\otimes id)(\sigma(m)))\\ &= \alpha(m_{[0](-1)})\otimes(m_{[0](0)}\otimes\alpha^{-1}(m_{[1]})),\end{aligned}$$

which purports that $\rho(m_{[0]}) = 1 \otimes \mu^{-1}(m_{[0]})$.

Since it has been proven in Proposition (3.3.4) that the morphism $\theta : H \otimes {}^{coH}M \to M,\ h \otimes m \mapsto hm$ in (3.33) is an isomorphism of left-covariant (H,α)-Hom-bimodules, we next show that it is right (H,α)-colinear to conclude that it is an isomorphism of bicovariant (H,α)-Hom-bimodules:

$$\sigma(\theta(h \otimes m)) = h_1 m_{(0)} \otimes h_2 m_{(1)} = (\theta \otimes id)((h_1 \otimes m_{[0]}) \otimes h_2 m_{[1]}) = (\theta \otimes id)(\sigma^{H \otimes {}^{coH}M}(h \otimes m)),$$

where $\sigma^{H \otimes {}^{coH}M} : H \otimes {}^{coH}M \to (H \otimes {}^{coH}M) \otimes H,\ h \otimes m \mapsto (h_1 \otimes m_{[0]}) \otimes h_2 m_{[1]}$, for $h \in H$ and $m \in {}^{coH}M$, by the equation (3.63).

Due to the fact that (M,μ) is a bicovariant (H,α)-Hom-bimodule, the left-hand sides of (3.64) and (3.65) are equal: Thus, by applying $(\varepsilon \otimes id_{N \otimes H}) \circ \tilde{a}_{H,N,H}$ to the right-hand sides of (3.64) and (3.65), we acquire the compatibility condition (3.62). □

Hence, by Propositions (3.5.2) and (3.5.3), we acquire

Theorem 3.5.4 *There is a one-to-one correspondence, given by (3.29)-(3.31), (3.63) and (3.34) , between bicovariant (H,α)-Hom-bimodules (M,μ) and pairs of a right (H,α)-Hom-module and a right (H,α)-Hom-comodule structures on $({}^{coH}M, \mu|_{{}^{coH}M})$ fulfilling the compatibility condition (3.62).*

We indicate by ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)^H_H$ the category of bicovariant (H,α)-Hom-bimodules; the objects are the bicovariant Hom-bimodules with those morphisms that are (H,α)-linear and (H,α)-colinear on both sides.

Proposition 3.5.5 *Let (H,α) be a monoidal Hom-Hopf algebra and (M,μ), (N,ν) be two bicovariant (H,α)-Hom-bimodules. Then, with the Hom-module and Hom-comodule structures given by (3.36), (3.37), (3.38) and (3.60), $(M \otimes_H N, \mu \otimes_H \nu)$ becomes a bicovariant Hom-bimodule over (H,α).*

Proof: The only condition left to be proven to finish the proof of the statement is the Hom-commutativity of ρ and σ :

$$
\begin{aligned}
&(\tilde{a}_{H,M\otimes_H N,H}\circ(\rho\otimes id))(\sigma(m\otimes_H n)) \\
&\quad= \tilde{a}((\rho\otimes id)((m_{[0]}\otimes_H n_{[0]})\otimes m_{[1]}n_{[1]})) \\
&\quad= \tilde{a}((m_{[0](-1)}n_{[0](-1)}\otimes(m_{[0](0)}\otimes_H n_{[0](0)}))\otimes m_{[1]}n_{[1]}) \\
&\quad= \alpha(m_{[0](-1)})\alpha(n_{[0](-1)})\otimes((m_{[0](0)}\otimes_H n_{[0](0)})\otimes\alpha^{-1}(m_{[1]})\alpha^{-1}(m_{[1]})) \\
&\quad= m_{(-1)}n_{(-1)}\otimes((m_{(0)[0]}\otimes_H n_{(0)[0]})\otimes m_{(0)[1]}n_{(0)[1]}) \\
&\quad= (id\otimes\sigma)(m_{(-1)}n_{(-1)}\otimes(m_{(0)}\otimes_H n_{(0)})) \\
&\quad= ((id\otimes\sigma)\circ\rho)(m\otimes_H n),
\end{aligned}
$$

where the fourth equality follows from the Hom-commutativity of the Hom-coactions of (H,α) on (M,μ) and (N,ν). □

Lemma 3.5.6 *Let (H,α) be a monoidal Hom-Hopf algebra.Then the k-linear map $c_{M,N}$: $M\otimes_H N\to N\otimes_H M$ given by, for $m\in M$ and $n\in N$,*

$$
\begin{aligned}
c_{M,N}(m\otimes_H n) &= m_{(-1)}P_R(n_{[0]})\otimes_H P_L(m_{(0)})n_{[1]} && (3.66)\\
&= m_{(-1)}(n_{[0][0]}S(n_{[0][1]}))\otimes_H(S(m_{(0)(-1)})m_{(0)(0)})n_{[1]} && (3.67)
\end{aligned}
$$

is a morphism in ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)^H_H$.

Proof: Let (M,μ) and (N,ν) be bicovariant (H,α)-Hom-bimodules. Since $M\otimes_H N$ is linearly spanned by each of the sets $\{hu\otimes_H v\}$, $\{w\otimes_H zh\}$, $\{hu\otimes_H z\}$, where $h\in H$, $u\in {}^{coH}M$, $v\in {}^{coH}N$, $w\in M^{coH}$ and $z\in N^{coH}$, we prove the statement of the lemma for such elements: Since ${}^M\rho(hu)=\Delta(h)\,{}^M\rho(u)=h\cdot{}^M\rho(u)=h_1 1_H\otimes h_2\mu^{-1}(u)=\alpha(h_1)\otimes h_2\mu^{-1}(u)$ and thus

$$(id\otimes{}^M\rho)({}^M\rho(hu))=\alpha(h_1)\otimes(\alpha(h_{21})\otimes h_{22}\mu^{-2}(u)),$$

we have

$$
\begin{aligned}
c_{M,N}(hu\otimes_H v) &= \alpha(h_1)(v_{[0][0]}S(v_{[0][1]}))\otimes_H (S(\alpha(h_{21}))(h_{22}\mu^{-2}(u)))v_{[1]}\\
&= \alpha(h_1)(v_{[0][0]}S(v_{[0][1]}))\otimes_H ((\alpha^{-1}(S(\alpha(h_{21})))h_{22})\mu^{-1}(u))v_{[1]}\\
&= \alpha(h_1)(v_{[0][0]}S(v_{[0][1]}))\otimes_H ((\varepsilon(h_2)1_H)\mu^{-1}(u))v_{[1]}\\
&= \alpha(h_1\varepsilon(h_2))(v_{[0][0]}S(v_{[0][1]}))\otimes_H (1_H\mu^{-1}(u))v_{[1]}\\
&= h(v_{[0][0]}S(v_{[0][1]}))\otimes_H uv_{[1]}\\
&= h(\nu^{-1}(v_{[0]})S(v_{[1]1}))\otimes_H u\alpha(v_{[1]2})\\
&= (\alpha^{-1}(h)\nu^{-1}(v_{[0]}))\alpha(S(v_{[1]1}))\otimes_H u\alpha(v_{[1]2})\\
&= \nu(\alpha^{-1}(h)\nu^{-1}(v_{[0]}))\otimes_H \alpha(S(v_{[1]1}))\mu^{-1}(u\alpha(v_{[1]2}))\\
&= hv_{[0]}\otimes_H \alpha(S(v_{[1]1}))(\mu^{-1}(u)v_{[1]2})\\
&= hv_{[0]}\otimes_H (S(v_{[1]1})\mu^{-1}(u))\alpha(v_{[1]2})\\
&= hv_{[0]}\otimes_H \widetilde{ad}_R(v_{[1]})u\\
&= hv_{[0]}\otimes_H u\triangleleft v_{[1]}. \qquad (3.68)
\end{aligned}
$$

Similarly, we obtain the following equations

$$c_{M,N}(w\otimes_H zh) = w_{(-1)}\triangleright z\otimes_H w_{(0)}h, \qquad (3.69)$$

$$c_{M,N}(hu\otimes_H z) = h\nu^{-1}(z)\otimes_H \mu(u). \qquad (3.70)$$

By using the formula (3.69), we now prove the right (H,α)-linearity and then in the sequel the right (H,α)-colinearity of $c_{M,N}$:

$$
\begin{aligned}
c_{M,N}((w\otimes_H zh)g) &= c_{M,N}(\mu(w)\otimes_H (zh)\alpha^{-1}(g))\\
&= c_{M,N}(\mu(w)\otimes_H \nu(z)(h\alpha^{-2}(g)))\\
&= \mu(w)_{(-1)}\triangleright \nu(z)\otimes_H \mu(w)_{(0)}(h\alpha^{-2}(g))\\
&= \alpha(w_{(-1)})\triangleright \nu(z)\otimes_H \mu(w_{(0)})(h\alpha^{-2}(g))\\
&= \nu(w_{(-1)}\triangleright z)\otimes_H (w_{(0)}h)\alpha^{-1}(g))\\
&= c_{M,N}(w\otimes_H zh)g,
\end{aligned}
$$

$$
\begin{aligned}
(c_{M,N}\otimes id_H)(\sigma^{M\otimes_H N}(w\otimes_H zh)) &= (c_{M,N}\otimes id_H)((w_{[0]}\otimes_H (zh)_{[0]})\otimes w_{[1]}(zh)_{[1]}) \\
&= c_{M,N}(w_{[0]}\otimes_H z_{[0]}h_1)\otimes w_{[1]}(z_{[1]}h_2) \\
&= c_{M,N}(\mu^{-1}(w)\otimes_H \nu^{-1}(z)h_1)\otimes 1_H(1_H h_2) \\
&= (\alpha^{-1}(w_{(-1)})\triangleright \nu^{-1}(z)\otimes_H \mu^{-1}(w_{(0)})h_1)\otimes 1_H(1_H h_2) \\
&= (\nu^{-1}(w_{(-1)}\triangleright z)\otimes_H \mu^{-1}(w_{(0)})h_1)\otimes 1_H(1_H h_2) \\
&= (\nu^{-1}(w_{(-1)}\triangleright z)\otimes_H w_{(0)[0]}h_1)\otimes 1_H(w_{(0)[1]}h_2) \\
&= ((w_{(-1)}\triangleright z)_{[0]}\otimes_H (w_{(0)}h)_{[0]})\otimes (w_{(-1)}\triangleright z)_{[1]}(w_{(0)}h)_{[1]} \\
&= \sigma^{N\otimes_H M}(w_{(-1)}\triangleright z\otimes_H w_{(0)}h) \\
&= \sigma^{N\otimes_H M}(c_{M,N}(w\otimes_H zh)),
\end{aligned}
$$

where the sixth equality follows from the fact that ${}^{M}\rho(w)\in H\otimes M^{coH}$ and the seventh one results from $w_{(-1)}\triangleright z\in N^{coH}$. Analogously, one can also show that $c_{M,N}$ is both left (H,α)-linear and left (H,α)-colinear, which finishes the proof. □

Proposition 3.5.7 *Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode. Then the k-linear map $c_{M,N}: M\otimes_H N\to N\otimes_H M$ given by (3.66) in the above lemma is an isomorphism in ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)^H_H$.*

Proof: In the above lemma, it has already been proven that $c_{M,N}$, where (M,μ) and (N,ν) are bicovariant (H,α)-Hom-bimodules, is a morphism in ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)^H_H$. Hereby we prove that it is an invertible linear map to finish the proof of the proposition: Define the k-linear map $c^{-1}_{N,M}: N\otimes_H M\to M\otimes_H N$ by

$$
c^{-1}_{N,M}(n\otimes_H m) = n_{[1]}(m_{(0)(0)}S^{-1}(m_{(0)(-1)}))\otimes_H (S^{-1}(n_{[0][1]})n_{[0][0]})m_{(-1)}. \tag{3.71}
$$

For $h \in H$, $u \in {}^{coH}M$, $v \in {}^{coH}N$, we get

$$\begin{aligned}
c_{M,N}^{-1}(hv \otimes_H u) &= (h_2 v_{[1]})(\mu^{-2}(u)S^{-1}(1_H)) \otimes_H (S^{-1}(h_{12}v_{[0][1]})(h_{11}v_{[0][0]}))1_H \\
&= (h_2 v_{[1]})\mu^{-1}(u) \otimes_H (S^{-1}(v_{[0][1]})S^{-1}(h_{12}))(h_{11}v_{[0][0]}))1_H \\
&= \alpha(h_2)(v_{[1]}\mu^{-2}(u)) \otimes_H ((\alpha^{-1}(S^{-1}(v_{[0][1]})S^{-1}(h_{12}))h_{11})\nu(v_{[0][0]}))1_H \\
&= \alpha(h_2)(v_{[1]}\mu^{-2}(u)) \otimes_H ((S^{-1}(v_{[0][1]})\alpha^{-1}(S^{-1}(h_{12})h_{11}))\nu(v_{[0][0]}))1_H \\
&= \alpha(\varepsilon(h_1)h_2)(v_{[1]}\mu^{-2}(u)) \otimes_H ((S^{-1}(v_{[0][1]})1_H)\nu(v_{[0][0]}))1_H \\
&= h(\alpha(v_{[1]2})\mu^{-2}(u)) \otimes_H (\alpha(S^{-1}(v_{[1]1}))v_{[0]})1_H \\
&= h(\alpha(v_{[1]2})\mu^{-2}(u)) \otimes_H \alpha^2(S^{-1}(v_{[1]1}))\nu(v_{[0]}) \\
&= (\alpha^{-1}(h)(v_{[1]2}\mu^{-3}(u)))\alpha^2(S^{-1}(v_{[1]1})) \otimes_H \nu^2(v_{[0]}) \\
&= h((S(S^{-1}(v_{[1]})_1)\mu^{-1}(\mu^{-2}(u)))\alpha(S^{-1}(v_{[1]})_2)) \otimes_H \nu^2(v_{[0]}) \\
&= h((v_{[1]2}\mu^{-3}(u))\alpha(S^{-1}(v_{[1]1}))) \otimes_H \nu^2(v_{[0]}) \\
&= h(\widetilde{ad}_R(S^{-1}(v_{[1]}))\mu^{-2}(u)) \otimes_H \nu^2(v_{[0]}) \\
&= h(\mu^{-2}(u) \triangleleft S^{-1}(v_{[1]})) \otimes_H \nu^2(v_{[0]}),
\end{aligned} \tag{3.72}$$

and we now verify that $c_{M,N}^{-1}$ is the inverse of $c_{M,N}$ in this case;

$$\begin{aligned}
c_{M,N}^{-1}(c_{M,N}(hu \otimes_H v)) &= c_{M,N}^{-1}(hv_{[0]} \otimes_H u \triangleleft v_{[1]}) \\
&= h(\mu^{-2}(u \triangleleft v_{[1]}) \triangleleft S^{-1}(v_{[0][1]})) \otimes_H \nu^2(v_{[0][0]}) \\
&= h((\mu^{-2}(u) \triangleleft \alpha^{-2}(v_{[1]})) \triangleleft S^{-1}(v_{[0][1]})) \otimes_H \nu^2(v_{[0][0]}) \\
&= h(\mu^{-1}(u) \triangleleft \alpha^{-1}(\alpha^{-1}(v_{[1]})S^{-1}(v_{[0][1]}))) \otimes_H \nu^2(v_{[0][0]}) \\
&= h(\mu^{-1}(u) \triangleleft \alpha^{-1}(v_{[1]2}S^{-1}(v_{[1]1}))) \otimes_H \nu(v_{[0]}) \\
&= h(\mu^{-1}(u) \triangleleft 1_H) \otimes_H \nu(v_{[0]}\varepsilon(v_{[1]})) \\
&= hu \otimes_H v,
\end{aligned}$$

and

$$\begin{aligned}
c_{M,N}(c_{M,N}^{-1}(hv\otimes_H u)) &= c_{M,N}(h(\mu^{-2}(u)\triangleleft S^{-1}(v_{[1]}))\otimes_H \nu^2(v_{[0]}))\\
&= h\nu^2(v_{[0][0]})\otimes_H (\mu^{-2}(u)\triangleleft S^{-1}(v_{[1]}))\triangleleft \alpha^2(v_{[0][1]})\\
&= h\nu^2(v_{[0][0]})\otimes_H \mu^{-1}(u)\triangleleft (S^{-1}(v_{[1]})\alpha(v_{[0][1]}))\\
&= h\nu(v_{[0]})\otimes_H \mu^{-1}(u)\triangleleft \alpha(S^{-1}(v_{[1]2})v_{[1]1})\\
&= h\nu(v_{[0]}\varepsilon(v_{[1]}))\otimes_H \mu^{-1}(u)\triangleleft 1_H\\
&= hv\otimes_H u.
\end{aligned}$$

By a similar reasoning we obtain, for $h\in H$, $w\in M^{coH}$, $z\in N^{coH}$ and $u\in {}^{coH}M$, the following formulas:

$$c_{M,N}^{-1}(z\otimes_H wh) = \mu^2(w_{(0)})\otimes_H (S^{-1}(w_{(-1)})\triangleright \nu^{-2}(z))h, \tag{3.73}$$

$$c_{M,N}^{-1}(hz\otimes_H u) = h\mu^{-1}(u)\otimes_H \nu(z), \tag{3.74}$$

and thus for each of the sets $\{w\otimes_H zh\}$, $\{hu\otimes_H z\}$ linearly spanning $M\otimes_H N$, we also get $c_{M,N}^{-1}(c_{M,N}(w\otimes_H zh)) = w\otimes_H zh$, $c_{M,N}(c_{M,N}^{-1}(z\otimes_H wh)) = z\otimes_H wh$, $c_{M,N}^{-1}(c_{M,N}(hu\otimes_H z)) = hu\otimes_H z$ and $c_{M,N}(c_{M,N}^{-1}(hz\otimes_H u)) = hz\otimes_H u$. □

Theorem 3.5.8 *${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)^H_H$ is a prebraided tensor category. It is a braided monoidal category if (H,α) has an invertible antipode.*

Proof: We have already verified that ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)^H_H$ is a tensor category, with tensor product $\otimes_H$ is defined in Proposition (3.5.5), and associativity constraint $\tilde{a}$, left unit constraint $\tilde{l}$ and right unit constraint $\tilde{r}$ are given by (3.39), (3.40) and (3.41), respectively. Thereby, together with the Proposition (3.5.7), to demonstrate that the Hexagon Axioms for $c_{M,N}$ hold finishes the proof the statement. Since $(M\otimes_H N)\otimes_H P$ is generated as a left (H,α)-Hom-module by the elements $(u\otimes_H z)\otimes_H p$ where $u\in {}^{coH}M$, $z\in N^{coH}$ and $p\in P^{coH}$, it is sufficient to prove the hexagonal relations for such elements. One can first note that $c_{M,N}(u\otimes_H z) = z\otimes_H u$ and thus

$$
\begin{aligned}
&(\tilde{a}_{N,P,M} \circ c_{M,N\otimes_H P} \circ \tilde{a}_{M,N,P})((u \otimes_H z) \otimes_H p) \\
&\quad = \nu(z) \otimes_H (\pi^{-1}(p) \otimes_H u) \\
&\quad = (id_N \otimes c_{M,P})(\nu(z) \otimes_H (u \otimes_H \pi^{-1}(p)) \\
&\quad = (id_N \otimes_H c_{M,P} \circ \tilde{a}_{N,M,P})((z \otimes_H u) \otimes_H p) \\
&\quad = (id_N \otimes_H c_{M,P} \circ \tilde{a}_{N,M,P} \circ (c_{M,N} \otimes_H id_P))((u \otimes_H z) \otimes_H p),
\end{aligned}
$$

which asserts the first hexagon axiom and the second one is obtained by a similar reasoning. □

Remark 5 *The (pre)braiding $c_{M,N}$ defined by (3.66) is called Woronowicz' (pre)braiding.*

Lemma 3.5.9 *Let (H,α) be a monoidal Hom-Hopf algebra with bijective antipode and (M,μ) be a bicovariant Hom-bimodule with left Hom-coaction $m \mapsto m_{(-1)} \otimes m_{(0)}$ and right Hom-coaction $m \mapsto m_{[0]} \otimes m_{[1]}$. Then the morphism $\Phi : M \to M$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, given by*

$$\Phi(m) = (S(m_{[0](-1)})m_{[0](0)})S(m_{[1]}) = S(m_{(-1)})(m_{(0)[0]}S(m_{(0)[1]}))$$

is bijective. Furthermore, it restricts to an isomorphism of the subobjects ${}^{coH}M$ and M^{coH}.

Proof: Let us set $\Psi(m) = (S^{-1}(m_{(0)[1]})m_{(0)[0]})S^{-1}(m_{(-1)})$. Since the following equality holds:

$$
\begin{aligned}
&\Phi(m)_{(-1)} \otimes \Phi(m)_{(0)[0]} \otimes \Phi(m)_{(0)[1]} \\
&\quad = S(\alpha(m_{[1]2})) \otimes (S(\alpha^{-1}(m_{[0](-1)2}))\mu^{-2}(m_{[0](0)}))S(\alpha^{-1}(m_{[1]1})) \otimes S(\alpha(m_{[0](-1)1})),
\end{aligned}
$$

we compute

$$
\begin{aligned}
& \Psi(\Phi(m)) \\
= & (S^{-1}(\Phi(m)_{(0)[1]})\Phi(m)_{(0)[0]})S^{-1}(\Phi(m)_{(-1)}) \\
= & (S^{-1}(S(\alpha(m_{[0](-1)1})))[(S(\alpha^{-1}(m_{[0](-1)2}))\mu^{-2}(m_{[0](0)}))S(\alpha^{-1}(m_{[1]1}))])S^{-1}(S(\alpha(m_{[1]2}))) \\
= & (\alpha(m_{[0](-1)1})[(S(\alpha^{-1}(m_{[0](-1)2}))\mu^{-2}(m_{[0](0)}))S(\alpha^{-1}(m_{[1]1}))])\alpha(m_{[1]2}) \\
= & ([m_{[0](-1)1}(S(\alpha^{-1}(m_{[0](-1)2}))\mu^{-2}(m_{[0](0)}))]S(m_{[1]1}))\alpha(m_{[1]2}) \\
= & ([(\alpha^{-1}(m_{[0](-1)1})S(\alpha^{-1}(m_{[0](-1)2})))\mu^{-1}(m_{[0](0)})]S(m_{[1]1}))\alpha(m_{[1]2}) \\
= & ([\alpha^{-1}(m_{[0](-1)1}S(m_{[0](-1)2}))\mu^{-1}(m_{[0](0)})]S(m_{[1]1}))\alpha(m_{[1]2}) \\
= & (\varepsilon(m_{[0](-1)})m_{[0](0)}S(m_{[1]1}))\alpha(m_{[1]2}) \\
= & (\mu^{-1}(m_{[0]})S(m_{[1]1}))\alpha(m_{[1]2}) \\
= & m_{[0]}(S(m_{[1]1})m_{[1]2}) = m.
\end{aligned}
$$

In a similar way, one can easily get $\Phi(\Psi(m)) = m$ for any $m \in M$ meaning Φ is bijective with inverse Ψ. It can also be shown that $\mu \circ \Phi = \Phi \circ \mu$ and $\mu \circ \Psi = \Psi \circ \mu$. To prove the second statement in the lemma, we next show that $\Phi : {}^{coH}M \to M^{coH}$ and $\Psi : M^{coH} \to {}^{coH}M$: For any $m \in {}^{coH}M$, we obtain

$$
\begin{aligned}
\Phi(m) &= S(m_{(-1)})(m_{(0)[0]}S(m_{(0)[1]})) \\
&= S(1)(\mu^{-1}(m_{[0]})S(\alpha^{-1}(m_{[1]}))) = m_{[0]}S(m_{[1]}) = P_R(m),
\end{aligned}
$$

that is, $\Phi(m) \in M^{coH}$, and for any $n \in M^{coH}$, we have

$$
\begin{aligned}
\Psi(n) &= (S^{-1}(n_{(0)[1]})n_{(0)[0]})S^{-1}(n_{(-1)}) \\
&= S^{-1}(n_{[1]})(n_{[0](0)}S^{-1}(n_{[0](-1)})) = S^{-1}(1)(\mu^{-1}(n_{(0)})S^{-1}(\alpha^{-1}(n_{(-1)}))) \\
&= n_{(0)}S^{-1}(n_{(-1)}) = (n_{(0)(-1)}P_L(n_{(0)(0)}))S^{-1}(n_{(-1)}) \\
&= (n_{(0)(-1)}S^{-1}(n_{(-1)})_1)(P_L(n_{(0)(0)}) \triangleleft S^{-1}(n_{(-1)})_2) \\
&= (n_{(-1)2}S^{-1}(\alpha(n_{(-1)1}))_1)(P_L(\mu^{-1}(n_{(0)})) \triangleleft S^{-1}(\alpha(n_{(-1)1}))_2) \\
&= (n_{(-1)2}S^{-1}(\alpha(n_{(-1)12})))(P_L(\mu^{-1}(n_{(0)})) \triangleleft S^{-1}(\alpha(n_{(-1)11}))) \\
&= (\alpha(n_{(-1)22})S^{-1}(\alpha(n_{(-1)21})))(P_L(\mu^{-1}(n_{(0)})) \triangleleft S^{-1}(n_{(-1)1})) \\
&= \alpha(\varepsilon(n_{(-1)2})1)(P_L(\mu^{-1}(n_{(0)})) \triangleleft S^{-1}(n_{(-1)1})) \\
&= 1(P_L(\mu^{-1}(n_{(0)})) \triangleleft S^{-1}(\alpha^{-1}(n_{(-1)}))) = P_L(n_{(0)}) \triangleleft S^{-1}(n_{(-1)}),
\end{aligned}
$$

i.e., $\Psi(n) \in {}^{coH}M$ for all $n \in M^{coH}$. □

We now restate the structure theory of bicovariant Hom-bimodules in the coordinate form as follows, here we assume that the scalars belong to a field k,

Theorem 3.5.10 *Let (M,μ) be a bicovariant (H,α)-Hom-bimodule with right (H,α)-Hom-coaction $\varphi: M\to M\otimes H$, $m\mapsto m_{[0]}\otimes m_{[1]}$ and $\{m_i\}_{i\in I}$ be a linear basis of ${}^{coH}M$. Then there exist a pointwise finite matrices $(f^i_j)_{i,j\in I}$ and $(v^i_j)_{i,j\in I}$ of linear functionals $f^i_j\in H'$ and elements $v^i_j\in H$ such that for any $h,g\in H$ and $i,j,k\in I$ we have*

1. $\mu^i_j f^j_k(hg)=f^i_j(h)f^j_k(\alpha(g))$, $f^i_j(1)=\mu^i_j$; $m_ih=(\bar{\mu}^i_j f^j_k\bullet\alpha^{-1}(h))m_k$,
2. $\varphi(m_i)=m_j\otimes v^j_i$, *where $v^i_j\in H$, $i,j\in I$, satisfy the relations*

$$\Delta(v^l_i)=\mu^j_l v^j_k\otimes\alpha^{-1}(v^k_i),\ \varepsilon(v^k_i)=\bar{\mu}^i_k,$$

3. *the equality*

$$v^k_i(h\bullet(f^k_j\circ\alpha))=((f^i_k\circ\alpha^2)\bullet h)\alpha^{-1}(v^j_k) \tag{3.75}$$

holds. Moreover, $\{n_i:=m_jS(v^j_i)\}_{i\in I}$ is a linear basis of M^{coH}. $\{m_i\}_{i\in I}$ and $\{n_i\}_{i\in I}$ are both free left (H,α)-Hom-module bases and free right (H,α)-Hom-module bases of M.

Proof: (1) had already been proven in Theorem (3.3.11). Since $\varphi({}^{coH}M)\subseteq{}^{coH}M\otimes H$, there exists a pointwise finite matrix $(v^i_j)_{i,j\in I}$ of elements $v^i_j\in H$ such that $\varphi(m_i)=m_k\otimes v^k_i$. Let us write $\varphi(m_i)=m_{i,[0]}\otimes m_{i,[1]}$. Then, by the Hom-coassociativity and Hom-unity of φ we have

$$\begin{aligned}(\bar{\mu}^k_j m_j)\otimes\Delta(v^k_i) &= \mu^{-1}(m_k)\otimes\Delta(v^k_i)=\mu^{-1}(m_{i,[0]})\otimes\Delta(m_{i,[1]})\\ &= m_{i,[0][0]}\otimes m_{i,[0][1]}\otimes\alpha^{-1}(m_{i,[1]})\\ &= m_j\otimes v^j_k\otimes\alpha^{-1}(v^k_i),\end{aligned}$$

which implies $\Delta(v^l_i)=\mu^j_l v^j_k\otimes\alpha^{-1}(v^k_i)$ by the relation $\mu^j_l\bar{\mu}^k_j=\delta_{lk}$, and

$$\bar{\mu}^i_k m_k=\mu^{-1}(m_i)=m_{i,[0]}\varepsilon(m_{i,[1]})=m_k\varepsilon(v^k_i),$$

which finishes the proof of item (2). To prove (3), let $i \in I$ and $h \in H$. Then

$$\begin{aligned} m_{i,[0]} \triangleleft \alpha^{-1}(h_1) \otimes m_{i,[1]}\alpha^{-1}(h_2) &= m_k \triangleleft \alpha^{-1}(h_1) \otimes v_i^k \alpha^{-1}(h_2) \\ &= f_j^k(\alpha^{-1}(h_1))m_j \otimes v_i^k \alpha^{-1}(h_2) \\ &= m_j \otimes v_i^k(f_j^k(\alpha^{-1}(h_1))\alpha^{-1}(h_2)) \\ &= m_j \otimes v_i^k \alpha^{-1}(f_j^k(\alpha^{-1}(h_1))h_2) \\ &= m_j \otimes v_i^k \alpha^{-1}(\alpha^{-2}(h) \bullet f_j^k), \end{aligned}$$

$$\begin{aligned} (m_i \triangleleft h_2)_{[0]} \otimes \alpha^{-1}(h_1)\alpha^{-1}((m_i \triangleleft h_2)_{[1]}) &= m_k \otimes \alpha^{-1}(h_1) f_j^i(h_2)\alpha^{-1}(v_j^k) \\ &= m_k \otimes \alpha^{-1}(h_1 f_j^i(h_2))\alpha^{-1}(v_j^k) \\ &= m_k \otimes \alpha^{-1}((f_j^i \circ \alpha) \bullet \alpha^{-2}(h))\alpha^{-1}(v_j^k). \end{aligned}$$

Thus, by Hom-Yetter-Drinfeld condition (3.62), we acquire

$$v_i^k \alpha^{-1}(\alpha^{-2}(h) \bullet f_j^k) = \alpha^{-1}((f_k^i \circ \alpha) \bullet \alpha^{-2}(h))\alpha^{-1}(v_k^j),$$

that is, $v_i^k(\alpha^{-3}(h) \bullet (f_j^k \circ \alpha)) = ((f_k^i \circ \alpha^2) \bullet \alpha^{-3}(h))\alpha^{-1}(v_k^j)$ holds. If we replace $\alpha^{-3}(h)$ by h, we get the required equality $v_i^k(h \bullet (f_j^k \circ \alpha)) = ((f_k^i \circ \alpha^2) \bullet h)\alpha^{-1}(v_k^j)$. By the above Lemma, we obtain $n_i = \Phi(m_i) = m_{i,[0]}S(m_{i,[1]}) = m_k S(v_i^k)$ for all $i \in I$. In Theorem (3.3.11), we have shown that $\{m_i\}_{i \in I}$ is both free left (H,α)-Hom-module basis and free right (H,α)-Hom-module basis of M. Similarly, one can prove that this statement also holds for $\{n_i\}_{i \in I}$. □

3.6 Yetter-Drinfeld Modules over Monoidal Hom-Hopf Algebras

In this section, we present and study the category of Yetter-Drinfeld modules over a monoidal Hom-bialgebra (H,α), and then demonstrate that if (H,α) is a monoidal Hom-Hopf algebra with an invertible antipode it is a braided monoidal category.

Definition 3.6.1 *Let (H,α) be a monoidal Hom-bialgebra, (N,ν) be a right (H,α)-Hom-module with Hom-action $N \otimes H \to N$, $n \otimes h \mapsto n \triangleleft h$ and a right (H,α)-Hom-comodule* *with Hom-coaction $N \to N \otimes H$, $n \mapsto n_{(0)} \otimes n_{(1)}$. Then (N,ν) is called a right-right (H,α)-Hom-Yetter-Drinfeld module if the condition (3.62) holds, that is,*

$$n_{(0)} \triangleleft \alpha^{-1}(h_1) \otimes n_{(1)}\alpha^{-1}(h_2) = (n \triangleleft h_2)_{(0)} \otimes \alpha^{-1}(h_1(n \triangleleft h_2)_{(1)}),$$

for all $h \in H$ and $n \in N$.

We denote by $\widetilde{\mathcal{H}}(\mathcal{YD})_H^H$ the category of (H,α)-Hom-Yetter-Drinfeld modules whose objects are Yetter-Drinfeld modules over (H,α) and morphisms are the ones that are right (H,α)-linear and right (H,α)-colinear.

