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Normal Forms for the Transverse Poisson Structure to a Coadjoint Orbit

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To state a theorem and then to show examples of it is literally to teach backwards.

E. Kim Nebeuts

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Abstract

In this thesis we investigate the existence of normal forms for the transverse Poisson structure to the coadjoint orbit of a point μ in the dual of a Lie algebra. We start by establishing a formula for the transverse Poisson structure to a coadjoint orbit which is easily programmable. After that, we prove a new sufficient condition for linearity of the transverse Poisson structure to a coadjoint orbit and apply it to important classes of Lie algebras. We also establish a necessary condition for linearity and use it to show, for example, that there are no linear transverse Poisson structures to any coadjoint orbit of $\mathfrak{se}(3)^*$, apart from the trivial ones. After, we prove that there are also no polynomial transverse Poisson structures to coadjoint orbits of $\mathfrak{se}(3)^*$. Nevertheless, it turns out that in this specific case there are transverse Poisson structures that are "polynomializable", i.e., Poisson-diffeomorphic to a polynomial Poisson structure. In order to illustrate the presented results, several examples of transverse Poisson structures to coadjoint orbits are computed, using the formula referred to above.

Resumo

Nesta tese investigamos a existência de formas normais para a estrutura de Poisson transversa à órbita coadjunta de um ponto μ no dual de uma álgebra de Lie. Em primeiro lugar, desenvolvemos uma formula facilmente programável para a estrutura de Poisson transversa a uma órbita coadjunta. Depois, demonstramos uma nova condição suficiente para a linearidade da estrutura de Poisson transversa a uma órbita coadjunta, que se aplica a classes importantes de álgebras de Lie. Demonstramos também uma condição necessária para a linearidade da transversa e usamo-la para mostrar que, por exemplo, não existem estruturas de Poisson transversas a nenhuma órbita coadjunta de $\mathfrak{se}(3)^*$ que sejam lineares, para além das triviais. Por fim, provamos que também não existem estruturas de Poisson transversas a órbitas coadjuntas de $\mathfrak{se}(3)^*$ que sejam polinomiais. No entanto, acontece que no caso de $\mathfrak{se}(3)^*$ existem estruturas de Poisson transversas que são "polinomializáveis", ou seja, Poisson-difeomorfas a estruturas de Poisson polinomiais. Ilustrámos os resultados apresentados através do cálculo explícito de vários exemplos de estruturas de Poisson transversas a órbitas coadjuntas, utilizando a formula referida acima.

Résumé

Dans cette thèse, nous étudions l'existence de formes normales pour la structure de Poisson transverse à l'orbite coadjointe en un point du dual d'une algèbre de Lie. En premier lieu, on prouve une formule pour la structure de Poisson transverse à une orbite coadjointe qui peut être facilement programmée. Ensuite, on exhibe une nouvelle condition suffisante pour la linéarité de la structure de Poisson transverse à une orbite coadjointe, que l'on applique à des classes importantes d'algèbres de Lie. On démontre aussi une condition nécessaire pour la linéarité d'une structure transverse que on l'utilise, par exemple, pour montrer qu'il n'existe pas de structure de Poisson transverse à aucune orbite coadjointe de $\mathfrak{se}(3)^*$ qui soit linéaire, sauf pour les cas triviaux. Finalement, on montre qu'il n'y a pas de structure de Poisson transverse à aucune orbite coadjointe de $\mathfrak{se}(3)^*$ qui soit polynomiale. Il existe cependant dans ce cas des structures de Poisson transverses qui sont "polynomializables", c'est-à-dire difféomorphes à une structure de Poisson polynomiale. Pour illustrer les résultats présentés, on calcule des exemples de structures de Poisson transverses à des orbites coadjointes, en utilisant notre formule.

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Introduction

Poisson manifolds appear naturally in many Classical Mechanics problems. Singular Poisson structures (i.e., Poisson structures in which the rank varies pointwise) appear less frequently, and are greater in complexity and interest than the more common constant rank Poisson structures. Some examples in Mechanics where such singular structures arise are:

1. the Euler equations for the angular velocity vector for a rigid body with a fixed point;

2. equations for a mechanical system in which a given parameter approaches a limit value.

The simpler Poisson structures are the symplectic structures, which are the basic layout for the equations in Classical Mechanics.

Lie-Poisson structures, defined on the dual of any real Lie algebra, are examples of non-symplectic Poisson structures. They appear in the Euler equations referred to above. In this case, the Poisson structure involved is the Lie-Poisson structure on the dual of the Lie algebra $\mathfrak{so}(3)$. These singular structures are precisely the linear ones, and constitute the simplest possible example of singular Poisson structures.

Given a Poisson manifold and a singular point (of arbitrary rank), we consider two associated (sub)structures:

1. the symplectic leaf through the singular point;

2. the transverse Poisson structure to the symplectic leaf through the same point.

The latter is defined on any transverse manifold to the symplectic leaf, and was introduced by A. Weinstein [18]. It depends on the chosen transverse manifold, but two transverse Poisson structures to a symplectic leaf at the same point are always Poisson-diffeomorphic.

In the particular case where the original Poisson manifold is the dual of a Lie algebra, the symplectic leaf through the singular point is the coadjoint orbit through the same point. In this case, there is a "canonic family" of transverse manifolds which are actually affine subspaces in the dual of the Lie algebra (see [18]). These affine subspaces are related to a specific supplement of the isotropy subalgebra of the given point, and different choices of that supplement lead to different (although Poisson-diffeomorphic) transverse Poisson structures.

There are conditions on such supplements of the isotropy subalgebra which imply linearity of the transverse Poisson structure (Molino [12]) or that it is quadratic (Oh [13]). Cushman & Roberts [3] proved that if the initial Lie algebra is semisimple, then there is a polynomial transverse Poisson structure. This result had already been conjectured by Damianou [4]. However, the conditions by Molino and Oh are often difficult to prove in a particular case.

In [1] we have presented a simple formula for computing the transverse Poisson structure to a coadjoint orbit and applied it to some specific cases. We verified that different choices of the supplement referred above may result in transverse Poisson structures substancially different in nature. That is the case of the dual of the Lie algebra $\mathfrak{so}(4)$, where we obtained both a linear and a non-polynomial transverse Poisson structure, at the same singular point.

It is important to notice that all the results we referred to up to this point were included in the author's masters degree dissertation, finished in February of 2003. These results correspond to the majority of Chapters 1 and 2 of this thesis. In Chapter 1, the exceptions are Lemma 9 and Examples 13, 23 and 25. In Chapter 2, the exceptions are the proofs of Proposition 36 and Theorem 38, as well as the whole Section 2.3.

Chapter 3 begins with a proof that, if there is a linear transverse Poisson structure to a coadjoint orbit at a certain point, then there will also be a linear transverse Poisson structure at any point in the same coadjoint orbit. Hence, the expression "linear Poisson structure *to a coadjoint orbit*" makes sense. After that, we prove a sufficient condition (on the isotropy subalgebra of the given point) so that there is an affine subspace on which the transverse Poisson structure is linear. This condition (which is sufficient but not necessary) implies that there is a supplement of the isotropy subalgebra in $\mathfrak{g}$ which satisfies the condition by Molino, which in turn guarantees the linearity of the transverse Poisson structure. Several corollaries of this result are then derived, establishing linear transverse Poisson structures to coadjoint orbits in several classes of Lie-Poisson manifolds. More specifically, there is a linear transverse Poisson structure to a coadjoint orbit when:

1. the Lie algebra $\mathfrak{g}$ is of compact type;

2. the Lie algebra $\mathfrak{g}$ considered is semisimple and the *splitting* point con-

sidered is a semisimple element of $\mathfrak{g}^*$;

3. the isotropy subalgebra of the splitting point is semisimple or the isotropy subgroup is compact.

We provide several examples that illustrate these corollaries. We also give an example that shows that the Molino condition is not necessary for linearity. Furthermore, we show that our condition for linearity is equivalent to the Molino condition for some classes of Lie algebras, but the two are not equivalent in general.

In Chapter 4, we work on necessary conditions for linearity of the transverse Poisson structure, using the notion of Taylor approximation (of degree 1) of a Poisson structure at a zero rank point. We were able to infer the linearity or non-linearity of transverse Poisson structures on several Lie-Poisson manifolds using a necessary condition for linearity. The main idea used throughout this chapter was to find out exactly what Poisson structures have in common with their linear Taylor approximations.

Chapter 5 was originated by two questions raised by the referee of [2], regarding transverse Poisson structures to coadjoint orbits of $\mathfrak{se}(3)^*$. Apart from their (non-)linearity, he was interested in two issues:

1. investigating the existence of polynomial transverse Poisson structures;
2. investigating if they are Poisson-diffeomorphic to polynomial Poisson structures (or "polynomializable").

In general, the first issue is much more difficult to address than the linearity issue, because a Taylor approximation of a Poisson tensor is not necessarily a Poisson tensor. However, in the case of $\mathfrak{se}(3)^*$, we were able to establish that there are no polynomial transverse Poisson structures. Regarding the second issue, we have indeed found polynomializable transverse Poisson structures to coadjoint orbits of $\mathfrak{se}(3)^*$ (apart from the trivial ones).

We include in appendices the computations of several transverse Poisson structures, referred to in examples throughout this thesis.

Chapter 1

Preliminary Results

1.1 Notation

We begin by stating some conventions followed in this work:

- All manifolds are supposed to be real, connected, finite dimensional and differentiable, even if we omit that fact in the text.
- If $\mathcal{M}$ is a finite-dimensional differentiable manifold, we denote its tangent bundle by $\mathrm{T}\mathcal{M}$ and its cotangent bundle by $\mathrm{T}^*\mathcal{M}$.
- Throughout this work, smoothness is intended in the C^∞ sense. The set of all smooth functions $f : \mathcal{M} \to \mathbb{R}$ is hence denoted by $C^\infty(\mathcal{M})$, while $\mathfrak{X}(\mathcal{M})$ is the set of all smooth vector fields on $\mathcal{M}$.
- We denote by $\Omega^k(\mathcal{M})$ the set of all smooth k-forms over $\mathcal{M}$. In particular,
$$\Omega^0(\mathcal{M}) = C^\infty(\mathcal{M}).$$
- We denote by $\mathcal{A}^k(\mathcal{M})$ the set of all smooth contravariant skew-symmetric k-tensors on $\mathcal{M}$.
- If $\varphi : \mathcal{M} \to \mathcal{N}$ is a differentiable application, then φ^* is the *pull-back* by φ.
- If φ_t is the flow of a vector field $X \in \mathfrak{X}(\mathcal{M})$ and p is a point of $\mathcal{M}$, then the set
$$\{\varphi_t(p) : t \in I\}$$
is the *integral curve of X through p* (where I is the maximal interval where the curve $\varphi_t(p)$ is defined).

- All vector spaces (including Lie algebras) are assumed to be real and finite-dimensional.
- Given a vector space V we denote by $\langle \alpha, v\rangle$ the evaluation of $\alpha \in V^*$ at $v \in V$.
- Given a vector subspace $S \subset V$, the symbol S° stands for the *annihilator* of S in V^*, which is the set defined as follows:

$$S = \{\alpha \in V^* : \langle \alpha, v\rangle = 0, \quad \forall v \in S\}.$$

- If $\mathcal{N} \subset \mathcal{M}$ is a submanifold and $y \in \mathcal{N}$, we denote by $\mathrm{T}_y^\circ\mathcal{N}$ the annihilator of the tangent space $\mathrm{T}_y\mathcal{N}$ in $\mathrm{T}_y^*\mathcal{M}$.
- We will frequently use the isomorphism

$$\begin{array}{rccc} \Psi : & V & \to & V^{**} \\ & v & \to & \Psi_v \end{array},$$

where Ψ_v is the linear map defined by

$$\begin{array}{rccc} \Psi_v : & V^* & \to & \mathbb{R} \\ & \alpha & \to & \langle \alpha, v\rangle \end{array}.$$

This identification between V and V^{**} will be made with no further comment.

1.2 Poisson Manifolds

Definition 1 Let $\mathcal{M}$ be a manifold. A *Poisson bracket* in $\mathcal{M}$ is a map

$$\begin{array}{rccc} \{,\} : & C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) & \to & C^\infty(\mathcal{M}) \\ & (f, g) & \mapsto & \{f, g\} \end{array}$$

with the following properties:

1. Skew-symmetry:

$$\{f, g\} = -\{g, f\}, \quad \forall f, g \in C^\infty(\mathcal{M}).$$

2. $\mathbb{R}-$Bilinearity:

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \forall a, b \in \mathbb{R}, \quad \forall f, g, h \in C^\infty(\mathcal{M}).$$

3. Jacobi Identity:

$$\{f,\,\{g,\,h\}\}+\{g,\,\{h,\,f\}\}+\{h,\,\{f,\,g\}\}=0,\quad \forall\, f,\,g,\,h\in C^\infty(\mathcal{M}).$$

4. Leibniz Identity:

$$\{fg,\,h\}=f\{g,\,h\}+\{f,\,h\}g,\quad \forall\, f,\,g,\,h\in C^\infty(\mathcal{M}).$$

We say that $(\mathcal{M};\,\{,\,\})$ is a *Poisson manifold.*

It follows from properties 2 and 4 that, fixed $f\in C^\infty(\mathcal{M})$, the map

$$\begin{array}{rccc}\{f,\,\cdot\}: & C^\infty(\mathcal{M}) & \to & C^\infty(\mathcal{M}) \\ & g & \mapsto & \{f,\,g\}\end{array}$$

is a derivation in $C^\infty(\mathcal{M})$. Then the following definition makes sense.

Definition 2 Let $(\mathcal{M};\,\{,\,\})$ be a Poisson manifold and $f\in C^\infty(\mathcal{M})$. The *hamiltonian vector field* associated to f is defined as follows:

$$\begin{array}{rccc}X_f: & C^\infty(\mathcal{M}) & \to & C^\infty(\mathcal{M}) \\ & g & \mapsto & \{f,\,g\}\end{array}.$$

The set of all hamiltonian fields in $\mathcal{M}$ is denoted by $\mathfrak{X}_H(\mathcal{M})$.

Lemma 3 *Let $(\mathcal{M};\,\{,\,\})$ be a Poisson manifold, $x_1,\,\ldots,\,x_n$ local coordinates in $\mathcal{M}$ and $f,\,g\in C^\infty(\mathcal{M})$. Then*

$$\{f,\,g\}=\sum_{i=1}^n\sum_{j=1}^n\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\{x_i,\,x_j\}=\sum_{1\le i<j\le n}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}-\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)\{x_i,\,x_j\}.$$

Consequently, the Poisson structure on $\mathcal{M}$ is completely determined by the skew-symmetric matrix of functions

$$\mathrm{P}(x)=[\mathrm{P}_{ij}(x)]_{i,j=1}^n\,,$$

where

$$\mathrm{P}_{ij}(x)=\{x_i,\,x_j\}(x).$$

Definition 4 The matrix P is the *Poisson matrix* of the structure $(\mathcal{M};\ \{,\})$ in coordinates $x_1,\ \ldots,\ x_n$.

Lemma 5 *Given a Poisson manifold* $(\mathcal{M};\ \{,\})$, *it is possible to define a bundle map*

$$\mathcal{P}^{\#} : \mathrm{T}^*\mathcal{M} \to \mathrm{T}\mathcal{M}$$

such that

$$\{f,\ g\} = \left\langle \mathrm{d}g,\ \mathcal{P}^{\#}(\mathrm{d}f) \right\rangle, \quad \forall f,\ g \in C^\infty(\mathcal{M}).$$

It is assumed that at each point x, $\mathcal{P}^{\#}_x$ is an $\mathbb{R}$-linear map with image in a fiber over x, i.e.

$$\mathcal{P}^{\#}_x : \mathrm{T}^*_x\mathcal{M} \to \mathrm{T}_x\mathcal{M}.$$

Remark 6 The skew-symmetry of $\{,\}$ and the Jacobi Identity lead to additional restrictions on $\mathcal{P}^{\#}$. Furthermore, with respect to the usual bases of $\mathrm{T}^*\mathcal{M}$ and $\mathrm{T}\mathcal{M}$, $\mathcal{P}^{\#}$ is represented by the transpose of P (which is also $-\mathrm{P}$ due to skew-symmetry).