Proposition 3.6.2 *Let (H,α) be a monoidal Hom-bialgebra and (M,μ),(N,ν) be two (H,α)-Hom-Yetter-Drinfeld modules. Then $(M \otimes N, \mu \otimes \nu)$ becomes a (H,α)-Hom-Yetter-Drinfeld module with the following structure maps*

$$(M \otimes N) \otimes H \to M \otimes N, \ (m \otimes n) \otimes h \mapsto m \triangleleft h_1 \otimes n \triangleleft h_2 = (m \otimes n) \triangleleft h, \tag{3.76}$$

$$M \otimes N \to (M \otimes N) \otimes H, \ m \otimes n \mapsto (m_{(0)} \otimes n_{(0)}) \otimes m_{(1)}n_{(1)}. \tag{3.77}$$

Proof: $(M \otimes N, \mu \otimes \nu)$ is both right (H,α)-Hom-module and a right (H,α)-Hom-comodule; to verify this one can see Propositions 2.6 and 2.8 in [21] for the left case. We only prove that the Hom-Yetter-Drinfeld condition is fulfilled for $(M \otimes N, \mu \otimes \nu)$: For $h \in H$, $m \in M$ and $n \in N$,

$$\begin{aligned}
&((m \otimes n) \triangleleft h_2)_{(0)} \otimes \alpha^{-1}(h_1)\alpha^{-1}(((m \otimes n) \triangleleft h_2)_{(1)}) \\
&\quad = (m \triangleleft h_{21} \otimes n \triangleleft h_{22})_{(0)} \otimes \alpha^{-1}(h_1)\alpha^{-1}((m \triangleleft h_{21} \otimes n \triangleleft h_{22})_{(1)}) \\
&\quad = (m \triangleleft h_{12} \otimes n \triangleleft \alpha^{-1}(h_2))_{(0)} \otimes h_{11}\alpha^{-1}((m \triangleleft h_{12} \otimes n \triangleleft \alpha^{-1}(h_2))_{(1)}) \\
&\quad = (m \triangleleft h_{12})_{(0)} \otimes (n \triangleleft \alpha^{-1}(h_2))_{(0)} \otimes h_{11}\alpha^{-1}((m \triangleleft h_{12})_{(1)}(n \triangleleft \alpha^{-1}(h_2))_{(1)}) \\
&\quad = (m \triangleleft h_{12})_{(0)} \otimes (n \triangleleft \alpha^{-1}(h_2))_{(0)} \otimes h_{11}\alpha^{-1}((m \triangleleft h_{12})_{(1)})\alpha^{-1}((n \triangleleft \alpha^{-1}(h_2))_{(1)}) \\
&\quad = (m \triangleleft h_{12})_{(0)} \otimes (n \triangleleft \alpha^{-1}(h_2))_{(0)} \otimes (\alpha^{-1}(h_{11})\alpha^{-1}((m \triangleleft h_{12})_{(1)}))(n \triangleleft \alpha^{-1}(h_2))_{(1)} \\
&\quad = m_{(0)} \triangleleft \alpha^{-1}(h_{11}) \otimes (n \triangleleft \alpha^{-1}(h_2))_{(0)} \otimes (m_{(1)} \triangleleft \alpha^{-1}(h_{12}))(n \triangleleft \alpha^{-1}(h_2))_{(1)} \\
&\quad = m_{(0)} \triangleleft \alpha^{-2}(h_1) \otimes (n \triangleleft h_{22})_{(0)} \otimes \alpha(m_{(1)})(\alpha^{-1}(h_{21})\alpha^{-1}((n \triangleleft h_{22})_{(1)})) \\
&\quad = m_{(0)} \triangleleft \alpha^{-2}(h_1) \otimes n_{(0)} \triangleleft \alpha^{-1}(h_{21}) \otimes \alpha(m_{(1)}(n_{(1)}\alpha^{-1}(h_{22})) \\
&\quad = m_{(0)} \triangleleft \alpha^{-1}(h_{11}) \otimes n_{(0)} \triangleleft \alpha^{-1}(h_{12}) \otimes (m_{(1)}n_{(1)})\alpha^{-1}(h_2) \\
&\quad = (m_{(0)} \otimes n_{(0)}) \triangleleft \alpha^{-1}(h_1) \otimes (m_{(1)}n_{(1)})\alpha^{-1}(h_2) \\
&\quad = (m \otimes n)_{(0)} \triangleleft \alpha^{-1}(h_1) \otimes (m \otimes n)_{(1)} \triangleleft \alpha^{-1}(h_2).
\end{aligned}$$

□

Proposition 3.6.3 *Let (H,α) be a monoidal Hom-bialgebra and (M,μ), (N,ν), (P,π) be (H,α)-Hom-Yetter-Drinfeld modules. Then the k-linear map $\tilde{a}_{M,N,P}:(M\otimes N)\otimes P\to M\otimes(N\otimes P), \tilde{a}_{M,N,P}((m\otimes n)\otimes p)=(\mu(m)\otimes(n\otimes\pi^{-1}(p)))$ is a right (H,α)-linear and right (H,α)-colinear isomorphism.*

Proof: The bijectivity of $\tilde{a}_{M,N,P}$ is obvious with the inverse $\tilde{a}^{-1}_{M,N,P}(m\otimes(n\otimes p))=((\mu^{-1}(m)\otimes n)\otimes\pi(p))$.

$$\begin{aligned}
\tilde{a}_{M,N,P}(((m\otimes n)\otimes p)\triangleleft h) &= \tilde{a}_{M,N,P}((m\triangleleft h_{11}\otimes n\triangleleft h_{12})\otimes p\triangleleft h_2)\\
&= \mu(m\triangleleft h_{11})\otimes(n\triangleleft h_{12}\otimes\pi^{-1}(p\triangleleft h_2))\\
&= \mu(m)\triangleleft\alpha(h_{11})\otimes(n\triangleleft h_{12}\otimes\pi^{-1}(p)\triangleleft\alpha^{-1}(h_2))\\
&= \mu(m)\triangleleft h_1\otimes(n\triangleleft h_{21}\otimes\pi^{-1}(p)\triangleleft h_{22})\\
&= \mu(m)\triangleleft h_1\otimes((n\otimes\pi^{-1}(p))\triangleleft h_2)\\
&= (\mu(m)\otimes(n\otimes\pi^{-1}(p)))\triangleleft h=\tilde{a}_{M,N,P}((m\otimes n)\otimes p)\triangleleft h,
\end{aligned}$$

which proves the (H,α)-linearity. Below we show the (H,α)-colinearity:

$$\begin{aligned}
\rho^{M\otimes(N\otimes P)}(\tilde{a}_{M,N,P}((m\otimes n)\otimes p)) &= \rho^{M\otimes(N\otimes P)}(\mu(m)\otimes(n\otimes\pi^{-1}(p)))\\
&= (\mu(m)_{(0)}\otimes(n\otimes\pi^{-1}(p))_{(0)})\otimes\mu(m)_{(1)}(n\otimes\pi^{-1}(p))_{(1)}\\
&= (\mu(m_{(0)})\otimes(n_{(0)}\otimes\pi^{-1}(p_{(0)})))\otimes\alpha(m_{(1)})(n_{(1)}\alpha^{-1}(p_{(1)}))\\
&= (\mu(m_{(0)})\otimes(n_{(0)}\otimes\pi^{-1}(p_{(0)})))\otimes(m_{(1)}n_{(1)})p_{(1)},
\end{aligned}$$

$$\begin{aligned}
&(\tilde{a}_{M,N,P}\otimes id_H)(\rho^{(M\otimes N)\otimes P}((m\otimes n)\otimes p))\\
&\quad= (\tilde{a}_{M,N,P}\otimes id_H)(((m\otimes n)_{(0)}\otimes p_{(0)})\otimes(m\otimes n)_{(1)}p_{(1)})\\
&\quad= (\tilde{a}_{M,N,P}\otimes id_H)(((m_{(0)}\otimes n_{(0)})\otimes p_{(0)})\otimes(m_{(1)}n_{(1)})p_{(1)})\\
&\quad= (\mu(m_{(0)})\otimes(n_{(0)}\otimes\pi^{-1}(p_{(0)})))\otimes(m_{(1)}n_{(1)})p_{(1)},
\end{aligned}$$

where ρ^Q denotes the right (H,α)-Hom-comodule structure of a Hom-Yetter-Drinfeld module Q. □

Proposition 3.6.4 *Let (H,α) be a monoidal Hom-bialgebra. Then the unit constraints in $\widetilde{\mathcal{H}}(\mathcal{YD})_H^H$ are given by the k-linear maps*

$$\tilde{l}_M : k\otimes M \to M,\ x\otimes m \mapsto x\mu(m), \tag{3.78}$$

$$\tilde{r}_M : M\otimes k \to M,\ m\otimes x \mapsto x\mu(m) \tag{3.79}$$

with respect to (k, id_k).

Proof: In the category $\mathcal{M}_k$ of k-modules, k itself is the unit object; so one can easily show that (k, id_k) is the unit object in $\widetilde{\mathcal{H}}(\mathcal{YD})_H^H$ with the trivial right Hom-action $k\otimes H \to k, x\otimes h \mapsto \varepsilon(h)x$ and the right Hom-coaction $k \to k\otimes H, x \mapsto x\otimes 1_H$ for any x in k and h in H. It is obvious that $\tilde{l}_M$ is a k-isomorphism with the inverse $\tilde{l}_M^{-1} : M \to k\otimes M, m \mapsto 1\otimes\mu^{-1}(m)$. It can easily be shown that the relation $\mu\circ\tilde{l}_M = \tilde{l}_M\circ(id_k\otimes\mu)$ holds. Now we prove the right (H,α)-linearity and right (H,α)-colinearity of $\tilde{l}_M$: For all $x\in k$, $h\in H$ and $m\in M$,

$$\begin{aligned}
\tilde{l}_M((x\otimes m)\triangleleft h) &= \tilde{l}_M(\varepsilon(h_1)x\otimes m\triangleleft h_2) = \varepsilon(h_1)x\mu(m\triangleleft h_2)\\
&= x\mu(m)\triangleleft\alpha(\varepsilon(h_1)h_2) = x\mu(m)\triangleleft\alpha(\alpha^{-1}(h))\\
&= \tilde{l}_M(x\otimes m)\triangleleft h,
\end{aligned}$$

$$\begin{aligned}
((\tilde{l}_M\otimes id_H)\circ\rho^{k\otimes M})(x\otimes m) &= (\tilde{l}_M\otimes id_H)((x\otimes m_{(0)})\otimes 1_H m_{(1)})\\
&= x\mu(m_{(0)})\otimes\alpha(m_{(1)})\\
&= x((\mu\otimes\alpha)\circ\rho^M)(m)\\
&= \rho^M(x\mu(m))\\
&= (\rho^M\circ\tilde{l}_M)(x\otimes m).
\end{aligned}$$

The same argument holds for $\tilde{r}$. □

Proposition 3.6.5 *Let (H,α) be a monoidal Hom-bialgebra and (M,μ), (N,ν) be (H,α)-Hom-Yetter-Drinfeld modules. Then the k-linear map*

$$c_{M,N} : M\otimes N \to N\otimes M,\ m\otimes n \mapsto \nu(n_{(0)})\otimes\mu^{-1}(m)\triangleleft n_{(1)} \tag{3.80}$$

is a right (H,α)-linear and right (H,α)-colinear morphism. In case (H,α) is a monoidal Hom-Hopf-algebra with an invertible antipode it is also a bijection.

Proof: We have the relation $(\nu\otimes\mu)\circ c_{M,N}=c_{M,N}\circ(\mu\otimes\nu)$ by the computation

$$\begin{aligned}
(\nu\otimes\mu)(c_{M,N}(m\otimes n)) &= (\nu\otimes\mu)(\nu(n_{(0)})\otimes\mu^{-1}(m)\lhd n_{(1)})\\
&= \nu(\nu(n)_{(0)})\otimes m\lhd\alpha(n_{(1)})\\
&= \nu(\nu(n)_{(0)})\otimes\mu^{-1}(\mu(m))\lhd\nu(n)_{(1)}\\
&= c_{M,N}(\mu(m)\otimes\nu(n)).
\end{aligned}$$

The (H,α)-linearity holds as follows

$$\begin{aligned}
c_{M,N}((m\otimes n)\lhd h) &= c_{M,N}(m\lhd h_1\otimes n\lhd h_2)\\
&= \nu((n\lhd h_2)_{(0)})\otimes\mu^{-1}(m\lhd h_1)\lhd(n\lhd h_2)_{(1)}\\
&= \nu((n\lhd h_2)_{(0)})\otimes(\mu^{-1}(m)\lhd\alpha^{-1}(h_1))\lhd(n\lhd h_2)_{(1)}\\
&= \nu((n\lhd h_2)_{(0)})\otimes m\lhd(\alpha^{-1}(h_1)\alpha^{-1}((n\lhd h_2)_{(1)}))\\
&= \nu(n_{(0)}\lhd\alpha^{-1}(h_1))\otimes m\lhd(n_{(1)}\alpha^{-1}(h_2))\\
&= \nu(n_{(0)}\lhd h_1\otimes(\mu^{-1}(m)\lhd n_{(1)})\lhd h_2\\
&= (\nu(n_{(0)})\otimes\mu^{-1}(m)\lhd n_{(1)})\lhd h\\
&= c_{M,N}(m\otimes n)\lhd h,
\end{aligned}$$

where in the fifth equality the twisted Yetter-Drinfeld condition has been used. We now show that $c_{M,N}$ is (H,α)-colinear: In fact,

$$\begin{aligned}
(\rho^{N\otimes M}\circ c_{M,N})(m\otimes n) &= \rho^{N\otimes M}(\nu(n_{(0)})\otimes\mu^{-1}(m)\lhd n_{(1)})\\
&= (\nu(n_{(0)})_{(0)}\otimes(\mu^{-1}(m)\lhd n_{(1)})_{(0)})\otimes\nu(n_{(0)})_{(1)}(\mu^{-1}(m)\lhd n_{(1)})_{(1)}\\
&= (\nu(n_{(0)(0)})\otimes\mu^{-1}((m\lhd\alpha(n_{(1)}))_{(0)}))\otimes\alpha(n_{(0)(1)})\alpha^{-1}((m\lhd\alpha(n_{(1)}))_{(1)})\\
&= (n_{(0)}\otimes\mu^{-1}((m\lhd\alpha^2(n_{(1)2}))_{(0)}))\otimes\alpha(n_{(1)1})\alpha^{-1}((m\lhd\alpha^2(n_{(1)2}))_{(1)})\\
&= (n_{(0)}\otimes\mu^{-1}(m_{(0)}\lhd\alpha^{-1}(\alpha^2(n_{(1)1}))))\otimes m_{(1)}\alpha^{-1}(\alpha^2(n_{(1)2}))\\
&= (\nu(n_{(0)(0)})\otimes\mu^{-1}(m_{(0)}\lhd\alpha(n_{(0)(1)})))\otimes m_{(1)}n_{(1)}\\
&= (c_{M,N}\otimes id_H)((m_{(0)}\otimes n_{(0)})\otimes m_{(1)}n_{(1)})\\
&= (c_{M,N}\otimes id_H)(\rho^{M\otimes N}(m\otimes n)).
\end{aligned}$$

Let us define

$$c_{M,N}^{-1} : N \otimes M \to M \otimes N,\ n \otimes m \mapsto \mu^{-1}(m) \triangleleft S^{-1}(n_{(1)}) \otimes \nu(n_{(0)}).$$

We verify that $c_{M,N}^{-1}$ is the inverse of $c_{M,N}$:

$$\begin{aligned}
c_{M,N}^{-1}(c_{M,N}(m \otimes n)) &= c_{M,N}^{-1}(\nu(n_{(0)}) \otimes \mu^{-1}(m) \triangleleft n_{(1)}) \\
&= \mu^{-1}(\mu^{-1}(m) \triangleleft n_{(1)}) \triangleleft S^{-1}(\nu(n_{(0)})_{(1)}) \otimes \nu(\nu(n_{(0)})_{(0)}) \\
&= (\mu^{-2}(m) \triangleleft \alpha^{-1}(n_{(1)})) \triangleleft S^{-1}(\alpha(n_{(0)(1)})) \otimes \nu^{2}(n_{(0)(0)}) \\
&= \mu^{-1}(m) \triangleleft (\alpha^{-1}(n_{(1)}) S^{-1}(n_{(0)(1)})) \otimes \nu^{2}(n_{(0)(0)}) \\
&= \mu^{-1}(m) \triangleleft (n_{(1)2} S^{-1}(n_{(1)1})) \otimes \nu(n_{(0)}) \\
&= \mu^{-1}(m) \triangleleft 1_H \otimes \nu(n_{(0)} \varepsilon(n_{(1)})) \\
&= m \otimes n,
\end{aligned}$$

and on the other hand we have

$$\begin{aligned}
c_{M,N}(c_{M,N}^{-1}(n \otimes m)) &= c_{M,N}(\mu^{-1}(m) \triangleleft S^{-1}(n_{(1)}) \otimes \nu(n_{(0)})) \\
&= \nu(\nu(n_{(0)})_{(0)}) \otimes \mu^{-1}(\mu^{-1}(m) \triangleleft S^{-1}(n_{(1)})) \triangleleft \nu(n_{(0)})_{(1)} \\
&= \nu^{2}(n_{(0)(0)}) \otimes (\mu^{-2}(m) \triangleleft \alpha^{-1}(S^{-1}(n_{(1)}))) \triangleleft \alpha(n_{(0)(1)}) \\
&= \nu(n_{(0)}) \otimes \mu^{-1}(m) \triangleleft (S^{-1}(n_{(1)2}) n_{(1)1}) \\
&= n \otimes m.
\end{aligned}$$

□

Theorem 3.6.6 *Let (H, α) be a monoidal Hom-bialgebra. Then $\widetilde{\mathcal{H}}(\mathcal{YD})_H^H$ is a prebraided monoidal category. It is a braided monoidal one under the requirement (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode.*

Proof: The definition of tensor product is given in Proposition (3.6.2), the associativity constraint is given in Proposition (3.6.3) and the (pre-)braiding is defined in Proposition (3.6.5). The Hexagon Axiom for c are left to be verified to finish the proof.

Let (M, μ),(N, ν), (P, π) be in $\widetilde{\mathcal{H}}(\mathcal{YD})_H^H$; we show that the first hexagon axiom holds for c:

$$
\begin{aligned}
&((id_N\otimes c_{M,P})\circ\tilde{a}_{N,M,P}\circ(c_{M,N}\otimes id_P))((m\otimes n)\otimes p)\\
&\quad= ((id_N\otimes c_{M,P})\circ\tilde{a}_{N,M,P})((\nu(n_{(0)})\otimes\mu^{-1}(m)\triangleleft n_{(1)})\otimes p)\\
&\quad= (id_N\otimes c_{M,P})(\nu^2(n_{(0)})\otimes(\mu^{-1}(m)\triangleleft n_{(1)}\otimes\pi^{-1}(p)))\\
&\quad= \nu^2(n_{(0)})\otimes(\pi(\pi^{-1}(p)_{(0)})\otimes\mu^{-1}(\mu^{-1}(m)\triangleleft n_{(1)})\triangleleft\pi^{-1}(p)_{(1)})\\
&\quad= \nu^2(n_{(0)})\otimes(p_{(0)}\otimes(\mu^{-2}(m)\triangleleft\alpha^{-1}(n_{(1)}))\triangleleft\alpha^{-1}(p_{(1)}))\\
&\quad= \nu^2(n_{(0)})\otimes(p_{(0)}\otimes\mu^{-1}(m)\triangleleft(\alpha^{-1}(n_{(1)})\alpha^{-2}(p_{(1)})))\\
&\quad= \nu^2(n_{(0)})\otimes(p_{(0)}\otimes\mu^{-1}(m\triangleleft(n_{(1)}\alpha^{-1}(p_{(1)}))))\\
&\quad= \tilde{a}_{N,P,M}((\nu(n_{(0)})\otimes p_{(0)})\otimes m\triangleleft(n_{(1)}\alpha^{-1}(p_{(1)})))\\
&\quad= \tilde{a}_{N,P,M}((\nu\otimes\pi)((n\otimes\pi^{-1}(p))_{(0)})\otimes\mu^{-1}(\mu(m))\triangleleft(n\otimes\pi^{-1}(p))_{(1)})\\
&\quad= (\tilde{a}_{N,P,M}\circ c_{M,N\otimes P})(\mu(m)\otimes(n\otimes\pi^{-1}(p)))\\
&\quad= (\tilde{a}_{N,P,M}\circ c_{M,N\otimes P}\circ\tilde{a}_{M,N,P})((m\otimes n)\otimes p).
\end{aligned}
$$

Lastly, we prove the second hexagon axiom:

$$
\begin{aligned}
&\tilde{a}^{-1}_{P,M,N}\circ c_{M\otimes N,P}\circ\tilde{a}^{-1}_{M,N,P}(m\otimes(n\otimes p))\\
&\quad= (\tilde{a}^{-1}_{P,M,N}\circ c_{M\otimes N,P})((\mu^{-1}(m)\otimes n)\otimes\pi(p))\\
&\quad= \tilde{a}^{-1}_{P,M,N}(\pi(\pi(p)_{(0)})\otimes((\mu^{-1}\otimes\nu^{-1})(\mu^{-1}(m)\otimes n)\triangleleft\pi(p)_{(1)}))\\
&\quad= \tilde{a}^{-1}_{P,M,N}(\pi^2(p_{(0)})\otimes(\mu^{-2}(m)\otimes\nu^{-1}(n))\triangleleft\alpha(p_{(1)}))\\
&\quad= \tilde{a}^{-1}_{P,M,N}(\pi^2(p_{(0)})\otimes(\mu^{-2}(m)\triangleleft\alpha(p_{(1)})_1\otimes\nu^{-1}(n)\triangleleft\alpha(p_{(1)})_2))\\
&\quad= \tilde{a}^{-1}_{P,M,N}(\pi^3(p_{(0)(0)})\otimes(\mu^{-2}(m)\triangleleft\alpha(p_{(0)(1)})\otimes\nu^{-1}(n)\triangleleft p_{(1)}))\\
&\quad= (\pi^2(p_{(0)(0)})\otimes\mu^{-2}(m)\triangleleft\alpha(p_{(0)(1)}))\otimes n\triangleleft p_{(1)}\\
&\quad= (\pi(\pi(p_{(0)})_{(0)})\otimes\mu^{-1}(\mu^{-1}(m))\triangleleft\pi(p_{(0)})_{(1)})\otimes n\triangleleft\alpha(p_{(1)}))\\
&\quad= (c_{M,P}\otimes id_N)((\mu^{-1}(m)\otimes\pi(p_{(0)}))\otimes n\triangleleft\alpha(p_{(1)}))\\
&\quad= (c_{M,P}\otimes id_N)((\mu^{-1}(m)\otimes\pi(p_{(0)}))\otimes\nu(\nu^{-1}(n)\triangleleft p_{(1)}))\\
&\quad= ((c_{M,P}\otimes id_N)\circ\tilde{a}^{-1}_{M,P,N})(m\otimes(\pi(p_{(0)})\otimes\nu^{-1}(n)\triangleleft p_{(1)}))\\
&\quad= ((c_{M,P}\otimes id_N)\circ\tilde{a}^{-1}_{M,P,N}\circ(id_M\otimes c_{N,P}))(m\otimes(n\otimes p)).
\end{aligned}
$$

□

Together with Theorem 3.3.5 and Theorem 3.5.4, Theorem (3.2.5) provides:

Theorem 3.6.7 *Let (H,α) be a monoidal Hom-Hopf algebra. Then the equivalences in (3.2.5):*

$$F = (H\otimes -, \alpha\otimes -) : \widetilde{\mathcal{H}}(\mathcal{M}_k) \to {}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k),$$
$$G = {}^{coH}(-) : {}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k) \to \widetilde{\mathcal{H}}(\mathcal{M}_k),$$

induce tensor equivalences between

1. *the category of right (H,α)-Hom-modules and the category of left-covariant (H,α)-Hom-bimodules,*
2. *the category of right-right (H,α)-Hom-Yetter-Drinfeld modules and the category of bicovariant (H,α)-Hom-bimodules.*

Proof: The right (H,α)-Hom-module structure on $(H\otimes V, \alpha\otimes\mu)$ for a right (H,α)-Hom-module (V,μ) is given in Proposition (3.3.3) and the right (H,α)-Hom-comodule structure on $(H\otimes W, \alpha\otimes\nu)$ for a right (H,α)-Hom-comodule (W,ν) is given in Proposition (3.5.2). It remains only to prove that one of the inverse equivalences, say F together with $\varphi_2(V,W) : (H\otimes V)\otimes_H (H\otimes W) \to H\otimes (V\otimes W)$ given by

$$\varphi_2(V,W)((g\otimes v)\otimes_H (h\otimes w)) = g\alpha(h_1)\otimes(\mu^{-1}(v)\triangleleft h_2\otimes w)$$

for all $g,h\in H$, $v\in V$, $w\in W$, is a tensor functor in each case. Define

$$\varphi_2(V,W)^{-1} : H\otimes(V\otimes W)\to (H\otimes V)\otimes_H (H\otimes W),\ h\otimes(v\otimes w)\mapsto (\alpha^{-1}(h)\otimes v)\otimes_H (1_H\otimes w),$$

which is an inverse of $\varphi_2(V,W)$: For $h,g\in H$, $v\in V$ and $w\in W$,

$$\begin{aligned}
\varphi_2(V,W)(\varphi_2(V,W)^{-1}(h\otimes(v\otimes w))) &= \varphi_2(V,W)((\alpha^{-1}(h)\otimes v)\otimes_H (1_H\otimes w))\\
&= \alpha^{-1}(h)1_H\otimes(\mu^{-1}(m)\triangleleft 1_H\otimes w)\\
&= h\otimes(v\otimes w),
\end{aligned}$$

$$\begin{aligned}
\varphi_2(V,W)^{-1}(\varphi_2(V,W)((g\otimes v)\otimes_H (h\otimes w))) &= \varphi_2(V,W)^{-1}(g\alpha(h_1)\otimes(\mu^{-1}(v)\triangleleft h_2\otimes w))\\
&= (\alpha^{-1}(g\alpha(h_1))\otimes\mu^{-1}(v)\triangleleft h_2)\otimes_H (1_H\otimes w)\\
&= (\alpha^{-1}(g)\otimes\mu^{-1}(v))h\otimes_H (1_H\otimes w)\\
&= (g\otimes v)\otimes_H h(1_H\otimes\nu^{-1}(w))\\
&= (g\otimes v)\otimes_H (h\otimes w),
\end{aligned}$$

and one can also show that the relation $(\alpha\otimes(\mu\otimes\nu))\circ\varphi_2(V,W)=\varphi_2(V,W)\circ((\alpha\otimes\mu)\otimes_H(\alpha\otimes\nu))$ holds. We now verify that the coherence condition on F is fulfilled:

$$
\begin{aligned}
&(\varphi_2(U,V\otimes W)\circ(id\otimes\varphi_2(V,W))\circ\tilde{a}_Q)(((g\otimes u)\otimes_H(h\otimes v))\otimes_H(k\otimes w))\\
&= (\varphi_2(U,V\otimes W)\circ(id\otimes\varphi_2(V,W)))((\alpha(g)\otimes\mu(u))\otimes_H((h\otimes v)\otimes_H(\alpha^{-1}(k)\otimes\pi^{-1}(w))))\\
&= \varphi_2(U,V\otimes W)((\alpha(g)\otimes\mu(u))\otimes_H(hk_1\otimes(\nu^{-1}(v)\lhd\alpha^{-1}(k_2)\otimes\pi^{-1}(w))))\\
&= \alpha(g)\alpha(h_1k_{11})\otimes(u\lhd h_2k_{12}\otimes(\nu^{-1}(v)\lhd\alpha^{-1}(k_2)\otimes\pi^{-1}(w)))\\
&= \alpha(g)(\alpha(h_1)k_1)\otimes(u\lhd h_2k_{21}\otimes(\nu^{-1}(v)\lhd k_{22}\otimes\pi^{-1}(w)))\\
&= (id\otimes\tilde{a}_{Q'})(\alpha(g)(\alpha(h_1)k_1)\otimes((\mu^{-1}(u)\lhd\alpha^{-1}(h_2k_{21})\otimes\nu^{-1}(v)\lhd k_{22})\otimes w))\\
&= (id\otimes\tilde{a}_{Q'})(\alpha(g)(\alpha(h_1)k_1)\otimes(((\mu^{-2}(u)\lhd\alpha^{-1}(h_2))\lhd k_{21}\otimes\nu^{-1}(v)\lhd k_{22})\otimes w))\\
&= (id\otimes\tilde{a}_{Q'})((g\alpha(h_1))\alpha(k_1)\otimes((\mu^{-2}(u)\lhd\alpha^{-1}(h_2)\otimes\nu^{-1}(v))\lhd k_2\otimes w))\\
&= ((id\otimes\tilde{a}_{Q'})\circ\varphi_2(U\otimes V,W))((g\alpha(h_1)\otimes(\mu^{-1}\lhd h_2\otimes v))\otimes_H(k\otimes w))\\
&= ((id\otimes\tilde{a}_{Q'})\circ\varphi_2(U\otimes V,W)\circ(\varphi_2(U,V)\otimes id))(((g\otimes u)\otimes_H(h\otimes v))\otimes_H(k\otimes w)).
\end{aligned}
$$

For (1) we verify that the k-isomorphism $\varphi_2(V,W)$ is a morphism of left-covariant (H,α)-Hom-bimodules, that is, we prove its left (H,α)-linearity, (H,α)-colinearity, and right (H,α)-linearity, respectively:

$$
\begin{aligned}
\varphi_2(V,W)(k((g\otimes v)\otimes_H(h\otimes w))) &= \varphi_2(V,W)(\alpha^{-1}(k)(g\otimes v)\otimes_H(\alpha(h)\otimes\nu(w))))\\
&= \varphi_2(V,W)((\alpha^{-2}(k)g\otimes\mu(v))\otimes_H(\alpha(h)\otimes\nu(w)))\\
&= (\alpha^{-2}(k)g)\alpha^2(h_1)\otimes(v\lhd\alpha(h_2)\otimes\nu(w))\\
&= \alpha^{-1}(k)(g\alpha(h_1))\otimes((\mu\otimes\nu)(\mu^{-1}(v)\lhd h_2\otimes w))\\
&= k(g\alpha(h_1)\otimes(\mu^{-1}(v)\lhd h_2\otimes w))\\
&= k\varphi_2(V,W)((g\otimes v)\otimes_H(h\otimes w)),
\end{aligned}
$$

$$
\begin{aligned}
&(id\otimes\varphi_2(V,W))({}^{Q}\rho((g\otimes v)\otimes_H(h\otimes w)))\\
&\quad= (id\otimes\varphi_2(V,W))(\alpha(g_1)\alpha(h_1)\otimes((g_2\otimes\mu^{-1}(v))\otimes_H(h_2\otimes\nu^{-1}(w))))\\
&\quad= \alpha(g_1)\alpha(h_1)\otimes(g_2\alpha(h_{21})\otimes(\mu^{-2}(v)\triangleleft h_{22}\otimes\nu^{-1}(w)))\\
&\quad= \alpha(g_1)\alpha^2(h_{11})\otimes(g_2\alpha(h_{12})\otimes(\mu^{-2}(v)\triangleleft\alpha^{-1}(h_2)\otimes\nu^{-1}(w)))\\
&\quad= \alpha((g\alpha(h_1))_1)\otimes((g\alpha(h_1))_2\otimes(\mu^{-1}(\mu^{-1}(v)\triangleleft h_2)\nu^{-1}(w)))\\
&\quad= {}^{Q'}\rho(g\alpha(h_1)\otimes(\mu^{-1}(v)\triangleleft h_2\otimes w))\\
&\quad= {}^{Q'}\rho(\varphi_2(V,W)((g\otimes v)\otimes_H(h\otimes w))),
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_2(V,W)(((g\otimes v)\otimes_H(h\otimes w))k)\\
&\quad= \varphi_2(V,W)((\alpha(g)\otimes\mu(v))\otimes_H(h\otimes w)\alpha^{-1}(k))\\
&\quad= \varphi_2(V,W)((\alpha(g)\otimes\mu(v))\otimes_H(h\alpha^{-1}(k_1)\otimes w\triangleleft\alpha^{-1}(k_2)))\\
&\quad= \alpha(g)(\alpha(h_1)k_{11})\otimes(v\triangleleft(h_2\alpha^{-1}(k_{12}))\otimes w\triangleleft\alpha^{-1}(k_2))\\
&\quad= \alpha(g)(\alpha(h_1)\alpha^{-1}(k_1))\otimes(v\triangleleft(h_2\alpha^{-1}(k_{21}))\otimes w\triangleleft k_{22})\\
&\quad= \alpha(g)(\alpha(h_1)\alpha^{-1}(k_1))\otimes(\mu(\mu^{-1}(v))\triangleleft(h_2\alpha^{-1}(k_{21}))\otimes w\triangleleft k_{22})\\
&\quad= (g\alpha(h_1))k_1\otimes((\mu^{-1}(v)\triangleleft h_2)\triangleleft k_{21}\otimes w\triangleleft k_{22})\\
&\quad= (g\alpha(h_1))k_1\otimes(\mu^{-1}(v)\triangleleft h_2\otimes w)k_2\\
&\quad= \varphi_2(V,W)((g\otimes v)\otimes_H(h\otimes w))k.
\end{aligned}
$$

For (2) we need only to check that $\varphi_2(V,W)$ is right (H,α)-colinear. Let us denote by $\sigma^{Q'}$ and σ^{Q} the right (H,α)-Hom-comodule structures on $Q'=H\otimes(V\otimes W)$ and $Q=(H\otimes V)\otimes_H(H\otimes W)$. Then

$$
\begin{aligned}
&\sigma^{Q'}(\varphi_2(V,W)((g\otimes v)\otimes_H (h\otimes w)))\\
&= \sigma^{Q'}(g\alpha(h_1)\otimes(\mu^{-1}(v)\triangleleft h_2\otimes w))\\
&= ((g\alpha(h_1))_1\otimes((\mu^{-1}(v)\triangleleft h_2)_{[0]}\otimes w_{[0]}))\otimes(g\alpha(h_1))_2((\mu^{-1}(v)\triangleleft h_2)_{[1]}w_{[1]})\\
&= (g_1\alpha(h_{11})\otimes((\mu^{-1}(v)\triangleleft h_2)_{[0]}\otimes w_{[0]}))\otimes\alpha(g_2)(\alpha(h_{12})(\alpha^{-1}((\mu^{-1}(v)\triangleleft h_2)_{[1]})\alpha^{-1}(w_{[1]})))\\
&= (g_1\alpha(h_{11})\otimes((\mu^{-1}(v)\triangleleft h_2)_{[0]}\otimes w_{[0]}))\otimes\alpha(g_2)((h_{12}\alpha^{-1}((\mu^{-1}(v)\triangleleft h_2)_{[1]}))w_{[1]})\\
&= (g_1h_1\otimes((\mu^{-1}(v)\triangleleft\alpha(h_{22}))_{[0]}\otimes w_{[0]}))\otimes\alpha(g_2)((h_{21}\alpha^{-1}((\mu^{-1}(v)\triangleleft\alpha(h_{22}))_{[1]}))w_{[1]})\\
&= (g_1h_1\otimes((\mu^{-1}(v)\triangleleft\alpha(h_2)_2)_{[0]}\otimes w_{[0]}))\otimes\alpha(g_2)(\alpha^{-1}(\alpha(h_2)_1(\mu^{-1}(v)\triangleleft\alpha(h_2)_2)_{[1]})w_{[1]})\\
&= (g_1h_1\otimes(\mu^{-1}(v)_{[0]}\triangleleft\alpha^{-1}(\alpha(h_2)_1)\otimes w_{[0]}))\otimes\alpha(g_2)((\mu^{-1}(v)_{[1]}\alpha^{-1}(\alpha(h_2)_2))w_{[1]})\\
&= (g_1h_1\otimes(\mu^{-1}(v_{[0]})\triangleleft h_{21}\otimes w_{[0]}))\otimes\alpha(g_2)((\alpha^{-1}(v_{[1]})h_{22})w_{[1]})\\
&= (g_1\alpha(h_{11})\otimes(\mu^{-1}(v_{[0]})\triangleleft h_{12}\otimes w_{[0]}))\otimes\alpha(g_2)((\alpha^{-1}(v_{[1]})\alpha^{-1}(h_2))w_{[1]})\\
&= (g_1\alpha(h_{11})\otimes(\mu^{-1}(v_{[0]})\triangleleft h_{12}\otimes w_{[0]}))\otimes(g_2v_{[1]})(h_2w_{[1]})\\
&= (\varphi_2(V,W)\otimes id_H)(((g_1\otimes v_{[0]})\otimes_H(h_1\otimes w_{[0]}))\otimes(g_2v_{[1]})(h_2w_{[1]}))\\
&= (\varphi_2(V,W)\otimes id_H)(\sigma^{Q}((g\otimes v)\otimes_H(h\otimes w))),
\end{aligned}
$$

where we have used the twisted Yetter-Drinfeld condition in the seventh equality. □

Corollary 3.6.8 *Let (H,α) be a monoidal Hom-Hopf algebra. The categories ${}^H_H\widetilde{\mathcal{H}}(\mathcal{M}_k)^H_H$ and $\widetilde{\mathcal{H}}(\mathcal{YD})^H_H$ are equivalent as prebraided monoidal categories. The tensor equivalence between them is braided whenever (H,α) has a bijective antipode.*

Proof: It suffices to regard the case of bicovariant (H,α)-Hom-bimodules $(M,\mu')=(H\otimes V,\alpha\otimes\mu)$ and $(N,\nu')=(H\otimes W,\alpha\otimes\nu)$ with (V,μ) and (W,ν) (H,α)-Hom-Yetter-Drinfeld modules. Thus for $h\in H$, $v\in V$ and $w\in W$ we have

$$
\begin{aligned}
& (\varphi_2(W,V)\circ c_{M,N}\circ \varphi_2(V,W)^{-1})(h\otimes(v\otimes w)) \\
= \;& \varphi_2(W,V)(c_{M,N}((\alpha^{-1}(h)\otimes v)\otimes_H (1_H\otimes w))) \\
= \;& \varphi_2(W,V)(h_1((1_H\otimes w_{[0][0]})S(\alpha(w_{[0][1]})))\otimes_H (S(h_{21})(\alpha^{-1}(h_{22})\otimes \mu^{-2}(v)))\alpha(w_{[1]})) \\
= \;& \varphi_2(W,V)(h_1((1_H\otimes \nu^{-1}(w_{[0]}))S(\alpha(w_{[1]1})))\otimes_H (\alpha^{-1}(S(h_{21})h_{22})\otimes \mu^{-1}(v))\alpha^2(w_{[1]2})) \\
= \;& \varphi_2(W,V)((h_1\varepsilon(h_2))((1_H\otimes \nu^{-1}(w_{[0]}))S(\alpha(w_{[1]1})))\otimes_H (1_H\otimes \mu^{-1}(v))\alpha^2(w_{[1]2})) \\
= \;& \varphi_2(W,V)((\alpha^{-2}(h)(1_H\otimes \nu^{-1}(w_{[0]})))S(\alpha^2(w_{[1]1}))\otimes_H (1_H\otimes \mu^{-1}(v))\alpha^2(w_{[1]2})) \\
= \;& \varphi_2(W,V)((\alpha^{-3}(h)1_H\otimes \nu(\nu^{-1}(w_{[0]})))S(\alpha^2(w_{[1]1}))\otimes_H (1_H\alpha^2(w_{[1]2})_1\otimes \mu^{-1}(v)\triangleleft \alpha^2(w_{[1]2})_2)) \\
= \;& \varphi_2(W,V)((\alpha^{-2}(h)\otimes w_{[0]})S(\alpha^2(w_{[1]1}))\otimes_H (\alpha^3(w_{[1]21})\otimes \mu^{-1}(v)\triangleleft \alpha^2(w_{[1]22}))) \\
= \;& \varphi_2(W,V)((\alpha^{-1}(h)\otimes \nu(w_{[0]}))\otimes_H S(\alpha^2(w_{[1]1}))(\alpha^2(w_{[1]21})\otimes \mu^{-2}(v)\triangleleft \alpha(w_{[1]22}))) \\
= \;& \varphi_2(W,V)((\alpha^{-1}(h)\otimes \nu(w_{[0]}))\otimes_H (S(\alpha(w_{[1]1}))\alpha^2(w_{[1]21})\otimes \mu^{-1}(v)\triangleleft \alpha^2(w_{[1]22}))) \\
= \;& \varphi_2(W,V)((\alpha^{-1}(h)\otimes \nu(w_{[0]}))\otimes_H (S(\alpha^2(w_{[1]11}))\alpha^2(w_{[1]12})\otimes \mu^{-1}(v)\triangleleft \alpha(w_{[1]2}))) \\
= \;& \varphi_2(W,V)((\alpha^{-1}(h)\otimes \nu(w_{[0]}))\otimes_H (\alpha^2(S(w_{[1]11})w_{[1]12})\otimes \mu^{-1}(v)\triangleleft \alpha(w_{[1]2}))) \\
= \;& \varphi_2(W,V)((\alpha^{-1}(h)\otimes \nu(w_{[0]}))\otimes_H (1_H\otimes \mu^{-1}(v)\triangleleft \alpha(\varepsilon(w_{[1]1})w_{[1]2}))) \\
= \;& \varphi_2(W,V)((\alpha^{-1}(h)\otimes \nu(w_{[0]}))\otimes_H (1_H\otimes \mu^{-1}(v)\triangleleft w_{[1]})) \\
= \;& \alpha^{-1}(h)1_H\otimes(\nu^{-1}(\nu(w_{[0]}))\triangleleft 1_H\otimes \mu^{-1}(v)\triangleleft w_{[1]}) \\
= \;& h\otimes(\nu(w_{[0]})\otimes \mu^{-1}(v)\triangleleft w_{[1]}) \\
= \;& (id_H\otimes c_{V,W})(h\otimes(v\otimes w)),
\end{aligned}
$$

which demonstrates that F is a (pre-)braided tensor equivalence. □

Chapter 4

Hom-Entwining Structures And Hom-Hopf-Type Modules

4.1 Introduction

Motivated by the study of symmetry properties of noncommutative principal bundles, *entwining structures* (over a commutative ring k with unit) were introduced in [11] as a triple $(A,C)_\psi$ consisting of a k-algebra A, a k-coalgebra C and a k-module map $\psi : C\otimes A \to A\otimes C$ satisfying, for all $a, a' \in A$ and $c \in C$,

$$(aa')_\kappa \otimes c^\kappa = a_\kappa a'_\lambda \otimes c^{\kappa\lambda}\,,\ 1_\kappa \otimes c^\kappa = 1\otimes c,$$

$$a_\kappa \otimes c^\kappa{}_1 \otimes c^\kappa{}_2 = a_{\lambda\kappa} \otimes c_1{}^\kappa \otimes c_2{}^\lambda\,,\ a_\kappa \varepsilon(c^\kappa) = a\varepsilon(c),$$

where the notation $\psi(c\otimes a) = a_\kappa \otimes c^\kappa$ (summation over κ is understood) is used. Given an entwining structure $(A,C)_\psi$, the notion of $(A,C)_\psi$-entwined module M was first defined in [12] as a right A-module with action $m\otimes a \mapsto m\cdot a$ and a right C-comodule with coaction $\rho^M : m \mapsto m_{(0)} \otimes m_{(1)}$ (summation understood) such that the following compatibility condition holds:

$$\rho^M(m\cdot a) = m_{(0)}\cdot a_\kappa \otimes m_{(1)}{}^\kappa\,,\ \forall a\in A, m\in M.$$

Hopf-type modules are typically the objects with an action of an algebra and a coaction of a coalgebra which satisfy some compatibility condition. The family of Hopf-type modules includes well known examples such as Hopf modules of Sweedler [75], relative Hopf modules of Doi and Takeuchi [35], [77], Long dimodules [59], Yetter-Drinfeld modules [71], [95], Doi-Koppinen Hopf modules [36], [52] and alternative Doi-Koppinen Hopf modules of Schauenburg [73]. All these modules except alternative Doi-Koppinen modules are special cases of Doi-Koppinen modules. As newer special cases of them, the family of Hopf-type modules also includes anti-Yetter-Drinfeld modules which were obtained as coefficients for the cyclic cohomology of Hopf algebras [40], [41], [44], and their generalizations termed (α,β)-Yetter-Drinfeld modules [69] (also called (α,β)-equivariant C-comodules in [50]). Basically, entwining structures and modules associated to them enable us to unify several categories of Hopf modules in the sense that the compatibility conditions for all of them can be restated in the form of the above condition required for entwined modules. One can refer to [14] and [20] for more information on the relationship between entwining structures and Hopf-type modules.