A Poisson structure on $\mathcal{M}$ may also be defined through a contravariant skew-symmetric 2-tensor $\mathcal{P}$. Given $\mathcal{P} \in \mathrm{A}^2(\mathcal{M})$, we may define a bracket of functions f, g in $C^\infty(\mathcal{M})$ by

$$\{f, g\} = \mathcal{P}(\mathrm{d}f, \mathrm{d}g).$$

This bracket is bilinear, skew-symmetric and satisfies the Leibniz Identity. Nevertheless, in order to satisfy Jacobi Identity, the 2-tensor $\mathcal{P}$ has to be such that

$$[\mathcal{P},\ \mathcal{P}]_S \equiv 0,$$

where

$$[\cdot,\cdot]_S : \mathrm{A}^k(\mathcal{M}) \times \mathrm{A}^l(\mathcal{M}) \to \mathrm{A}^{k+l-1}(\mathcal{M})$$

stands for the Schouten bracket. For further details see [16]. Such a $\mathcal{P}$ will be called a Poisson tensor on $\mathcal{M}$. Its relation with the bundle map $\mathcal{P}^{\#}$, defined in Lemma 5, is the following:

$$\begin{array}{rccc} \mathcal{P}: & \Omega^1(\mathcal{M}) \times \Omega^1(\mathcal{M}) & \to & C^\infty(\mathcal{M}) \\ & (\alpha,\ \beta) & \mapsto & \left\langle \beta, \mathcal{P}^{\#}(\alpha) \right\rangle \end{array}.$$

Let $x_1, \ldots, x_m$ be local coordinates em $\mathcal{M}$. Then, we can write

$$\mathcal{P} = \sum_{i<j} \mathcal{P}_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

and the coefficients $\mathcal{P}_{ij}$ coincide with the entries P_{ij} of the Poisson matrix, since:

$$\begin{aligned} \mathcal{P}_{ij} &= \mathcal{P}(\mathrm{d}x_i, \mathrm{d}x_j) \\ &= \{x_i, x_j\}. \end{aligned}$$

Notation 7 *From now on, we may denote a generic Poisson manifold* $(\mathcal{M}; \{,\})$ *by* $(\mathcal{M}, \mathcal{P})$*,* $(\mathcal{M}, \mathcal{P}^{\#})$ *or* $(\mathcal{M}, \mathrm{P})$*.*

Definition 8 Let $(\mathcal{M}; \{,\})$ be a Poisson manifold and $x \in \mathcal{M}$.

1. The *rank* of the bracket $\{,\}$ at x is the rank of the map

   $$\mathcal{P}^{\#}_x : \mathrm{T}^*_x\mathcal{M} \to \mathrm{T}_x\mathcal{M}.$$

   Equivalently, it is the rank of the Poisson matrix $\mathrm{P}(x)$ and is denoted by $\operatorname{rank} \mathrm{P}_x$ or $\operatorname{rank} \mathcal{P}^{\#}_x$.

2. The point x is said to be *regular* if $\operatorname{rank} \mathrm{P}_x$ is constant in a neighborhood of x in $\mathcal{M}$. Otherwise, x is said to be *singular*.

3. The Poisson structure is *non-degenerate* if, for any $x \in \mathcal{M}$,

   $$\operatorname{rank} \mathrm{P}_x = \dim \mathcal{M}.$$

There are some restrictions to the rank of a Poisson structure at a given point x. The matrix P is always skew-symmetric, so $\operatorname{rank} \mathrm{P}_x$ is always even. Moreover, we know from Linear Algebra that $\operatorname{rank} \mathrm{P}_x$ is a lower semicontinuous function of x. Hence, if x is such that

$$\operatorname{rank} \mathrm{P}_x = \max_{y \in \mathcal{M}} \operatorname{rank} \mathrm{P}_y$$

then x is regular. If, in addition, all entries P_{ij} of the Poisson matrix are analytic functions, then there are no regular points whose rank is not maximal. To see that, consider a point x of maximal rank, say $2r$. Since x is regular, the rank of $\{,\}$ is $2r$ in a neighborhood of x. Furthermore, there is a submatrix Q of P such that

$$\det \mathrm{Q}(z) \neq 0$$

for all points z in a neighborhood of x. Now suppose there is a regular point y whose rank is lower than $2r$. Then there is a neighborhood of y such that no submatrix of P has rank $2r$. Therefore,

$$\det \mathrm{Q}(w) = 0$$

for all points w in a neighborhood of y. On the other hand, $\det \mathrm{Q}$ is an analytic function, because it is the result of products and sums of the analytic entries of the Poisson matrix. Being analytic and constant in an open set, $\det \mathrm{Q}$ must be constant, absurd. We have thus proved the following Lemma:

Lemma 9 *Given an analytic Poisson structure, a point is regular if and only if it has maximal rank.*

We remark that this does not mean that, in analytic Poisson structures, all regular points have rank equal to $\dim \mathcal{M}$. For example, $\mathcal{M}$ could be odd-dimensional.

Example 10 If $(\mathcal{M}, \omega)$ is a symplectic manifold, then

$$\{f, g\} = \omega(X_f, X_g)$$

is a Poisson bracket on $\mathcal{M}$. For example, considering $\mathbb{R}^{2n}$ endowed with the usual symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

we get

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}.$$

Hence, the Poisson matrix is

$$\mathrm{P} = \begin{pmatrix} 0_{n\times n} & \mathbb{I}_n \\ -\mathbb{I}_n & 0_{n\times n} \end{pmatrix}, \forall\, (x, y) \in \mathbb{R}^{2n}.$$

This matrix will be denoted by $\mathbb{J}_0$. We remark that this Poisson structure is non-degenerate. In fact, it would be so for any symplectic structure we had chosen. Conversely, all non-degenerate Poisson structures are symplectic.

In fact, non-degeneracy of the Poisson structure on $\mathcal{M}$ implies that the bundle map $\mathcal{P}^{\#}$ is such that

$$\operatorname{rank} \mathcal{P}^{\#}_x = \dim \mathrm{T}_x\mathcal{M}, \quad \forall x \in \mathcal{M}.$$

Hence, the linear map $\mathcal{P}_x^{\#} : \mathrm{T}_x^*\mathcal{M} \to \mathrm{T}_x\mathcal{M}$ is invertible, for every x in $\mathcal{M}$. One can check that

$$\omega(X, Y) = \left\langle \left(\mathcal{P}^{\#}\right)^{-1}(X), Y \right\rangle$$

is a closed, non-degenerate differential two-form, hence a symplectic form on $\mathcal{M}$.

Example 11 Consider $\mathbb{R}^m$, with cartesian coordinates

$$\{x_1, \ldots, x_l, y_1, \ldots, y_l, z_1, \ldots, z_{m-2l}\}.$$

When equipped with the bracket

$$\{f, g\}^{m-2l} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i},$$

$\mathbb{R}^m$ is a Poisson manifold, which is not symplectic for $m \neq 2l$. The Poisson matrix in these coordinates is

$$\mathrm{P} = \begin{pmatrix} \mathbb{J}_0 & 0_{2l\times(m-2l)} \\ 0_{(m-2l)\times 2l} & 0_{(m-2l)\times(m-2l)} \end{pmatrix}, \forall\, (x, \, y\, z) \in \mathbb{R}^{2n+m}.$$

We notice that all points in this Poisson manifold are regular.

Example 12 In $\mathcal{M} = \mathbb{R}^3$, we may define a Poisson bracket as follows. Given f, g in $C^\infty(\mathbb{R}^3)$, $\{f, g\}$ is the only function such that

$$\{f, g\}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \mathrm{d}f \wedge \mathrm{d}g \wedge (\mathrm{d}x + \mathrm{d}y + \mathrm{d}z),$$

or alternatively,

$$\{f, g\} = \frac{\partial f}{\partial x}\left(\frac{\partial g}{\partial y} - \frac{\partial g}{\partial z}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial z} - \frac{\partial g}{\partial x}\right) + \frac{\partial f}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\right).$$

We can easily verify, by computing the Poisson matrix, that all points have rank two. Also in this example, all points in $\mathcal{M}$ are regular but the Poisson structure is not symplectic.

Example 13 Consider $\mathbb{R}^2$ equipped with the Poisson bracket given by the matrix

$$\begin{pmatrix} 0 & f(x,y) \\ -f(x,y) & 0 \end{pmatrix},$$

where

$$f(x,y) = \begin{cases} 0 & if \quad x^2+y^2 \leq 1 \\ e^{\frac{-1}{x^2+y^2-1}} & if \quad x^2+y^2 > 1 \end{cases}.$$

then all points in the open disk with radius 1 (centered at the origin) have rank 0 and are regular. Points on the circle of radius 1 have rank zero and are singular. Points outside the circle of radius 1 are regular but of rank 2. We remark that such an f is a smooth function, but not analytic.

1.3 The Dual of a Lie Algebra

Definition 14 Let V be a vector space. A Poisson bracket on V,

$$\{,\} : C^\infty(V) \times C^\infty(V) \to C^\infty(V)$$

is called *linear* if, for any α, $\beta \in V^*$ we have:

$$\{\alpha, \beta\} \in V^*.$$

Let $(\mathfrak{g}; [,])$ be a real and finite dimensional Lie algebra. Then $\mathfrak{g}^*$ is a Poisson manifold when equipped with the *Lie-Poisson bracket*:

$$\{f, g\}^L(x) = \langle x, [\mathrm{d}f_x, \mathrm{d}g_x] \rangle,$$

where x is an element of $\mathfrak{g}^*$, f and g are smooth functions in $\mathfrak{g}^*$. We remark that

$$\mathrm{T}_x(\mathfrak{g}^*) \cong \mathfrak{g}^*.$$

Furthermore,

$$f, g : \mathfrak{g}^* \to \mathbb{R},$$

hence

$$\mathrm{d}f_x, \mathrm{d}g_x \in (\mathrm{T}_x\mathfrak{g}^*)^* = \mathfrak{g}^{**} \cong \mathfrak{g}.$$

Now we take X, Y elements of $\mathfrak{g}^{**}$. The bundle map $\mathcal{P}^{\#} : \mathfrak{g} \to \mathfrak{g}^*$, associated to the Lie-Poisson structure, is such that

$$\begin{aligned} \left\langle Y, \mathcal{P}^{\#}_x(X) \right\rangle &= \{X,\ Y\}^L(x) \\ &= \langle x,\ [\mathrm{d}X_x, \mathrm{d}\,Y_x] \rangle \\ &\cong \langle x,\ [X, Y] \rangle \,, \end{aligned}$$

i.e.

$$\begin{array}{rccc} \mathcal{P}^{\#}_x : & \mathfrak{g} & \to & \mathfrak{g}^* \\ & X & \mapsto & \mathrm{ad}^*_X(x) \end{array} .$$

To compute the Lie-Poisson matrix, we consider a basis of $\mathfrak{g}$, $\{X_1,\ \ldots,\ X_n\}$, and c^k_{ij} the structure constants of $\mathfrak{g}$ with respect to the given basis, i.e., such that

$$[X_i,\ X_j] = \sum_k c^k_{ij} X_k.$$

Then $x_1,\ \ldots,\ x_n$, the elements of $\mathfrak{g}^{**}$ identified with $X_1,\ \ldots,\ X_n$, are linear coordinates in $\mathfrak{g}^*$. The entries of the Poisson matrix, in these coordinates, are given by:

$$\begin{aligned} \mathrm{P}_{ij}(x) &= \{x_i,\ x_j\}^L(x) \\ &\cong \langle x,\ [X_i, X_j] \rangle \\ &\cong \sum_k c^k_{ij} x_k(x). \end{aligned}$$

The coefficients c^k_{ij} are real numbers, therefore this is a linear Poisson structure. In particular, the entries of the Poisson matrix are analytic functions of the coordinates $x_1,\ \ldots,\ x_n$, so Lemma 9 guarantees that the regular points are exactly those of maximal rank. This implies that the origin is always a singular point (unless $\mathfrak{g}$ is abelian).

Remark 15 All linear Poisson structures (in a vector space V) are Lie-Poisson structures on the dual of some Lie algebra. This derives from the fact that the space of linear functions in V^* is a subalgebra of the Lie algebra $(C^\infty(V^*);\ \{,\})$ and is hence a Lie algebra itself.

1.4 The Symplectic Foliation

Definition 16 *Let* $(\mathcal{M}_1; \{,\}_1)$, $(\mathcal{M}_2; \{,\}_2)$ *be two Poisson manifolds.*

1. *A differentiable map* $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ *is a* Poisson map *if*

$$\phi^*\{f,\, g\}_2 = \{\phi^* f,\, \phi^* g\}_1, \quad \forall\, f,\, g \in C^\infty(\mathcal{M}_2).$$

If ϕ *is also a diffeomorphism (resp., local diffeomorphism) then* ϕ *is a* Poisson diffeomorphism *(resp.,* local Poisson diffeomorphism*).*

2. *Suppose that* $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$*. An* infinitesimal automorphism *in* $(\mathcal{M}; \{,\})$ *is a vector field* X *of* $\mathcal{M}$ *whose flow* σ_t *has the following property: Given* $t \in \mathbb{R}$,

$$\sigma_t : D_t \to D_{-t}$$

is a Poisson diffeomorphism, where D_t *is the subset (eventually empty) of* $\mathcal{M}$ *where the flow* σ_t *is defined.*

We remark that if ϕ is a Poisson diffeomorphism in $\mathcal{M}$, then so is its inverse. Moreover, the set of all Poisson diffeomorphisms in $\mathcal{M}$ equipped with map composition has a group structure. Poisson diffeomorphisms also preserve the rank of the Poisson structures, i.e., if

$$\phi : \mathcal{M}_1 \to \mathcal{M}_2$$

is a Poisson diffeomorphism, then

$$\mathrm{rank}(\mathrm{P}_1)_x = \mathrm{rank}(\mathrm{P}_2)_{\phi(x)}, \quad \forall\, x \in \mathcal{M}_1.$$

We will now take a closer look at infinitesimal automorphisms and clarify their role in a Poisson structure.

Theorem 17 (Libermann, Marle [8]) *Let* $(\mathcal{M};\, \{,\})$ *be a Poisson manifold and* $X \in \mathfrak{X}(\mathcal{M})$*. The following statements are equivalent:*

1. X *is a derivation of the algebra* $(C^\infty(\mathcal{M});\, \{,\})$*, i.e.,*

$$\forall\, f,\, g \in C^\infty(\mathcal{M}) : \; X(\{f,\, g\}) = \{X(f),\, g\} + \{f,\, X(g)\};$$

2. X *is an infinitesimal automorphism.*

We refer the reader to [8] for a proof of this result. Any hamiltonian vector field X_f is therefore an infinitesimal automorphism, because for any g, h smooth functions in $\mathcal{M}$,

$$X_f(\{g,\, h\}) = \{X_f(g),\, h\} + \{g,\, X_f(h)\}.$$

Hence, the flow of an hamiltonian field is a (local) Poisson diffeomorphism.

Definition 18 Given a Poisson manifold $(\mathcal{M};\{,\})$, a submanifold $\mathcal{N}$ is a *Poisson submanifold* if there is a Poisson bracket $\{,\}_{\mathcal{N}}$ in $\mathcal{N}$ such that the inclusion

$$i:\mathcal{N}\to\mathcal{M}$$

is a Poisson map, i.e.,

$$i^*\{f,\,g\}=\{i^*f,\,i^*g\}_{\mathcal{N}}\,.$$

In other words, the function $\{f,\,g\}|_{\mathcal{N}}$ depends only on the restrictions of f and g to $\mathcal{N}$.

Let us now see another characterization of Poisson submanifolds, due to Weinstein.

Theorem 19 (Weinstein [18]) *Given a Poisson manifold* $(\mathcal{M};\{,\})$, *a submanifold* $\mathcal{N}\subset\mathcal{M}$ *is a* Poisson submanifold *if and only if the following holds, for every point* y *of* $\mathcal{N}$:

$$\mathcal{P}^{\#}_y(\mathrm{T}^*_y\mathcal{M})\subset\mathrm{T}_y\mathcal{N},$$

i.e., if and only if all hamiltonian vector fields in $\mathcal{M}$ *are tangent to* $\mathcal{N}$.

We recall that given $\alpha\in\mathrm{T}^*_y\mathcal{M}$, there is a smooth function f such that $\alpha=\mathrm{d}f_y$. Then, the statement

$$\mathcal{P}^{\#}_y(\mathrm{T}^*_y\mathcal{M})\subset\mathrm{T}_y\mathcal{N}$$

is the same as

$$\left(X_f\right)_y\in\mathrm{T}_y\mathcal{N},\quad\forall f\in C^\infty(\mathcal{M}).$$

Proposition 20 (Weinstein [18]) *Given a Poisson manifold* $(\mathcal{M};\{,\})$, *we may define the following relation in* $\mathcal{M}$:

$x\sim y$ *if there is a curve* $\gamma:[-\epsilon,\,\epsilon]\to\mathcal{M}$ *such that*

(i) $\gamma(0)=x$, $\gamma(\epsilon)=y$;

(ii) γ *is piecewise smooth;*

(iii) *Every smooth piece of* γ *is an integral curve of an hamiltonian field in* $\mathcal{M}$.

Then $\sim$ *is an equivalence relation and the equivalence class* $[x]$ *of each point* x *is a Poisson submanifold of* $\mathcal{M}$. *Moreover,*

$$dim([x])=\operatorname{rank}\mathrm{P}_x.$$

As a consequence of Proposition 20, the structure $\{,\}|_{[x]}$ is non-degenerate or symplectic. The symplectic form is given by

$$\omega(X,Y) = \left\langle \left(\mathcal{P}^{\#}\right)^{-1}(X), Y \right\rangle,$$

where X and Y are vector fields tangent to $[x]$.