Entwining structures have been generalized to weak entwining structures in [19] to define Doi-Koppinen data for a weak Hopf algebra, motivated by [9]. Thereafter, it has been proven in [13] that both entwined modules and weak entwined modules are comodules of certain type of corings which built on a tensor product of an algebra and a coalgebra, and shown that various properties of entwined modules can be obtained from properties of comodules of a coring. Here we recall from [76] that for an associative algebra A with unit, an A-*coring* is an A-bimodule $\mathcal{C}$ with A-bilinear maps $\Delta_{\mathcal{C}}:\mathcal{C}\to\mathcal{C}\otimes_A\mathcal{C},\ c\mapsto c_1\otimes c_2$ called coproduct and $\varepsilon_{\mathcal{C}}:\mathcal{C}\to A$ called counit, such that

$$\Delta_{\mathcal{C}}(c_1)\otimes c_2=c_1\otimes\Delta_{\mathcal{C}}(c_2),\ \varepsilon_{\mathcal{C}}(c_1)c_2=c=c_1\varepsilon_{\mathcal{C}}(c_2),\ \forall c\in\mathcal{C}.$$

Given an A-coring $\mathcal{C}$, a *right* $\mathcal{C}$*-comodule* is a right A-module M with a right A-linear map $\rho^M:M\to M\otimes\mathcal{C},\ m\mapsto m_{(0)}\otimes m_{(1)}$ called coaction, such that

$$\rho^M(m_{(0)})\otimes m_{(1)}=m_{(0)}\otimes\Delta_{\mathcal{C}}(m_{(1)}),\ m=m_{(0)}\varepsilon_{\mathcal{C}}(m_{(1)}),\ \forall m\in M.$$

The main aim of the present chapter is to generalize the entwining structures, entwined modules and the associated corings within the framework of monoidal Hom-structures and then to study Hopf-type modules in the Hom-setting. The idea is to replace algebra and coalgebra in a classical entwining structure with a monoidal Hom-algebra and a monoidal Hom-coalgebra to make a definition of Hom-entwining structures and associated entwined Hom-modules. Following [13], these entwined Hom-modules are identified with Hom-comodules of the associated Hom-coring. The dual algebra of this Hom-coring is proven to be the Koppinen smash. Furthermore, we give a construction regarding Hom-Doi-Kopinen datum and Doi-Koppinen Hom-Hopf modules as special cases of Hom-entwining structures and associated entwined Hom-modules. Besides, we introduce alternative Hom-Doi-Koppinen datum. By using these constructions, we get Hom-versions of the aforementioned Hopf-type modules as special cases of entwined Hom-modules, and give examples of Hom-corings in addition to trivial Hom-coring and canonical Hom-coring.

4.2 Hom-corings and Hom-Entwining structures

Definition 4.2.1 *1. Let (A,α) be a monoidal Hom-algebra. An (A,α)-Hom-coring consists of an (A,α)-Hom-bimodule $(\mathcal{C},\chi)$ together with (A,α)-bilinear maps $\Delta_{\mathcal{C}}: \mathcal{C}\to\mathcal{C}\otimes_A\mathcal{C}$, $c\mapsto c_1\otimes c_2$ and $\varepsilon_{\mathcal{C}}:\mathcal{C}\to A$ called comultiplication and counit such that*

$$\chi^{-1}(c_1)\otimes\Delta_{\mathcal{C}}(c_2)=c_{11}\otimes(c_{12}\otimes\chi^{-1}(c_2));\ \ \varepsilon_{\mathcal{C}}(c_1)c_2=c=c_1\varepsilon_{\mathcal{C}}(c_2);\ \ \varepsilon_{\mathcal{C}}(\chi(c))=\alpha(\varepsilon_{\mathcal{C}}(c)). \tag{4.1}$$

For any $c\in\mathcal{C}$, the equality

$$\Delta_{\mathcal{C}}(\chi(c))=\chi(c_1)\otimes\chi(c_2) \tag{4.2}$$

is a consequence of (4.1) in a similar manner as in the Remark (3) of Chapter 3.

2. A right $(\mathcal{C},\chi)$-Hom-comodule (M,μ) is defined as a right (A,α)-Hom-module with a right A-linear map $\rho: M\to M\otimes_A\mathcal{C}$, $m\mapsto m_{(0)}\otimes m_{(1)}$ satisfying

$$\mu^{-1}(m_{(0)})\otimes\Delta_{\mathcal{C}}(m_{(1)})=m_{(0)(0)}\otimes(m_{(0)(1)}\otimes\chi^{-1}(m_{(1)}));\ \ m=m_{(0)}\varepsilon_{\mathcal{C}}(m_{(1)}). \tag{4.3}$$

The equation

$$\mu(m)_{(0)}\otimes\mu(m)_{(1)}=\mu(m_{(0)})\otimes\chi(m_{(1)}) \tag{4.4}$$

can be obtained in the same way as Hom-comodule setting over a monoidal Hom-coalgebra.

Theorem 4.2.2 *Let $\phi:(A,\alpha)\to(B,\beta)$ be a morphism of monoidal Hom-algebras. Then, for an (A,α)-Hom-coring $(\mathcal{C},\chi)$, $(B\mathcal{C})B=((B\otimes_A\mathcal{C})\otimes_A B,(\beta\otimes\chi)\otimes\beta)$ is a (B,β)-Hom-coring, called a base ring extension of the (A,α)-Hom-coring $(\mathcal{C},\chi)$, with a comultiplication and a counit,*

$$\Delta_{(B\mathcal{C})B}((b\otimes_A c)\otimes_A b') = ((\beta^{-1}(b)\otimes_A c_1)\otimes_A 1_B)\otimes_B ((1_B\otimes_A c_2)\otimes_A \beta^{-1}(b')), \tag{4.5}$$

$$\varepsilon_{(B\mathcal{C})B}((b\otimes_A c)\otimes_A b') = (b\phi(\varepsilon_{\mathcal{C}}(c)))b'. \tag{4.6}$$

Proof: For $b, b', b'' \in B$ and $c \in \mathcal{C}$,

$$\begin{aligned}
&\Delta_{(B\mathcal{C})B}(((b'\otimes_A c)\otimes_A b'')b) \\
&= \Delta_{(B\mathcal{C})B}((\beta(b')\otimes_A \chi(c))\otimes_A b''\beta^{-1}(b)) \\
&= ((b'\otimes_A \chi(c)_1)\otimes_A 1_B)\otimes_B ((1_B\otimes_A \chi(c)_2)\otimes_A \beta^{-1}(b''\beta^{-1}(b))) \\
&\overset{(4.2)}{=} ((b'\otimes_A \chi(c_1))\otimes_A 1_B)\otimes_B ((1_B\otimes_A \chi(c_2))\otimes_A \beta^{-1}(b'')\beta^{-2}(b)) \\
&= ((b'\otimes_A \chi(c_1))\otimes_A 1_B)\otimes_B ((\beta\otimes\chi)(1_B\otimes_A c_2)\otimes_A \beta^{-1}(b'')\beta^{-1}(\beta^{-1}(b))) \\
&= ((\beta\otimes\chi)\otimes\beta)((\beta^{-1}(b')\otimes_A c_1)\otimes_A 1_B)\otimes_B ((1_B\otimes_A c_2)\otimes_A \beta^{-1}(b''))\beta^{-1}(b) \\
&= \Delta_{(B\mathcal{C})B}((b'\otimes_A c)\otimes_A b'')b,
\end{aligned}$$

which proves the right (B,β)-linearity of $\Delta_{(B\mathcal{C})B}$. It can also be shown that $\Delta_{(B\mathcal{C})B}\circ\bar{\chi} = (\bar{\chi}\otimes\bar{\chi})\circ\Delta_{(B\mathcal{C})B}$, where $\bar{\chi} = (\beta\otimes\chi)\otimes\beta$. And as well, the left (B,β)-linearity of $\Delta_{(B\mathcal{C})B}$ and the fact that it preserves the compatibility condition between the left and right (B,β)-Hom-actions on $(B\mathcal{C})B$ can be checked similarly, that is,

$$\Delta_{(B\mathcal{C})B}(b((b'\otimes_A c)\otimes_A b'')) = b\Delta_{(B\mathcal{C})B}((b'\otimes_A c)\otimes_A b''),$$

$$(b\Delta_{(B\mathcal{C})B}((b''\otimes_A c)\otimes_A b'''))\beta(b') = \beta(b)(\Delta_{(B\mathcal{C})B}((b''\otimes_A c)\otimes_A b''')b').$$

Next we prove the Hom-coassociativity of $\Delta_{(B\mathcal{C})B}$:

$$
\begin{aligned}
&((\beta^{-1}\otimes\chi^{-1})\otimes\beta^{-1})(((b\otimes_A c)\otimes_A b')_1)\otimes_B \Delta_{(BC)B}(((b\otimes_A c)\otimes_A b')_2)\\
&\quad= ((\beta^{-2}(b)\otimes_A \chi^{-1}(c_1))\otimes_A 1_B)\otimes_B (((1_B\otimes_A c_{21})\otimes_A 1_B)\\
&\qquad \otimes_B((1_B\otimes_A c_{22})\otimes_A \beta^{-2}(b')))\\
&\quad\overset{(4.1)}{=} ((\beta^{-2}(b)\otimes_A c_{11})\otimes_A 1_B)\otimes_B (((1_B\otimes_A c_{12})\otimes_A 1_B)\\
&\qquad \otimes_B((1_B\otimes_A \chi^{-1}(c_2))\otimes_A \beta^{-2}(b')))\\
&\quad= ((\beta^{-1}(b)\otimes_A c_1)\otimes_A 1_B)_1\otimes_B (((\beta^{-1}(b)\otimes_A c_1)\otimes_A 1_B)_2\\
&\qquad \otimes_B((1_B\otimes_A \chi^{-1}(c_2))\otimes_A \beta^{-2}(b')))\\
&\quad= ((b\otimes_A c)\otimes_A b')_{11}\otimes_B (((b\otimes_A c)\otimes_A b')_{12}\\
&\qquad \otimes_B((\beta^{-1}\otimes\chi^{-1})\otimes\beta^{-1})((b\otimes_A c)\otimes_A b')_2).
\end{aligned}
$$

Now we demonstrate that $\varepsilon_{(BC)B}$ is left (B,β)-linear:

$$
\begin{aligned}
&\varepsilon_{(BC)B}(b((b'\otimes_A c)\otimes_A b''))\\
&\quad= \varepsilon_{(BC)B}((\beta^{-2}(b)b'\otimes_A \chi(c))\otimes_A \beta(b''))\\
&\quad= ((\beta^{-2}(b)b')\phi(\varepsilon_{\mathcal{C}}(\chi(c))))\beta(b'') \overset{(4.2)}{=} ((\beta^{-2}(b)b')\phi(\alpha(\varepsilon_{\mathcal{C}}(c))))\beta(b'')\\
&\quad= (\beta^{-1}(b)(b'\beta^{-1}(\phi(\alpha(\varepsilon_{\mathcal{C}}(c))))))\beta(b'') = (\beta^{-1}(b)(b'\phi(\varepsilon_{\mathcal{C}}(c))))\beta(b'')\\
&\quad= b((b'\phi(\varepsilon_{\mathcal{C}}(c)))b'') = b\varepsilon_{(BC)B}((b'\otimes_A c)\otimes_A b''),
\end{aligned}
$$

where $\phi\circ\alpha=\beta\circ\phi$ was used in the fifth equality. Additionally, we have

$$
\begin{aligned}
(\varepsilon_{(BC)B}\circ\bar{\chi})((b\otimes_A c)\otimes_A b') &= (\beta(b)\phi(\varepsilon_{\mathcal{C}}(\chi(c))))\beta(b')\\
&= \beta((b\phi(\varepsilon_{\mathcal{C}}(c)))b') = (\beta\circ\varepsilon_{\mathcal{C}})((b\otimes_A c)\otimes_A b'),
\end{aligned}
$$

meaning $\varepsilon_{(BC)B}\in\widetilde{\mathcal{H}}(\mathcal{M}_k)$. In the same manner, one can show that $\varepsilon_{(BC)B}$ is right (B,β)-linear and it preserves the compatibility condition between the left and right (B,β)-Hom-actions on $(B\mathcal{C})B$, i.e.,

$$
\varepsilon_{(BC)B}(((b'\otimes_A c)\otimes_A b'')b) = \varepsilon_{(BC)B}((b'\otimes_A c)\otimes_A b'')b,
$$

$$
(b\varepsilon_{(BC)B}((b''\otimes_A c)\otimes_A b'''))\beta(b') = \beta(b)(\varepsilon_{(BC)B}((b''\otimes_A c)\otimes_A b''')b').
$$

Below, we prove the counity condition:

$$
\begin{aligned}
((\beta^{-1}(b)\otimes_A c_1)\otimes_A 1_B)\varepsilon_{(BC)B}((1_B\otimes_A c_2)\otimes_A \beta^{-1}(b')) \\
&= ((\beta^{-1}(b)\otimes_A c_1)\otimes_A 1_B)((1_B\phi(\varepsilon_{\mathcal{C}}(c_2)))\beta^{-1}(b')) \\
&= ((\beta^{-1}(b)\otimes_A c_1)\otimes_A 1_B)(\beta(\phi(\varepsilon_{\mathcal{C}}(c_2)))\beta^{-1}(b')) \\
&= (b\otimes_A \chi(c_1))\otimes_A \beta(\phi(\varepsilon_{\mathcal{C}}(c_2)))\beta^{-1}(b') \\
&= (b\otimes_A \chi(c_1))\otimes_A \phi(\alpha(\varepsilon_{\mathcal{C}}(c_2)))\beta^{-1}(b') \\
&\overset{(3.22)}{=} (\beta^{-1}(b)\otimes c_1)\alpha(\varepsilon_{\mathcal{C}}(c_2))\otimes_A b' \\
&= (b\otimes_A c_1\varepsilon_{\mathcal{C}}(c_2))\otimes_A b' \\
&\overset{(4.2)}{=} (b\otimes_A c)\otimes_A b' \\
&\overset{(4.2)}{=} (b\otimes_A \varepsilon_{\mathcal{C}}(c_1)c_2)\otimes_A b' \\
&\overset{(3.22)}{=} (\beta^{-1}(b)\phi(\varepsilon_{\mathcal{C}}(c_1))\otimes_A \chi(c_2))\otimes_A b' \\
&= (\beta^{-2}(b\phi(\alpha(\varepsilon_{\mathcal{C}}(c_1))))1_B\otimes_A \chi(c_2))\otimes_A \beta(\beta^{-1}(b')) \\
&= (b\phi(\alpha(\varepsilon_{\mathcal{C}}(c_1))))((1_B\otimes_A c_2)\otimes_A \beta^{-1}(b')) \\
&= ((\beta^{-1}(b)\phi(\varepsilon_{\mathcal{C}}(c_1)))1_B)((1_B\otimes_A c_2)\otimes_A \beta^{-1}(b')) \\
&= \varepsilon_{(BC)B}((\beta^{-1}(b)\otimes_A c_1)\otimes_A 1_B)((1_B\otimes_A c_2)\otimes_A \beta^{-1}(b')),
\end{aligned}
$$

which completes the proof that given a morphism of monoidal Hom-algebras $\phi:(A,\alpha)\to(B,\beta)$, $((B\otimes_A \mathcal{C})\otimes_A B,(\beta\otimes\chi)\otimes\beta)$ is a (B,β)-Hom-coring. □

Example 4.2.3 *A monoidal Hom-algebra (A,α) has a natural (A,α)-Hom-bimodule structure with its Hom-multiplication. (A,α) is an (A,α)-Hom-coring by the canonical isomorphism $A\to A\otimes_A A$, $a\mapsto \alpha^{-1}(a)\otimes 1_A$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, as a comultiplication and the identity $A\to A$ as a counit. This Hom-coring is called a trivial (A,α)-Hom-coring.*

Example 4.2.4 *Let $\phi:(B,\beta)\to(A,\alpha)$ be a morphism of monoidal Hom-algebras. Then $(\mathcal{C},\chi)=(A\otimes_B A,\alpha\otimes\alpha)$ is an (A,α)-Hom-coring with comultiplication*

$$\Delta_{\mathcal{C}}(a\otimes_B a')=(\alpha^{-1}(a)\otimes_B 1_A)\otimes_A(1_A\otimes_B \alpha^{-1}(a'))=(\alpha^{-1}(a)\otimes_B 1_A)\otimes_B a'$$

and counit

$$\varepsilon_{\mathcal{C}}(a\otimes_B a')=aa'.$$

Proof: By Theorem (4.2.2), for $\phi:(B,\beta)\to(A,\alpha)$ and the trivial (B,β)-Hom-coring (B,β) with $\Delta_B(b)=\beta^{-1}(b)\otimes_B 1_B$ and $\varepsilon_B(b)=b$, we have the base ring extension of the trivial (B,β)-Hom-coring (B,β) to (A,α)-Hom-coring $(AB)A=((A\otimes_B B)\otimes_B A,(\alpha\otimes\beta)\otimes\alpha)$ with

$$\Delta_{(AB)A}((a\otimes_B b)\otimes_B a')=((\alpha^{-1}(a)\otimes_B\beta^{-1}(b))\otimes_B 1_A)\otimes_A((1_A\otimes_B 1_B)\otimes_B\alpha^{-1}(a')),$$

$$\varepsilon_{(AB)A}((a\otimes_B b)\otimes_B a')=(a\phi(b))a'.$$

On the other hand we have the isomorphism $\varphi:A\to A\otimes_B B,\ a\mapsto\alpha^{-1}(a)\otimes_B 1_B$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, with the inverse $\psi:A\otimes_B B\to A,\ a\otimes_B b\mapsto a\phi(b)$: For $a\in A$ and $b\in B$,

$$\psi(\varphi(a))=\alpha^{-1}(a)\phi(1_B)=\alpha^{-1}(a)1_A=a,$$

$$\begin{aligned}\varphi(\psi(a\otimes_B b)) &= \varphi(a\phi(b))=\alpha^{-1}(a\phi(b))\otimes_B 1_B\\ &= \alpha^{-1}(a)\alpha^{-1}(\phi(b))\otimes_B 1_B=\alpha^{-1}(a)\phi(\beta^{-1}(b))\otimes_B 1_B\\ &= a\otimes_B\beta^{-1}(b)1_B=a\otimes_B b,\end{aligned}$$

in addition one can check that $\alpha\circ\psi=\psi\circ(\alpha\otimes\beta)$ and $(\alpha\otimes\beta)\circ\varphi=\varphi\circ\alpha$. Thus, $(AB)A\overset{\psi\otimes 1}{\simeq}A\otimes_B A=\mathcal{C}$ and

$$\Delta_{\mathcal{C}}(a\otimes_B b)=((\psi\otimes id)\otimes(\psi\otimes id))\circ\Delta_{(AB)A}\circ(\varphi\otimes id)(a\otimes_B b)=(\alpha^{-1}(a)\otimes_B 1_A)\otimes_A(1_A\otimes_B\alpha^{-1}(a')),$$

$$\varepsilon_{\mathcal{C}}(a\otimes_B a')=\varepsilon_{(AB)A}\circ(\varphi\otimes id)(a\otimes_B a')=aa'.$$

$(A\otimes_B A,\alpha\otimes\alpha)$ is called the *Sweedler* or *canonical* (A,α)*-Hom-coring* associated to a monoidal Hom-algebra extension $\phi:(B,\beta)\to(A,\alpha)$. □

For the monoidal Hom-algebra (A,α) and the (A,α)-Hom-coring $(\mathcal{C},\chi)$, let us put ${}^*\mathcal{C}={}_A Hom^{\mathcal{H}}(\mathcal{C},A)$, consisting of left (A,α)-linear morphisms $f:(\mathcal{C},\chi)\to(A,\alpha)$, that is, $f(ac)=af(c)$ for $a\in A$, $c\in\mathcal{C}$ and $f\circ\chi=\alpha\circ f$. Similarly, $\mathcal{C}^*=Hom_A^{\mathcal{H}}(\mathcal{C},A)$ and ${}^*\mathcal{C}^*={}_A Hom_A^{\mathcal{H}}(\mathcal{C},A)$ consist of right (A,α)-Hom-module maps and (A,α)-Hom-bimodule maps, respectively. Now we prove that these modules of (A,α)-linear morphisms $\mathcal{C}\to A$ have ring structures.

Proposition 4.2.5 *1. ${}^*\mathcal{C}$ is an associative algebra with unit $\varepsilon_{\mathcal{C}}$ and multiplication*

$$(f *^l g)(c) = f(c_1 g(c_2))$$

for $f, g \in {}^*\mathcal{C}$ and $c \in \mathcal{C}$.

2. $\mathcal{C}^*$ is an associative algebra with unit $\varepsilon_{\mathcal{C}}$ and multiplication

$$(f *^r g)(c) = g(f(c_1)c_2)$$

for $f, g \in \mathcal{C}^*$ and $c \in \mathcal{C}$.

3. ${}^*\mathcal{C}^*$ is an associative algebra with unit $\varepsilon_{\mathcal{C}}$ and multiplication

$$(f * g)(c) = f(c_1)g(c_2)$$

for $f, g \in {}^*\mathcal{C}^*$ and $c \in \mathcal{C}$.

Proof:

1. For $f, g, h \in {}^*\mathcal{C}$ and $c \in \mathcal{C}$,

$$\begin{aligned}((f *^l g) *^l h)(c) &= f((c_1 h(c_2))_1 g((c_1 h(c_2))_2)) = f(\chi(c_{11}) g(c_{12}\alpha^{-1}(h(c_2))))\\ &= f(\chi(c_{11}) g(c_{12} h(\chi^{-1}(c_2)))) \overset{(4.1)}{=} f(c_1 g(c_{21} h(c_{22})))\\ &= (f *^l (g *^l h))(c),\end{aligned}$$

where the second equality comes from the fact that $\Delta_{\mathcal{C}}$ is right (A,α)-linear, i.e., $\Delta_{\mathcal{C}}(ca) = (ca)_1 \otimes_A (ca)_2 = \Delta_{\mathcal{C}}(c)a = (c_1 \otimes_A c_2)a = \chi(c_1) \otimes_A c_2\alpha^{-1}(a)$, $\forall c \in \mathcal{C}$, $a \in A$.

$$(f *^l \varepsilon_{\mathcal{C}})(c) = f(c_1 \varepsilon_{\mathcal{C}}(c_2)) = f(c),$$

$$(\varepsilon_{\mathcal{C}} *^l f)(c) = \varepsilon_{\mathcal{C}}(c_1 f(c_2)) = \varepsilon_{\mathcal{C}}(c_1) f(c_2) = f(\varepsilon_{\mathcal{C}}(c_1)c_2) = f(c).$$

By similar computations one can prove (2) and (3).

□

Definition 4.2.6 *A (right-right) Hom-entwining structure is a triple* $[(A,\alpha),(C,\gamma)]_\psi$ *consisting of a monoidal Hom-algebra* (A,α)*, a monoidal Hom-coalgebra* (C,γ) *and a* k*-linear map* $\psi: C\otimes A \to A\otimes C$ *satisfying the following conditions for all* $a,a'\in A$, $c\in C$*:*

$$(aa')_\kappa \otimes \gamma(c)^\kappa = a_\kappa a'_\lambda \otimes \gamma(c^{\kappa\lambda}), \tag{4.7}$$

$$\alpha^{-1}(a_\kappa)\otimes c^\kappa{}_1 \otimes c^\kappa{}_2 = \alpha^{-1}(a)_{\kappa\lambda}\otimes c_1{}^\lambda \otimes c_2{}^\kappa, \tag{4.8}$$

$$1_\kappa \otimes c^\kappa = 1\otimes c, \tag{4.9}$$

$$a_\kappa \varepsilon(c^\kappa) = a\varepsilon(c), \tag{4.10}$$

where we have used the notation $\psi(c\otimes a) = a_\kappa \otimes c^\kappa$, $a\in A$, $c\in C$, *for the so-called entwining map* ψ*. It is said that* (C,γ) *and* (A,α) *are entwined by* ψ*.* ψ *is in* $\widetilde{\mathcal{H}}(\mathcal{M}_k)$*, that is, the relation holds:*

$$\alpha(a)_\kappa \otimes \gamma(c)^\kappa = \alpha(a_\kappa)\otimes\gamma(c^\kappa), \tag{4.11}$$

which follows from (4.7), (4.9) and Hom-unity of (A,α)*:*

$$\begin{aligned} \alpha(a_\kappa)\otimes\gamma(c^\kappa) &= a_\kappa 1\otimes\gamma(c^\kappa) = a_\kappa 1_\lambda \otimes \gamma(c^{\kappa\lambda}) \\ &= (a1)_\kappa \otimes \gamma(c)^\kappa = \alpha(a)_\kappa \otimes \gamma(c)^\kappa. \end{aligned}$$

It can also be obtained from (4.8) and (4.10).

Definition 4.2.7 *A* $[(A,\alpha),(C,\gamma)]_\psi$*-entwined Hom-module is an object* $(M,\mu)\in\widetilde{\mathcal{H}}(\mathcal{M}_k)$ *which is a right* (A,α)*-Hom-module with action* $\rho_M: M\otimes A\to M$, $m\otimes a\mapsto ma$ *and a right* (C,γ)*-Hom-comodule with coaction* $\rho^M: M\to M\otimes C$, $m\mapsto m_{(0)}\otimes m_{(1)}$ *fulfilling the condition, for all* $m\in M$, $a\in A$,

$$\rho^M(ma) = m_{(0)}\alpha^{-1}(a)_\kappa \otimes \gamma(m_{(1)}{}^\kappa). \tag{4.12}$$

By $\widetilde{\mathcal{M}}_A^C(\psi)$, we denote the category of $[(A,\alpha),(C,\gamma)]_\psi$-entwined Hom-modules together with the morphisms in which are both right (A,α)-linear and right (C,γ)-colinear.

With the following theorem, we construct a Hom-coring associated to an entwining Hom-structure and show an identification of entwined Hom-modules with Hom-comodules of this Hom-coring, pursuing the Proposition 2.2 in [13].

Theorem 4.2.8 *Let (A,α) be a monoidal Hom-algebra and (C,γ) be a monoidal Hom-coalgebra.*

1. *For a Hom-entwining structure $[(A,\alpha),(C,\gamma)]_\psi$, $(A\otimes C,\alpha\otimes\gamma)$ is an (A,α)-Hom-bimodule with a left Hom-module structure $a(a'\otimes c)=\alpha^{-1}(a)a'\otimes\gamma(c)$ and a right Hom-module structure $(a'\otimes c)a=a'\alpha^{-1}(a)_\kappa\otimes\gamma(c^\kappa)$, for all $a,a'\in A$, $c\in C$. Furthermore, $(\mathcal{C},\chi)=(A\otimes C,\alpha\otimes\gamma)$ is an (A,α)-Hom-coring with the comultiplication and counit*

$$\Delta_{\mathcal{C}}:\mathcal{C}\to\mathcal{C}\otimes_A\mathcal{C},\ a\otimes c\mapsto(\alpha^{-1}(a)\otimes c_1)\otimes_A(1\otimes c_2), \tag{4.13}$$

$$\varepsilon_{\mathcal{C}}:\mathcal{C}\to A,\ a\otimes c\mapsto\alpha(a)\varepsilon(c). \tag{4.14}$$

2. *If $\mathcal{C}=(A\otimes C,\alpha\otimes\gamma)$ is an (A,α)-Hom-coring with the comultiplication and counit given above, then $[(A,\alpha),(C,\gamma)]_\psi$ is a Hom-entwining structure, where*

$$\psi:C\otimes A\to A\otimes C,\ c\otimes a\mapsto(1\otimes\gamma^{-1}(c))a.$$

3. *Let $(\mathcal{C},\chi)=(A\otimes C,\alpha\otimes\gamma)$ be the (A,α)-Hom-coring associated to $[(A,\alpha),(C,\gamma)]_\psi$ as in (1). Then the category of $[(A,\alpha),(C,\gamma)]_\psi$-entwined Hom-modules is isomorphic to the category of right $(\mathcal{C},\chi)$-Hom-comodules.*

Proof:

1. We first show that the right Hom-action of (A,α) on $(A\otimes C,\alpha\otimes\gamma)$ is Hom-associative and Hom-unital, for all $a,d,e\in A$ and $c\in C$:

$$\begin{aligned}
(\alpha(a)\otimes\gamma(c))(de) &= \alpha(a)\alpha^{-1}(de)_\kappa\otimes\gamma(\gamma(c)^\kappa)\\
&= \alpha(a)(\alpha^{-1}(d)\alpha^{-1}(e))_\kappa\otimes\gamma(\gamma(c)^\kappa)\\
&\overset{(4.7)}{=} \alpha(a)(\alpha^{-1}(d)_\kappa\alpha^{-1}(e)_\lambda)\otimes\gamma^2(c^{\kappa\lambda})\\
&= (a\alpha^{-1}(d)_\kappa)\alpha(\alpha^{-1}(e)_\lambda)\otimes\gamma(\gamma(c^{\kappa\lambda}))\\
&\overset{(4.11)}{=} (a\alpha^{-1}(d)_\kappa)\alpha(\alpha^{-1}(e))_\lambda\otimes\gamma(\gamma(c^\kappa)^\lambda)\\
&= (a\alpha^{-1}(d)_\kappa\otimes\gamma(c^\kappa))\alpha(e)\\
&= ((a\otimes c)d)\alpha(e),
\end{aligned}$$

$$
\begin{aligned}
(a\otimes c)1 &= a\alpha^{-1}(1)_\kappa \otimes \gamma(c^\kappa) = a1_\kappa \otimes \gamma(c^\kappa)\\
&= \alpha^{-1}(\alpha(a))1_\kappa \otimes \gamma(c^\kappa) = \alpha(a)(1_\kappa \otimes c^\kappa)\\
&\overset{(4.9)}{=} \alpha(a)(1\otimes c) = a1\otimes\gamma(c)\\
&= (\alpha\otimes\gamma)(a\otimes c).
\end{aligned}
$$

One can also show that the left Hom-action, too, satisfies the Hom-associativity and Hom-unity. For any $a, b, d \in A$ and $c \in C$,

$$
\begin{aligned}
(b(a\otimes c))\alpha(d) &= (\alpha^{-1}(b)a\otimes\gamma(c))\alpha(d) = (\alpha^{-1}(b)a)\alpha^{-1}(\alpha(d))_\kappa \otimes \gamma(\gamma(c)^\kappa)\\
&= (\alpha^{-1}(b)a)\alpha(\alpha^{-1}(d))_\kappa \otimes \gamma(\gamma(c)^\kappa) \overset{(4.11)}{=} (\alpha^{-1}(b)a)\alpha(\alpha^{-1}(d)_\kappa)\otimes\gamma^2(c^\kappa)\\
&= b(a\alpha^{-1}(d)_\kappa)\otimes\gamma^2(c^\kappa) = \alpha^{-1}(\alpha(b))(a\alpha^{-1}(d)_\kappa)\otimes\gamma(\gamma(c^\kappa))\\
&= \alpha(b)(a\alpha^{-1}(d)_\kappa\otimes\gamma(c^\kappa)) = \alpha(b)((a\otimes c)d),
\end{aligned}
$$

proves the compatibility condition between left and right (A,α)-Hom-actions.