Definition 21 *The Poisson submanifold* $[x]$ *of* $\mathcal{M}$ *is called the* symplectic leaf of $(\mathcal{M}; \{,\})$ through x *and is denoted by* $\mathcal{S}_x$ *or simply* $\mathcal{S}$*. The partition of* $\mathcal{M}$ *by these equivalence classes is called the* symplectic foliation *of the Poisson manifold* $\mathcal{M}$*.*

The simplest example of a symplectic foliation is the case of a connected symplectic manifold (Example 10), where the only symplectic leaf is the entire manifold.

In fact, we can characterize the symplectic foliation of a Poisson manifold in terms of distribution theory. First we define a generalized distribution $\mathcal{D}$ in $\mathcal{M}$ as follows:

$$\mathcal{D}_x = \mathrm{Im}(\mathcal{P}_x^{\#}).$$

This distribution is differentiable because it is generated by the hamiltonian vector fields, which are differentiable sections defined in $\mathcal{M}$. It can be shown that this distribution is completely integrable, and its induced foliation is the symplectic foliation (see [8]).

In some cases, the notion of Casimir function is important to determine the symplectic foliation.

Definition 22 A function c in $C^\infty(\mathcal{M})$ is a *Casimir function* of $\mathcal{M}$ if it satisfies any of the following three (equivalent) conditions:

1. $\{c, f\} = 0, \quad \forall f \in C^\infty(\mathcal{M})$;

2. c is constant along the flows of all hamiltonian vector fields;

3. $X_c \equiv 0$.

Notice that Casimir functions are constant on any symplectic leaf. Therefore, in a connected symplectic manifold, the only Casimirs are the constant functions. Now we will see some more interesting examples of symplectic foliations.

Example 23 Consider $\mathbb{R}^3$, with the same Poisson bracket as in Example 12. We have already established that all points have rank two and hence are regular. Then Proposition 20 implies that all symplectic leaves are two-dimensional. Now we remark that the linear function $c = -x + y + z$ is a (global) Casimir, because

$$\{-x+y+z,\, f\} = -\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\right) + \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial z}\right) + \left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right) = 0,$$

for all f in $\mathbb{R}^3$. Hence, all symplectic leaves are contained in planes of the form

$$A_k = \{(x, y, z) : -x + y + z = k\},$$

which are the level sets of the Casimir function f. The A_k's are two-dimensional and form a partition of $\mathbb{R}^3$. We will now check that these planes are indeed the symplectic foliation by proving that each one of them is a single symplectic leaf.

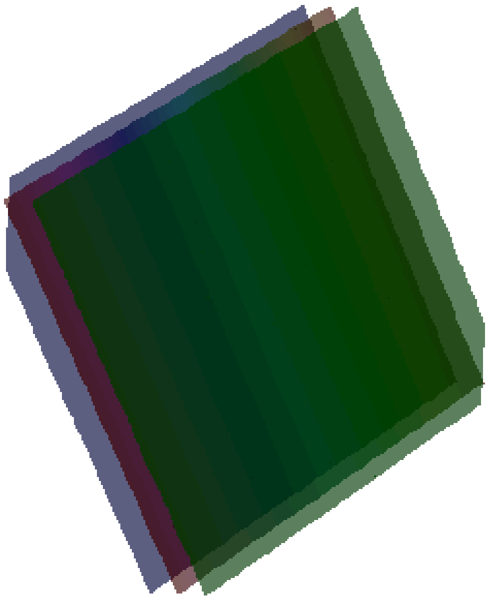

Figure 1.1: symplectic foliation by planes

We consider, for example, the following hamiltonian vector fields and respective integral curves through an arbitrary point (x_0, y_0, z_0): We consider, for example, the following hamiltonian field:

$$X_x(\cdot) = \{x, \cdot\} \Leftrightarrow X_x = \frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

Let $\gamma_1(t) = (x, y, z)$ be its integral curve through the point (x_0, y_0, z_0) of $\mathbb{R}^3$. We know it satisfies

$$\begin{cases} \dot{x}(t) = 0 \\ \dot{y}(t) = 1 \\ \dot{z}(t) = -1 \end{cases},$$

so

$$\gamma_1(t) = (x_0, y_0 + t, z_0 - t).$$

Analogously, the hamiltonian vector field of y is

$$X_y = -\frac{\partial}{\partial x} - \frac{\partial}{\partial z}$$

and its integral curve through the same point is

$$\gamma_2(t) = (x_0 - t, y_0 - t, z_0).$$

It is easy to see that any point of the plane A_{k_0} containing (x_0, y_0, z_0) can be reached using combinations of the flows of X_x and X_y. Therefore, the A_k's are the symplectic foliation of this Poisson structure.

Example 24 Consider $\mathfrak{sl}(2, \mathbb{R})^*$ endowed with the Lie-Poisson bracket $\{,\}^L$, described in Section 1.3. We consider the basis $\{X_1, X_2, X_3\}$ of $\mathfrak{sl}(2, \mathbb{R})$, where

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In the linear coordinates x_1, x_2, x_3 (see section 1.3), the Poisson matrix is

$$\mathrm{P} = \begin{pmatrix} . & -2x_2 & 2x_3 \\ 2x_2 & . & -x_1 \\ -2x_3 & x_1 & . \end{pmatrix}.$$

Now we investigate the existence of Casimirs. These funtions are exactly the ones whose differentials are in the kernel of the bundle morphism $\mathcal{P}^\#$. We have that

$$\ker \mathcal{P}^\# = \mathrm{span}\{x_1 \mathrm{d}x_1 + 2x_3 \mathrm{d}x_2 + 2x_2 \mathrm{d}x_3\},$$

and a simple integration shows that the following function is a Casimir:

$$c(x) = x_1^2 + 4x_2x_3, \ a \in \mathbb{R}.$$

The symplectic leaves are contained in its level sets, which are a cone, one-leaf hyperboloids or two-leaves hyperboloids.

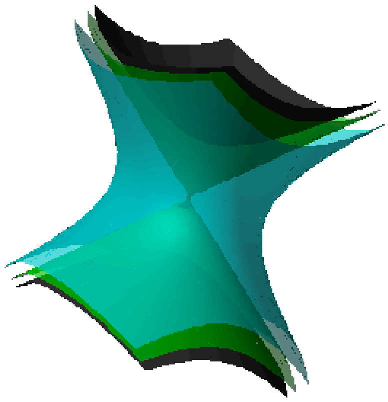

Figure 1.2: symplectic foliation of $\mathfrak{sl}(2,\mathbb{R})^*$

The origin is itself a symplectic leaf, since it is a point of rank zero. Therefore, the cone of equation

$$x_1^2 = -4x_2x_3$$

encloses three symplectic leaves: the origin, the component $x_1 > 0$ and the component $x_1 < 0$. The other symplectic leaves are:

1. one-leaf hyperboloids,

$$x_1^2 = -4x_2x_3 + k,\ \ k > 0;$$

2. each of the connected components of two-leaves hyperboloids,

$$x_1^2 = -4x_2x_3 - k,\ \ k > 0.$$

In some well-behaved cases, one can determine the symplectic foliation without using Casimir functions.

Example 25 Consider the Poisson manifold $(\mathbb{R}^4,\ \mathrm{P})$, with coordinates x, y, z, w, where

$$\mathrm{P} = \begin{pmatrix} . & y & . & . \\ -y & . & . & . \\ . & . & . & 1 \\ . & . & -1 & . \end{pmatrix},$$

where dots stand for zeros. In this case, there are no Casimir functions, because almost every point in $\mathcal{M}$ has rank 4, equal to the dimension of $\mathcal{M}$.

In order to find out which kind of sets are the symplectic leaves for this Poisson manifold, we compute the hamiltonian vector fields associated with the coordinate functions and their corresponding flows. We have

$$X_x = y\frac{\partial}{\partial y}.$$

Let $\gamma_1(t) = (x, y, z, w)$ be its integral curve through the point (x_0, y_0, z_0, w_0) of $\mathbb{R}^4$, i.e. $\gamma_1(t) = (x_0, y_0 e^t, z_0, w_0)$.