First, it can easily be proven that the morphisms $A\otimes(\mathcal{C}\otimes_A\mathcal{C})\to\mathcal{C}\otimes_A\mathcal{C}$,

$$
a\otimes((a'\otimes c)\otimes_A(a''\otimes c'))\mapsto \alpha^{-1}(a)(a'\otimes c)\otimes_A(\alpha(a'')\otimes\gamma(c')) \tag{4.15}
$$

and $(\mathcal{C}\otimes_A\mathcal{C})\otimes A\to\mathcal{C}\otimes_A\mathcal{C}$,

$$
((a'\otimes c)\otimes_A(a''\otimes c'))a\mapsto(\alpha(a')\otimes\gamma(c))\otimes_A(a''\otimes c')\alpha^{-1}(a) \tag{4.16}
$$

define a left Hom-action and a right Hom-action of (A,α) on $(\mathcal{C}\otimes_A\mathcal{C},\chi\otimes\chi)$, respectively. Next it is shown that the comultiplication $\Delta_\mathcal{C}$ is (A,α)-bilinear, that is, $\Delta_\mathcal{C}$ preserves the left and right (A,α)-Hom-actions and the compatibility condition between them as follows: Let $a, a', b, d \in A$ and $c \in C$, then we have the following computations

$$\begin{aligned}
\Delta_{\mathcal{C}}(a(a'\otimes c)) &= (\alpha^{-1}(\alpha^{-1}(a)a')\otimes\gamma(c)_1)\otimes_A(1\otimes\gamma(c)_2)\\
&\overset{(3.6)}{=} (\alpha^{-2}(a)\alpha^{-1}(a')\otimes\gamma(c_1))\otimes_A(1\otimes\gamma(c_2))\\
&= \alpha^{-1}(a)(\alpha^{-1}(a')\otimes c_1)\otimes_A(\alpha(1)\otimes\gamma(c_2))\\
&\overset{(4.15)}{=} a((\alpha^{-1}(a')\otimes c_1)\otimes_A(1\otimes c_2))\\
&= a\Delta_{\mathcal{C}}(a'\otimes c),
\end{aligned}$$

$$\begin{aligned}
\Delta_{\mathcal{C}}((a'\otimes c)a) &= \Delta_{\mathcal{C}}(a'\alpha^{-1}(a)_\kappa\otimes\gamma(c^\kappa))\\
&= (\alpha^{-1}(a'\alpha^{-1}(a)_\kappa)\otimes\gamma(c^\kappa)_1)\otimes_A(1\otimes\gamma(c^\kappa)_2)\\
&\overset{(3.6)}{=} (\alpha^{-1}(a')\alpha^{-1}(\alpha^{-1}(a)_\kappa)\otimes\gamma(c^\kappa{}_1))\otimes_A(1\otimes\gamma(c^\kappa{}_2))\\
&\overset{(4.8)}{=} (\alpha^{-1}(a')\alpha^{-2}(a)_{\kappa\lambda}\otimes\gamma(c_1{}^\lambda))\otimes_A(1\otimes\gamma(c_2{}^\kappa))\\
&= (\alpha^{-1}(a')\otimes c_1)\alpha(\alpha^{-2}(a)_\kappa)\otimes_A(1\otimes\gamma(c_2{}^\kappa))\\
&\overset{(3.22)}{=} (a'\otimes\gamma(c_1))\otimes_A\alpha(\alpha^{-2}(a)_\kappa)(1\otimes c_2{}^\kappa)\\
&= (a'\otimes\gamma(c_1))\otimes_A(\alpha(\alpha^{-2}(a)_\kappa)\otimes\gamma(c_2{}^\kappa))\\
&= (a'\otimes\gamma(c_1))\otimes_A(1\alpha^{-1}(\alpha^{-1}(a))_\kappa\otimes\gamma(c_2{}^\kappa))\\
&= (a'\otimes\gamma(c_1))\otimes_A(1\otimes c_2)\alpha^{-1}(a)\\
&\overset{(4.16)}{=} ((\alpha^{-1}(a')\otimes c_1)\otimes_A(1\otimes c_2))a\\
&= \Delta_{\mathcal{C}}(a'\otimes c)a,
\end{aligned}$$

$$
\begin{aligned}
\alpha(b)(\Delta_{\mathcal{C}}(a\otimes c)d) &= \alpha(b)(((\alpha^{-1}(a)\otimes c_1)\otimes_A(1\otimes c_2))d)\\
&= \alpha(b)((a\otimes\gamma(c_1))\otimes_A(1\otimes c_2)\alpha^{-1}(d))\\
&\overset{(4.16)}{=} \alpha(b)((a\otimes\gamma(c_1))\otimes_A(1\alpha^{-1}(\alpha^{-1}(d))_\kappa\otimes\gamma(c_2{}^\kappa)))\\
&= \alpha(b)((a\otimes\gamma(c_1))\otimes_A(\alpha(\alpha^{-2}(d)_\kappa)\otimes\gamma(c_2{}^\kappa)))\\
&\overset{(4.15)}{=} b(a\otimes\gamma(c_1))\otimes_A(\alpha^2(\alpha^{-2}(d)_\kappa)\otimes\gamma^2(c_2{}^\kappa))\\
&= (\alpha^{-1}(b)a\otimes\gamma^2(c_1))\otimes_A\alpha^2(\alpha^{-2}(d)_\kappa)(1\otimes\gamma(c_2{}^\kappa))\\
&\overset{(3.22)}{=} (\alpha^{-1}(\alpha^{-1}(b)a)\otimes\gamma(c_1))\alpha^2(\alpha^{-2}(d)_\kappa)\otimes_A(1\otimes\gamma^2(c_2{}^\kappa))\\
&= ((\alpha^{-2}(b)\alpha^{-1}(a))\alpha(\alpha^{-2}(d)_\kappa)_\lambda\otimes\gamma(\gamma(c_1)^\lambda))\otimes_A(1\otimes\gamma^2(c_2{}^\kappa))\\
&\overset{(4.11)}{=} ((\alpha^{-2}(b)\alpha^{-1}(a))\alpha(\alpha^{-2}(d)_{\kappa\lambda})\otimes\gamma(\gamma(c_1{}^\lambda)))\otimes_A(1\otimes\gamma^2(c_2{}^\kappa))\\
&= (\alpha^{-1}(b)(\alpha^{-1}(a)\alpha^{-2}(d)_{\kappa\lambda})\otimes\gamma^2(c_1{}^\lambda))\otimes_A(1\otimes\gamma^2(c_2{}^\kappa))\\
&= (\alpha^{-1}(b)(\alpha^{-1}(a)\alpha^{-1}(\alpha^{-1}(d))_{\kappa\lambda})\otimes\gamma^2(c_1{}^\lambda))\otimes_A(1\otimes\gamma^2(c_2{}^\kappa))\\
&\overset{(4.8)}{=} (\alpha^{-1}(b)(\alpha^{-1}(a)\alpha^{-1}(\alpha^{-1}(d)_\kappa))\otimes\gamma^2(c^\kappa{}_1))\otimes_A(1\otimes\gamma^2(c^\kappa{}_2))\\
&= ((\alpha^{-2}(b)\alpha^{-1}(a))\alpha^{-1}(d)_\kappa\otimes\gamma^2(c^\kappa{}_1))\otimes_A(1\otimes\gamma^2(c^\kappa{}_2))\\
&\overset{(3.6)}{=} (\alpha^{-1}((\alpha^{-1}(b)a)\alpha(\alpha^{-1}(d)_\kappa))\otimes\gamma^2(c^\kappa)_1)\otimes_A(1\otimes\gamma^2(c^\kappa)_2)\\
&= \Delta_{\mathcal{C}}((\alpha^{-1}(b)a)\alpha(\alpha^{-1}(d)_\kappa)\otimes\gamma^2(c^\kappa))\\
&= \Delta_{\mathcal{C}}(b(a\otimes c))\alpha(d).
\end{aligned}
$$

One easily checks that the counit $\varepsilon_{\mathcal{C}}$ is both left and right (A,α)-linear. For any $a,b,d\in A$ and $c\in C$ we have

$$
\begin{aligned}
\varepsilon_{\mathcal{C}}((b(a\otimes c))\alpha(d)) &= \varepsilon_{\mathcal{C}}(b(a\alpha^{-1}(d)_\kappa)\otimes\gamma^2(c^\kappa))\\
&= \alpha(b(a\alpha^{-1}(d)_\kappa))\varepsilon(\gamma^2(c^\kappa))\\
&\overset{(3.6)}{=} \alpha(b)(\alpha(a)\alpha(\alpha^{-1}(d)_\kappa))\varepsilon(c^\kappa)\\
&= \alpha(b)(\alpha(a)\alpha(\alpha^{-1}(d)_\kappa\varepsilon(c^\kappa)))\\
&\overset{(4.10)}{=} \alpha(b)(\alpha(a)\alpha(\alpha^{-1}(d)\varepsilon(c)))\\
&= \alpha(b)(\alpha(a)\varepsilon(c)d)\\
&= \alpha(b)(\varepsilon_{\mathcal{C}}(a\otimes c)d).
\end{aligned}
$$

This finishes the proof that $\varepsilon_{\mathcal{C}}$ is (A,α)-bilinear. Let us put

$$\Delta_{\mathcal{C}}(a\otimes c)=(a\otimes c)_1\otimes_A(a\otimes c)_2=(\alpha^{-1}(a)\otimes c_1)\otimes_A(1\otimes c_2).$$

Then we get the following

$$\begin{aligned}
&(\alpha^{-1}\otimes\gamma^{-1})((a\otimes c)_1)\otimes_A\Delta_{\mathcal{C}}((a\otimes c)_2)\\
&\quad= (\alpha^{-2}(a)\otimes\gamma^{-1}(c_1))\otimes_A((1\otimes c_{21})\otimes_A(1\otimes c_{22}))\\
&\quad= (\alpha^{-2}(a)\otimes c_{11})\otimes_A((1\otimes c_{12})\otimes_A(1\otimes\gamma^{-1}(c_2)))\\
&\quad= (\alpha^{-1}(a)\otimes c_1)_1\otimes_A((\alpha^{-1}(a)\otimes c_1)_2\otimes_A(1\otimes\gamma^{-1}(c_2)))\\
&\quad= (a\otimes c)_{11}\otimes_A((a\otimes c)_{12}\otimes_A(\alpha^{-1}\otimes\gamma^{-1})((a\otimes c)_2)),
\end{aligned}$$

where in the second step the Hom-coassociativity of (C,γ) is used.

$$\begin{aligned}
\varepsilon_{\mathcal{C}}((a\otimes c)_1)(a\otimes c)_2 &= \varepsilon_{\mathcal{C}}((\alpha^{-1}(a)\otimes c_1))(1\otimes c_2)\\
&= \alpha(\alpha^{-1}(a)\varepsilon(c_1))(1\otimes c_2)=a(1\otimes\varepsilon(c_1)c_2)\\
&= a(1\otimes\gamma^{-1}(c))=a\otimes c,
\end{aligned}$$

on the other hand we have

$$\begin{aligned}
(a\otimes c)_1\varepsilon_{\mathcal{C}}((a\otimes c)_2) &= (\alpha^{-1}(a)\otimes c_1)\alpha(1)\varepsilon(c_2)\\
&= (\alpha^{-1}(a)\otimes c_1\varepsilon(c_2))1\\
&= (\alpha^{-1}(a)\otimes\gamma^{-1}(c))1\\
&= a\otimes c.
\end{aligned}$$

We also show that the following relations

$$\begin{aligned}
\Delta_{\mathcal{C}}(\alpha(a)\otimes\gamma(c)) &= (\alpha^{-1}(\alpha(a))\otimes\gamma(c)_1)\otimes_A(1\otimes\gamma(c)_2)\\
&= (\alpha(\alpha^{-1}(a))\otimes\gamma(c_1))\otimes_A(\alpha(1)\otimes\gamma(c_2))\\
&= ((\alpha\otimes\gamma)\otimes(\alpha\otimes\gamma))(\Delta_{\mathcal{C}}(a\otimes c)),
\end{aligned}$$

$$
\begin{aligned}
\varepsilon_{\mathcal{C}}(\alpha(a)\otimes\gamma(c)) &= \alpha(\alpha(a))\varepsilon(\gamma(c)) \\
&= \alpha(\alpha(a))\varepsilon(c) \\
&= \alpha(\varepsilon_{\mathcal{C}}(a\otimes c))
\end{aligned}
$$

hold, which completes the proof that $(A\otimes C,\alpha\otimes\gamma)$ is an (A,α)-Hom-coring.

2. Let us denote $\psi(c\otimes a)=(1\otimes\gamma^{-1}(c))a=a_\kappa\otimes c^\kappa$. ψ is in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$:

$$
\begin{aligned}
(\alpha\otimes\gamma)(\psi(c\otimes a)) &= \alpha(a_\kappa)\otimes\gamma(c^\kappa)=(\alpha\otimes\gamma)((1\otimes\gamma^{-1}(c))a) \\
&= (\alpha(1)\otimes\gamma(\gamma^{-1}(c)))\alpha(a)=(1\otimes c)\alpha(a) \\
&= (1\otimes\gamma^{-1}(\gamma(c)))\alpha(a)=\alpha(a)_\kappa\otimes\gamma(c)^\kappa \\
&= \psi(\gamma(c)\otimes\alpha(a)),
\end{aligned}
$$

where in the third equality the fact that the right Hom-action of (A,α) on $(A\otimes C,\alpha\otimes\gamma)$ is a morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ was used. Now, let $a,a'\in A$ and $c\in C$, then

$$
\begin{aligned}
\psi(c\otimes aa') &= (aa')_\kappa\otimes c^\kappa=(1\otimes\gamma^{-1}(c))(aa') \\
&= ((\alpha^{-1}(1)\otimes\gamma^{-1}(\gamma^{-1}(c)))a)\alpha(a')=((1\otimes\gamma^{-1}(\gamma^{-1}(c)))a)\alpha(a') \\
&= (a_\kappa\otimes\gamma^{-1}(c)^\kappa)\alpha(a')=(\alpha^{-1}(a_\kappa)1\otimes\gamma(\gamma^{-1}(\gamma^{-1}(c)^\kappa)))\alpha(a') \\
&= (a_\kappa(1\otimes\gamma^{-1}(\gamma^{-1}(c)^\kappa)))\alpha(a') \\
&= \alpha(a_\kappa)((1\otimes\gamma^{-1}(\gamma^{-1}(c)^\kappa))a')=\alpha(a_\kappa)\psi(\gamma^{-1}(c)^\kappa\otimes a') \\
&= \alpha(a_\kappa)(a'_\lambda\otimes\gamma^{-1}(c)^{\kappa\lambda})=\alpha^{-1}(\alpha(a_\kappa))a'_\lambda\otimes\gamma(\gamma^{-1}(c)^{\kappa\lambda}) \\
&= a_\kappa a'_\lambda\otimes\gamma(\gamma^{-1}(c)^{\kappa\lambda}).
\end{aligned}
$$

In the above equality, if we replace c by $\gamma(c)$ we obtain $(aa')_\kappa\otimes\gamma(c)^\kappa=a_\kappa a'_\lambda\otimes\gamma(c^{\kappa\lambda})$. Next, by using the right (A,α)-linearity of $\Delta_{\mathcal{C}}$ we prove the following

$$
\begin{aligned}
& \alpha^{-1}(a)_{\kappa\lambda}\otimes c_1{}^{\lambda}\otimes c_2{}^{\kappa} \\
= \;& \psi(c_1\otimes\alpha^{-1}(a)_{\kappa})\otimes c_2{}^{\kappa} \\
= \;& (1\otimes\gamma^{-1}(c_1))\alpha^{-1}(a)_{\kappa}\otimes c_2{}^{\kappa} \\
= \;& (1\otimes\gamma^{-1}(c_1))\alpha^{-1}(a)_{\kappa}\otimes_A(1\otimes\gamma^{-1}(c_2{}^{\kappa})) \\
\overset{(3.22)}{=} \;& (1\otimes c_1)\otimes_A\alpha^{-1}(a)_{\kappa}(1\otimes\gamma^{-2}(c_2{}^{\kappa})) \\
= \;& (1\otimes c_1)\otimes_A(\alpha^{-1}(a)_{\kappa}\otimes\gamma^{-1}(c_2{}^{\kappa})) \\
= \;& (id_{A\otimes C}\otimes id_A\otimes\gamma^{-1})((1\otimes c_1)\otimes_A\psi(c_2\otimes\alpha^{-1}(a))) \\
= \;& (id_{A\otimes C}\otimes id_A\otimes\gamma^{-1})((1\otimes c_1)\otimes_A((1\otimes\gamma^{-1}(c_2))\alpha^{-1}(a))) \\
\overset{(4.16)}{=} \;& (id_{A\otimes C}\otimes id_A\otimes\gamma^{-1})(((\alpha^{-1}(1)\otimes\gamma^{-1}(c_1))\otimes_A(1\otimes\gamma^{-1}(c_2)))a) \\
\overset{(3.6)}{=} \;& (id_{A\otimes C}\otimes id_A\otimes\gamma^{-1})(((1\otimes\gamma^{-1}(c)_1)\otimes_A(1\otimes\gamma^{-1}(c)_2))a) \\
= \;& (id_{A\otimes C}\otimes id_A\otimes\gamma^{-1})(\Delta_{\mathcal{C}}(1\otimes\gamma^{-1}(c))a) \\
= \;& (id_{A\otimes C}\otimes id_A\otimes\gamma^{-1})(\Delta_{\mathcal{C}}((1\otimes\gamma^{-1}(c))a)) \\
= \;& (id_{A\otimes C}\otimes id_A\otimes\gamma^{-1})(\Delta_{\mathcal{C}}(a_{\kappa}\otimes c^{\kappa})) \\
= \;& (id_{A\otimes C}\otimes id_A\otimes\gamma^{-1})((\alpha^{-1}(a_{\kappa})\otimes c^{\kappa}{}_1)\otimes_A(1\otimes c^{\kappa}{}_2)) \\
= \;& (\alpha^{-1}(a_{\kappa})\otimes c^{\kappa}{}_1)\otimes_A(1\otimes\gamma^{-1}(c^{\kappa}{}_2)) \\
= \;& (\alpha^{-1}(\alpha^{-1}(a_{\kappa}))\otimes\gamma^{-1}(c^{\kappa}{}_1))1\otimes\gamma(\gamma^{-1}(c^{\kappa}{}_2)) \\
= \;& \alpha^{-1}(a_{\kappa})\otimes c^{\kappa}{}_1\otimes c^{\kappa}{}_2.
\end{aligned}
$$

We also find

$$\psi(c\otimes 1) = 1_{\kappa}\otimes c^{\kappa} = (1\otimes\gamma^{-1}(c))1 = 1\otimes c.$$

Finally, the fact of $\varepsilon_{\mathcal{C}}$ being right (A,α)-linear gives

$$
\begin{aligned}
\alpha(a_{\kappa})\varepsilon(c^{\kappa}) &= \varepsilon_{\mathcal{C}}(a_{\kappa}\otimes c^{\kappa}) = \varepsilon_{\mathcal{C}}((1\otimes\gamma^{-1}(c))a) \\
&= \varepsilon_{\mathcal{C}}(1\otimes\gamma^{-1}(c))a = \alpha(1)\varepsilon(\gamma^{-1}(c))a = 1a\varepsilon(c) \\
&= \alpha(a)\varepsilon(c),
\end{aligned}
$$

which means that $a_{\kappa}\varepsilon(c^{\kappa}) = a\varepsilon(c)$. Therefore $[(A,\alpha),(C,\gamma)]_{\psi}$ is a Hom-entwining structure.

3. The essential point is that if (M,μ) is a right (A,α)-Hom-module, then $(M\otimes C,\mu\otimes\gamma)$ is a right (A,α)-Hom-module with the Hom-action $\rho_{M\otimes C}:(M\otimes C)\otimes A\to M\otimes C$,

$(m\otimes c)\otimes a \mapsto (m\otimes c)a = m\alpha^{-1}(a)_\kappa \otimes \gamma(c^\kappa)$. $\rho_{M\otimes C}$ indeed satisfies Hom-associativity and Hom-unity as follows. For all $m \in M$, $a, a' \in A$ and $c \in C$,

$$\begin{aligned}
(\mu(m)\otimes\gamma(c))(aa') &= \mu(m)\alpha^{-1}(aa')_\kappa \otimes \gamma(\gamma(c)^\kappa) \\
&\overset{(4.7)}{=} \mu(m)(\alpha^{-1}(a)_\kappa \alpha^{-1}(a')_\lambda) \otimes \gamma(\gamma(c^{\kappa\lambda})) \\
&= (m\alpha^{-1}(a)_\kappa)\alpha(\alpha^{-1}(a')_\lambda) \otimes \gamma(\gamma(c^{\kappa\lambda})) \\
&\overset{(4.11)}{=} (m\alpha^{-1}(a)_\kappa)\alpha(\alpha^{-1}(a'))_\lambda \otimes \gamma(\gamma(c^\kappa)^\lambda) \\
&= (m\alpha^{-1}(a)_\kappa)\alpha^{-1}(\alpha(a'))_\lambda \otimes \gamma(\gamma(c^\kappa)^\lambda) \\
&= (m\alpha^{-1}(a)_\kappa \otimes \gamma(c^\kappa))\alpha(a') \\
&= ((m\otimes c)a)\alpha(a'),
\end{aligned}$$

$$\begin{aligned}
(m\otimes c)1 &= m\alpha^{-1}(1)_\kappa \otimes \gamma(c^\kappa) = m1_\kappa \otimes \gamma(c^\kappa) \\
&\overset{(4.9)}{=} m1 \otimes \gamma(c) = \mu(m) \otimes \gamma(c).
\end{aligned}$$

With respect to this Hom-action of (A,α) on $(M\otimes C, \mu\otimes\gamma)$, becoming an $[(A,\alpha),(C,\gamma)]_\psi$-entwined Hom-module is equivalent to the fact that the Hom-coaction of (C,γ) on (M,μ) is right (A,α)-linear.

Let $(M,\mu) \in \widetilde{\mathcal{M}}_A^C(\psi)$ with the right (C,γ)-Hom-comodule structure $m \mapsto m_{(0)} \otimes m_{(1)}$. Then $(M,\mu) \in \widetilde{\mathcal{M}}^{\mathcal{C}}$ with the Hom-coaction $\rho^M : M \to M \otimes_A \mathcal{C}$, $m \mapsto m_{(0)} \otimes_A (1 \otimes \gamma^{-1}(m_{(1)}))$, which actually is

$$\begin{aligned}
\rho^M(m) &= m_{(0)} \otimes_A (1 \otimes \gamma^{-1}(m_{(1)})) = \mu^{-1}(m)1 \otimes \gamma(\gamma^{-1}(m_{(1)})) \\
&= m_{(0)} \otimes m_{(1)},
\end{aligned}$$

where in the second equality we have used the canonical identification

$$\phi : M \otimes_A (A \otimes C) \simeq M \otimes C,\ m \otimes_A (a \otimes c) \mapsto \mu^{-1}(m)a \otimes \gamma(c),$$

and ρ^M is (A,α)-linear since

$$\rho^M(ma) = (ma)_{(0)} \otimes (ma)_{(1)} = m_{(0)}\alpha^{-1}(a)_\kappa \otimes \gamma({m_{(1)}}^\kappa) = (m_{(0)} \otimes m_{(1)})a.$$

Conversely, if (M,μ) is a right $(A\otimes C,\alpha\otimes\gamma)$-Hom-comodule with the coaction $\rho^M : M \to M\otimes_A (A\otimes C)$, by using the canonical identification above, one gets the (C,γ)-Hom-comodule structure $\bar{\rho}^M = \phi\circ\rho^M : M \to M\otimes C$ on (M,μ). One can also check that ϕ is right (A,α)-linear once the following (A,α)-Hom-module structure on $M\otimes_A \mathcal{C}$ is given:

$$\rho_{M\otimes_A\mathcal{C}} : (M\otimes_A \mathcal{C})\otimes A \to M\otimes_A \mathcal{C},\ (m\otimes_A (a\otimes c))\otimes a' \mapsto \mu(m)\otimes_A (a\otimes c)\alpha^{-1}(a'),$$

thus $\bar{\rho}^M$ is (A,α)-linear since by definition ρ^M is (A,α)-linear. Therefore (M,μ) has an $[(A,\alpha),(C,\gamma)]_\psi$-entwined Hom-module structure.

□

One should refer to both [20, Proposition 25] and [14, Item 32.9] for the classical version of the following theorem.

Theorem 4.2.9 *Let* $[(A,\alpha),(C,\gamma)]_\psi$*be an entwining Hom-structure and* $(\mathcal{C},\chi) = (A\otimes C,\alpha\otimes\gamma)$ *be the associated* (A,α)*-Hom-coring. Then the so-called Koppinen smash or* ψ*-twisted convolution algebra* $Hom^{\mathcal{H}}_\psi(C,A) = (Hom^{\mathcal{H}}(C,A),*_\psi,\eta_A\circ\varepsilon_C)$*, where* $(f *_\psi g)(c) = f(c_2)_\kappa g(c_1{}^\kappa)$ *for any* $f,g\in Hom^{\mathcal{H}}(C,A)$*, is anti-isomorphic to the algebra* $({}^*\mathcal{C},*^l,\varepsilon_{\mathcal{C}})$ *in Proposition (4.2.5).*

Proof: For $f,g,h\in Hom^{\mathcal{H}}(C,A)$ and $c\in C$,

$$\begin{aligned}
&((f *_\psi g) *_\psi h)(c)\\
&\quad= (f *_\psi g)(c_2)_\kappa h(c_1{}^\kappa) = (f(c_{22})_\lambda g(c_{21}{}^\lambda))h(c_1{}^\kappa)\\
&\quad\overset{(4.7)}{=} (f(c_{22})_{\lambda\kappa} g(c_{21}{}^\lambda)_\sigma)h(\gamma(\gamma^{-1}(c_1)^{\kappa\sigma})) = (f(c_{22})_{\lambda\kappa} g(c_{21}{}^\lambda)_\sigma)\alpha(h(\gamma^{-1}(c_1)^{\kappa\sigma}))\\
&\quad= \alpha(f(c_{22})_{\lambda\kappa})(g(c_{21}{}^\lambda)_\sigma h(\gamma^{-1}(c_1)^{\kappa\sigma})) \overset{\kappa\leftrightarrow\lambda}{=} \alpha(f(c_{22})_{\kappa\lambda})(g(c_{21}{}^\kappa)_\sigma h(\gamma^{-1}(c_1)^{\lambda\sigma}))\\
&\quad\overset{(3.5)}{=} \alpha(f(\gamma^{-1}(c_2))_{\kappa\lambda})(g(c_{12}{}^\kappa)_\sigma h(c_{11}{}^{\lambda\sigma})) = \alpha(\alpha^{-1}(f(c_2))_{\kappa\lambda})(g(c_{12}{}^\kappa)_\sigma h(c_{11}{}^{\lambda\sigma}))\\
&\quad\overset{(4.8)}{=} f(c_2)_\kappa(g(c_1{}^\kappa{}_2)_\sigma h(c_1{}^\kappa{}_1{}^\sigma))\\
&\quad= f(c_2)_\kappa(g *_\psi h)(c_1{}^\kappa)\\
&\quad= (f *_\psi (g *_\psi h))(c),
\end{aligned}$$

proving that $*_\psi$ is associative. Now we show that $\eta\varepsilon$ is the unit for $*_\psi$:

$$
\begin{aligned}
(\eta\varepsilon *_{\psi} f)(c) &= \eta\varepsilon(c_2)_{\kappa} f({c_1}^{\kappa}) = \varepsilon(c_2)1_{\kappa} f({c_1}^{\kappa}) \\
&= 1_{\kappa} f(\gamma^{-1}(c)^{\kappa}) \overset{(4.9)}{=} 1 f(\gamma^{-1}(c)) \\
&= f(c) \\
&= f(\gamma^{-1}(c))1 = f(c_2)\varepsilon(c_1)1 \\
&\overset{(4.10)}{=} f(c_2)_{\kappa}\varepsilon({c_1}^{\kappa})1 = f(c_2)_{\kappa}\eta\varepsilon({c_1}^{\kappa}) \\
&= (f *_{\psi} \eta\varepsilon)(c).
\end{aligned}
$$

The map $\phi : {}^*\mathcal{C} = {}_AHom^{\mathcal{H}}(A \otimes C, A) \to Hom^{\mathcal{H}}(C, A)$ given by

$$\phi(\xi)(c) = \xi(1 \otimes \gamma^{-1}(c)) \tag{4.17}$$

for any $\xi \in {}^*\mathcal{C}$ and $c \in C$, is a k-module isomorphism with the inverse $\varphi : Hom^{\mathcal{H}}(C, A) \to {}^*\mathcal{C}$ given by $\varphi(f)(a \otimes c) = af(c)$ for all $f \in Hom^{\mathcal{H}}(C, A)$ and $a \otimes c \in A \otimes C$: Let $a \in A$, $a' \otimes c \in A \otimes C$ and $f \in Hom^{\mathcal{H}}(C, A)$. Then

$$
\begin{aligned}
\varphi(f)(a(a' \otimes c)) &= \varphi(f)(\alpha^{-1}(a)a' \otimes \gamma(c)) = (\alpha^{-1}(a)a')f(\gamma(c)) \\
&= (\alpha^{-1}(a)a')\alpha(f(c)) = a(a'f(c)) = a\varphi(f)(a' \otimes c)
\end{aligned}
$$

and

$$\varphi(f)(\alpha(a) \otimes \gamma(c)) = \alpha(a)f(\gamma(c)) = \alpha(af(c)) = \alpha(\varphi(f)(a \otimes c)),$$

showing that $\varphi(f)$ is (A, α)-linear. On the other hand,

$$\varphi(\phi(\xi))(a \otimes c) = a\phi(\xi)(c) = a\xi(1 \otimes \gamma^{-1}(c)) = \xi(a(1 \otimes \gamma^{-1}(c))) = \xi(a \otimes c),$$

$$\phi(\varphi(f))(c) = \varphi(f)(1 \otimes \gamma^{-1}(c)) = 1f(\gamma^{-1}(c)) = f(c).$$

Now if we put $\phi(\xi) = f$ and $\phi(\xi') = f'$, we have $f(c) = \xi(1 \otimes \gamma^{-1}(c))$, $f'(c) = \xi'(1 \otimes \gamma^{-1}(c))$ for $c \in C$, and then

$$
\begin{aligned}
(\xi *^l \xi')(a\otimes c) &= \xi((a\otimes c)_1\xi'((a\otimes c)_2)) \\
&\overset{(4.13)}{=} \xi((\alpha^{-1}(a)\otimes c_1)\xi'(1\otimes c_2)) \\
&= \xi((\alpha^{-1}(a)\otimes c_1)f'(\gamma(c_2))) = \xi((\alpha^{-1}(a)\otimes c_1)\alpha(f'(c_2))) \\
&= \xi(\alpha^{-1}(a)\alpha^{-1}(\alpha(f'(c_2)))_\kappa \otimes \gamma(c_1{}^\kappa)) = \xi(\alpha^{-1}(a)f'(c_2)_\kappa \otimes \gamma(c_1{}^\kappa)) \\
&= (\alpha^{-1}(a)f'(c_2)_\kappa)f(\gamma(c_1{}^\kappa)) = (\alpha^{-1}(a)f'(c_2)_\kappa)\alpha(f(c_1{}^\kappa)) \\
&= a(f'(c_2)_\kappa f(c_1{}^\kappa)) = a(f' *_\psi f)(c),
\end{aligned}
$$

which induces the following

$$
\begin{aligned}
\phi(\xi *^l \xi')(c) &= (\xi *^l \xi')(1\otimes\gamma^{-1}(c)) = 1(f' *_\psi f)(\gamma^{-1}(c)) \\
&= \alpha((f' *_\psi f)(\gamma^{-1}(c))) = (f' *_\psi f)(\gamma(\gamma^{-1}(c))) \\
&= (f' *_\psi f)(c) = (\phi(\xi') *_\psi \phi(\xi))(c).
\end{aligned}
$$

Moreover, $\phi(\varepsilon_{\mathcal{C}})(c) = \varepsilon_{\mathcal{C}}(1\otimes\gamma^{-1}(c)) = \alpha(1)\varepsilon(\gamma^{-1}(c)) = \eta\varepsilon(c)$. Therefore ϕ is the anti-isomorphism of the algebras ${}^*\mathcal{C}$ and $Hom_\psi^{\mathcal{H}}(C,A)$. □

4.3 Entwinings and Hom-Hopf-type Modules

Definition 4.3.1 *Let (B,β) be a monoidal Hom-bialgebra. A right (B,β)-Hom-module coalgebra (C,γ) is a monoidal Hom-coalgebra and a right (B,β)-Hom-module with the Hom-action $\rho_C : C\otimes B \to C$, $c\otimes b \mapsto cb$ such that ρ_C is a Hom-coalgebra morphism, that is, for any $c\in C$ and $b\in B$*

$$(cb)_1\otimes(cb)_2 = c_1b_1\otimes c_2b_2,\ \varepsilon_C(cb) = \varepsilon_C(c)\varepsilon_B(b). \tag{4.18}$$

The equality

$$\rho_C\circ(\gamma\otimes\beta) = \gamma\circ\rho_C$$

is a consequence of (4.18) and the properties of (B,β) and (C,γ).

By the following construction, we show that a Hom-Doi-Koppinen datum comes from a Hom-entwining structure and that the Doi-Koppinen Hom-Hopf modules are the same as the associated entwined Hom-modules, and give the structure of Hom-coring corresponding to the relevant Hom-entwining structure.

Proposition 4.3.2 *Let (B,β) be a monoidal Hom-bialgebra. Let (A,α) be a right (B,β)-Hom-comodule algebra with Hom-coaction $\rho^A : A \to A \otimes B$, $a \mapsto a_{(0)} \otimes a_{(1)}$ and (C,γ) be a right (B,β)-Hom-module coalgebra with Hom-action $\rho_C : C \otimes B \to C$, $c \otimes b \mapsto cb$. Define the morphism*

$$\psi : C \otimes A \to A \otimes C,\ c \otimes a \mapsto \alpha(a_{(0)}) \otimes \gamma^{-1}(c)a_{(1)} = a_\kappa \otimes c^\kappa. \tag{4.19}$$

Then the following assertions hold.

1. *$[(A,\alpha),(C,\gamma)]_\psi$ is an Hom-entwining structure.*
2. *(M,μ) is an $[(A,\alpha),(C,\gamma)]_\psi$-entwined Hom-module if and only if it is a right (A,α)-Hom-module with $\rho_M : M \otimes A \to M$, $m \otimes a \mapsto ma$ and a right (C,γ)-Hom-comodule with $\rho^M : M \to M \otimes C$, $m \mapsto m_{(0)} \otimes m_{(1)}$ such that*

$$\rho^M(ma) = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)} \tag{4.20}$$

   for any $m \in M$ and $a \in A$.
3. *$(\mathcal{C},\chi) = (A \otimes C, \alpha \otimes \gamma)$ is an (A,α)-Hom-coring with comultiplication and counit given by (4.13) and (4.14), respectively, and it has the (A,α)-Hom-bimodule structure $a(a' \otimes c) = \alpha^{-1}(a)a' \otimes \gamma(c)$, $(a' \otimes c)a = a'a_{(0)} \otimes ca_{(1)}$ for $a, a' \in A$ and $c \in C$.*
4. *$Hom^{\mathcal{H}}(C,A)$ is an associative algebra with the unit $\eta\varepsilon$ and the multiplication $*_\psi$ defined by*

$$(f *_\psi g)(c) = \alpha(f(c_2)_{(0)})g(\gamma^{-1}(c_1)f(c_2)_{(1)}) = \alpha(f(c_2))_{(0)}\alpha^{-1}(g(c_1\alpha(f(c_2)_{(1)}))), \tag{4.21}$$

   for all $f, g \in Hom^{\mathcal{H}}(C,A)$ and $c \in C$.

Proof:

1. By (4.19) we have $a_\kappa \otimes \gamma(c)^\kappa = \alpha(a_{(0)}) \otimes ca_{(1)}$, and thus

$$
\begin{aligned}
(aa')_\kappa \otimes \gamma(c)^\kappa &= \alpha((aa')_{(0)}) \otimes c((aa')_{(1)}) \\
&\overset{(3.18)}{=} \alpha(a_{(0)}a'_{(0)}) \otimes c(a_{(1)}a'_{(1)}) = \alpha(a_{(0)})\alpha(a'_{(0)}) \otimes (\gamma^{-1}(c)a_{(1)})\beta(a'_{(1)}) \\
&\overset{(4.19)}{=} a_\kappa \alpha(a'_{(0)}) \otimes c^\kappa \beta(a'_{(1)}) \\
&= a_\kappa \alpha(a'_{(0)}) \otimes \gamma(\gamma^{-1}(c^\kappa)a'_{(1)}) \\
&\overset{(4.19)}{=} a_\kappa a'_\lambda \otimes \gamma(c^{\kappa\lambda}),
\end{aligned}
$$

which shows that ψ satisfies (4.7). To prove that ψ fulfills (4.8) we have the computation

$$
\begin{aligned}
\alpha^{-1}(a_\kappa) \otimes c^\kappa{}_1 \otimes c^\kappa{}_2 &= \alpha^{-1}(\alpha(a_{(0)})) \otimes (\gamma^{-1}(c)a_{(1)})_1 \otimes (\gamma^{-1}(c)a_{(1)})_2 \\
&\overset{(4.18)}{=} a_{(0)} \otimes \gamma^{-1}(c)_1 a_{(1)1} \otimes \gamma^{-1}(c)_2 a_{(1)2} \\
&= a_{(0)} \otimes \gamma^{-1}(c_1) a_{(1)1} \otimes \gamma^{-1}(c_2) a_{(1)2} \\
&\overset{(3.9)}{=} \alpha(a_{(0)(0)}) \otimes \gamma^{-1}(c_1) a_{(0)(1)} \otimes \gamma^{-1}(c_2)\beta^{-1}(a_{(1)}) \\
&\overset{(4.19)}{=} a_{(0)_\kappa} \otimes c_1{}^\kappa \otimes \gamma^{-1}(c_2)\beta^{-1}(a_{(1)}) \\
&= \alpha(\alpha^{-1}(a_{(0)}))_\kappa \otimes c_1{}^\kappa \otimes \gamma^{-1}(c_2)\beta^{-1}(a_{(1)}) \\
&\overset{(3.10)}{=} \alpha(\alpha^{-1}(a)_{(0)})_\kappa \otimes c_1{}^\kappa \otimes \gamma^{-1}(c_2)\alpha^{-1}(a)_{(1)} \\
&\overset{(4.19)}{=} \alpha^{-1}(a)_{\lambda\kappa} \otimes c_1{}^\kappa \otimes c_2{}^\lambda.
\end{aligned}
$$

To finish the proof of (1) we finally verify that ψ satisfies (4.9) and (4.10) as follows,

$$
1_\kappa \otimes c^\kappa = \alpha(1_{(0)}) \otimes \gamma^{-1}(c)1_{(1)} = \alpha(1_A) \otimes \gamma^{-1}(c)1_B = 1 \otimes c,
$$

$$
\begin{aligned}
a_\kappa \varepsilon(c^\kappa) &= \alpha(a_{(0)})\varepsilon(\gamma^{-1}(c)a_{(1)}) = \alpha(a_{(0)})\varepsilon(\gamma^{-1}(c\beta(a_{(1)}))) \\
&\overset{(3.6)}{=} \alpha(a_{(0)})\varepsilon(c\beta(a_{(1)})) \overset{(4.18)}{=} \alpha(a_{(0)})\varepsilon(c)\varepsilon_B(\beta(a_{(1)})) \\
&= \alpha(a_{(0)}\varepsilon_B(a_{(1)}))\varepsilon(c) \overset{(3.9)}{=} \alpha(\alpha^{-1}(a))\varepsilon(c) \\
&= a\varepsilon(c).
\end{aligned}
$$

2. We see that the condition for entwined Hom-modules,i.e., $\rho^M(ma) = m_{(0)}\alpha^{-1}(a)_\kappa \otimes \gamma(m_{(1)}{}^\kappa)$ and the condition in (4.20) are equivalent by the following, for $m \in M$ and $a \in A$,

$$\begin{aligned} m_{(0)}\alpha^{-1}(a)_\kappa \otimes \gamma(m_{(1)}{}^\kappa) &= m_{(0)}\alpha(\alpha^{-1}(a)_{(0)}) \otimes \gamma(\gamma^{-1}(m_{(1)})\alpha^{-1}(a)_{(1)}) \\ &= m_{(0)}\alpha(\alpha^{-1}(a_{(0)})) \otimes \gamma(\gamma^{-1}(m_{(1)})\beta^{-1}(a_{(1)})) \\ &= m_{(0)}a_{(0)} \otimes \gamma(\gamma^{-1}(m_{(1)}a_{(1)})) \\ &= m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}. \end{aligned}$$

3. We only prove that the right (A,α)-Hom-module structure holds as is given in the assertion. The rest of the structure of the corresponding Hom-coring can be seen at once from Theorem (4.2.8). For $a, a' \in A$ and $c \in C$,

$$\begin{aligned} (a' \otimes c)a &= a'\alpha^{-1}(a)_\kappa \otimes \gamma(c^\kappa) \\ &= a'\alpha(\alpha^{-1}(a)_{(0)}) \otimes \gamma(\gamma^{-1}(c)\alpha^{-1}(a)_{(1)}) = a'a_{(0)}) \otimes \gamma(\gamma^{-1}(c)\beta^{-1}(a_{(1)})) \\ &= a'a_{(0)} \otimes ca_{(1)}. \end{aligned}$$

4. By the definition of product $*_\psi$ given in Theorem (4.2.9) and the definition of ψ given in (4.19) we have, for $f, g \in Hom^{\mathcal{H}}(C, A)$ and $c \in C$,

$$\begin{aligned} (f *_\psi g)(c) &= f(c_2)_\kappa g(c_1{}^\kappa) \\ &= \alpha(f(c_2)_{(0)})g(\gamma^{-1}(c_1)f(c_2)_{(1)}) = \alpha(f(c_2)_{(0)})g(\gamma^{-1}(c_1\beta(f(c_2)_{(1)}))) \\ &= \alpha(f(c_2)_{(0)})\alpha^{-1}(g(c_1\beta(f(c_2)_{(1)}))) = \alpha(f(c_2))_{(0)}\alpha^{-1}(g(c_1\alpha(f(c_2))_{(1)})). \end{aligned}$$

□

Definition 4.3.3 *A triple $[(A,\alpha),(B,\beta),(C,\gamma)]$ is called a (right-right) Hom-Doi-Koppinen datum if it satisfies the conditions of Proposition (4.3.2), that is, if (A,α) is a right (B,β)-Hom-comodule algebra and (C,γ) is a right (B,β)-Hom-module coalgebra for a monoidal Hom-bialgebra (B,β).*

$[(A,\alpha),(C,\gamma)]_\psi$ in Proposition (4.3.2) is called a Hom-entwining structure associated to a Hom-Doi-Koppinen datum.

A Doi-Koppinen Hom-Hopf module or a unifying Hom-Hopf module is a Hom-module satisfying the condition (4.20).

Now we give the following collection of examples. Each of them is a special case of the construction given above.

Example 4.3.4 ***Hom-bialgebra entwinings and Hom-Hopf modules*** *Let (B,β) be a monoidal Hom-bialgebra with Hom-multiplication $m_B : B\otimes B \to B$, $b\otimes b' \mapsto bb'$ and Hom-comutiplication $\Delta_B : B \to B\otimes B$, $b \mapsto b_1 \otimes b_2$.*

1. *$[(B,\beta),(B,\beta)]_\psi$, with $\psi : B\otimes B \to B\otimes B, b'\otimes b \mapsto \beta(b_1)\otimes\beta^{-1}(b')b_2$, is an Hom-entwining structure.*

2. *(M,μ) is an $[(B,\beta),(B,\beta)]_\psi$-entwined Hom-module if and only if it is a right (B,β)-Hom-module with $\rho_M : M\otimes B \to M$, $m\otimes b \mapsto mb$ and a right (B,β)-Hom-comodule with $\rho^M : M \to M\otimes B$, $m \mapsto m_{(0)}\otimes m_{(1)}$ such that*
$$\rho^M(mb) = m_{(0)}b_1 \otimes m_{(1)}b_2 \tag{4.22}$$
for all $m \in M$ and $b \in B$. Such Hom-modules are called Hom-Hopf modules (see [21]).

3. *$(\mathcal{C},\chi) = (B\otimes B, \beta\otimes\beta)$ is a (B,β)-Hom-coring with comultiplication $\Delta_{\mathcal{C}}(b\otimes b') = (\beta^{-1}(b)\otimes b'_1)\otimes_B(1_B\otimes b'_2)$ and counit $\varepsilon_{\mathcal{C}}(b\otimes b') = \beta(b)\varepsilon_B(b')$, and (B,β)-Hom-bimodule structure*
$$b(b'\otimes b'') = \beta^{-1}(b)b'\otimes\beta(b'),\ (b'\otimes b'')b = b'b_1\otimes b''b_2$$
for all $b,b',b'' \in B$.

Proof: Since Δ_B is a Hom-algebra morphism, (B,β) is a right (B,β)-Hom-comodule algebra with Hom-coaction
$$\rho^B = \Delta_B : B \to B\otimes B,\ b \mapsto b_{(0)}\otimes b_{(1)} = b_1\otimes b_2,$$
and since m_B is a Hom-coalgebra morphism, (B,β) is a right (B,β)-Hom-module coalgebra with Hom-action $\rho_B = m_B : B\otimes B \to B$, $b\otimes b' \mapsto bb'$. So, we have the triple

$[(B,\beta),(B,\beta),(B,\beta)]$ as Hom-Doi-Koppinen datum, and the associated Hom-entwining structure is $[(B,\beta),(B,\beta)]_\psi$, where $\psi(b'\otimes b)=\beta(b_{(0)})\otimes\beta^{-1}(b')b_{(1)}=\beta(b_1)\otimes\beta^{-1}(b')b_2$. The rest of the assertions are immediately obtained by the above proposition.

□

Example 4.3.5 ***Relative entwinings and relative Hom-Hopf modules*** *Let (B,β) be a monoidal Hom-bialgebra and let (A,α) be a (B,β)-Hom-comodule algebra with Hom-coaction $\rho^A: A\to A\otimes B, a\mapsto a_{(0)}\otimes a_{(1)}$.*

1. *$[(A,\alpha),(B,\beta)]_\psi$, with $\psi: B\otimes A\to A\otimes B$, $b\otimes a\mapsto \alpha(a_{(0)})\otimes\beta^{-1}(b)a_{(1)}$, is an Hom-entwining structure.*

2. *(M,μ) is an $[(A,\alpha),(B,\beta)]_\psi$-entwined Hom-module if and only if it is a right (A,α)-Hom-module with $\rho_M: M\otimes A\to M$, $m\otimes a\mapsto ma$ and a right (B,β)-Hom-comodule with $\rho^M: M\to M\otimes B$, $m\mapsto m_{[0]}\otimes m_{[1]}$ such that*

$$\rho^M(ma)=m_{[0]}a_{(0)}\otimes m_{[1]}a_{(1)} \tag{4.23}$$

   for all $m\in M$ and $a\in A$. Hom-modules fulfilling the above condition are called relative Hom-Hopf modules (see [39]).

3. *$(\mathcal{C},\chi)=(A\otimes B,\alpha\otimes\beta)$ is a (A,α)-Hom-coring with comultiplication $\Delta_{\mathcal{C}}(a\otimes b)=(\alpha^{-1}(a)\otimes b_1)\otimes_A(1_A\otimes b_2)$ and counit $\varepsilon_{\mathcal{C}}(a\otimes b)=\alpha(a)\varepsilon_B(b)$, and (A,α)-Hom-bimodule structure*

$$a(a'\otimes b)=\alpha^{-1}(a)a'\otimes\beta(b),\ (a'\otimes b)a=a'a_{(0)}\otimes ba_{(1)}$$

   for all $a,a'\in A$ and $b\in B$.

Proof: The relevant Hom-Doi-Koppinen datum is $[(A,\alpha),(B,\beta),(B,\beta)]$, where the first object (A,α) is assumed to be a right (B,β)-Hom-comodule algebra with the Hom-coaction $\rho^A: a\mapsto a_{(0)}\otimes a_{(1)}$ and the third object (B,β) is a right (B,β)-Hom-module coalgebra with Hom-action given by its Hom-multiplication. Hence, $[(A,\alpha),(B,\beta)]_\psi$ is the associated Hom-entwining structure, where $\psi(b\otimes a)=\alpha(a_{(0)})\otimes\beta^{-1}(b)a_{(1)}$. Assertions (2) and (3) can be seen at once from Proposition (4.3.2). □

Remark 6 *(A,α) itself is a relative Hom-Hopf-module by its Hom-multiplication and the (B,β)-Hom-coaction ρ^A.*

Example 4.3.6 ***Dual-relative entwinings and*** $[(C,\gamma),(A,\alpha)]$***-Hom-Hopf modules*** *Let (A,α) be a monoidal Hom-bialgebra and let (C,γ) be a right (A,α)-Hom-module coalgebra with Hom-action $\rho_C : C\otimes A \to C, c\otimes a \mapsto ca$.*

1. *$[(A,\alpha),(C,\gamma)]_\psi$, with $\psi : C\otimes A \to A\otimes C, c\otimes a \mapsto \alpha(a_1)\otimes\beta^{-1}(c)a_2$, is an Hom-entwining structure.*

2. *(M,μ) is an $[(A,\alpha),(C,\gamma)]_\psi$-entwined Hom-module if and only if it is a right (A,α)-Hom-module with $\rho_M : M\otimes A \to M$, $m\otimes a \mapsto ma$ and a right (C,γ)-Hom-comodule with $\rho^M : M \to M\otimes B$, $m \mapsto m_{(0)}\otimes m_{(1)}$ such that*

$$\rho^M(ma) = m_{(0)}a_1 \otimes m_{(1)}a_2 \tag{4.24}$$

   for all $m\in M$ and $a\in A$. Such a Hom-module is called $[(C,\gamma),(A,\alpha)]$-Hom-Hopf module.

3. *$(\mathcal{C},\chi) = (A\otimes C, \alpha\otimes\gamma)$ is a (A,α)-Hom-coring with comultiplication $\Delta_{\mathcal{C}}(a\otimes c) = (\alpha^{-1}(a)\otimes c_1)\otimes_A(1_A\otimes c_2)$ and counit $\varepsilon_{\mathcal{C}}(a\otimes b) = \alpha(a)\varepsilon_C(c)$, and (A,α)-Hom-bimodule structure*

$$a(a'\otimes b) = \alpha^{-1}(a)a'\otimes\gamma(c),\ (a'\otimes c)a = a'a_1\otimes ca_2$$

   for all $a, a'\in A$ and $c\in C$.

Proof: (A,α) is a right (A,α)-Hom-comodule algebra with Hom-coaction given by the Hom-comultiplication

$$\rho^A = \Delta_A : A \to A\otimes A,\ a\mapsto a_{(0)}\otimes a_{(1)} = a_1\otimes a_2,$$

since Δ_A is a Hom-algebra morphism. Besides (C,γ) is assumed to be a right (A,α)-Hom-module coalgebra with Hom-action $\rho_C(c\otimes a) = ca$. Thus, the related Hom-Doi-Koppinen datum is $[(A,\alpha),(A,\alpha),(C,\gamma)]$. Then $[(A,\alpha),(C,\gamma)]_\psi$ is the Hom-entwining structure associated to the datum, where

$$\psi(c\otimes a) = \alpha(a_{(0)})\otimes\gamma^{-1}(c)a_{(1)} = \alpha(a_1)\otimes\gamma^{-1}(c)a_2.$$

The assertions (2) and (3) are also immediate by Proposition (4.3.2). □

Remark 7 *(C,γ) itself is a $[(C,\gamma),(A,\alpha)]$-Hom-Hopf-module by the (A,α)-Hom-action ρ_C and its Hom-comultiplication.*

The following example gives a Hom-generalization of the so-called (α,β)-Yetter-Drinfeld modules introduced in [69] as an entwined Hom-module:

Example 4.3.7 ***Generalized Yetter-Drinfeld entwinings and*** (ϕ,φ)***-Hom-Yetter-Drinfeld modules*** *Let* (H,α) *be a monoidal Hom-Hopf algebra and let* $\phi,\varphi : H \to H$ *be two monoidal Hom-Hopf algebra automorphisms. Define the map, for all* $h,g \in H$

$$\psi : H\otimes H \to H\otimes H,\ g\otimes h \mapsto \alpha^2(h_{21})\otimes\varphi(S(h_1))(\alpha^{-2}(g)\phi(h_{22})), \tag{4.25}$$

where S *is the antipode of* H*.*

1. $[(H,\alpha),(H,\alpha)]_\psi$ *is an Hom-entwining structure.*
2. (M,μ) *is an* $[(H,\alpha),(H,\alpha)]_\psi$*-entwined Hom-module if and only if it is a right* (H,α)*-Hom-module with* $\rho_M : M\otimes H \to M,\ m\otimes h \mapsto mh$ *and a right* (H,α)*-Hom-comodule with* $\rho^M : M \to M\otimes H,\ m \mapsto m_{(0)}\otimes m_{(1)}$ *such that*

$$\rho^M(mh) = m_{(0)}\alpha(h_{21})\otimes\varphi(S(h_1))(\alpha^{-1}(m_{(1)})\phi(h_{22})) \tag{4.26}$$

   for all $m\in M$ *and* $h\in H$*. A Hom-module* (M,μ) *satisfying this condition is called* (ϕ,φ)*-Hom-Yetter-Drinfeld module .*

3. $(\mathcal{C},\chi) = (H\otimes H,\alpha\otimes\alpha)$ *is an* (H,α)*-Hom-coring with comultiplication* $\Delta_{\mathcal{C}}(h\otimes h') = (\alpha^{-1}(h)\otimes h'_1)\otimes_H(1_H\otimes h'_2)$ *and counit* $\varepsilon_{\mathcal{C}}(h\otimes h') = \alpha(h)\varepsilon_H(h')$*, and* (H,α)*-Hom-bimodule structure*

$$g(h\otimes h') = \alpha^{-1}(g)h\otimes\alpha(h'),\ (h\otimes h')g = h\alpha(g_{21})\otimes\varphi(S(g_1))(\alpha^{-1}(h')\phi(g_{22}))$$

   for all $h,h',g\in H$*.*

Proof: In the first place, we prove that the map

$$\rho^H : H\to H\otimes(H^{op}\otimes H),\ h\mapsto h_{(0)}\otimes h_{(1)} := \alpha(h_{21})\otimes(\alpha^{-1}(\varphi(S(h_1)))\otimes h_{22})$$

defines a $(H^{op}\otimes H,\alpha\otimes\alpha)$-Hom-comodule algebra structure on (H,α). Let us put $(H^{op}\otimes H,\alpha\otimes\alpha) = (\widetilde{H},\tilde{\alpha})$. Then

$$
\begin{aligned}
&h_{(0)(0)}\otimes(h_{(0)(1)})\otimes\tilde{\alpha}^{-1}(h_{(1)})\\
&\quad= \alpha(\alpha(h_{21})_{21})\otimes((\alpha^{-1}(\varphi(S(\alpha(h_{21})_1)))\otimes\alpha(h_{21})_{22})\otimes(\alpha^{-2}(\varphi(S(h_1)))\otimes\alpha^{-1}(h_{22})))\\
&\quad= \alpha^2(h_{2121})\otimes((\alpha^{-1}(\varphi(S(\alpha(h_{211}))))\otimes\alpha(h_{2122}))\otimes(\alpha^{-2}(\varphi(S(h_1)))\otimes\alpha^{-1}(h_{22})))\\
&\quad= \alpha^2(h_{2121})\otimes((\varphi(S(h_{211}))\otimes\alpha(h_{2122}))\otimes(\alpha^{-2}(\varphi(S(h_1)))\otimes\alpha^{-1}(h_{22})))\\
&\quad= h_{21}\otimes((\alpha^{-1}(\varphi(S(h_{12})))\otimes h_{221})\otimes(\alpha^{-1}(\varphi(S(h_{11})))\otimes h_{222}))\\
&\quad= h_{21}\otimes((\alpha^{-1}(\varphi(S(h_1)))_1\otimes h_{221})\otimes(\alpha^{-1}(\varphi(S(h_1)))_2\otimes h_{222}))\\
&\quad= \alpha^{-1}(h_{(0)})\otimes\Delta_{\widetilde{H}}(h_{(1)}),
\end{aligned}
$$

where in the fourth step we used

$$\alpha(h_{11})\otimes\alpha^{-1}(h_{12})\otimes\alpha^{-2}(h_{21})\otimes\alpha^{-1}(h_{221})\otimes\alpha(h_{222})=h_1\otimes h_{211}\otimes h_{2121}\otimes h_{2122}\otimes h_{22},$$

which can be obtained by applying the Hom-coassociativity of Δ_H three times. We also have

$$
\begin{aligned}
h_{(0)}\varepsilon_{\widetilde{H}}(h_{(0)}) &= \alpha(h_{21})\varepsilon(\alpha^{-1}(\varphi(S(h_1))))\varepsilon(h_{22})\\
&= \alpha(h_{21}\varepsilon(h_{22}))\varepsilon(\alpha^{-1}(\varphi(S(h_1))))=\alpha(\alpha^{-1}(h_2))\varepsilon(h_1)\\
&= \alpha^{-1}(h),
\end{aligned}
$$

where in the third equality we used the relations $\varepsilon\circ\alpha^{-1}=\varepsilon$, $\varepsilon\circ\varphi=\varepsilon$ and $\varepsilon\circ S=\varepsilon$. One can easily check that the relations $\rho^H\circ\alpha=(\alpha\otimes\tilde{\alpha})\circ\rho^H$ and $\rho^H(1_H)=1_H\otimes 1_{\widetilde{H}}$ hold. For $g,h\in H$,

$$
\begin{aligned}
\rho^H(g)\rho^H(g) &= (\alpha(g_{21})\otimes(\alpha^{-1}(\varphi(S(g_1)))\otimes g_{22}))(\alpha(h_{21})\otimes(\alpha^{-1}(\varphi(S(h_1)))\otimes h_{22}))\\
&= \alpha(g_{21})\alpha(h_{21})\otimes(\alpha^{-1}(\varphi(S(h_1)))\alpha^{-1}(\varphi(S(g_1)))\otimes g_{22}h_{22})\\
&= \alpha(g_{21}h_{21})\otimes(\alpha^{-1}(\varphi(S(h_1)S(g_1)))\otimes g_{22}h_{22})\\
&= \alpha((gh)_{21})\otimes(\alpha^{-1}(\varphi(S((gh)_1)))\otimes(gh)_{22})\\
&= \rho^H(gh),
\end{aligned}
$$

which completes the proof of the statement that ρ^H makes (H,α) an $(\widetilde{H},\tilde{\alpha})$-Hom-comodule algebra. We next consider the map, for all $g,h,k\in H$

$$\rho_H: H\otimes\widetilde{H}\to H,\ g\cdot(h\otimes k):=(h\alpha^{-1}(g))\phi(\alpha(k))$$

and we claim that it defines an $(\widetilde{H},\tilde{\alpha})$-Hom-module coalgebra structure on (H,α): Indeed,

$$\begin{aligned}
(g\cdot(h\otimes k))\cdot(\alpha(h')\otimes\alpha(k')) &= ((h\alpha^{-1}(g))\phi(\alpha(k)))\cdot(\alpha(h')\otimes\alpha(k'))\\
&= (\alpha(h')((\alpha^{-1}(h)\alpha^{-2}(g))\alpha^{-1}(\phi(\alpha(k)))))\phi(\alpha^2(k'))\\
&= (\alpha(h')((\alpha^{-1}(h)\alpha^{-2}(g))\phi(k)))\phi(\alpha^2(k'))\\
&= ((h'(\alpha^{-1}(h)\alpha^{-2}(g)))\alpha(\phi(k)))\phi(\alpha^2(k'))\\
&= (\alpha^{-1}((h'h)g)\alpha(\phi(k)))\phi(\alpha^2(k'))\\
&= ((h'h)g)(\alpha(\phi(k))\alpha^{-1}(\phi(\alpha^2(k')))) = ((h'h)g)(\phi(\alpha(k))\phi(\alpha(k')))\\
&= ((h'h)g)\phi(\alpha((kk'))) = ((h'h)\alpha^{-1}(\alpha(g)))\phi(\alpha((kk')))\\
&= \alpha(g)\cdot(h'h\otimes kk') = \alpha(g)\cdot((h\otimes k)(h'\otimes k')),
\end{aligned}$$

$$h\cdot(1_H\otimes 1_H) = (1_H\alpha^{-1}(h))\phi(\alpha(1_H)) = \alpha(h),$$

$$\begin{aligned}
(g\cdot(h\otimes k))_1\otimes(g\cdot(h\otimes k))_2 &= ((h\alpha^{-1}(g))\phi(\alpha(k)))_1\otimes((h\alpha^{-1}(g))\phi(\alpha(k)))_2\\
&= (h\alpha^{-1}(g))_1\phi(\alpha(k))_1\otimes(h\alpha^{-1}(g))_2\phi(\alpha(k))_2\\
&= (h_1\alpha^{-1}(g_1))\phi(\alpha(k_1))\otimes(h_2\alpha^{-1}(g_2))\phi(\alpha(k_2))\\
&= g_1\cdot(h_1\otimes k_1)\otimes g_2\cdot(h_2\otimes k_2)\\
&= g_1\cdot(h\otimes k)_1\otimes g_2\cdot(h\otimes k)_2,
\end{aligned}$$

$$\varepsilon(g\cdot(h\otimes k)) = \varepsilon((h\alpha^{-1}(g))\phi(\alpha(k))) = \varepsilon(h)\varepsilon(\alpha^{-1}(g))\varepsilon(\phi(\alpha(k))) = \varepsilon(h)\varepsilon(g)\varepsilon(k) = \varepsilon(h)\varepsilon_{\widetilde{H}}(g\otimes k),$$

proving that (H,α) is an $(\widetilde{H},\tilde{\alpha})$-Hom-module coalgebra with the Hom-action ρ_H. Hence, the Hom-Doi-Koppinen datum is given by $[(H,\alpha),(H^{op}\otimes H,\alpha\otimes\alpha),(H,\alpha)]$ to which the Hom-entwining structure $[(H,\alpha),(H,\alpha)]_\psi$ is associated, where we have the entwining map $\psi: H\otimes H\to H\otimes H$ as

$$\begin{aligned}
\psi(g\otimes h) &= \alpha(h_{(0)})\alpha^{-1}(g)\cdot h_{(1)} = \alpha(\alpha(h_{21}))\otimes\alpha^{-1}(g)\cdot(\alpha^{-1}(\varphi(S(h_1)))\otimes h_{22})\\
&= \alpha^2(h_{21})\otimes(\alpha^{-1}(\varphi(S(h_1)))\alpha^{-2}(g))\phi(\alpha(h_{22}))\\
&= \alpha^2(h_{21})\otimes\varphi(S(h_1))(\alpha^{-2}(g)\phi(h_{22})).
\end{aligned}$$

For $m \in M$ and $h \in H$, we have the condition (4.26)

$$\begin{aligned}\rho^M(mh) &= m_{(0)}h_{(0)} \otimes m_{(1)} \cdot h_{(1)} \\ &= m_{(0)}\alpha(h_{21}) \otimes m_{(1)} \cdot (\alpha^{-1}(\varphi(S(h_1))) \otimes h_{22}) \\ &= m_{(0)}\alpha(h_{21}) \otimes \alpha^{-1}(\varphi(S(h_1))m_{(1)})\phi(\alpha(h_{22})) \\ &= m_{(0)}\alpha(h_{21}) \otimes \varphi(S(h_1))(\alpha^{-1}(m_{(1)})\phi(h_{22})).\end{aligned}$$

By the above proposition, the (H,α)-Hom-coring structure of $(H\otimes H, \alpha\otimes\alpha)$ is immediate. Here we only write down the right Hom-module condition

$$\begin{aligned}(h\otimes h')g &= hg_{(0)} \otimes h' \cdot g_{(1)} \\ &= h\alpha(g_{21}) \otimes h' \cdot (\alpha^{-1}(\varphi(S(g_1))) \otimes g_{22}) \\ &= h\alpha(g_{21}) \otimes \varphi(S(g_1))(\alpha^{-1}(h')\phi(g_{22})),\end{aligned}$$

completing the proof. □

Remark 8 *1. By putting $\phi = id_H = \varphi$ in the compatibility condition (4.26) we get the usual condition for (right-right) Hom-Yetter-Drinfeld modules, which is*

$$\rho^M(mh) = m_{(0)}\alpha(h_{21}) \otimes S(h_1)(\alpha^{-1}(m_{(1)})h_{22}). \tag{4.27}$$

2. If the antipode S of (H,α) is a bijection , then by taking $\phi = id_H$ and $\varphi = S^{-2}$, we have the compatibility condition for (right-right) anti-Hom-Yetter-Drinfeld modules as follows

$$\rho^M(mh) = m_{(0)}\alpha(h_{21}) \otimes S^{-1}(h_1)(\alpha^{-1}(m_{(1)})h_{22}). \tag{4.28}$$

We get an equivalent condition for the generalized Hom-Yetter-Drinfeld modules by the following

Proposition 4.3.8 *The compatibility condition (4.26) for (ϕ,φ)-Hom-Yetter-Drinfeld modules is equivalent to the equation*

$$m_{(0)}\alpha^{-1}(h_1) \otimes m_{(1)}\phi(\alpha^{-1}(h_2)) = (mh_2)_{(0)} \otimes \alpha^{-1}(\varphi(h_1)(mh_2)_{(1)}). \tag{4.29}$$

Proof: Assume that (4.29) holds, then

$$
\begin{aligned}
& m_{(0)}\alpha(h_{21})\otimes\varphi(S(h_1))(\alpha^{-1}(m_{(1)})\phi(h_{22})) \\
&\quad = m_{(0)}\alpha^{-1}(\alpha^2(h_{21}))\otimes\varphi(S(h_1))(\alpha^{-1}(m_{(1)})\alpha^{-2}(\phi(\alpha^2(h_{22})))) \\
&\quad = m_{(0)}\alpha^{-1}(\alpha^2(h_2)_1)\otimes\varphi(S(h_1))\alpha^{-1}(m_{(1)}\alpha^{-1}(\phi(\alpha^2(h_2)_2))) \\
&\quad \overset{(4.29)}{=} (m\alpha^2(h_2)_2)_{(0)}\otimes\varphi(S(h_1))(\alpha^{-2}(\varphi(\alpha^2(h_2)_1))\alpha^{-2}((m\alpha^2(h_2)_2)_{(1)})) \\
&\quad = (m\alpha^2(h_{22}))_{(0)}\otimes\varphi(S(h_1))(\varphi(h_{21})\alpha^{-2}((m\alpha^2(h_{22}))_{(1)})) \\
&\quad \overset{(3.5)}{=} (m\alpha(h_2))_{(0)}\otimes\varphi(S(\alpha(h_{11})))(\varphi(h_{12})\alpha^{-2}((m\alpha(h_2))_{(1)})) \\
&\quad = (m\alpha(h_2))_{(0)}\otimes\varphi(S(h_{11})h_{12})\alpha^{-1}((m\alpha(h_2))_{(1)}) \\
&\quad = (m\alpha(h_2))_{(0)}\otimes\varphi(\varepsilon(h_1)1_H)\alpha^{-1}((m\alpha(h_2))_{(1)})) \\
&\quad = \varepsilon(h_1)(m\alpha(h_2))_{(0)}\otimes(m\alpha(h_2))_{(1)} \\
&\quad = \varepsilon(h_1)\rho^M(m\alpha(h_2)) = \rho^M(mh),
\end{aligned}
$$

which gives us (4.26). One can easily show that by applying the Hom-coassociativity condition (3.5) twice we have

$$\alpha^{-1}(h_1)\otimes h_{21}\otimes\alpha(h_{221})\otimes\alpha(h_{222}) = h_{11}\otimes h_{12}\otimes h_{21}\otimes h_{22}, \tag{4.30}$$

which is used in the below computation. Thus, if we suppose that (4.26) holds, then

$$
\begin{aligned}
& (mh_2)_{(0)}\otimes\alpha^{-1}(\varphi(h_1)(mh_2)_{(1)}) \\
&\quad \overset{(4.26)}{=} m_{(0)}\alpha(h_{221})\otimes\alpha^{-1}(\varphi(h_1)(\varphi(S(h_{21}))(\alpha^{-1}(m_{(1)})\phi(h_{222})))) \\
&\quad = m_{(0)}\alpha(h_{221})\otimes\alpha^{-1}((\alpha^{-1}(\varphi(h_1))\varphi(S(h_{21})))(m_{(1)}\alpha(\phi(h_{222})))) \\
&\quad \overset{(4.30)}{=} m_{(0)}h_{21}\otimes\alpha^{-1}((\varphi(h_{11})\varphi(S(h_{12})))(m_{(1)}\phi(h_{22}))) \\
&\quad = m_{(0)}h_{21}\otimes(\varepsilon(h_1)1_H)\alpha^{-1}(m_{(1)}\phi(h_{22})) \\
&\quad = m_{(0)}h_{21}\otimes\varepsilon(h_1)m_{(1)}\phi(h_{22}) \\
&\quad \overset{(3.5)}{=} m_{(0)}h_{12}\varepsilon(h_{11})\otimes m_{(1)}\phi(\alpha^{-1}(h_2)) \\
&\quad = m_{(0)}\alpha^{-1}(h_1)\otimes m_{(1)}\phi(\alpha^{-1}(h_2)),
\end{aligned}
$$

finishing the proof. □

Remark 9 *The above result implies that the equations (4.27) and (4.28) are equivalent to*

$$m_{(0)}\alpha^{-1}(h_1)\otimes m_{(1)}\alpha^{-1}(h_2) = (mh_2)_{(0)}\otimes\alpha^{-1}(h_1(mh_2)_{(1)})$$

and

$$m_{(0)}\alpha^{-1}(h_1)\otimes m_{(1)}\alpha^{-1}(h_2)=(mh_2)_{(0)}\otimes\alpha^{-1}(S^{-2}(h_1)(mh_2)_{(1)}),$$

respectively.

Example 4.3.9 ***The flip and Hom-Long dimodule*** *Let (H,α) be a monoidal Hom-bialgebra. Then:*

1. *$[(H,\alpha),(H,\alpha)]_\psi$, where $\psi: H\otimes H\to H\otimes H$, $g\otimes h\mapsto h\otimes g$, is an Hom-entwining structure.*
2. *(M,μ) is an $[(H,\alpha),(H,\alpha)]_\psi$-entwined Hom-module if and only if it is a right (H,α)-Hom-module with $\rho_M: M\otimes H\to M$, $m\otimes h\mapsto mh$ and a right (H,α)-Hom-comodule with $\rho^M: M\to M\otimes H$, $m\mapsto m_{(0)}\otimes m_{(1)}$ such that*
$$\rho^M(mh)=m_{(0)}\alpha^{-1}(h)\otimes\alpha(m_{(1)}) \tag{4.31}$$
for all $m\in M$ and $h\in H$. Such Hom-modules (M,μ) are called (right-right) (H,α)-Hom-Long dimodules (see [27]).
3. *$(\mathcal{C},\chi)=(H\otimes H,\alpha\otimes\alpha)$ is an (H,α)-Hom-coring with comultiplication $\Delta_{\mathcal{C}}(h\otimes h')=(\alpha^{-1}(h)\otimes h'_1)\otimes_H(1_H\otimes h'_2)$ and counit $\varepsilon_{\mathcal{C}}(h\otimes h')=\alpha(h)\varepsilon_H(h')$, and (H,α)-Hom-bimodule structure*
$$g(h\otimes h')=\alpha^{-1}(g)h\otimes\alpha(h'),\ (h\otimes h')g=h\alpha^{-1}(g)\otimes\alpha(h')$$
for all $h,h',g\in H$.

Proof: (H,α) itself is a right (H,α)-Hom-comodule algebra with Hom-coaction $\rho^H=\Delta_H: H\to H\otimes H$, $h\mapsto h_{(0)}\otimes h_{(1)}=h_1\otimes h_2$. In addition, (H,α) becomes a right (H,α)-Hom-module coalgebra with the trivial Hom-action $\rho_H: H\otimes H\to H$, $g\otimes h\mapsto g\cdot h=\alpha(g)\varepsilon(h)$. Hence we have $[(H,\alpha),(H,\alpha),(H,\alpha)]$ as Hom-Doi-Koppinen datum with the associated Hom-entwining structure $[(H,\alpha),(H,\alpha)]_\psi$, where $\psi(h'\otimes h)=\alpha(h_{(0)})\otimes\alpha^{-1}(h')\cdot h_{(1)}=\alpha(h_1)\otimes\alpha^{-1}(h')\cdot h_2=\alpha(h_1)\otimes\alpha(\alpha^{-1}(h'))\varepsilon(h_2)=h\otimes h'$. □

Definition 4.3.10 *Let (B,β) be a monoidal Hom-bialgebra. A left (B,β)-Hom-comodule coalgebra (C,γ) is a monoidal Hom-coalgebra and a left (B,β)-Hom-comodule with a Hom-coaction $\rho: C\to B\otimes C$, $c\mapsto c_{(-1)}\otimes c_{(0)}$ such that, for any $c\in C$*

$$c_{(-1)}\otimes c_{(0)1}\otimes c_{(0)2}=c_{1(-1)}c_{2(-1)}\otimes c_{1(0)}\otimes c_{2(0)},\ c_{(-1)}\varepsilon_C(c_{(0)})=1_B\varepsilon_C(c). \tag{4.32}$$

The equation

$$\rho \circ \gamma = (\beta \otimes \gamma) \circ \rho$$

follows from (4.32) and the properties of (B,β) *and* (C,γ)*.*

We lastly introduce the below construction regarding the Hom-version of the so-called alternative Doi-Koppinen datum given in [73].

Proposition 4.3.11 *Let* (B,β) *be a monoidal Hom-bialgebra. Let* (A,α) *be a left* (B,β)*-Hom-module algebra with Hom-action* ${}_A\rho : B \otimes A \to A,\ b \otimes a \mapsto b \cdot a$ *and* (C,γ) *be a left* (B,β)*-Hom-comodule coalgebra with Hom-coaction* ${}^C\rho : C \to B \otimes C,\ c \mapsto c_{(-1)} \otimes c_{(0)}$*. Define the map*

$$\psi : C \otimes A \to A \otimes C,\ c \otimes a \mapsto c_{(-1)} \cdot \alpha^{-1}(a) \otimes \gamma(c_{(0)}) \tag{4.33}$$

Then the following statements hold.