Analogously we obtain the following hamiltonian vector fields and respective integral curves through the point (x_0, y_0, z_0, w_0):

$$\begin{aligned} X_y = -y\frac{\partial}{\partial x} &\quad\longrightarrow\quad \gamma_2(t) = (x_0 - y_0 t, y_0, z_0, w_0) \\ X_z = -\frac{\partial}{\partial w} &\quad\longrightarrow\quad \gamma_3(t) = (x_0, y_0, z_0, w_0 - t) \\ X_w = \frac{\partial}{\partial z} &\quad\longrightarrow\quad \gamma_4(t) = (x_0, y_0, z_0 + t, w_0) \end{aligned}$$

First we consider the case $y_0 = 0$ (notice that these points have rank 2). From $(x_0, 0, z_0, w_0)$, the flows of X_x and X_y take us nowhere. However, piecewise smooth combinations of the flows of X_z and X_t take us to any point in the two-dimensional subspace

$$A_{x_0} = \{(x_0, 0, z, w) : z \in \mathbb{R},\ w \in \mathbb{R}\},$$

which is therefore contained in a symplectic leaf.

If $y_0 > 0$, any point in the set

$$B = \{(x, y, z, w) : x \in \mathbb{R},\ y > 0,\ z \in \mathbb{R},\ w \in \mathbb{R}\}$$

can be reached using combinations of the flows of X_x, X_y, X_z and X_w. If $y_0 < 0$ we reach in the same way any point in

$$C = \{(x, y, z, w) : x \in \mathbb{R},\ y < 0,\ z \in \mathbb{R},\ w \in \mathbb{R}\}.$$

Therefore, we conclude that each of the sets B, C and A_{x_0}, for all real values of x_0, is contained in a different symplectic leaf. Since they form a partition of $\mathbb{R}^4$, we know that each one of them is indeed a symplectic leaf.

Now we present a result which gives a general description of the symplectic leaves of a Lie-Poisson structure:

Lemma 26 (Libermann & Marle [8]) *Let* $\mathfrak{g}$ *be the Lie algebra of a connected Lie group* $\mathbf{G}$ *and* X *an element of* $\mathfrak{g}$*. Then the fundamental vector field of* X *for the coadjoint action* Ad^* *is the hamiltonian vector field of* X *with respect to the Lie-Poisson structure on* $\mathfrak{g}^*$ *(we regard* X *as a linear function in* $\mathfrak{g}^*$*).*

As a consequence we obtain the following:

Corollary 27 (Libermann & Marle [8]) *The symplectic leaf of the Lie-Poisson structure through* μ *in* $\mathfrak{g}^*$ *is the coadjoint orbit of* μ *for the action of* $\mathbf{G}$ *in* $\mathfrak{g}^*$*.*

1.5 Weinstein's Splitting Theorem

Theorem 28 (Weinstein [18]) *Let* $(\mathcal{M}_1, \{,\}_1)$, $(\mathcal{M}_2, \{,\}_2)$ *be two Poisson manifolds. Then*

$$\mathcal{M}_1 \times \mathcal{M}_2$$

admits a Poisson bracket $\{,\}$ *such that:*

1. $\pi_i : \mathcal{M} \to \mathcal{M}_i$ *(canonic projection on the i-th factor) is a Poisson map for* $i = 1, 2$*;*
2. $\{f_1 \circ \pi_1, f_2 \circ \pi_2\} = 0, \quad \forall\, f_i \in C^\infty(\mathcal{M}_i)$.

Definition 29 The structure defined in the previous theorem is called the *Poisson product* of $(\mathcal{M}_1, \{,\}_1)$ and $(\mathcal{M}_2, \{,\}_2)$.

Theorem 30 (Weinstein [18]) *Let* $(\mathcal{M}, \mathcal{P})$ *be an* m*-dimensional Poisson manifold and* $x_0 \in \mathcal{M}$ *such that* $\mathrm{rank}\,\mathcal{P}_{x_0} = 2l$*. Then there exist a* $2l$*-dimensional symplectic manifold* $\mathcal{S} \subset \mathcal{M}$*, a Poisson manifold* $(\mathcal{N}, \mathcal{P}_\mathcal{N})$*, of dimension* $m - 2l$*, and a neighborhood* U *of* x_0 *in* $\mathcal{M}$ *where we can define a Poisson diffeomorphism*

$$\begin{array}{rccl} \varphi : & U & \to & \mathcal{S} \times \mathcal{N} \\ & x & \mapsto & (\varphi^\mathcal{S}(x), \varphi^\mathcal{N}(x)). \end{array}$$

Moreover,

$$\mathrm{rank}\,(\mathcal{P}_\mathcal{N})_{\varphi^\mathcal{N}(x_0)} = 0.$$

We note that $\mathcal{S} \times \mathcal{N}$ is equipped with the Poisson product structure. On the other hand, Darboux-Weinstein's theorem implies that if $\mathcal{S}$, $\mathcal{N}$ and $\mathcal{S}'$, $\mathcal{N}'$ are two pairs of submanifolds of $\mathcal{M}$ in the conditions of Splitting Theorem, then $\mathcal{S}$ is locally symplectomorphic (and consequently, locally Poisson-diffeomorphic) to $\mathcal{S}'$ (see [8]). Therefore we can choose, as a *representative* of $\mathcal{S}$, the symplectic leaf through x.

Definition 31 The Poisson manifold $(\mathcal{N}, \mathcal{P}_{\mathcal{N}})$ of the Splitting Theorem is called a *transverse Poisson manifold* (or a *transverse Poisson structure*) *to the symplectic leaf* $\mathcal{S}$ *at* x_0.

Remark 32 Another way to state this theorem is to say that there are local coordinates $p_1, \ldots, p_l, q_1, \ldots, q_l, z_1, \ldots, z_{m-2l}$ around x_0 such that the Poisson matrix is block diagonal:

$$\begin{pmatrix} \mathbb{J}_0 & 0_{2l\times m} \\ 0_{m\times 2l} & \mathrm{Q} \end{pmatrix},$$

where the submatrix Q, which depends only on the coordinates $z_1, \ldots, z_{m-2l}$, is zero at x_0 (although possibly not identically zero on any neighborhood of x_0).

If x is a regular point of $(\mathcal{M}, \mathrm{P})$ then the rank of P is constant in a neighborhood of x. Since the rank at x is $2l$, the submatrix Q must be zero in the same neighborhood. Thus, we have just proved the next Corollary:

Corollary 33 (Weinstein [18]) *If x is a regular point, then $(\mathcal{M}; \{,\})$ is locally Poisson-diffeomorphic to $\left(\mathbb{R}^m; \{,\}^{m-2l}\right)$, the Poisson manifold described in Example 11.*

In other words, if x is regular, the Poisson structure has $m-2l$ local Casimir functions, which would be the z_i's. Consider the application

$$\begin{array}{rccl} c: & \mathcal{U} & \to & \mathbb{R}^{m-2l} \\ & y & \to & (z_1(y), \ldots, z_{m-2l}(y)) \end{array},$$

where $\mathcal{U}$ is a neighborhood of x and $z_1, \ldots, z_{m-2l}$ are independent Casimir functions defined in $\mathcal{U}$, i.e.

$$\{\mathrm{d}z_1, \ldots, \mathrm{d}z_{m-2l}\}$$

is a free set. It is possible to prove that, around x, the distribution generated by the hamiltonian vector fields is

$$\mathcal{D}_y = \ker(\mathrm{d}c_y).$$

Therefore, the level sets of c (or their connected components) are integral manifolds of $\mathcal{D}$, and hence open subsets of symplectic leaves.

Notice that Corollary 33 completely characterizes the Poisson structure on a neighborhood of any of its regular points. Therefore, in the next chapters we will restrict our study to the case in which x is singular.