1. $[(A,\alpha),(C,\gamma)]_\psi$ *is an Hom-entwining structure.*
2. (M,μ) *is an* $[(A,\alpha),(C,\gamma)]_\psi$*-entwined Hom-module iff it is a right* (A,α)*-Hom-module with* $\rho_M : M\otimes A \to M,\ m\otimes a \mapsto ma$ *and a right* (C,γ)*-Hom-comodule with* $\rho^M : M \to M \otimes C,\ m \mapsto m_{[0]} \otimes m_{[1]}$ *such that*

$$\rho^M(ma) = (ma)_{[0]} \otimes (ma)_{[1]} = m_{[0]}(m_{[1](-1)} \cdot \alpha^{-2}(a)) \otimes \gamma^2(m_{[1](0)}) \tag{4.34}$$

   for any $m \in M$ *and* $a \in A$*.*

3. $(\mathcal{C},\chi) = (A\otimes C, \alpha\otimes\gamma)$ *is an* (A,α)*-Hom-coring with comultiplication and counit given by (4.13) and (4.14), respectively, and the* (A,α)*-Hom-bimodule structure* $a(a'\otimes c) = \alpha^{-1}(a)a' \otimes \gamma(c)$, $(a' \otimes c)a = a'(c_{(-1)} \cdot \alpha^{-2}(a)) \otimes \gamma^2(c_{(0)})$ *for* $a,a' \in A$ *and* $c \in C$*.*

A triple $[(A,\alpha),(B,\beta),(C,\gamma)]$ *satisfying the above assumptions of the proposition is called an alternative Hom-Doi-Koppinen datum.*

Proof: The first two conditions for Hom-entwining structures will be checked and the rest of the proof can be completed by performing similar computations as in Proposition (4.3.2). For $a,a' \in A$ and $c \in C$,

$$
\begin{aligned}
(aa')_\kappa \otimes \gamma(c)^\kappa &= \gamma(c)_{(-1)} \cdot \alpha^{-1}(aa') \otimes \gamma(\gamma(c)_{(0)}) \\
&= \beta(c_{(-1)}) \cdot (\alpha^{-1}(a)\alpha^{-1}(a')) \otimes \gamma^2(c_{(0)}) \\
&= (\beta(c_{(-1)})_1 \cdot \alpha^{-1}(a))(\beta(c_{(-1)})_2 \cdot \alpha^{-1}(a')) \otimes \gamma^2(c_{(0)}) \\
&= (\beta(c_{(-1)1}) \cdot \alpha^{-1}(a))(\beta(c_{(-1)2}) \cdot \alpha^{-1}(a')) \otimes \gamma^2(c_{(0)}) \\
&= (\beta(\beta^{-1}(c_{(-1)})) \cdot \alpha^{-1}(a))(\beta(c_{(0)(-1)}) \cdot \alpha^{-1}(a')) \otimes \gamma^2(\gamma(c_{(0)(0)})) \\
&= (c_{(-1)} \cdot \alpha^{-1}(a))(\gamma(c_{(0)})_{(-1)} \cdot \alpha^{-1}(a')) \otimes \gamma^2(\gamma(c_{(0)})_{(0)}) \\
&= (c_{(-1)} \cdot \alpha^{-1}(a))a'_\lambda \otimes \gamma(\gamma(c_{(0)})^\lambda) \\
&= a_\kappa a'_\lambda \otimes \gamma(c^{\kappa\lambda}),
\end{aligned}
$$

$$
\begin{aligned}
\alpha^{-1}(a_\kappa) \otimes c^\kappa{}_1 \otimes c^\kappa{}_2 &= \alpha^{-1}(c_{(-1)} \cdot \alpha^{-1}(a)) \otimes \gamma(c_{(0)})_1 \otimes \gamma(c_{(0)})_2 \\
&= \beta^{-1}(c_{(-1)}) \cdot \alpha^{-2}(a) \otimes \gamma(c_{(0)1}) \otimes \gamma(c_{(0)2}) \\
&= \beta^{-1}(c_{1(-1)}c_{2(-1)}) \cdot \alpha^{-2}(a) \otimes \gamma(c_{1(0)}) \otimes \gamma(c_{2(0)}) \\
&= (\beta^{-1}(c_{1(-1)})\beta^{-1}(c_{2(-1)})) \cdot \alpha^{-2}(a) \otimes \gamma(c_{1(0)}) \otimes \gamma(c_{2(0)}) \\
&= c_{1(-1)} \cdot (\beta^{-1}(c_{2(-1)}) \cdot \alpha^{-3}(a)) \otimes \gamma(c_{1(0)}) \otimes \gamma(c_{2(0)}) \\
&= c_{1(-1)} \cdot \alpha^{-1}(c_{2(-1)} \cdot \alpha^{-2}(a)) \otimes \gamma(c_{1(0)}) \otimes \gamma(c_{2(0)}) \\
&= (c_{2(-1)} \cdot \alpha^{-1}(\alpha^{-1}(a)))_\kappa \otimes c_1{}^\kappa \otimes \gamma(c_{2(0)}) \\
&= \alpha^{-1}(a)_{\lambda\kappa} \otimes c_1{}^\kappa \otimes c_2{}^\lambda.
\end{aligned}
$$

□

Chapter 5

Covariant Hom-Differential Calculus

The general theory of covariant differential calculi on quantum groups was introduced by S. L. Woronowicz in [80], [81],[82]. Many results obtained in this chapter in the Hom-setting follow from the classical results appear in the fundamental reference [82]. In Section 5.1, after the notions of first order differential calculus (FODC) on a monoidal Hom-algebra and left-covariant FODC over a left Hom-quantum space with respect to a monoidal Hom-Hopf algebra are presented, the left-covariance of a Hom-FODC is characterized. The extension of the universal FODC over a monoidal Hom-algebra to a universal Hom-differential calculus (Hom-DC) is described as well (for the classical case, that is, for the extension of a FODC over an algebra A to the differential envelope of A one should refer to [32], [31]). In the rest of the chapter, the concepts of left-covariant and bicovariant FODC over a monoidal Hom-Hopf algebra (H, α) are studied in detail. A subobject $\mathcal{R}$ of $ker\varepsilon$, which is a right Hom-ideal of (H, α), and a quantum Hom-tangent space are associated to each left-covariant (H, α)-Hom-FODC: It is indicated that left-covariant Hom-FODCs are in one-to one correspondence with these right Hom-ideals $\mathcal{R}$, and that the quantum Hom-tangent space and the left coinvariant of the monoidal Hom-Hopf algebra on Hom-FODC form a nondegenerate dual pair. The quantum Hom-tangent space associated to a bicovariant Hom-FODC is equipped with an analogue of Lie bracket (or commutator) through Woronowicz' braiding and it is proven that this commutator satisfies quantum (or generalized) versions of the anti-symmetry relation and Hom-Jacobi identity, which is therefore called the quantum (or generalized) Hom-Lie algebra of that bicovariant Hom-FODC. Throughout the chapter, we work with vector spaces over a field k.

5.1 Left-Covariant FODC over Hom-quantum spaces

Definition 5.1.1 *A first order differential calculus over a monoidal Hom-algebra* (A,α) *is an* (A,α)*-Hom-bimodule* (Γ,γ) *with a linear map* $d: A \to \Gamma$ *such that*

1. d *satisfies the Leibniz rule, i.e.,* $d(ab) = a\cdot db + da\cdot b, \forall a,b\in A$,
2. $d\circ\alpha = \gamma\circ d$, *which means that* d *is in* $\widetilde{\mathcal{H}}(\mathcal{M}_k)$,
3. Γ *is linearly spanned by the elements of the form* $(a\cdot db)\cdot c$ *with* $a,b,c\in A$.

We call (Γ,γ) an (A,α)-Hom-FODC for short.

Remark 10 1. *In the above definition, the second condition, i.e.* $d\circ\alpha = \gamma\circ d$, *is equivalent to the equality* $d1 = 0$.

2. *By the compatibility condition for Hom-bimodule structure of* (Γ,γ), *we have* $(a\cdot db)\cdot c = \alpha(a)\cdot(db\cdot\alpha^{-1}(c))$, *which implies that* Γ *is also linearly spanned by the elements* $a\cdot(db\cdot c)$ *for all* $a,b,c\in A$. *Thus we denote* $\Gamma = (A\cdot dA)\cdot A = A\cdot(dA\cdot A)$.

3. *By using the Leibniz rule and the fact that* $d(\alpha(a)) = \gamma(da)$ *for any* $a\in A$, *we get*

$$\begin{aligned}(a\cdot db)\cdot c &= (d(ab) - da\cdot b)\cdot c = d(ab)\cdot c - (da\cdot b)\cdot c\\ &= d(ab)\cdot c - \gamma(d(a))\cdot(b\alpha^{-1}(c)) = d(ab)\cdot c - d(\alpha(a))\cdot(b\alpha^{-1}(c)),\end{aligned}$$

and

$$\begin{aligned}\alpha(a)\cdot(db\cdot\alpha^{-1}(c)) &= \alpha(a)\cdot(d(b\alpha^{-1}(c)) - b\cdot d(\alpha^{-1}(c)))\\ &= \alpha(a)\cdot d(b\alpha^{-1}(c)) - \alpha(a)\cdot(b\cdot d(\alpha^{-1}(c)))\\ &= \alpha(a)\cdot d(b\alpha^{-1}(c)) - (ab)\cdot d(c).\end{aligned}$$

Hence, $\Gamma = A\cdot dA = dA\cdot A$.

Definition 5.1.2 *Let* (H,β) *be a monoidal Hom-bialgebra and* (A,α) *be a left Hom-quantum space for* (H,β) *(i.e. a left* (H,β)*-Hom-comodule algebra) with the left Hom-coaction* $\varphi: A\to H\otimes A$, $a\mapsto a_{(-1)}\otimes a_{(0)}$. *An* (A,α)*-Hom-FODC* (Γ,γ) *is called left-covariant with respect to* (H,β) *if there is a left Hom-coaction* $\phi:\Gamma\to H\otimes\Gamma$, $\omega\mapsto\omega_{(-1)}\otimes\omega_{(0)}$ *of* (H,β) *on* (Γ,γ) *such that*

1. $\phi(\alpha(a)\cdot(\omega\cdot b))=\varphi(\alpha(a))(\phi(\omega)\varphi(b))$, $\forall a,b\in A$, $\omega\in\Gamma$,

2. $\phi(da)=(id\otimes d)\varphi(a)$, $\forall a\in A$

Condition (1) can equivalently be written as $\phi((a\cdot\omega)\cdot\alpha(b))=(\varphi(a)\phi(\omega))\varphi(\alpha(b))$ by using the Hom-bimodule compatibility conditions for (Γ,γ) and $(H\otimes\Gamma,\beta\otimes\gamma)$, where left and right $(H\otimes A,\beta\otimes\alpha)$-Hom-module structures of $(H\otimes\Gamma,\beta\otimes\gamma)$ are respectively given by

$$(h\otimes a)(h'\otimes\omega)=hh'\otimes a\cdot\omega,$$

$$(h'\otimes\omega)(h\otimes a)=hh'\otimes\omega\cdot a$$

for $h,h\in H$, $a\in A$ and $\omega\in\Gamma$. Condition (2) means that $d:A\to\Gamma$ is left (H,β)-colinear, since the equality $d\circ\alpha=\gamma\circ d$ holds too.

One can see that for a given (A,α)-Hom-FODC (Γ,γ) there exists at most one morphism $\phi:\Gamma\to H\otimes\Gamma$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ which makes (Γ,γ) left-covariant: Indeed, if there is one such ϕ, then by the conditions (1) and (2) in Definition 5.1.2 we do the following computation

$$\begin{aligned}
\phi\left(\sum_i a_i\cdot db_i\right) &= \sum_i \phi(\gamma^{-1}(a_i\cdot db_i)\cdot 1_A)=\sum_i \phi((\alpha^{-1}(a_i)\cdot\gamma^{-1}(db_i))\cdot 1_A)\\
&= \sum_i (\varphi(\alpha^{-1}(a_i))\phi(\gamma^{-1}(db_i)))\varphi(1_A)\\
&= \sum_i ([(\beta^{-1}\otimes\alpha^{-1})(\varphi(a_i))][(\beta^{-1}\otimes\gamma^{-1})(\phi(db_i))])(1_H\otimes 1_A)\\
&= \sum_i [(\beta^{-1}\otimes\gamma^{-1})(\varphi(a_i)\phi(db_i))](1_H\otimes 1_A)\\
&= \sum_i \varphi(a_i)\phi(db_i)=\sum_i \varphi(a_i)(id\otimes d)(\varphi(b_i)),
\end{aligned}$$

showing that φ and d describe ϕ uniquely.

Proposition 5.1.3 *Let (Γ,γ) be an (A,α)-Hom-FODC. Then the following statements are equivalent:*

1. *(Γ,γ) is left-covariant.*

2. *There is a morphism $\phi:\Gamma\to H\otimes\Gamma$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that $\phi(a\cdot db)=\varphi(a)(id\otimes d)(\varphi(b))$ for all $a,b\in A$.*

3. $\sum_i a_i \cdot db_i = 0$ *in* Γ *implies that* $\sum_i \varphi(a_i)(id \otimes d)(\varphi(b_i)) = 0$ *in* $H \otimes \Gamma$*.*

Proof:$(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are trivial.

$(3) \Rightarrow (1)$: Let $\phi : \Gamma \to H \otimes \Gamma$ be defined by the equation

$$\phi(\sum_i a_i \cdot db_i) = \sum_i \varphi(a_i)(id \otimes d)(\varphi(b_i))$$

as was obtained in the above computation. By using hypothesis (3) it is immediate to see that ϕ is well-defined. If we write $\varphi(a) = a_{(-1)} \otimes a_{(0)}$ for any $a \in A$ and $\phi(\omega) = \omega_{(-1)} \otimes \omega_{(0)}$ for all $\omega \in \Gamma$, then for $\omega = \sum_i a_i \cdot db_i \in \Gamma$ we have

$$\phi(\omega) = \omega_{(-1)} \otimes \omega_{(0)} = \sum_i a_{i,(-1)} b_{i,(-1)} \otimes a_{i,(0)} \cdot db_{i,(0)},$$

where we have used the notation $\varphi(a_i) = a_{i,(-1)} \otimes a_{i,(0)}$. Now we prove that ϕ is a left Hom-coaction of (H, β) on (Γ, γ):

$$\begin{aligned}
\beta^{-1}(\omega_{(-1)}) \otimes \phi(\omega_{(0)}) &= \sum_i \beta^{-1}(a_{i,(-1)} b_{i,(-1)}) \otimes \phi(a_{i,(0)} \cdot db_{i,(0)}) \\
&= \sum_i \beta^{-1}(a_{i,(-1)}) \beta^{-1}(b_{i,(-1)}) \otimes a_{i,(0)(-1)} b_{i,(0)(-1)} \otimes a_{i,(0)(0)} \cdot db_{i,(0)(0)} \\
&= \sum_i a_{i,(-1)1} b_{i,(-1)1} \otimes a_{i,(-1)2} b_{i,(-1)2} \otimes \alpha^{-1}(a_{i,(0)}) \cdot d(\alpha^{-1}(b_{i,(0)})) \\
&= \sum_i (a_{i,(-1)} b_{i,(-1)})_1 \otimes (a_{i,(-1)} b_{i,(-1)})_2 \otimes \gamma^{-1}(a_{i,(0)} \cdot db_{i,(0)}) \\
&= \Delta(\omega_{(-1)}) \otimes \gamma^{-1}(\omega_{(0)}),
\end{aligned}$$

$$\begin{aligned}
\varepsilon(\omega_{(-1)})\omega_{(0)} &= \sum_i \varepsilon(a_{i,(-1)} b_{i,(-1)}) a_{i,(0)} \cdot db_{i,(0)} \\
&= \sum_i \varepsilon(a_{i,(-1)}) a_{i,(0)} \cdot d(\varepsilon(b_{i,(-1)}) b_{i,(0)}) \\
&= \sum_i \alpha^{-1}(a_i) \cdot d(\alpha^{-1}(b_i)) = \gamma^{-1}(\omega),
\end{aligned}$$

$$
\begin{aligned}
\phi(\gamma(\sum_i a_i \cdot db_i)) &= \phi(\sum_i \alpha(a_i) \cdot d(\alpha(b_i))) \\
&= \sum_i \varphi(\alpha(a_i))(id \otimes d)(\varphi(\alpha(b_i))) \\
&= (\beta \otimes \alpha)(\varphi(a_i))(id \otimes d)((\beta \otimes \alpha)(\varphi(b_i))) \\
&= \sum_i \beta(a_{i,(-1)})\beta(b_{i,(-1)}) \otimes \alpha(a_{i,(0)}) \cdot d(\alpha(b_{i,(0)})) \\
&= (\beta \otimes \gamma)(\phi(\sum_i a_i \cdot db_i)).
\end{aligned}
$$

Let $\omega = \sum_i a_i \cdot db_i \in \Gamma$, and $a, b \in A$. Then we have

$$
\begin{aligned}
&\phi(\alpha(a) \cdot (\omega \cdot b)) \\
&= \phi(\alpha(a) \cdot (\sum_i (a_i \cdot db_i) \cdot b)) \\
&= \phi(\alpha(a) \cdot (\sum_i (\alpha(a_i) \cdot d(b_i \alpha^{-1}(b)) - (a_i b_i) \cdot db))) \\
&= \phi(\sum_i [(a\alpha(a_i)) \cdot d(\alpha(b_i)b) - (a(a_i b_i)) \cdot d(\alpha(b))]) \\
&= \sum_i \varphi(a\alpha(a_i))(id \otimes d)(\varphi(\alpha(b_i)b)) - \sum_i \varphi(a(a_i b_i))(id \otimes d)(\varphi(\alpha(b))) \\
&= \sum_i (\varphi(a)\varphi(\alpha(a_i)))(id \otimes d)(\varphi(\alpha(b_i)b)) - \sum_i (\varphi(a)\varphi(a_i b_i))(id \otimes d)(\varphi(\alpha(b))) \\
&= \sum_i \varphi(\alpha(a))(\varphi(\alpha(a_i))(id \otimes d)(\varphi(b_i \alpha^{-1}(b)))) - \sum_i \varphi(\alpha(a))(\varphi(a_i b_i)(id \otimes d)(\varphi(b))) \\
&= \varphi(\alpha(a))(\sum_i \varphi(\alpha(a_i))(id \otimes d)(\varphi(b_i \alpha^{-1}(b))) - \sum_i \varphi(a_i b_i)(id \otimes d)(\varphi(b))) \\
&= \varphi(\alpha(a))(\sum_i \varphi(\alpha(a_i))(id \otimes d)(\varphi(b_i \alpha^{-1}(b))) - \sum_i (\varphi(a_i)\varphi(b_i))(id \otimes d)(\varphi(b))) \\
&= \varphi(\alpha(a))(\sum_i \varphi(\alpha(a_i))[(id \otimes d)(\varphi(b_i \alpha^{-1}(b))) - \varphi(b_i)(id \otimes d)(\varphi(\alpha^{-1}(b)))]) \\
&= \varphi(\alpha(a))(\sum_i \varphi(\alpha(a_i))[((id \otimes d)(\varphi(b_i)))\varphi(\alpha^{-1}(b))]) \\
&= \varphi(\alpha(a))([\sum_i \varphi(a_i)(id \otimes d)(\varphi(b_i))]\varphi(b)) \\
&= \varphi(\alpha(a))(\phi(\omega)\varphi(b)),
\end{aligned}
$$

which is the first condition of Definition 5.1.2. For any $a \in A$, we get

$$\begin{aligned}
\phi(da) &= \phi(1_A \cdot \gamma^{-1}(da)) = \phi(1_A \cdot d(\alpha^{-1}(a))) \\
&= \varphi(1_A)(id \otimes d)(\varphi(\alpha^{-1}(a))) \\
&= (1_H \otimes 1_A)[((id \otimes d) \circ (\beta^{-1} \otimes \alpha^{-1}))(\varphi(a))] \\
&= (1_H \otimes 1_A)[((\beta^{-1} \otimes \gamma^{-1}) \circ (id \otimes d))(\varphi(a))] = (id \otimes d)(\varphi(a)),
\end{aligned}$$

which is the second condition of Definition 5.1.2. □

5.2 Universal Differential Calculus of a Monoidal Hom-Hopf Algebra

In the theory of quantum groups, a differential calculus is a substitute of the de Rham complex of a smooth manifold for arbitrary algebras. In this section, the definition of differential calculus over a monoidal Hom-algebra (abbreviated, Hom-DC) is given and the construction of the universal differential calculus of a monoidal Hom-algebra (universal Hom-DC) is outlined.

Definition 5.2.1 *A graded monoidal Hom-algebra is a monoidal Hom-algebra* (A, α) *together with subobjects* $A_n, n \geq 0$ *(that is, for each* k*-submodule* $A_n \subseteq A$, $(A_n, \alpha|_{A_n}) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$*) such that*

$$A = \bigoplus_{n \geq 0} A_n,$$

$1 \in A_0$, *and* $A_n A_m \subseteq A_{n+m}$ *for all* $n, m \geq 0$.

Definition 5.2.2 *A differential calculus over a monoidal Hom-algebra* (A, α) *is a graded monoidal Hom-algebra* $(\Gamma = \bigoplus_{n \geq 0} \Gamma^n, \gamma)$ *with a linear map* $d : \Gamma \to \Gamma$, *in* $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, *of degree one (i.e.,* $d : \Gamma^n \to \Gamma^{n+1}$*) such that*

1. $d^2 = 0$,
2. $d(\omega\omega') = d(\omega)\omega' + (-1)^n \omega d(\omega')$ *for* $\omega \in \Gamma^n, \omega' \in \Gamma$ *(graded Leibniz rule),*
3. $\Gamma^0 = A$, $\gamma|_{\Gamma^0} = \alpha$, *and* Γ^n *is a linear span of the elements of the form* $a_0(da_1(\cdots(da_{n-1}da_n)\cdots))$ *with* $a_0, \cdots, a_n \in A$, $n \geq 0$.

A differential Hom-ideal of (Γ,γ) *is a Hom-ideal* $\mathcal{I}$ *of the monoidal Hom-algebra* (Γ,γ) *(that is,* $\mathcal{I}$ *is a subobject of* (Γ,γ) *such that* $(\Gamma\mathcal{I})\Gamma=\Gamma(\mathcal{I}\Gamma)\subset\mathcal{I}$*) such that* $\mathcal{I}\cap\Gamma^0=\{0\}$ *and* $\mathcal{I}$ *is invariant under the differentiation* d*.*

Let us write γ^n for $\gamma|_{\Gamma^n}$ for all $n\geq 0$. Then, the map $d\in\widetilde{\mathcal{H}}(\mathcal{M}_k)$ means that $d\circ\gamma^n=\gamma^{n+1}\circ d$ for all $n\geq 0$. Let $\mathcal{I}$ be a differential Hom-ideal of a (A,α)-Hom-DC (Γ,γ). Then, γ induces an automorphism $\bar{\gamma}$ of $\bar{\Gamma}:=\Gamma/\mathcal{I}$ and $(\bar{\Gamma},\bar{\gamma})$ is a monoidal Hom-algebra. Since the condition $\mathcal{I}\cap\Gamma^0=\{0\}$ holds, $\bar{\Gamma}^0=\Gamma^0=A$. On the other hand, let $\pi:\Gamma\to\bar{\Gamma}$ be the canonical surjective map and define $\bar{d}:\bar{\Gamma}\to\bar{\Gamma}$ by $\bar{d}(\pi(\omega)):=\pi(d(\omega))$ for any $\omega\in\Gamma$. Thus, $(\bar{\Gamma},\bar{\gamma})$ is again a Hom-DC on (A,α) with differentiation $\bar{d}$.

In the rest of the section, the construction of the universal differential calculus on a monoidal Hom-algebra (A,α) is discussed. Let (A,α) be a monoidal Hom-algebra with Hom-multiplication $m_A:A\otimes A\to A$. The linear map $d:A\to A\otimes A$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, given by

$$da:=1\otimes\alpha^{-1}(a)-\alpha^{-1}(a)\otimes 1,\ \forall a\in A$$

satisfies the Leibniz rule: For $a,b\in A$,

$$\begin{aligned}
a\cdot db+da\cdot b &= a\cdot(1\otimes\alpha^{-1}(b)-\alpha^{-1}(b)\otimes 1)+(1\otimes\alpha^{-1}(a)-\alpha^{-1}(a)\otimes 1)\cdot b\\
&= \alpha^{-1}(a)1\otimes\alpha(\alpha^{-1}(b))-\alpha^{-1}(a)\alpha^{-1}(b)\otimes 1\\
&+ 1\otimes\alpha^{-1}(a)\alpha^{-1}(b)-\alpha(\alpha^{-1}(a))\otimes 1\alpha^{-1}(b)\\
&= a\otimes b-\alpha^{-1}(ab)\otimes 1+1\otimes\alpha^{-1}(ab)-a\otimes b=1\otimes\alpha^{-1}(ab)-\alpha^{-1}(ab)\otimes 1\\
&= d(ab).
\end{aligned}$$

For any $a\in A$, we get

$$\begin{aligned}
(d\circ\alpha)(a) &= d(\alpha(a))=1\otimes a-a\otimes 1\\
&= (\alpha\otimes\alpha)(1\otimes\alpha^{-1}(a)-\alpha^{-1}(a)\otimes 1)=(\alpha\otimes\alpha)(da),
\end{aligned}$$

meaning d is in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Let $\Omega^1(A)$ be the (A,α)-Hom-subbimodule of $(A\otimes A,\alpha\otimes\alpha)$ generated by elements of the form $a\cdot db$ for $a,b\in A$. Then we have

$$\Omega^1(A)=ker\, m_A.$$

Indeed, if $a\cdot db\in\Omega^1(A)$, then

$$m_A(a\cdot db)=m_A(a\otimes b-\alpha^{-1}(ab)\otimes 1)=ab-\alpha^{-1}(ab)1=0.$$

On the other hand, if $\sum_i a_i \otimes b_i \in ker\, m_A$ ($\sum_i$ denotes a finite sum), then $\sum_i a_i b_i = 0$, thus we write

$$\sum_i a_i \otimes b_i = \sum_i (a_i \otimes b_i - \alpha^{-1}(a_i b_i) \otimes 1) = \sum_i a_i \cdot (1 \otimes \alpha^{-1}(b_i) - \alpha^{-1}(b_i) \otimes 1) = \sum_i a_i \cdot db_i.$$

The left and right (A, α)-Hom-module structures of $(\Omega^1(A), \beta) = (\Omega^1(A), (\alpha \otimes \alpha)|_{ker\, m_A})$ are respectively given by

$$a \cdot (b \cdot dc) = (\alpha^{-1}(a)b) \cdot d(\alpha(c)) \qquad (a \cdot db) \cdot c = \alpha(a) \cdot d(b\alpha^{-1}(c)) - (ab) \cdot dc,$$

for any $a, b, c \in A$. $(\Omega^1(A), \beta)$ is called the *universal first order differential calculus* of monoidal Hom-algebra (A, α).

Let $\bar{A} := A/k \cdot 1$ be the quotient space of A by the scalar multiples of the Hom-unit and let $\bar{a}$ denote the equivalence class $a + k \cdot 1$ for any $a \in A$. α induces an automorphism $\bar{\alpha} : \bar{A} \to \bar{A}$, $\bar{a} \mapsto \bar{\alpha}(\bar{a}) = \overline{\alpha(a)}$ and $(\bar{A}, \bar{\alpha}) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$. Let $A \otimes \bar{A} = \Omega^1(A)$ by the identification $a_0 \otimes \overline{a_1} \mapsto a_0 da_1$. This identification is well-defined since $d1 = 0$, and one can easily show that it is an (A, α)-Hom-bimodule isomorphism once the Hom-bimodule structure of $(A \otimes \bar{A}, \alpha \otimes \bar{\alpha})$ is given by, for $b \in A$,

$$b(a_0 \otimes \overline{a_1}) = \alpha^{-1}(b)a_0 \otimes \overline{\alpha(a_1)}, \ (a_0 \otimes \overline{a_1})b = \alpha(a_0) \otimes \overline{a_1 \alpha^{-1}(b)} - a_0 a_1 \otimes \bar{b}.$$

Now, we set

$$\Omega^n(A) := \otimes_A^{(n)}(\Omega^1(A)) = \Omega^1(A) \otimes_A (\otimes_A^{(n-1)}(\Omega^1(A))).$$

Above, $\otimes_A^{(n)}(\Omega^1(A))$ has been put for

$$T_A^n(\Omega^1(A)) = \otimes_A^{t^n}(\Omega^1(A), \cdots, \Omega^1(A)) = \Omega^1(A) \otimes_A (\Omega^1(A) \otimes_A (\cdots (\Omega^1(A) \otimes_A \Omega^1(A)) \cdots)),$$

where t^n is a fixed element in the set T_n of planar binary trees with n leaves and one root, which corresponds to the parenthesized monomial $x_1(x_2(\cdots(x_{n-1}x_n)\cdots))$ in n non-commuting variables (see [83] e.g.). One should also refer to [21, Section 6] for the construction of tensor Hom-algebra applied to an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$). So, we have, for any $n \geq 0$,

$$\Omega^n(A) = A \otimes (\otimes^{(n)}(\bar{A})) = A \otimes (\bar{A} \otimes (\bar{A} \otimes (\cdots (\bar{A} \otimes \bar{A}) \cdots)))$$

by the correspondence $(A\otimes\bar{A})\otimes_A(A\otimes(\otimes^{(n-1)}(\bar{A})))=A\otimes(\otimes_A^{(n)}(\bar{A}))$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$,

$$\begin{aligned}(a_0\otimes\overline{a_1})\otimes_A(a_2\otimes(\otimes^{(n-1)}(\overline{a_3},\cdots,\overline{a_{n+1}}))) \quad\mapsto\quad & \alpha(a_0)\otimes(\otimes^n(\overline{\alpha^{-1}(a_1a_2)},\overline{a_3},\cdots,\overline{a_{n+1}}))\\ & -a_0a_1\otimes(\otimes^n(\overline{a_2},\cdots,\overline{a_{n+1}})),\end{aligned}$$

where we have used the notation $\otimes^{(n)}(\overline{a_1},\cdots,\overline{a_n})$ for $\overline{a_1}\otimes(\overline{a_2}\otimes(\cdots(\overline{a_{n-1}}\otimes\overline{a_n})\cdots))$. To the object $A\otimes(\otimes^{(n)}(\bar{A}))$ we associate the automorphism $\alpha\otimes(\otimes^{(n)}(\bar{\alpha})):A\otimes(\otimes^{(n)}(\bar{A}))\to A\otimes(\otimes^{(n)}(\bar{A}))$ given by

$$a_0\otimes(\otimes^{(n)}(\overline{a_1},\cdots,\overline{a_n}))\mapsto\alpha(a_0)\otimes(\otimes^{(n)}(\overline{\alpha(a_1)},\cdots,\overline{\alpha(a_n)})),$$

for $a_0\in A$ and $\overline{a_i}\in\bar{A}$, $i=1,\cdots,n$.

On $\bigoplus_{n=0}^{\infty}\Omega^n(A)$, we define the differential by the linear mapping $d:A\otimes(\otimes^{(n)}(\bar{A}))\to A\otimes(\otimes^{(n+1)}(\bar{A}))$ of degree one by

$$d(a_0\otimes(\overline{a_1}\otimes(\cdots(\overline{a_{n-1}}\otimes\overline{a_n})\cdots)))=1\otimes(\overline{\alpha^{-1}(a_0)}\otimes(\cdots(\overline{\alpha^{-1}(a_{n-1})}\otimes\overline{\alpha^{-1}(a_n)})\cdots)). \tag{5.1}$$

We immediately obtain $d^2=0$ from the fact that $\bar{1}=0$. If we start with $a_n\in A$, multiplying on the left and applying d repeatedly gives us the following

$$a_0\otimes(\overline{a_1}\otimes(\cdots(\overline{a_{n-1}}\otimes\overline{a_n})\cdots))=a_0(da_1(da_2(\cdots(da_{n-1}da_n)\cdots))),$$

where $a_0(da_1(da_2(\cdots(da_{n-1}da_n)\cdots)))=a_0\otimes_A(da_1\otimes_A(da_2\otimes_A(\cdots(da_{n-1}\otimes_A da_n)\cdots)))$.

We make $\bigoplus_{n=0}^{\infty}\Omega^n(A)$ an (A,α)-Hom-bimodule as follows. The left (A,α)-Hom-module structure is given by, for $b\in A$ and $a_0(da_1(da_2(\cdots(da_{n-1}da_n)\cdots)))\in\Omega^n(A)$, $n\geq 1$,

$$\begin{aligned}& b(a_0(da_1(da_2(\cdots(da_{n-1}da_n)\cdots))))\\ &\qquad = (\alpha^{-1}(b)a_0)(d(\alpha(a_1))(d(\alpha(a_2))(\cdots(d(\alpha(a_{n-1}))d(\alpha(a_n))\cdots))).\end{aligned}$$

We now get the right (A,α)-Hom-module structure: One can show that, for $b\in A$, $a_0da_1\in\Omega^1(A)$, $a_0(da_1da_2)\in\Omega^2(A)$ and $a_0(da_1(da_2da_3))\in\Omega^3(A)$, the following equations hold:

$$(a_0da_1)b=\alpha(a_0)d(a_1\alpha^{-1}(b))-(a_0a_1)db,$$

$$
\begin{aligned}
(a_0(da_1da_2))b &= \alpha(a_0)(d(\alpha(a_1))d(a_2\alpha^{-2}(b))) - \alpha(a_0)(d(a_1a_2)d(\alpha^{-1}(b)))\\
&+ (a_0\alpha(a_1))(d(\alpha(a_2))d(\alpha^{-1}(b))),
\end{aligned}
$$

$$
\begin{aligned}
&(a_0(da_1(da_2da_3)))b\\
&= \alpha(a_0)(d(\alpha(a_1))(d(\alpha(a_2))d(a_3\alpha^{-3}(b)))) - \alpha(a_0)(d(\alpha(a_1))(d(a_2a_3)d(\alpha^{-2}(b))))\\
&+ \alpha(a_0)(d(a_1\alpha(a_2))(d(\alpha(a_3))d(\alpha^{-2}(b)))) - (a_0\alpha(a_1))(d(\alpha^2(a_2))(d(\alpha(a_3))d(\alpha^{-2}(b)))).
\end{aligned}
$$

By induction, one can also prove that the equation

$$
\begin{aligned}
&(a_0(da_1(da_2(\cdots(da_{n-1}da_n)\cdots))))b\\
=\ & (-1)^n(a_0\alpha(a_1))(d(\alpha^2(a_2))(d(\alpha^2(a_3))(\cdots d(\alpha^2(a_{n-1}))(d(\alpha(a_n))d(\alpha^{-(n-1)}(b)))\cdots)))\\
+\ & \sum_{i=1}^{n-3}(-1)^{n-i}\alpha(a_0)(d(\alpha(a_1))(\cdots d(\alpha(a_{i-1}))(d(a_i\alpha(a_{i+1}))(d(\alpha^2(a_{i+2}))(\cdots\\
& d(\alpha^2(a_{n-1}))(d(\alpha(a_n))d(\alpha^{-(n-1)}(b)))\cdots)))\cdots))\\
+\ & \alpha(a_0)(d(\alpha(a_1))(\cdots d(\alpha(a_{n-3}))(d(a_{n-2}\alpha(a_{n-1}))(d(\alpha(a_n))d(\alpha^{-(n-1)}(b))))\cdots))\\
-\ & \alpha(a_0)(d(\alpha(a_1))(\cdots d(\alpha(a_{n-2}))(d(a_{n-1}a_n)d(\alpha^{-(n-1)}(b)))\cdots))\\
+\ & \alpha(a_0)(d(\alpha(a_1))(\cdots(d(\alpha(a_{n-1}))d(a_n\alpha^{-n}(b)))\cdots))
\end{aligned}
$$

holds for $a_0(da_1(da_2(\cdots(da_{n-1}da_n)\cdots))) \in \Omega^n(A)$, $n \geq 4$.