Chapter 2

The Transverse Poisson Structure to a Symplectic Leaf

2.1 The General Case

Following [18] and [3], we now describe the construction of the transverse Poisson structure to a symplectic leaf at a point x. Let $(\mathcal{M}; \{,\})$ be a finite-dimensional Poisson manifold, $\mathcal{P}^{\#} : \mathrm{T}^*\mathcal{M} \to \mathrm{T}\mathcal{M}$ the bundle map corresponding to the bracket $\{,\}$, x a point in $\mathcal{M}$ and $\mathcal{S}$ the symplectic leaf through x. Then

$$\mathrm{T}_x\mathcal{S} = \mathrm{Im}(\mathcal{P}^{\#}_x).$$

Definition 34 Consider a submanifold $\mathcal{N} \subset \mathcal{M}$ which intersects the symplectic leaf at x transversally, i.e. such that

$$\mathrm{T}_x\mathcal{M} = \mathrm{T}_x\mathcal{N} \oplus \mathrm{T}_x\mathcal{S}. \tag{2.1}$$

Such an $\mathcal{N}$ will be called a *transverse submanifold to $\mathcal{S}$ (at x).*

We note that 2.1 is equivalent to

$$\mathrm{T}^*_x\mathcal{M} = \mathrm{T}^\circ_x\mathcal{N} \oplus \mathrm{T}^\circ_x\mathcal{S}. \tag{2.2}$$

Weinstein established in [18] that we can define a Poisson structure $\mathcal{P}^{\#}_{\mathcal{N}}$ in this submanifold $\mathcal{N}$, in such a way that the symplectic leaves in $\mathcal{N}$ are the intersections of the symplectic leaves of $\mathcal{M}$ with $\mathcal{N}$. Hence, $\mathcal{P}^{\#}_{\mathcal{N}}$ is said to be *naturally induced* from $\mathcal{P}^{\#}$. It is the transverse Poisson structure to $\mathcal{S}$ referred to in the Splitting Theorem. We remark that this structure is not obtained by restricting to $\mathcal{N}$ the Poisson structure on $\mathcal{M}$, as in Definition 18, because $\mathcal{N}$, unlike $\mathcal{S}$, is not a Poisson submanifold of $\mathcal{M}$. In the remainder of this chapter we will detail the construction of this kind of Poisson manifolds, following the work of Weinstein [18] and Cushman & Roberts [3].

In order to define this Poisson structure on $\mathcal{N}$, one must find a somehow "natural" way to project vectors of $\mathrm{T}_y\mathcal{M}$ onto $\mathrm{T}_y\mathcal{N}$. Unfortunately, we cannot use decomposition 2.1 for that purpose, because it is only valid at the "splitting" point x. Each point $y \neq x$ of $\mathcal{N}$ belongs to a symplectic leaf $[y]$, which may be quite different from $\mathcal{S}$. For example, the dimension of the leaf could change. The following lemma will be useful to define a splitting of $\mathrm{T}_y\mathcal{M}$ which is valid for all points in a neighborhood of x in $\mathcal{N}$.

Lemma 35 *Let $(\mathcal{M}; \{,\})$ be a Poisson manifold and $x \in \mathcal{M}$. Then*

$$\ker \mathcal{P}_x^{\#} = \mathrm{T}_x^{\circ}\mathcal{S}.$$

Proof. We already know that

$$\mathrm{Im}(\mathcal{P}_x^{\#}) = \mathrm{T}_x\mathcal{S}.$$

Combined with the skew-symmetry of the Poisson structure, that justifies the following:

$$\begin{aligned} \alpha \in \ker \mathcal{P}_x^{\#} &\Leftrightarrow \left\langle \beta,\, \mathcal{P}_x^{\#}(\alpha) \right\rangle = 0, \quad \forall \beta \in \mathrm{T}_x^*\mathcal{M} \\ &\Leftrightarrow \left\langle \alpha,\, \mathcal{P}_x^{\#}(\beta) \right\rangle = 0, \quad \forall \beta \in \mathrm{T}_x^*\mathcal{M} \\ &\Leftrightarrow \langle \alpha,\, v \rangle = 0, \quad \forall v \in \mathrm{T}_x\mathcal{S}, \end{aligned}$$

which concludes the proof. ■

Proposition 36 (Cushman & Roberts [3]) *Let $(\mathcal{M}; \{,\})$ be a Poisson manifold, $x \in \mathcal{M}$, $\mathcal{S}$ the symplectic leaf through x and $\mathcal{N}$ a transverse submanifold to $\mathcal{S}$. Then there is a neighborhood U of x in $\mathcal{N}$ where the following condition is satisfied:*

$$\mathrm{T}_y\mathcal{M} = \mathrm{T}_y\mathcal{N} \oplus \mathcal{P}_y^{\#}(\mathrm{T}_y^{\circ}\mathcal{N}), \quad \forall y \in U. \tag{2.3}$$

Proof. First we will check that $\mathcal{P}_x^{\#}$ maps $\mathrm{T}_x^{\circ}\mathcal{N}$ onto $\mathrm{T}_x\mathcal{S}$. Given $v \in \mathrm{T}_x\mathcal{S}$, let $\alpha \in \mathrm{T}_x^*\mathcal{M}$ be such that

$$\mathcal{P}_x^{\#}(\alpha) = v.$$

According to condition 2.2, there is a unique way to write

$$\alpha = \alpha_1 + \alpha_2,$$

with

$$\alpha_1 \in \mathrm{T}^\circ_x\mathcal{N},\ \alpha_2 \in \mathrm{T}^\circ_x\mathcal{S}.$$

Hence

$$\begin{aligned} v &= \mathcal{P}^\#_x(\alpha) \\ &= \mathcal{P}^\#_x(\alpha_1), \end{aligned}$$

because $\alpha_2 \in \ker(\mathcal{P}^\#_x)$, by Lemma 35. Therefore $\mathrm{T}_x\mathcal{S} \subset \mathcal{P}^\#_x\left(\mathrm{T}^\circ_x\mathcal{N}\right)$. In adition, the two vector spaces have the same dimension, thus

$$\mathrm{T}_x\mathcal{S} = \mathcal{P}^\#_x\left(\mathrm{T}^\circ_x\mathcal{N}\right).$$

Therefore, at x, decomposition 2.3 is just decomposition 2.1, and hence valid.

Now we prove that decomposition 2.3 is also valid in a neighborhood of x in $\mathcal{N}$. We have that $\mathcal{P}^\#_x\big|_{\mathrm{T}^\circ_x\mathcal{N}}$ is a bijection onto its image. In addition,

$$\mathcal{P}^\#_x\big|_{\mathrm{T}^\circ_x\mathcal{N}} : \mathrm{T}^\circ_x\mathcal{N} \longrightarrow \mathrm{T}_x\mathcal{M}$$

has maximum possible rank. On the other hand, the rank of a Poisson structure is a lower semicontinuous function and hence a continuity argument shows that there is a neighborhood of x (in $\mathcal{N}$) in which the rank of $\mathcal{P}^\#_y\big|_{\mathrm{T}^\circ_y\mathcal{N}}$ is constant and maximal. Therefore, a dimensional argument shows that condition 2.3 holds in a neighborhood of x if and only if

$$\mathrm{T}_y\mathcal{N} \cap \mathcal{P}^\#_y(\mathrm{T}^\circ_y\mathcal{N}) = \{0\} \tag{2.4}$$

holds in the same neighborhood. We know that condition 2.4 holds at x, and it indeed holds in a neighborhood of x by transversality. In fact, suppose that there is no such neighborhood. Then there is a differentiable section X of the vector subbundle $\mathcal{P}^\#\left(\mathrm{T}^\circ\mathcal{N}\right) \subset \mathrm{T}_\mathcal{N}\mathcal{M}$ and a curve

$$\alpha :]-\epsilon, \epsilon[\longrightarrow \mathcal{N}$$

such that:

1. $\alpha(0) = x$,
2. $X \circ \alpha(0) \notin \mathrm{T}_{\alpha(0)}\mathcal{N}$ (because 2.4 holds at x);
3. $X \circ \alpha(t) \in \mathrm{T}_{\alpha(t)}\mathcal{N}, \quad \forall\, t \neq 0.$

But then the differential section X is not continuous, which is absurd.

Thus, the sum in condition 2.3 is indeed a direct sum, and the proof is concluded. ■

The following Lemma from Linear Algebra will be useful in the subsequent construction of the transverse Poisson structure to a symplectic leaf:

Lemma 37 *Let* S *and* T *be vector subspaces of a vector space* V*. Then*

$$(\mathrm{S} \cap \mathrm{T})^{\circ} = \mathrm{S}^{\circ} + \mathrm{T}^{\circ}.$$

In particular,

$$\mathrm{S} \cap \mathrm{T} = \{0\} \Rightarrow \mathrm{S}^{\circ} + \mathrm{T}^{\circ} = V^{*}.$$

Theorem 38 (Weinstein [18]) *Let* $(\mathcal{M}, \mathcal{P})$ *be a Poisson manifold,* x *a point in* $\mathcal{M}$*,* $\mathcal{S}$ *the symplectic leaf through* x *and* $\mathcal{N}$ *a transverse submanifold to* $\mathcal{S}$ *at* x*. Then there is a naturally induced Poisson structure defined in* $\mathcal{N}$*.*

Proof. First we will establish that, at every point y in a neighborhood of x in $\mathcal{N}$, we have

$$\mathrm{T}_y^{\circ}\mathcal{N} \cap \ker \mathcal{P}_y^{\#} = \{0\}\,. \tag{2.5}$$