Next, we define the Hom-multiplication between any two parenthesized monomials, by using the right Hom-module structure given above, as

$$
\begin{aligned}
&[a_0(da_1(\cdots(da_{n-1}da_n)\cdots))][a_{n+1}(da_{n+2}(\cdots(da_{n+k-1}da_{n+k})\cdots))]\\
&\quad= [(\alpha^{-1}(a_0)(d(\alpha^{-1}(a_1))(\cdots(d(\alpha^{-1}(a_{n-1}))d(\alpha^{-1}(a_n))\cdots))))a_{n+1}]\\
&\quad\quad [d(\alpha(a_{n+2}))(\cdots(d(\alpha(a_{n+k-1}))d(\alpha(a_{n+k}))\cdots))], \qquad (5.2)
\end{aligned}
$$

for $\omega_n = a_0(da_1(\cdots(da_{n-1}da_n)\cdots)) \in \Omega^n(A)$ and $\omega_{k-1} = a_{n+1}(da_{n+2}(\cdots(da_{n+k-1}da_{n+k})\cdots)) \in \Omega^{k-1}(A)$. For any $n \geq 4$, we explicitly write the above multiplication:

$$
\begin{aligned}
& \omega_n\omega_{k-1} \\
= \ & [a_0(da_1(\cdots(da_{n-1}da_n)\cdots))][a_{n+1}(da_{n+2}(\cdots(da_{n+k-1}da_{n+k})\cdots))] \\
= \ & [(\alpha^{-1}(a_0)(d(\alpha^{-1}(a_1))(\cdots(d(\alpha^{-1}(a_{n-1}))d(\alpha^{-1}(a_n))\cdots))))a_{n+1}] \\
& [d(\alpha(a_{n+2}))(\cdots(d(\alpha(a_{n+k-1}))d(\alpha(a_{n+k}))\cdots))] \\
= \ & [(-1)^n(\alpha^{-1}(a_0)a_1)(d(\alpha(a_2))(d(\alpha(a_3))(\cdots d(\alpha(a_{n-1}))(d(a_n)d(\alpha^{-(n-1)}(a_{n+1})))\cdots))) \\
+ \ & \sum_{i=1}^{n-3}(-1)^{n-i}a_0(d(a_1)(\cdots d(a_{i-1})(d(\alpha^{-1}(a_i)a_{i+1})(d(\alpha(a_{i+2}))(\cdots \\
& d(\alpha(a_{n-1}))(d(a_n)d(\alpha^{-(n-1)}(a_{n+1})))\cdots)))\cdots)) \\
+ \ & a_0(d(a_1)(\cdots d(a_{n-3})(d(\alpha^{-1}(a_{n-2})a_{n-1})(d(a_n)d(\alpha^{-(n-1)}(a_{n+1}))))\cdots)) \\
- \ & a_0(d(a_1)(\cdots d(a_{n-2})(d(\alpha^{-1}(a_{n-1}a_n))d(\alpha^{-(n-1)}(a_{n+1})))\cdots)) \\
+ \ & a_0(d(a_1)(\cdots(d(a_{n-1})d(\alpha^{-1}(a_n)\alpha^{-n}(a_{n+1})))\cdots))] \\
& [d(\alpha(a_{n+2}))(\cdots(d(\alpha(a_{n+k-1}))d(\alpha(a_{n+k}))\cdots))] \\
= \ & (-1)^n(a_0\alpha(a_1))(d(\alpha^2(a_2))(\cdots d(\alpha^2(a_{n-1}))(d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots)))\cdots)) \\
+ \ & \sum_{i=1}^{n-3}(-1)^{n-i}\alpha(a_0)(d(\alpha(a_1))(\cdots d(\alpha(a_{i-1}))(d(a_i\alpha(a_{i+1}))(d(\alpha^2(a_{i+2}))(\cdots d(\alpha^2(a_{n-1})) \\
& (d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots(d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots)))\cdots)))\cdots)) \\
+ \ & \alpha(a_0)(d(\alpha(a_1))(\cdots d(\alpha(a_{n-3}))(d(a_{n-2}\alpha(a_{n-1}))(d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots)) \\
- \ & \alpha(a_0)(d(\alpha(a_1))\cdots d(\alpha(a_{n-2}))(d(a_{n-1}a_{n-1})(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))\cdots)) \\
+ \ & \alpha(a_0)(d(\alpha(a_1))(\cdots d(\alpha(a_{n-2}))(d(\alpha(a_{n-1}))(d(\alpha^{-1}(a_n)\alpha^{-n}(a_{n+1}))(d(\alpha^{-(n-1)}(a_{n+2}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots))
\end{aligned}
$$

On the other hand, we have the following computations for ω_n and ω_{k-1} given above:

$$
\begin{aligned}
& d\omega_n\omega_{k-1} \\
= \ & [da_0(da_1(\cdots(da_{n-1}da_n)\cdots))][a_{n+1}(da_{n+2}(\cdots(da_{n+k-1}da_{n+k})\cdots))] \\
= \ & [(d(\alpha^{-1}(a_0))(d(\alpha^{-1}(a_1))(\cdots(d(\alpha^{-1}(a_{n-1}))d(\alpha^{-1}(a_n))\cdots))))a_{n+1}] \\
& [d(\alpha(a_{n+2}))(\cdots(d(\alpha(a_{n+k-1}))d(\alpha(a_{n+k})))\cdots)] \\
= \ & (-1)^{n+1}\alpha(a_0)(d(\alpha(a_1))(\cdots(d(\alpha(a_{n-1}))(da_n(d(\alpha^{-n}(a_{n+1}))(\cdots \\
& (d(\alpha^{-n}(a_{n+k-1}))d(\alpha^{-n}(a_{n+k})))\cdots))))\cdots)) \\
+ \ & (-1)^n d(a_0\alpha(a_1))(d(\alpha^2(a_2))(\cdots(d(\alpha^2(a_{n-1}))(d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots)) \\
+ \ & \sum_{i=2}^{n-2}(-1)^{n+1-i}d(\alpha(a_0))(\cdots(d(\alpha(a_{i-2}))(d(a_{i-1}\alpha(a_i))(d(\alpha^2(a_{i+1}))(\cdots(d(\alpha^2(a_{n-1})) \\
& (d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots(d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots))))\cdots) \\
+ \ & d(\alpha(a_0))(\cdots(d(\alpha(a_{n-3}))(d(a_{n-2}\alpha(a_{n-1}))(d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots)))))\cdots) \\
- \ & d(\alpha(a_0))(\cdots(d(\alpha(a_{n-2}))(d(a_{n-1}a_n)(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots) \\
+ \ & d(\alpha(a_0))(\cdots(d(\alpha(a_{n-1}))(d(\alpha^{-1}(a_n)\alpha^{-n}(a_{n+1}))(d(\alpha^{-(n-1)}(a_{n+2}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots)
\end{aligned}
$$

and

$$
\begin{aligned}
& \omega_n d\omega_{k-1} \\
= \ & [a_0(da_1(\cdots(da_{n-1}da_n)\cdots))][da_{n+1}(\cdots(da_{n+k-1}da_{n+k})\cdots)] \\
= \ & \alpha(a_0)([da_1(\cdots(da_{n-1}da_n)\cdots)][d(\alpha^{-1}(a_{n+1}))(\cdots(d(\alpha^{-1}(a_{n+k-1}))d(\alpha^{-1}(a_{n+k})))\cdots)]) \\
= \ & \alpha(a_0)(d(\alpha(a_1))(\cdots(d(\alpha(a_{n-1}))(d(a_n)(d(\alpha^{-n}(a_{n+1}))(\cdots \\
& (d(\alpha^{-n}(a_{n+k-1}))d(\alpha^{-n}(a_{n+k})))\cdots))))\cdots)).
\end{aligned}
$$

Thus, the equation below holds:

$$
\begin{aligned}
& d\omega_n\omega_{k-1} + (-1)^n\omega_n d\omega_{k-1} \\
= \;& (-1)^n d(a_0\alpha(a_1))(d(\alpha^2(a_2))(\cdots(d(\alpha^2(a_{n-1}))(d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots)) \\
+ \;& \sum_{i=2}^{n-2}(-1)^{n+1-i} d(\alpha(a_0))(\cdots(d(\alpha(a_{i-2}))(d(a_{i-1}\alpha(a_i))(d(\alpha^2(a_{i+1}))(\cdots(d(\alpha^2(a_{n-1})) \\
& (d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots(d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots))))\cdots) \\
+ \;& d(\alpha(a_0))(\cdots(d(\alpha(a_{n-3}))(d(a_{n-2}\alpha(a_{n-1}))(d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots)))))\cdots) \\
- \;& d(\alpha(a_0))(\cdots(d(\alpha(a_{n-2}))(d(a_{n-1}a_n)(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots) \\
+ \;& d(\alpha(a_0))(\cdots(d(\alpha(a_{n-1}))(d(\alpha^{-1}(a_n)\alpha^{-n}(a_{n+1}))(d(\alpha^{-(n-1)}(a_{n+2}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots) \\
= \;& (-1)^n d(a_0\alpha(a_1))(d(\alpha^2(a_2))(\cdots(d(\alpha^2(a_{n-1}))(d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots)) \\
+ \;& \sum_{i=1}^{n-3}(-1)^{n-i} d(\alpha(a_0))(\cdots(d(\alpha(a_{i-1}))(d(a_i\alpha(a_{i+1}))(d(\alpha^2(a_{i+2}))(\cdots(d(\alpha^2(a_{n-1})) \\
& (d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots(d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots))))\cdots) \\
+ \;& d(\alpha(a_0))(\cdots(d(\alpha(a_{n-3}))(d(a_{n-2}\alpha(a_{n-1}))(d(\alpha(a_n))(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots)))))\cdots) \\
- \;& d(\alpha(a_0))(\cdots(d(\alpha(a_{n-2}))(d(a_{n-1}a_n)(d(\alpha^{-(n-1)}(a_{n+1}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots) \\
+ \;& d(\alpha(a_0))(\cdots(d(\alpha(a_{n-1}))(d(\alpha^{-1}(a_n)\alpha^{-n}(a_{n+1}))(d(\alpha^{-(n-1)}(a_{n+2}))(\cdots \\
& (d(\alpha^{-(n-1)}(a_{n+k-1}))d(\alpha^{-(n-1)}(a_{n+k})))\cdots))))\cdots) \\
= \;& d(\omega_n\omega_{k-1}),
\end{aligned}
$$

which the graded Leibniz rule. Next, we verify by induction that the following identity holds:

$$d(a_0(da_1(\cdots(da_{n-1}da_n)\cdots))) = da_0(da_1(\cdots(da_{n-1}da_n)\cdots)) \tag{5.3}$$

using the graded Leibniz rule and the equation $d^2 = d \circ d = 0$. For $a_0 da_1 \in \Omega^1(A)$,

$$d(a_0 da_1) = da_0 da_1 + (-1)^0 a_0 d(d(a_1)) = da_0 da_1.$$

Suppose now that the identity

$$d(a_0(da_1(\cdots(da_{n-2}da_{n-1})\cdots))) = da_0(da_1(\cdots(da_{n-2}da_{n-1})\cdots))$$

holds for $a_0(da_1(\cdots(da_{n-2}da_{n-1})\cdots)) \in \Omega^{n-1}(A)$, that is, if we replace a_i with a_{i+1} for $i = 0, \cdots, n-1$, we have $d(a_1(da_2(\cdots(da_{n-1}da_n)\cdots))) = da_1(da_2(\cdots(da_{n-1}da_n)\cdots))$. Thus, for $a_0(da_1(\cdots(da_{n-1}da_n)\cdots)) \in \Omega^n(A)$,

$$\begin{aligned}
& d(a_0(da_1(\cdots(da_{n-1}da_n)\cdots))) \\
= \; & da_0(da_1(da_2(\cdots(da_{n-1}da_n)\cdots))) + (-1)^0 a_0 d(da_1(da_2(\cdots(da_{n-1}da_n)\cdots))) \\
= \; & da_0(da_1(da_2(\cdots(da_{n-1}da_n)\cdots))) + (-1)^0 a_0 d(d(a_1(da_2(\cdots(da_{n-1}da_n)\cdots)))) \\
= \; & da_0(da_1(da_2(\cdots(da_{n-1}da_n)\cdots))).
\end{aligned}$$

Let (Γ, γ) be another Hom-DC on (A, α) with differential $\tilde{d}$ and let the morphism $\psi : \Omega(A) \to \Gamma$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, be given by

$$\psi(a) = a \text{ and } \psi(a_0(da_1(\cdots(da_{n-1}da_n)\cdots))) = a_0(\tilde{d}a_1(\cdots(\tilde{d}a_{n-1}\tilde{d}a_n)\cdots)),\ n \geq 1$$

for $a \in A$, $a_0(da_1(\cdots(da_{n-1}da_n)\cdots)) \in \Omega^n(A)$. Clearly, ψ is surjective by its definition. Now, let $\mathcal{N} := ker\psi$ be the kernel of ψ. From the equations (5.2) and (5.3) it is concluded that $\mathcal{N}$ is a differential Hom-ideal of $\Omega(A)$. Thus, Γ is identified with $\Omega(A)/\mathcal{N}$ showing the universality of $\Omega(A)$.

5.3 Left-Covariant FODC over Monoidal Hom-Hopf Algebras

5.3.1 Left-Covariant Hom-FODC and Their Right Hom-ideals

Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode throughout the section. (H, α) is a left Hom-quantum space for itself with respect to the Hom-comultiplication $\Delta : H \to H \otimes H, h \mapsto h_1 \otimes h_2$. Thus, by applying Definition 5.1.2 to the monoidal Hom-Hopf algebra (H, α) we obtain the following

Definition 5.3.1 *A FODC* (Γ,γ) *over the monoidal Hom-Hopf algebra* (H,α) *is said to be left-covariant if* (Γ,γ) *is a left-covariant FODC over the left Hom-quantum space* (H,α) *with left Hom-coaction* $\varphi=\Delta$ *in Definition 5.1.2.*

Remark 11 *According to Proposition 5.1.3, an* (H,α)*-Hom-FODC* (Γ,γ) *is left-covariant if and only if there exists a morphism* $\phi:\Gamma\to H\otimes\Gamma$ *in* $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ *such that, for* $h,g\in H$,

$$\phi(h\cdot dg)=\Delta(h)(id\otimes d)(\Delta(g)). \tag{5.4}$$

In the proof of Proposition 5.1.3, it has been shown that if there is such a morphism ϕ*, it defines a left Hom-comodule structure of* (Γ,γ) *on* (H,α) *and satisfies*

$$\phi(\alpha(h)\cdot(\omega\cdot g))=\Delta(\alpha(h))(\phi(\omega)\Delta(g))$$

for $h,g\in H$ *and* $\omega\in\Gamma$*. From this it follows that* (Γ,γ) *is a left-covariant* (H,α)*-Hom-bimodule.*

Let (Γ,γ) be a left-covariant (H,α)-Hom-FODC with derivation $d:H\to\Gamma$. By the above remark (Γ,γ) is a left-covariant (H,α)-Hom-bimodule, and then by adapting the structure theory of left-covariant Hom-bimodules, which is discussed in Lemma (3.3.2) and Proposition (3.3.4), to (Γ,γ) we summarize the following results. We have the unique projection $P_L:(\Gamma,\gamma)\to({}^{coH}\Gamma,\gamma|_{^{coH}\Gamma})$ given by $P_L(\rho)=S(\rho_{(-1)})\rho_{(0)}$, for all $\rho\in\Gamma$, such that

$$P_L(h\cdot\rho)=\varepsilon(h)\gamma(P_L(\rho)),\qquad \rho=\rho_{(-1)}P_L(\rho_{(0)})$$

and

$$P_L(\rho\cdot h)=\widetilde{ad}_R(h)(P_L(\rho))=:P_L(\rho)\triangleleft h$$

for any $h\in H$ and $\rho\in\Gamma$. Let us now define a linear mapping $\omega_\Gamma:H\stackrel{d}{\to}\Gamma\stackrel{P_L}{\to}{}^{coH}\Gamma$ by

$$\omega_\Gamma(h)=P_L(dh),\ \forall h\in H.$$

Obviously, it is in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, that is, $\omega_\Gamma\circ\alpha=\gamma\circ\omega_\Gamma$. Since $\phi(dh)=(dh)_{(-1)}\otimes(dh)_{(0)}=(id\otimes d)(\Delta(h))=h_1\otimes dh_2$ by the above remark, we obtain

$$\omega_\Gamma(h)=P_L(dh)=S(h_1)\cdot dh_2,\ \forall h\in H. \tag{5.5}$$

On the other hand, we can write $dh=(dh)_{(-1)}\cdot P_L((dh)_{(0)})=h_1\cdot P_L(dh_2)$, that is,

$$dh = h_1 \cdot \omega_\Gamma(h_2),\ \forall h \in H. \tag{5.6}$$

We will drop the subscript Γ from $\omega_\Gamma(\cdot)$. By definition, for any $h \in H$, $\omega(h) \in {}^{coH}\Gamma$. Conversely, let $\rho = \sum_i h_i \cdot dg_i \in {}^{coH}\Gamma$ for $h_i, g_i \in H$. Then

$$\rho = P_L(\rho) = \sum_i \varepsilon(h_i)\gamma(P_L(dg_i)) = \sum_i \varepsilon(h_i)\gamma(\omega(g_i)) = \sum_i \varepsilon(h_i)\omega(\alpha(g_i)),$$

showing that $\rho \in \omega(H)$. Thus, we get $\omega(H) = {}^{coH}\Gamma$ which implies that $\Gamma = H \cdot \omega(H) = \omega(H) \cdot H$ and hence any k-linear basis of $\omega(H)$ is a left (H,α)-Hom-module basis and a right (H,α)-Hom-module basis for (Γ,γ).

For $h, g \in H$, we get

$$\begin{aligned}
\omega(h) \triangleleft g &= P_L(\omega(h)\cdot g) = P_L((S(h_1)\cdot dh_2)\cdot g)\\
&= P_L(S(\alpha(h_1))\cdot(dh_2\cdot\alpha^{-1}(g))) = \varepsilon(S(\alpha(h_1)))\gamma(P_L(dh_2\cdot\alpha^{-1}(g)))\\
&= \varepsilon(h_1)\gamma(P_L(dh_2\cdot\alpha^{-1}(g))) = \gamma(P_L(d(\alpha^{-1}(h))\cdot\alpha^{-1}(g)))\\
&= \gamma(P_L(d(\alpha^{-1}(hg)) - \alpha^{-1}(h)\cdot d(\alpha^{-1}(g))))\\
&= \gamma(\omega(\alpha^{-1}(hg))) - \gamma(\varepsilon(\alpha^{-1}(h))\gamma(P_L(d(\alpha^{-1}(g)))))\\
&= \omega(hg) - \varepsilon(h)\gamma^2(\omega(\alpha^{-1}(g))) = \omega(hg) - \varepsilon(h)\omega(\alpha(g))\\
&= \omega(hg - \varepsilon(h)\alpha(g)) = \omega((h-\varepsilon(h)1)g).
\end{aligned}$$

Thus, by setting the notation $\bar{h} := h - \varepsilon(h)1$, we have

$$\omega(h) \triangleleft g = \widetilde{ad}_R(g)(\omega(h)) = \omega(\bar{h}g), \tag{5.7}$$

and we rewrite the (H,α)-Hom-bimodule structure as

$$g' \cdot (g \cdot \omega(h)) = (\alpha^{-1}(g')g) \cdot \omega(\alpha(h)), \tag{5.8}$$

$$(g' \cdot \omega(h)) \cdot g = (g'g_1) \cdot (\omega(h) \triangleleft g_2) = (g'g_1) \cdot \omega(\bar{h}g_2), \tag{5.9}$$

for $g, g', h \in H$.

In the following example we introduce the universal FODC over monoidal Hom-Hopf-algebra (H,α).

Example 5.3.2 *We define* $(\Omega^1(H),\beta) := (H \otimes ker\varepsilon, \alpha \otimes \alpha')$, *where* $\alpha' = \alpha|_{ker\varepsilon}$. *Let us denote the element* $1 \otimes \alpha^{-1}(\bar{g}) = 1 \otimes \overline{\alpha^{-1}(g)}$, *for* $g \in H$, *by* $\omega(g)$. *Thus we identify* $g \otimes \bar{h} \in$

$\Omega^1(H)$, *where* $g,h \in H$, *with* $g \cdot \omega(h)$. *We then introduce the Hom-bimodule structure of* $\Omega^1(H)$ *as in (5.8) and (5.9), for all* $g, g', h \in H$,

$$g' \cdot (g \cdot \omega(h)) := (\alpha^{-1}(g')g) \cdot \omega(\alpha(h)),$$

$$(g' \cdot \omega(h)) \cdot g := (g' g_1) \cdot \omega(\bar{h} g_2),$$

and a linear mapping

$$d : H \to \Omega^1(H),\ h \mapsto h_1 \otimes \overline{h_2} = h_1 \cdot \omega(h_2).$$

For any $g, h \in H$,

$$\begin{aligned} g \cdot dh + dg \cdot h &= g \cdot (h_1 \cdot \omega(h_2)) + (g_1 \cdot \omega(g_2)) \cdot h \\ &= (\alpha^{-1}(g)h_1) \cdot \omega(\alpha(h_2)) + (g_1 h_1) \cdot \omega(\overline{g_2} h_2) \\ &= (\alpha^{-1}(g)h_1) \cdot \omega(\alpha(h_2)) + (g_1 h_1) \cdot \omega(g_2 h_2) - (g_1 h_1)\omega((\varepsilon(g_2)1)h_2) \\ &= (\alpha^{-1}(g)h_1) \cdot \omega(\alpha(h_2)) + (g_1 h_1) \cdot \omega(g_2 h_2) - (\alpha^{-1}(g)h_1) \cdot \omega(\alpha(h_2)) \\ &= (g_1 h_1) \cdot \omega(g_2 h_2) = (gh)_1 \omega((gh)_2) \\ &= d(gh), \end{aligned}$$

showing that d *satisfies the Leibniz rule.*

$$d(\alpha(h)) = \alpha(h_1) \cdot \omega(\alpha(h_2)) = \alpha(h_1) \cdot \beta(\omega(h_2)) = \beta(h_1 \cdot \omega(h_2)) = \beta(dh),$$

which means that $d \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$.

$$\begin{aligned} \omega(h) &= \omega(\alpha(\varepsilon(h_1)h_2)) = \varepsilon(h_1)\omega(\alpha(h_2)) \\ &= \varepsilon(h_1)\beta(\omega(h_2)) = (\varepsilon(h_1)1) \cdot \omega(h_2) \\ &= (S(h_{11})h_{12}) \cdot \omega(h_2) = (S(\alpha^{-1}(h_1))h_{21}) \cdot \omega(\alpha(h_{22})) \\ &= \alpha(S(\alpha^{-1}(h_1))) \cdot (h_{21} \cdot \beta^{-1}(\omega(\alpha(h_{22})))) \\ &= S(h_1) \cdot (h_{21} \cdot \omega(h_{22})) = S(h_1) \cdot d(h_2), \end{aligned}$$

which proves that $\Omega^1(H) = H \cdot dH$. *Therefore,* $(\Omega^1(H), \beta)$ *is an* (H, α)*-Hom-FODC.*

For another (H, α)*-Hom-FODC* (Γ, γ) *with differentiation* $\bar{d} : H \to \Gamma$, *let us define the linear map* $\psi : \Omega^1(H) \to \Gamma$ *by* $\psi(h \cdot dg) = h \cdot \bar{d}g$, *where* $g, h \in H$. *It is well-defined: Suppose* *that* $\sum_i h_i \cdot dg_i = 0$ *in* $\Omega^1(H)$*, where* $h_i, g_i \in H$*. Then we have*

$$\begin{aligned}
\sum_i h_i \cdot dg_i &= \sum_i h_i \cdot (g_{i,1} \otimes \overline{g_{i,2}}) = \sum_i \alpha^{-1}(h_i) g_{i,1} \otimes \alpha(\overline{g_{i,2}}) \\
&= \sum_i \alpha^{-1}(h_i) g_{i,1} \otimes \overline{\alpha(g_{i,2})} = \sum_i \alpha^{-1}(h_i) g_{i,1} \otimes (\alpha(g_{i,2}) - \varepsilon(\alpha(g_{i,2}))1) \\
&= \sum_i [\alpha^{-1}(h_i) g_{i,1} \otimes \alpha(g_{i,2}) - \alpha^{-1}(h_i) g_{i,1} \varepsilon(g_{i,2}) \otimes 1)] \\
&= \sum_i [\alpha^{-1}(h_i) g_{i,1} \otimes \alpha(g_{i,2}) - \alpha^{-1}(h_i g_i) \otimes 1)] = 0.
\end{aligned}$$

So, by applying $(m \otimes id) \circ \tilde{a}^{-1} \circ (id \otimes S \otimes id) \circ (id \otimes \Delta)$ *to*

$$\sum_i h_i \cdot dg_i = \sum_i [\alpha^{-1}(h_i) g_{i,1} \otimes \alpha(g_{i,2}) - \alpha^{-1}(h_i g_i) \otimes 1)] = 0,$$

we acquire the equality $\sum_i (h_i \otimes g_i - \alpha^{-1}(h_i g_i) \otimes 1) = 0$*. Thus* $\sum_i h_i \cdot \bar{d} g_i = 0$ *in* Γ *concluding that* ψ *is well-defined. On the other hand we prove that* $\psi \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$*:*

$$\begin{aligned}
\psi(\beta(h \cdot dg)) &= \psi(\alpha(h) \cdot \beta(dg)) = \psi(\alpha(h) \cdot d(\alpha(g))) \\
&= \alpha(h) \cdot \bar{d}(\alpha(g)) = \alpha(h) \cdot \gamma(\bar{d}(g)) = \gamma(h \cdot \bar{d} g) = \gamma(\psi(h \cdot dg)).
\end{aligned}$$

The subobject $(ker\psi, \beta|_{ker\psi}) = (\mathcal{N}, \beta')$ *is an* (H,α)*-Hom-subbimodule of* $(\Omega^1(H), \beta)$*: Indeed, for* $h' \in H$ *and* $h \cdot dg \in \mathcal{N}$*,*

$$\begin{aligned}
\psi(h' \cdot (h \cdot dg)) &= \psi((\alpha^{-1}(h')h) \cdot d(\alpha(g))) = (\alpha^{-1}(h')h) \cdot \bar{d}(\alpha(g)) \\
&= h' \cdot (h \cdot \bar{d} g) = h' \cdot \psi(h \cdot dg) = 0,
\end{aligned}$$

$$\begin{aligned}
\psi((h \cdot dg) \cdot h') &= \psi(\alpha(h) \cdot d(g\alpha^{-1}(h')) - (hg) \cdot dh') \\
&= \alpha(h) \cdot \bar{d}(g\alpha^{-1}(h')) - (hg) \cdot \bar{d} h' = \alpha(h) \cdot \bar{d}(g\alpha^{-1}(h')) - \alpha(h) \cdot (g \cdot \bar{d}(\alpha^{-1}(h'))) \\
&= \alpha(h) \cdot (\bar{d} g \cdot \alpha^{-1}(h')) = (h \cdot \bar{d} g) \cdot h' = 0.
\end{aligned}$$

Hence we have the quotient object $(\Omega^1(H)/\mathcal{N}, \bar{\beta})$ *as* (H,α)*-Hom-bimodule, where the automorphism* $\bar{\beta}$ *is induced by* β *and define the* (H,α)*-bilinear map* $\bar{\psi} : \Omega^1(H)/\mathcal{N} \to \Gamma$*,* $h \cdot dg \mapsto h \cdot \bar{d} g$*, which is surjective by definition. Since* $ker\bar{\psi} = \mathcal{N}$*,* $\bar{\psi}$ *is 1-1, showing that* Γ *is isomorphic to the quotient* $\Omega^1(H)/\mathcal{N}$*. Therefore* $(\Omega^1(H), \beta)$ *is the universal Hom-FODC over* (H,α)*.*

We define the subobject

$$\mathcal{R}_\Gamma = \{h \in ker\varepsilon |\, \omega_\Gamma(h) = 0\} \tag{5.10}$$

of $(ker\varepsilon, \alpha|_{ker\varepsilon})$ for a given left-covariant (H,α)-Hom-FODC (Γ,γ), which is clearly a Hom-ideal of (H,α). We now prove that there is a one-to-one correspondence between left-covariant (H,α)-Hom-FODC and right Hom-ideals of $(ker\varepsilon,\alpha')$, where $\alpha' = \alpha|_{ker\varepsilon}$.

Proposition 5.3.3 *1. Let $(\mathcal{R},\alpha'')$ be a right Hom-ideal of (H,α) which is a subobject of $(ker\varepsilon,\alpha')$, where $\alpha'' = \alpha|_{\mathcal{R}}$. Then $\mathcal{N} := H\cdot\omega_{\Omega^1(H)}(\mathcal{R})$ is an (H,α)-Hom-subbimodule of $(\Omega^1(H),\beta)$. Furthermore, $(\Gamma,\gamma) := (\Omega^1(H)/\mathcal{N},\bar{\beta})$ is a left-covariant Hom-FODC over (H,α) such that $\mathcal{R}_\Gamma = \mathcal{R}$.*

2. For a given left-covariant (H,α)-Hom-FODC (Γ,γ), $\mathcal{R}_\Gamma$ is a right Hom-ideal of (H,α) and Γ is isomorphic to $\Omega^1(H)/H\cdot\omega_{\Omega^1(H)}(\mathcal{R}_\Gamma)$.

Proof:

1. For any $h\in\mathcal{R}$ and $g\in H$, we have

$$\begin{aligned}\omega(h)\cdot g &= (1g_1)\cdot(\beta^{-1}(\omega(h))\triangleleft g_2) = \alpha(g_1)\cdot(\omega(\alpha^{-1}(h))\triangleleft g_2)\\ &= \alpha(g_1)\cdot\omega(\alpha^{-1}(\bar{h})g_2) = \alpha(g_1)\cdot\omega(\alpha^{-1}(h)g_2),\end{aligned}$$

which is in $H\cdot\omega_{\Omega^1(H)}(\mathcal{R})$, and hence $\mathcal{N} = H\cdot\omega_{\Omega^1(H)}(\mathcal{R})$ is an (H,α)-Hom-subbimodule of $\Omega^1(H) = H\cdot\omega_{\Omega^1(H)}(H)$. So, $(\Gamma = \Omega^1(H)/\mathcal{N},\bar{\beta})$ is a (H,α)-Hom-FODC with differentiation $\bar{d}: H\to\Gamma$, $h\mapsto \bar{d}h = \pi(dh) = h_1\cdot\omega(h_2)+\mathcal{N}$, where $\pi:\Omega^1(H)\to\Omega^1(H)/\mathcal{N}$ is the natural projection.

Let $\phi:\Omega^1(H)\to H\otimes\Omega^1(H)$, $h\cdot\omega(g)\mapsto\alpha(h_1)\otimes h_2\cdot\omega(\alpha^{-1}(g))$ be the Hom-coaction for the left-covariant Hom-FODC $(\Omega^1(H),\beta)$. Since, for $h\cdot\omega(r)\in\mathcal{N}$ we have

$$\phi(h\cdot\omega(r)) = \alpha(h_1)\otimes h_2\cdot\omega(\alpha^{-1}(r))\in H\otimes\mathcal{N},$$

that is, $\phi(\mathcal{N})\subseteq H\otimes\mathcal{N}$, ϕ passes to a left Hom-action of (H,α) on $(\Gamma,\bar{\beta})$ as $\bar{\phi}(h\cdot\omega(g)+\mathcal{N}) = \alpha(h_1)\otimes(h_2\cdot\omega(\alpha^{-1}(g))+\mathcal{N})$. For $g,h\in H$, we get

$$
\begin{aligned}
&\Delta(g)(id\otimes\bar{d})(\Delta(h))\\
&= (g_1\otimes g_2)(h_1\otimes \bar{d}h_2)\\
&= g_1h_1\otimes g_2\cdot(h_{21}\cdot\omega(h_{22})+\mathcal{N}) = g_1h_1\otimes(g_2\cdot(h_{21}\cdot\omega(h_{22}))+\mathcal{N})\\
&= g_1h_1\otimes((\alpha^{-1}(g_2)h_{21})\cdot\omega(\alpha(h_{22}))+\mathcal{N})\\
&= g_1\alpha(h_{11})\otimes((\alpha^{-1}(g_2)h_{12})\cdot\omega(h_2)+\mathcal{N}) = \bar{\phi}(\alpha^{-1}(g)h_1\cdot\omega(\alpha(h_2))+\mathcal{N})\\
&= \bar{\phi}(g\cdot(h_1\cdot\omega(h_2))+\mathcal{N}) = \bar{\phi}(g\cdot\bar{d}h),
\end{aligned}
$$

proving the left-covariance of $(\Gamma,\bar{\beta})$ with respect to (H,α). Thus, we have the projection $\overline{P}_L:\Gamma\to {}^{coH}\Gamma$ given by

$$\overline{P}_L(h\cdot\omega(g)+\mathcal{N}) = \varepsilon(h)\omega(\alpha(g))+\mathcal{N}$$

for $h\cdot\omega(g)\in\Omega^1(H)$.

For $h\in\mathcal{R}$,

$$\omega_\Gamma(h) = \overline{P}_L(\bar{d}h) = \overline{P}_L(h_1\cdot\omega(h_2)+\mathcal{N}) = \varepsilon(h_1)\omega(\alpha(h_2))+\mathcal{N} = \omega(h)+\mathcal{N} = \mathcal{N} = 0_\Gamma,$$

implying that $\mathcal{R}\subseteq\mathcal{R}_\Gamma$. On the contrary, if $\omega_\Gamma(h) = 0_\Gamma$ for some $h\in ker\varepsilon$, then $\omega(h)\in\mathcal{N} = H\cdot\omega(\mathcal{R})$, that is, $h\in\mathcal{R}$, i.e., $\mathcal{R}_\Gamma\subseteq\mathcal{R}$. Therefore, $\mathcal{R}=\mathcal{R}_\Gamma$.

2. Since (Γ,γ) is a left-covariant Hom-FODC, $\widetilde{ad}_R(g)(\omega(h)) = \omega(\bar{h}g)$ holds for $g,h\in H$. Hence, for $h\in\mathcal{R}_\Gamma$ and $g\in ker\varepsilon$, we have $\omega_\Gamma(hg) = \omega_\Gamma(\bar{h}g) = \widetilde{ad}_R(g)(\omega_\Gamma(h)) = 0$ since $\omega_\Gamma(h) = 0$. Therefore, $\mathcal{R}_\Gamma$ is a right Hom-ideal of $ker\varepsilon$. Thus, $\Gamma\simeq\Omega^1(H)/H\cdot\omega_{\Omega^1(H)}(\mathcal{R}_\Gamma)$ by (1).

□

5.3.2 Quantum Hom-Tangent Space

In the theory of Lie groups, if $A = C^\infty(G)$ is the algebra of smooth functions on a Lie group G and $\mathcal{R}$ is the ideal of A consisting of all functions vanishing with first derivatives at the neutral element of G, then the vector space of all linear functionals on A is identified with the tangent space at the neutral element, i.e., with the Lie algebra of G. In the theory of quantum groups, this consideration gives rise to the notion of quantum tangent space associated to a left-covariant FODC Γ on a Hopf algebra A, which is defined as the vector space

$$\mathcal{T}_\Gamma = \{X \in A' | \, X(1) = 0, \, X(a) = 0, \forall a \in \mathcal{R}_\Gamma\},$$

where $\mathcal{R}_\Gamma = \{a \in ker \varepsilon_A | \, P_L(da) = 0\}$. In what follows, we study the Hom-version of the quantum tangent space.

We recall that the dual monoidal Hom-algebra $(H', \bar{\alpha})$ of (H, α) consists of functionals $X : H \to k$ and is equipped with the convolution product $(XY)(h) = X(h_1)Y(h_2)$, for $X, Y \in H'$ and $h \in H$, as Hom-multiplication and with the Hom-unit $\varepsilon : H \to k$, where automorphism $\bar{\alpha} : H' \to H'$ is given by $\bar{\alpha}(X) = X \circ \alpha^{-1}$. The morphism

$$H' \otimes H \to H, X \otimes h \mapsto X \bullet h := \alpha^2(h_1) X(\alpha(h_2)),$$

in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, makes (H, α) a left $(H', \bar{\alpha})$-Hom-module.

Definition 5.3.4 *Let (Γ, γ) be a left-covariant (H, α)-Hom-FODC. Then the subobject*

$$\mathcal{T}_\Gamma = \{X \in H' | \, X(1) = 0, \, X(h) = 0, \forall h \in \mathcal{R}_\Gamma\} \tag{5.11}$$

of $(H', \bar{\alpha})$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, is said to be the quantum Hom-tangent space to (Γ, γ).

Proposition 5.3.5 *Let (Γ, γ) be a left-covariant (H, α)-Hom-FODC and $(\mathcal{T}_\Gamma, \bar{\alpha}')$ be the quantum Hom-tangent space to it, where $\bar{\alpha}' = \bar{\alpha}|_{\mathcal{T}_\Gamma}$. Then, there is a unique bilinear form $< \cdot, \cdot >: \mathcal{T}_\Gamma \times \Gamma \to k$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that*

$$< X, h \cdot dg >= \varepsilon(h) X(g), \, \forall g, h \in H, \, X \in \mathcal{T}_\Gamma. \tag{5.12}$$

With respect to this bilinear form, $(\mathcal{T}_\Gamma, \bar{\alpha}')$ and $({}^{coH}\Gamma, \gamma') = (\omega(H), \gamma')$ form a nondegenerate dual pairing, where $\gamma' = \gamma|_{{}^{coH}\Gamma}$. Moreover, we have

$$< X, \omega(h) >= X(\alpha^{-1}(h)), \forall h \in H, X \in \mathcal{T}_\Gamma. \tag{5.13}$$

Proof: We define $< X, \rho >:= X(\sum_i \varepsilon(h_i) g_i) = \sum_i \varepsilon(h_i) X(g_i)$ for $X \in H'$ and $\rho = \sum_i h_i \cdot dg_i \in$

Γ. Suppose that $\rho = \sum_i h_i \cdot dg_i = 0$. Then

$$\begin{aligned} 0 &= P_L(\gamma^{-1}(\rho)) = \sum_i P_L(\alpha^{-1}(h_i)d(\alpha^{-1}(g_i))) \\ &= \sum_i \varepsilon(\alpha^{-1}(h_i))\gamma(P_L(d(\alpha^{-1}(g_i)))) \\ &= \sum_i \varepsilon(h_i)\gamma(\omega(\alpha^{-1}(g_i))) \\ &= \omega\left(\sum_i \varepsilon(h_i)g_i\right), \end{aligned}$$

hence $\omega(\sum_i \varepsilon(h_i)\overline{g_i}) = 0$, which implies that $\sum_i \varepsilon(h_i)\overline{g_i} \in \mathcal{R}_\Gamma$. Thus, by the definition of $\mathcal{T}_\Gamma$ we get

$$\begin{aligned} < X, \rho > &= X\left(\sum_i \varepsilon(h_i)g_i\right) = X\left(\sum_i (\varepsilon(h_i)\overline{g_i} + \varepsilon(h_i)\varepsilon(g_i)1)\right) \\ &= X\left(\sum_i \varepsilon(h_i)\overline{g_i}\right) + \sum_i \varepsilon(h_i)\varepsilon(g_i)X(1) = 0, \end{aligned}$$

which proves that the bilinear form $< \cdot, \cdot >$ is well-defined. Uniqueness comes immediately from the fact that $\Gamma = H \cdot dH$. Since

$$\begin{aligned} < \bar{\alpha}(X), \gamma(\rho) > &= (X \circ \alpha^{-1})\left(\sum_i \varepsilon(\alpha(h_i))\alpha(g_i)\right) = \sum_i \varepsilon(\alpha(h_i))(X \circ \alpha^{-1})(\alpha(g_i)) \\ &= X\left(\sum_i \varepsilon(h_i)g_i\right) = < X, \rho >, \end{aligned}$$

the bilinear form $< \cdot, \cdot >$ is in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. For any $h \in H$, $< X, \omega(h) >=< X, S(h_1) \cdot dh_2 >= \varepsilon(h_1)X(h_2) = X(\varepsilon(h_1)h_2) = X(\alpha^{-1}(h))$, which is the formula (5.13). For any $h \in ker\varepsilon$, if $< X, \omega(h) >= X(\alpha^{-1}(h)) = 0$, $\forall X \in \mathcal{T}_\Gamma$, then $\alpha^{-1}(h) \in \mathcal{R}_\Gamma$: Suppose that the element $0 \neq \alpha^{-1}(h) \in ker\varepsilon$ is not contained in $\mathcal{R}_\Gamma$. Then we can extend $\alpha^{-1}(h)$ to a basis of $ker\varepsilon$ and find a functional $X \in \mathcal{T}_\Gamma$ such that $X(\alpha^{-1}(h)) \neq 0$, which contradicts with the hypothesis of the statement. So we have $h \in \mathcal{R}_\Gamma$ since $\omega \circ \alpha^{-1} = \gamma^{-1} \circ \omega$. On the other hand $< X, \omega(h) >= X(\alpha^{-1}(h)) = \bar{\alpha}(X)(h) = 0$ for all $\omega(h) \in \omega(H)$ implies $\bar{\alpha}(X) = 0$, that is, $X = 0$. Hence, $(\mathcal{T}_\Gamma, \bar{\alpha}')$ and $({}^{coH}\Gamma, \gamma') = (\omega(H), \gamma')$ form a nondegenerate dual pairing with respect to $< \cdot, \cdot >$. □

Let $\{X_i\}_{i\in I}$ be a linear basis of $\mathcal{T}_\Gamma$ and $\{\omega_i\}_{i\in I}$ be the dual basis of ${}^{coH}\Gamma$, that is, $< X_i, \omega_j >= \delta_{ij}$ for $i, j \in I$. Also, from Theorem (3.3.11), recall the family of functionals $\{f^i_j\}_{i,j\in I}$ in the definition of the Hom-action ${}^{coH}\Gamma \otimes H \to {}^{coH}\Gamma, \omega_i \otimes h \mapsto \omega_i \triangleleft h = f^i_j(h)\omega_j$, where all but finitely many $f^i_j(h)$ vanish and Einstein summation convention is used. These functionals satisfy, for all $h, g \in H$ and $i, j \in I$,

$$f^i_j(hg) = (\bar{\gamma}^i_k f^k_l)(h) f^l_j(\alpha(g)),\ f^i_j(1) = \gamma^i_j,$$

where $\gamma'(\omega_i) = \gamma^i_j\omega_j$ and $\gamma'^{-1}(\omega_i) = \bar{\gamma}^i_j\omega_j$ such that $\gamma^i_j\bar{\gamma}^j_k = \delta_{ik} = \bar{\gamma}^i_j\gamma^j_k$.

Proposition 5.3.6 *For $h, g \in H$, we have*

$$dh = (X_i \bullet \alpha^{-2}(h)) \cdot \omega_i, \tag{5.14}$$

$$X_i(hg) = \varepsilon(h)(\gamma^j_i X_j)(g) + X_k(h)(\gamma^l_i \bar{f}^k_l)(g), \tag{5.15}$$

where $\bar{f}^k_l = \bar{\gamma}^k_p f^p_l$.

Proof: By the formula 5.13, we have $< X_i, \omega(h) >= X_i(\alpha^{-1}(h))$ implying $\omega(h) = X_i(\alpha^{-1}(h))\omega_i$. Thus, $dh = h_1 \cdot \omega(h_2) = h_1 \cdot (X_i(\alpha^{-1}(h_2)\omega_i)) = (X_i \bullet \alpha^{-2}(h)) \cdot \omega_i$ which is the formula 5.14. By using this formula and the Leibniz rule, we obtain

$$\begin{aligned}
(X_l \bullet \alpha^{-2}(hg)) \cdot \omega_l &= d(hg) = dh \cdot g + h \cdot dg \\
&= ((X_j \bullet \alpha^{-2}(h)) \cdot \omega_j) \cdot g + h \cdot ((X_i \bullet \alpha^{-2}(g)) \cdot \omega_i) \\
&= \alpha(X_j \bullet \alpha^{-2}(h)) \cdot (\omega_j \cdot \alpha^{-1}(g)) + (\alpha^{-1}(h)(X_i \bullet \alpha^{-2}(g))) \cdot \gamma'(\omega_i) \\
&= \alpha(X_j \bullet \alpha^{-2}(h)) \cdot ((\bar{f}^j_k \bullet \alpha^{-2}(g)) \cdot \omega_k) + (\alpha^{-1}(h)(X_i \bullet \alpha^{-2}(g))) \cdot (\gamma^i_k\omega_k) \\
&= ((X_j \bullet \alpha^{-2}(h))(\bar{f}^j_k \bullet \alpha^{-2}(g))) \cdot (\gamma^k_l\omega_l) + (\alpha^{-1}(h)(X_i \bullet \alpha^{-2}(g))) \cdot (\gamma^i_l\omega_l) \\
&= [(X_j \bullet \alpha^{-2}(h))((\gamma^k_l \bar{f}^j_k) \bullet \alpha^{-2}(g)) + \alpha^{-1}(h)((\gamma^i_l X_i) \bullet \alpha^{-2}(g))] \cdot \omega_l,
\end{aligned}$$

hence, by replacing $\alpha^{-2}(h)$ and $\alpha^{-2}(g)$ by h and g, respectively, we get

$$X_l \bullet (hg) = \alpha(h)((\gamma^i_l X_i) \bullet g) + (X_j \bullet h)((\gamma^k_l \bar{f}^j_k) \bullet g), l \in I.$$

By applying ε to the both sides of this equation we acquire

$$X_l(hg) = \varepsilon(h)(\gamma^l_i X_i)(g) + X_j(h)(\gamma^l_k \bar{f}^k_j)(g),$$

since, for any $h \in H$ and $f \in H'$, the equality $\varepsilon(f \bullet h) = \varepsilon(\alpha^2(h_1))f(\alpha(h_2)) = \varepsilon(h_1)f(\alpha(h_2)) = f(\alpha(\varepsilon(h_1)h_2)) = f(h)$ holds. □

Let (A,α) be a monoidal Hom-algebra. Then we consider $A'\otimes A'$, where $A' = Hom(A,k)$, as a linear subspace of $(A\otimes A)'$ by identifying $f\otimes g\in A'\otimes A'$ with the linear functional on $A\otimes A$ specified by $(f\otimes g)(a\otimes a') := f(a)g(a')$ for $a,a'\in A$. For $f\in H'$, let us define $\Delta(f)\in(A\otimes A)'$ by $\Delta(f)(a\otimes b) := f(ab)$ for $a,b\in A$. We now denote, by A°, the set of all functionals $f\in A'$ such that $\Delta(f)\in A'\otimes A'$, i.e., it is written as a finite sum

$$\Delta(f)=\sum_{p=1}^{P} f_p\otimes g_p$$

for some functionals $f_p, g_p\in A'$, $p=1,\ldots,P$, where P is a natural number so that we have $f(ab)=\sum_p f_p(a)g_p(b)$. Then (A°,α°) is a monoidal Hom-coalgebra with Hom-comultiplication given above and the Hom-counit is defined by $\varepsilon(f)=f(1_A)$, where $\alpha^\circ(f)=f\circ\alpha^{-1}$ for any $f\in A^\circ$: Let $f\in A^\circ$ and $\Delta(f)=\sum_p f_p\otimes g_p$ such that the functionals $\{f_p\}_{p=1}^P$ are chosen to be linearly independent. So, one can find $a_q\in A$ such that $f_p(a_q)=\delta_{pq}$. Thus we get

$$\begin{aligned} g_q(ab) &= \sum_p \delta_{qp} g_p(ab) = \sum_p f_p(a_q) g_p(ab) = f(a_q(ab)) \\ &= f((\alpha^{-1}(a_q)a)\alpha(b)) = \sum_p f_p(\alpha^{-1}(a_q)a) g_p(\alpha(b)), \end{aligned}$$

showing that $g_q\in A^\circ$, and analogously $f_q\in A^\circ$, and hence $\Delta(f)\in A^\circ\otimes A^\circ$. Let $f\in A^\circ$ and $a,b,c\in A$. Then we have the Hom-coassociativity of Δ:

$$(\bar{\alpha}^{-1}\otimes\Delta)(\Delta(f))(a\otimes b\otimes c) = f(\alpha(a)(bc)) = f((ab)\alpha(c)) = (\Delta\otimes\bar{\alpha}^{-1})(\Delta(f))(a\otimes b\otimes c).$$

On the other hand,

$$(id\otimes\varepsilon)(\Delta(f))(h) = \left(\sum_p \bar{\alpha}(f_p)g_p(1_A)\right)(h) = \sum_p f_p(\alpha^{-1}(h))g_p(1_A) = f(h)$$

shows that Hom-counity is satisfied.

Suppose that (A,α) is a monoidal Hom-bialgebra, then the monoidal Hom-coalgebra (A°,α°) endowed with the convolution product, as in the argument before Lemma (3.3.10), is as well a monoidal Hom-bialgebra with the Hom-unit given by the Hom-counit ε of the monoidal Hom-coalgebra (A,α): One can easily check the compatibility condition between Hom-comultiplication and Hom-multiplication of (A°,α°) from that of (A,α). So, it suffices to verify that for any $f,g\in A^\circ$, fg is also in A°: If we put $\Delta(f)=\sum_p f_p\otimes g_p$ and $\Delta(g)=\sum_q h_q\otimes k_q$, then we get

$$(fg)(ab)=\Delta(fg)(a\otimes b)=\sum_{p,q} f_p h_q(a) g_p k_q(b)=\left(\sum_{p,q} f_p h_q\otimes g_p k_q\right)(a\otimes b),$$

so that $fg\in A^\circ$.

If (A,α) is a monoidal Hom-Hopf algebra, then so is (A°,α°) with antipode defined by $S(f)(a)=f(S(a))$ for $f\in A^\circ$ and $a\in A$: Set $\Delta(f)=\sum_p f_p\otimes g_p$, and then we obtain

$$\Delta(S(f))(a\otimes b)=S(f)(ab)=f(S(ab))=\sum_p S(f_p)(b)S(g_p)(a)=\left(\sum_p S(g_p)\otimes S(f_p)\right)(a\otimes b),$$

implying $S(f)\in A^\circ$. Lastly, for $a\in A$, we have

$$((m(S\otimes id)\Delta)(f))(a)=\sum_p (S(f_p)g_p)(a)=\varepsilon(a)f(1)=1_{A^\circ}(a)\varepsilon_{A^\circ}(f)=((\eta\circ\varepsilon)(f))(a),$$

similarly we get $((m(id\otimes S)\Delta)(f))(a)=((\eta\circ\varepsilon)(f))(a)$.

We then call the monoidal Hom-coalgebra (respectively, Hom-bialgebra, Hom-Hopf algebra) A° above the *dual monoidal Hom-coalgebra* (respectively, *Hom-bialgebra*,*Hom-Hopf algebra*). Suppose now that the vector space $\mathcal{T}_\Gamma$ is finite dimensional. Then assert from (3.42) and (5.15) that the functionals f^i_j and X_l are in the dual monoidal Hom-Hopf algebra H° and we have the following equations, where there is summation over repeating indices,

$$\Delta(f^i_j)=\bar{f}^i_l\otimes f^l_j\circ\alpha, \tag{5.16}$$

$$\Delta(X_l)=X_j\otimes\gamma^k_l\bar{f}^j_k+\varepsilon\otimes\gamma^l_i X_i \tag{5.17}$$

in H°.

5.4 Bicovariant FODC over Monoidal Hom-Hopf Algebras

5.4.1 Right-Covariant Hom-FODC

Definition 5.4.1 *Let (H,β) be a monoidal Hom-bialgebra. A FODC (Γ,γ) over a right Hom-quantum space (A,α) with right Hom-coaction $\varphi: A\to A\otimes H$, $a\mapsto a_{[0]}\otimes a_{[1]}$ is called right-covariant with respect to (H,β) if there exists a right Hom-coaction $\phi:\Gamma\to\Gamma\otimes H$, $\omega\mapsto\omega_{[0]}\otimes\omega_{[1]}$ of (H,β) on (Γ,γ) such that*

1. $\phi(\alpha(a)\cdot(\omega\cdot b)) = \varphi(\alpha(a))(\phi(\omega)\varphi(b))[= (\varphi(a)\phi(\omega))\varphi(\alpha(b)) = \phi(a\cdot\omega)\cdot b]$, $\forall a,b \in A$, $\omega \in \Gamma$,

2. $\phi(da) = (d\otimes id)(\varphi(a))$, $\forall a \in A$

Let (H,α) be a monoidal Hom-Hopf algebra with an invertible antipode S. Since (H,α) a right Hom-quantum space for itself with respect to the Hom-comultiplication $\Delta : H \to H\otimes H, h \mapsto h_1 \otimes h_2$, the above definition induces

Definition 5.4.2 *A (H,α)-Hom-FODC (Γ,γ) is said to be right-covariant if (Γ,γ) is a right-covariant FODC over the right Hom-quantum space (H,α) with right Hom-action $\varphi = \Delta$ in the above definition, or in an equivalent way if there is a morphism $\phi : \Gamma \to \Gamma\otimes H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that, for $h,g \in H$,*

$$\phi(h\cdot dg) = \Delta(h)(d\otimes id)(\Delta(g)). \tag{5.18}$$

If we modify the Proposition 5.1.3 to the right-covariant case, we conclude that the right-covariant (H,α)-Hom-FODC (Γ,γ) is a right-covariant (H,α)-Hom-bimodule. Thus, by using the unique projection $P_R : (\Gamma,\gamma) \to (\Gamma^{coH}, \gamma|_{\Gamma^{coH}})$, $P_R(\rho) = \omega_{[0]}\cdot S(\omega_{[1]})$ we define the linear mapping

$$\eta_\Gamma : H \to \Gamma^{coH},\ \eta(h) := P_R(dh),$$

for any $h \in H$,in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, for which $\eta(H)\Gamma^{coH}$. Since $\phi(dh) = dh_1 \otimes h_2$, we have, for $h \in H$

$$\eta(h) = dh_1 \cdot S(h_2)\, and\, dh = \eta(h_1)\cdot h_2.$$

5.4.2 Bicovariant Hom-FODC

Definition 5.4.3 *A (H,α)-Hom-FODC (Γ,γ) is said to be bicovariant if it is both left-covariant and right-covariant FODC.*

Remark 12 *By the Remark 11 and the Definition 5.4.2, a (H,α)-Hom-FODC (Γ,γ) is bicovariant if and only if there exist morphisms $\phi_L : \Gamma \to H\otimes\Gamma$ and $\phi_R : \Gamma \to \Gamma\otimes H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, satisfying the equations 5.4 and 5.18, respectively. So, if (Γ,γ) is a bicovariant*

(H,α)*-Hom-FODC with Hom-coactions* ϕ_L *and* ϕ_R *satisfying 5.4 and 5.18 we get, for* $h, g \in H$,

$$(id \otimes \phi_R)(\phi_L(h \cdot dg)) = (id \otimes \phi_R)(h_1 g_1 \otimes h_2 \cdot dg_2) = h_1 g_1 \otimes (h_{21} \cdot dg_{21} \otimes h_{22} g_{22}),$$

$$\begin{aligned}(\tilde{a} \circ (\phi_L \otimes id))(\phi_R(h \cdot dg)) &= (\tilde{a} \circ (\phi_L \otimes))(h_1 \cdot dg_1 \otimes h_2 g_2) \\ &= \alpha(h_{11} g_{11}) \otimes (h_{12} \cdot dg_{12} \otimes \alpha^{-1}(h_2 g_2)) \\ &= h_1 g_1 \otimes (h_{21} \cdot dg_{21} \otimes h_{22} g_{22}).\end{aligned}$$

Thus, (Γ, γ) *is a bicovariant* (H,α)*-Hom-bimodule and the whole structure theory of bicovariant bimodules can be applied to it.*

Lemma 5.4.4 *Let* (H,α) *be a monoidal Hom-Hopf algebra. Then*

1. *The linear mapping* $\widetilde{Ad}_R : H \to H \otimes H$ *given by*

$$\widetilde{Ad}_R(h) = \alpha(h_{12}) \otimes S(h_{11})\alpha^{-1}(h_2) = \alpha(h_{21}) \otimes S(\alpha^{-1}(h_1))h_{22}$$

   is a right Hom-coaction of (H,α) *on itself.*

2. *The linear mapping* $\widetilde{Ad}_L : H \to H \otimes H$ *given by*

$$\widetilde{Ad}_L(h) = \alpha(h_{11})S(\alpha^{-1}(h_2)) \otimes \alpha(h_{12}) = \alpha^{-1}(h_1)S(h_{22}) \otimes \alpha(h_{21})$$

   is a left Hom-coaction of (H,α) *on itself.* $\widetilde{Ad}_R$ *and* $\widetilde{Ad}_L$ *are called adjoint right Hom-coaction and adjoint left Hom-coaction, respectively*

Proof:

1. If we write $\widetilde{Ad}_R(h) = h_{[0]} \otimes h_{[1]}$ for $h \in H$, then the Hom-coassociativity follows from

$$\begin{aligned}\alpha^{-1}(h_{[0]}) \otimes \Delta(h_{[1]}) &= \alpha^{-1}(\alpha(h_{12})) \otimes \Delta(S(h_{11})\alpha^{-1}(h_2)) \\ &= h_{12} \otimes S(h_{112})\alpha^{-1}(h_{21}) \otimes S(h_{111})\alpha^{-1}(h_{22}) \\ &= \alpha^2(h_{1212}) \otimes S(\alpha(h_{1211}))h_{122} \otimes S(\alpha^{-1}(h_{11}))\alpha^{-2}(h_{22}) \\ &= \alpha^2(h_{1212}) \otimes S(\alpha(h_{1211}))h_{122} \otimes \alpha^{-1}(S(h_{11})\alpha^{-1}(h_{22})) \\ &= h_{[0][0]} \otimes h_{[0][1]} \otimes \alpha^{-1}(h_{[1]}),\end{aligned}$$

   where in the third step we have used

$$h_{11} \otimes \alpha(h_{1211}) \otimes h_{1212} \otimes \alpha^{-1}(h_{122}) \otimes h_2 = \alpha(h_{111}) \otimes h_{112}) \otimes \alpha^{-2}(h_{12}) \otimes \alpha^{-2}(h_{21}) \otimes \alpha(h_{22}),$$

which results from

$$\begin{aligned}
& ((id_H \otimes \tilde{a}_{H,H,H}) \otimes id_H) \circ ((id_H \otimes (\Delta \otimes id_H)) \otimes id_H) \\
& \circ((id_H \otimes \Delta) \otimes id_H) \circ (\Delta \otimes id_H) \circ \Delta \\
= \; & (\tilde{a}_{H,H,H\otimes H} \otimes id_H) \circ ((\Delta \otimes id_{H\otimes H}) \otimes id_H) \circ (\tilde{a}_{H,H,H} \otimes id_H) \\
& \circ \tilde{a}^{-1}_{H\otimes H,H,H} \circ (id_{H\otimes H} \otimes \Delta) \circ (\Delta \otimes id_H) \circ \Delta.
\end{aligned}$$

Hom-unity condition: For any $h \in H$,

$$\begin{aligned}
h_{[0]}\varepsilon(h_{[1]}) &= \alpha(h_{12})\varepsilon(S(h_{11})\alpha^{-1}(h_2)) = \alpha(h_{12})\varepsilon(h_{11})\varepsilon(h_2) \\
&= \alpha(\varepsilon(h_{11})h_{12})\varepsilon(h_2) = h_1\varepsilon(h_2) = \alpha^{-1}(h),
\end{aligned}$$

and one can also easily show that $\widetilde{Ad}_R \circ \alpha = (\alpha \otimes \alpha) \circ \widetilde{Ad}_R$. Thus $\widetilde{Ad}_R$ is a right Hom-action of (H, α) onto itself.

2. In a similar manner, it can be proven that $\widetilde{Ad}_L$ is a left Hom-action of (H, α) onto itself.

□

With the next lemma we describe the right coaction ϕ_R on a left-invariant form $\omega_\Gamma(h)$ and the left coaction ϕ_L on a right-invariant form $\eta_\Gamma(h)$ by means of $\widetilde{Ad}_R$ and $\widetilde{Ad}_L$, respectively.

Lemma 5.4.5 *For $h \in H$, we have the formulas*

1. $\phi_R(\omega(h)) = (\omega \otimes id)(\widetilde{Ad}_R(h))$,
2. $\phi_L(\eta(h)) = (id \otimes \eta)(\widetilde{Ad}_L(h))$.

Proof:

1. For $h \in H$,

$$\begin{aligned}
\phi_R(\omega(h)) &= \Delta(S(h_1))(d \otimes id)(\Delta(h_2)) \\
&= (S(h_{12}) \otimes S(h_{11}))(dh_{21} \otimes h_{22}) = S(h_{12}) \cdot dh_{21} \otimes S(h_{11})h_{22} \\
&= S(\alpha(h_{121})) \cdot d(\alpha(h_{122})) \otimes S(h_{11})\alpha^{-1}(h_2) \\
&= \omega(\alpha(h_{12})) \otimes S(h_{11})\alpha^{-1}(h_2) = (\omega \otimes id)(\alpha(h_{12}) \otimes S(h_{11})\alpha^{-1}(h_2)) \\
&= (\omega \otimes id)(\widetilde{Ad}_R(h)).
\end{aligned}$$

2. Similarly, one can show that the equality $\phi_L(\eta(h)) = (id \otimes \eta)(\widetilde{Ad}_L(h))$ holds.

□

Proposition 5.4.6 *Suppose that* (Γ, γ) *is a left-covariant* (H, α)*-Hom-FODC with associated right Hom-ideal* $\mathcal{R}_\Gamma$. *Then* (Γ, γ) *is a bicovariant* (H, α)*-Hom-FODC if and only if* $\widetilde{Ad}_R(\mathcal{R}) \subseteq \mathcal{R} \otimes H$*, that is,* $\mathcal{R}$ *is* $\widetilde{Ad}_R$*-invariant.*

Proof: If (Γ, γ) is a bicovariant Hom-FODC, then the equation obtained in Lemma (5.4.5) holds. It implies that $\widetilde{Ad}_R(\mathcal{R}) \subseteq \mathcal{R} \otimes H$ since $\mathcal{R} = (h \in ker\varepsilon | \, \omega(h) = 0)$. On the contrary, suppose that $\widetilde{Ad}_R(\mathcal{R}) \subseteq \mathcal{R} \otimes H$. We know that the universal Hom-FODC $\Omega^1(H)$ is bicovariant. So, by applying Lemma (5.4.5) to the bicovariant Hom-FODC $\Omega^1(H)$ and using the Ad_R-invariance of $\mathcal{R}$, we conclude that theright Hom-action of $\Omega^1(H)$ passes to the quotient $\Omega^1(H)/\mathcal{N}$, where $\mathcal{N} := H\omega_{\Omega^1(H)}((R))$, which is right-covariant. Hence, from Proposition (5.3.3), (Γ, γ) is right-covariant as well. □

5.4.3 Quantum Monoidal Hom-Lie Algebra

Let (Γ, γ) be a bicovariant (H, α)-Hom-FODC with associated right Hom-ideal $\mathcal{R}$ and finite dimensional quantum Hom-tangent space $(\mathcal{T}, \tau)$, where $\tau = \bar{\alpha}|_{\mathcal{T}}$.

We define a linear mapping $[-,-] : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T}$ by setting, for $X, Y \in \mathcal{T}$,

$$[X, Y](h) = (X \otimes Y)(\widetilde{Ad}_R(h)), \; \forall h \in H. \tag{5.19}$$

$[X, Y] \in \mathcal{T}$: Indeed, since $\widetilde{Ad}_R(\mathcal{R}) \subseteq \mathcal{R} \otimes H$ by the previous proposition and any element of $\mathcal{T}$ annihilates $\mathcal{R}$ by the definition of quantum Hom-tangent space, $(X \otimes Y)(\widetilde{Ad}_R(h)) = 0$ for all $h \in \mathcal{R}$, i.e., $[X, Y](h) = 0, \; \forall h \in \mathcal{R}$. We also obtain $[X, Y](1) = 0$ since $X(1) = 0 = Y(1)$. Thus $[X, Y] \in \mathcal{T}$. Besides, we have

$$\begin{aligned}
[\tau(X), \tau(Y)](h) &= (X \circ \alpha^{-1} \otimes Y \circ \alpha^{-1})(\widetilde{Ad}_R(h)) \\
&= X(h_{12})Y(S(\alpha^{-1}(h_{11}))\alpha^{-2}(h_2)) \\
&= (X \otimes Y)(\widetilde{Ad}_R(\alpha^{-1}(h))) = [X, Y](\alpha^{-1}(h)) \\
&= \tau([X, Y])(h),
\end{aligned}$$

for any $h \in H$, which means $[-,-] : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T}$ is a morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

We now fix some notation. Suppose that $<\cdot,\cdot>: \mathcal{T} \times {}^{coH}\Gamma \to k$ is the bilinear form in the Proposition 5.3.5. There exists a unique bilinear form $<\cdot,\cdot>_2 : (\mathcal{T}\otimes\mathcal{T})\times{}^{coH}(\Gamma\otimes_H\Gamma) \to k$ defined by

$$< X\otimes Y, u\otimes v >_2 =< X, u >< Y, v > \tag{5.20}$$

for $X, Y \in \mathcal{T}$ and $u, v \in {}^{coH}\Gamma$, which is nondegenerate as the bilinear form $<\cdot,\cdot>$ is. If we put $B : \Gamma\otimes_H\Gamma \to \Gamma\otimes_H\Gamma$ for the Woronowicz' braiding, then, for $h, g \in H$, we compute

$$\begin{aligned} B(\omega(h)\otimes_H \omega(g)) &= \gamma(\omega(\alpha(g_{12})))\otimes_H \gamma^{-1}(\omega(h)) \triangleleft (S(g_{11})\alpha^{-1}(g_2)) \\ &= \omega(\alpha^2(g_{12}))\otimes_H \gamma^{-1}(\omega(h) \triangleleft (S(\alpha(g_{11}))g_2)) \\ &= \omega(\alpha^2(g_{12}))\otimes_H \gamma^{-1}(\omega(\bar{h}(S(\alpha(g_{11}))g_2))) \\ &= \omega(\alpha^2(g_{12}))\otimes_H \omega(\overline{\alpha^{-1}(h)}(S(g_{11})\alpha^{-1}(g_2))). \end{aligned} \tag{5.21}$$

With respect to the nondegenerate bilinear form $<\cdot,\cdot>_2$, we define the transpose B^t of B as a linear map acting on $\mathcal{T}\otimes\mathcal{T}$ such that

$$< B^t(X\otimes Y), u\otimes v >_2 =< X\otimes Y, B(u\otimes v) >_2 .$$

We now recall that the dual monoidal Hom-Hopf algebra (H°, α°) of (H, α) consists of functionals $f \in H'$ for which $\Delta_{H^\circ}(f) = f_1 \otimes f_2 \in H'\otimes H'$ and the Hom-counit is given by $\varepsilon_{H^\circ}(f) = f(1_H)$. Since, also $\Delta(f)(h\otimes g) := f(hg)$ for $\Delta(f) \in (H\otimes H)'$ and $h, g \in H$, we have $f(hg) = f_1(h)f_2(g)$. α° is given by $\alpha^\circ(f) = f \circ \alpha^{-1}$ for $f \in H^\circ$. Hom-multiplication m_{H° is the convolution, i.e., $m_{H^\circ}(f\otimes f')(h) = (ff')(h) = f(h_1)f'(h_2)$ for $f, f' \in H'$, $h \in H$ and the Hom-unit is ε_H. The antipode is given by $S(f)(h) = f(S(h))$ for $f \in H^\circ$ and $h \in H$. Since we assumed that $\mathcal{T}_\Gamma$ is finite dimensional, $\mathcal{T}_\Gamma$ is contained in H°. Thus we have the following theorem in $(H^\circ$.

Theorem 5.4.7 *For any $X, Y, Z \in \mathcal{T}_\Gamma$ we have*

1. $[X, Y] = \widetilde{ad}_R(Y)(X) = XY - m_{H^\circ}(B^t(X\otimes Y))$.

2. *Let $\chi = \sum_i X_i\otimes Y_i$ for $X_i, Y_i \in \mathcal{T}$ such that $B^t(\chi) = \chi$, then $\sum_i[X_i, Y_i] = 0$.*

3. $[\tau(X), [Y, \tau^{-1}(Z)]] = [[X, Y], Z] - \sum_i[[X, \tau^{-1}(Z_i)], \tau(Y_i)]$, *where $Y_i, Z_i \in \mathcal{T}$ such that $B^t(Y\otimes Z) = \sum_i Z_i\otimes Y_i$.*

Proof:

1. For $h \in H$,

$$\begin{aligned}
\widetilde{ad}_R(Y)(X)(h) &= ((S(Y_1)\tau^{-1}(X))\tau(Y_2))(h) \\
&= (S(Y_1)\tau^{-1}(X))(h_1)\tau(Y_2))(h_2) = S(Y_1)(h_{11})\tau^{-1}(X)(h_{12})\tau(Y_2))(h_2) \\
&= Y_1(S(h_{11}))X(\alpha(h_{12}))Y_2(\alpha^{-1}(h_2)) = X(\alpha(h_{12}))Y_1(S(h_{11}))Y_2(\alpha^{-1}(h_2)) \\
&= X(\alpha(h_{12}))Y(S(h_{11})\alpha^{-1}(h_2)) = (X \otimes Y)(\alpha(h_{12}) \otimes S(h_{11})\alpha^{-1}(h_2)) \\
&= (X \otimes Y)(\widetilde{Ad}_R(h)) = [X, Y](h),
\end{aligned}$$

which gives us the first equality. If we set the finite sum for $B^t(X \otimes Y) = \sum_i Y_i \otimes X_i$ with $X_i, Y_i \in \mathcal{T}$, then, for any $h, g \in H$,

$$\begin{aligned}
B^t(X \otimes Y)(h \otimes g) &= \sum_i Y_i(h)X_i(g) = \sum_i < \tau^{-1}(Y_i), \omega(h) >< \tau^{-1}(X_i), \omega(g) > \\
&= < \sum_i \tau^{-1}(Y_i) \otimes \tau^{-1}(X_i), \omega(h) \otimes_H \omega(g) >_2 \\
&= < B^t(\tau^{-1}(X) \otimes \tau^{-1}(Y)), \omega(h) \otimes_H \omega(g) >_2 \\
&= < \tau^{-1}(X) \otimes \tau^{-1}(Y), B(\omega(h) \otimes_H \omega(g)) >_2 \\
&= < \tau^{-1}(X) \otimes \tau^{-1}(Y), \omega(\alpha^2(g_{12})) \otimes_H \omega(\overline{\alpha^{-1}(h)}(S(g_{11})\alpha^{-1}(g_2))) >_2 \\
&= < \tau^{-1}(X), \omega(\alpha^2(g_{12})) >< \tau^{-1}(Y), \omega(\overline{\alpha^{-1}(h)}(S(g_{11})\alpha^{-1}(g_2))) > \\
&= X(\alpha^2(g_{12}))Y(\overline{\alpha^{-1}(h)}(S(g_{11})\alpha^{-1}(g_2))) \\
&= X(\alpha^2(g_{12}))Y_1(\overline{\alpha^{-1}(h)})Y_2(S(g_{11})\alpha^{-1}(g_2)) \\
&= Y_1(\overline{\alpha^{-1}(h)})X(\alpha^2(g_{12}))Y_{21}(S(g_{11}))Y_{22}(\alpha^{-1}(g_2)) \\
&= \overline{\tau(Y_1)}(h)S(Y_{21})(g_{11})X(\alpha^2(g_{12}))Y_{22}(\alpha^{-1}(g_2)) \\
&= \overline{\tau(Y_1)}(h)(S(Y_{21})\tau^{-2}(X))(g_1)\tau(Y_{22})(g_2) \\
&= \overline{\tau(Y_1)}(h)[(S(Y_{21})\tau^{-2}(X))\tau(Y_{22})](g) \\
&= (\overline{\tau(Y_1)} \otimes \widetilde{ad}_R(Y_2)(\tau^{-1}(X)))(h \otimes g),
\end{aligned}$$

where in the sixth equality we have used the equation 5.21. So, we have $B^t(X \otimes Y) = \overline{\tau(Y_1)} \otimes \widetilde{ad}_R(Y_2)(\tau^{-1}(X))$. Hence, we make the following computation

$$
\begin{aligned}
m_{H'}(B^t(X\otimes Y)) &= \overline{\tau(Y_1)}\widetilde{ad}_R(Y_2)(\tau^{-1}(X)) \\
&= (\tau(Y_1)-\varepsilon_{H^\circ}(\tau(Y_1))1_{H^\circ})\widetilde{ad}_R(Y_2)(\tau^{-1}(X)) \\
&= \tau(Y_1)[(S(Y_{21})\tau^{-2}(X))\tau(Y_{22})]-(\varepsilon_{H^\circ}(Y_1)1_{H^\circ})[(S(Y_{21})\tau^{-2}(X))\tau(Y_{22})] \\
&= \tau(Y_1)[S(\tau(Y_{21}))(\tau^{-2}(X)Y_{22})]-(\varepsilon_{H^\circ}(Y_1)1_{H^\circ})((S(Y_{21})\tau^{-2}(X))\tau(Y_{22})) \\
&= (Y_1S(\tau(Y_{21})))\tau(\tau^{-2}(X)Y_{22})-(\varepsilon_{H^\circ}(Y_1)1_{H^\circ})((S(Y_{21})\tau^{-2}(X))\tau(Y_{22})) \\
&= (\tau(Y_{11})S(\tau(Y_{12})))(\tau^{-1}(X)Y_2)-1_{H^\circ}([S(\varepsilon_{H^\circ}(Y_{11})Y_{12})\tau^{-2}(X)]Y_2) \\
&= (\varepsilon_{H^\circ}(Y_1)1_{H^\circ})(\tau^{-1}(X)Y_2)-(S(Y_1)\tau^{-1}(X))\tau(Y_2) \\
&= 1_{H^\circ}(\tau^{-1}(XY))-\widetilde{ad}_R(Y)(X) \\
&= XY-\widetilde{ad}_R(Y)(X),
\end{aligned}
$$

that is, we get $\widetilde{ad}_R(Y)(X)=XY-m_{H'}(B^t(X\otimes Y))$.

2. It immediately follows from (1) that $\sum_i[X_i,Y_i]=\sum_i X_iY_i-m_{H'}(B^t(\chi))=\sum_i X_iY_i-\sum_i X_iY_i=0$.

3. Let us first set $[X,Y]=\widetilde{ad}_R(Y)(X)=X\triangleleft Y$. Then,

$$[[X,Y],Z]=[X,Y]\triangleleft Z=(X\triangleleft Y)\triangleleft Z=\tau(X)\triangleleft(Y\tau^{-1}(Z))=[\tau(X),Y\tau^{-1}(Z)].$$

Since, by (1), $YZ=[Y,Z]+\sum_i Z_iY_i$ for $B^t(Y\otimes Z)=\sum_i Z_i\otimes Y_i$, we have

$$
\begin{aligned}
[[X,Y],Z] &= [\tau(X),Y\tau^{-1}(Z)]=[\tau(X),[Y,\tau^{-1}(Z)]+\sum_i\tau^{-1}(Z_i)Y_i] \\
&= [\tau(X),[Y,\tau^{-1}(Z)]]+\sum_i[\tau(X),\tau^{-1}(Z_i)Y_i] \\
&= [\tau(X),[Y,\tau^{-1}(Z)]]+\sum_i[[X,\tau^{-1}(Z_i)],\tau(Y_i)],
\end{aligned}
$$

that is, $[\tau(X),[Y,\tau^{-1}(Z)]]=[[X,Y],Z]-\sum_i[[X,\tau^{-1}(Z_i)],\tau(Y_i)]$ holds.

□

Remark 13 *If we take the braiding B as the flip operator, then B^t is the flip on $\mathcal{T}\otimes\mathcal{T}$ by its definition. In this case, we obtain*

$$[X,Y]=XY-YX,\qquad [X,Y]+[Y,X]=0,\forall X,Y\in\mathcal{T}$$

and

$$\begin{aligned}[\tau(X),[Y,\tau^{-1}(Z)]] &= [[X,Y],Z]-[[X,\tau^{-1}(Z)],\tau(Y)] = -[Z,[X,Y]]+[\tau(Y),[X,\tau^{-1}(Z)]] \\ &= -[Z,[X,Y]]-[\tau(Y),[\tau^{-1}(Z),X]].\end{aligned}$$

Then, by replacing Z with $\tau(Z)$ in the above equality, we get

$$[\tau(X),[Y,Z]]+[\tau(Y),[Z,X]]+[\tau(Z),[X,Y]]=0,$$

which is the Hom-Jacobi identity. In the above theorem, items (2) and (3) are the quantum versions of the antisymmetry and the Hom-Jacobi identity. Therefore, $(\mathcal{T}_\Gamma,\tau)$ is called the quantum Hom-Lie algebra of the bicovariant (H,α)-Hom-FODC.

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