Let $\alpha \in \mathrm{T}_y^{\circ}\mathcal{N} \cap \ker \mathcal{P}_y^{\#}$. Given $v \in \mathrm{T}_y\mathcal{M}$, condition 36 assures that we can write

$$v = v_1 + v_2,$$

with

$$v_1 \in \mathrm{T}_y\mathcal{N},\, v_2 \in \mathcal{P}_y^{\#}(\mathrm{T}_y^{\circ}\mathcal{N}).$$

Then

$$\begin{aligned}
\langle \alpha, v\rangle &= \langle \alpha, v_1 + v_2\rangle \\
&= \langle \alpha, v_2\rangle \quad (\alpha \in \mathrm{T}_y^{\circ}\mathcal{N}, v_1 \in \mathrm{T}_y\mathcal{N}) \\
&= \left\langle \alpha, \mathcal{P}_y^{\#}(\beta)\right\rangle, \quad \beta \in \mathrm{T}_y^{\circ}\mathcal{N} \\
&= \left\langle -\beta, \mathcal{P}_y^{\#}(\alpha)\right\rangle \\
&= 0 \quad (\alpha \in \ker \mathcal{P}_y^{\#}).
\end{aligned}$$

Because v is an arbitrary element of $\mathrm{T}_y\mathcal{M}$, we have $\alpha = 0$ and condition 2.5 holds at y. Now we apply Lemma 35 and Lemma 37 to this condition, obtaining

$$\mathrm{T}_y\mathcal{N} + \mathrm{Im}\mathcal{P}_y^{\#} = \mathrm{T}_y\mathcal{M}.$$

In other words, $\mathcal{N}$ intersects each symplectic leaf of $\mathcal{M}$ transversally. Therefore $\mathcal{N}$ is a union of manifolds,

$$\mathcal{N} = \bigcup_{y \in \mathcal{N}} \mathcal{N}_y,$$

each of which is a submanifold of a symplectic leaf of $\mathcal{M}$:

$$\mathcal{N}_y = \mathcal{N} \cap [y].$$

We have to check that each $\mathcal{N}_y$ is a symplectic submanifold. With that purpose in mind, we fix y and consider the symplectic form ω in $[y]$ (see Section 1.4). Given a point z in $\mathcal{N}_y$, we must restrict ω_z to $\mathrm{T}_z\mathcal{N}_y$ and verify that its null space is only the zero vector. First we note that this null space is exactly

$$(\mathrm{T}_z\mathcal{N}_y)^{\omega_z} \cap \mathrm{T}_z\mathcal{N}_y,$$

where $(\mathrm{T}_z\mathcal{N}_y)^{\omega_z}$ stands for the symplectic complement of $\mathrm{T}_z\mathcal{N}_y$ in $\mathrm{T}_z[y]$. Now, we will prove that

$$(\mathrm{T}_z\mathcal{N}_y)^{\omega_z} = \mathcal{P}_z^{\#}\left(\mathrm{T}_z^{\circ}\mathcal{N}\right). \tag{2.6}$$

Consider any vector tangent at z to the symplectic leaf $[y]$. We may express it as $\mathcal{P}_z^{\#}(\alpha)$. We have that

$$\begin{aligned}
\mathcal{P}_z^{\#}(\alpha) \in (\mathrm{T}_z\mathcal{N}_y)^{\omega_z} \quad &\Leftrightarrow \quad \omega_z\left(u, \mathcal{P}_z^{\#}(\alpha)\right) = 0 \quad \forall u \in \mathrm{T}_z\mathcal{N}_y \\
&\Leftrightarrow \quad \left\langle \left(\mathcal{P}_z^{\#}\right)^{-1}\left(\mathcal{P}_z^{\#}(\alpha)\right), u \right\rangle = 0 \quad \forall u \in \mathrm{T}_z\mathcal{N}_y \\
&\Leftrightarrow \quad \langle \alpha, u \rangle = 0 \quad \forall u \in \mathrm{T}_z\mathcal{N}_y \\
&\Leftrightarrow \quad \alpha \in \mathrm{T}_z^{\circ}\mathcal{N}_y
\end{aligned}$$

and thus

$$(\mathrm{T}_z\mathcal{N}_y)^{\omega_z} = \mathcal{P}_z^{\#}\left(\mathrm{T}_z^{\circ}\mathcal{N}_y\right).$$

We remark that, by transversality,

$$\mathrm{T}_z\mathcal{N}_y = \mathrm{T}_z\mathcal{N} \cap \mathrm{T}_z[y].$$

Together with Lemmas 35 and 37, this is all we need to establish the following:

$$\begin{aligned}
\mathcal{P}_z^{\#}\left(\mathrm{T}_z^{\circ}\mathcal{N}_y\right) &= \mathcal{P}_z^{\#}\left(\mathrm{T}_z^{\circ}\mathcal{N}\right) + \mathcal{P}_z^{\#}\left(\mathrm{T}_z^{\circ}[y]\right) \\
&= \mathcal{P}_z^{\#}\left(\mathrm{T}_z^{\circ}\mathcal{N}\right).
\end{aligned}$$

Then, (2.6) is proved and the null space of $\omega_z|_{\mathrm{T}_z\mathcal{N}_y}$ is

$$\mathcal{P}_z^{\#}\left(\mathrm{T}_z^{\circ}\mathcal{N}\right) \cap \mathrm{T}_z\mathcal{N}_y,$$

or, because $\mathrm{T}_z[y]$ is the image of the linear map $\mathcal{P}_z^{\#}$,

$$\mathcal{P}_z^{\#}\left(\mathrm{T}_z^{\circ}\mathcal{N}\right) \cap \mathrm{T}_z\mathcal{N}.$$

Hence, Proposition 36 implies that the null space above reduces to the zero vector and, indeed, the intersection manifolds which constitute $\mathcal{N}$ are symplectic. We turn this set of symplectic submanifolds into a symplectic foliation simply by gluing the Poisson brackets of each one of them, forming one global Poisson bracket on $\mathcal{N}$. In order to do that, we must guarantee that this naturally induced Poisson structure on $\mathcal{N}$ is smooth.

We remark that decomposition 2.3 gives a smooth bundle projection

$$\pi : \mathrm{T}_{\mathcal{N}}\mathcal{M} \longrightarrow \mathrm{T}\mathcal{N},$$

given by

$$\pi_y : \mathrm{T}_y\mathcal{M} \longrightarrow \mathrm{T}_y\mathcal{N}, \quad \forall y \in U \subset \mathcal{N}.$$

with

$$\ker \pi_y = \mathcal{P}_y^{\#}(\mathrm{T}_y^{\circ}\mathcal{N}).$$

This allows us to define the bundle map $\mathcal{P}_{\mathcal{N}}^{\#}$ as follows (the diagram commutes):

$$\begin{array}{ccc} \mathrm{T}_y^*\mathcal{N} & \xrightarrow{\left(\mathcal{P}_{\mathcal{N}}^{\#}\right)_y} & \mathrm{T}_y\mathcal{N} \\ \pi_y^* \downarrow & & \uparrow \pi_y \\ \mathrm{T}_y^*\mathcal{M} & \xrightarrow[\mathcal{P}_y^{\#}]{} & \mathrm{T}_y\mathcal{M} \end{array}$$

or, equivalently,

$$\left(\mathcal{P}_{\mathcal{N}}^{\#}\right)_y = \pi_y \circ \mathcal{P}_y^{\#} \circ \pi_y^*.$$

■

Remark 39 We remark that for each y in $\mathcal{N}$, given $\alpha \in \mathrm{T}_y^*\mathcal{N}$, we have simply

$$\pi_y^*\alpha = \alpha \circ \pi_y.$$

Given a Poisson manifold $(\mathcal{M}; \{,\})$, x a point in $\mathcal{M}$, $\mathcal{S}$ the symplectic leaf through x and $\mathcal{N}$ a transverse submanifold to $\mathcal{S}$, a *transverse Poisson* *structure to $\mathcal{S}$ at x* (as in Definition 31) may be represented by $\left(\mathcal{N}, \mathcal{P}^{\#}_{\mathcal{N}}\right)$, where $\mathcal{P}^{\#}_{\mathcal{N}}$ is the bundle map given above.

The transverse Poisson structure to a symplectic leaf is unique up to Poisson diffeomorphisms, in the sense of the following Lemma:

Lemma 40 (Weinstein [18]) *Suppose $\mathcal{N}_1$ and $\mathcal{N}_2$ are two submanifolds having dimension complementary to the symplectic leaf $\mathcal{S}$. Suppose also that each of them intersects $\mathcal{S}$ at a single point, transversally. Then there is an automorphism of $\mathcal{M}$ which maps a neighborhood of $\mathcal{N}_1 \cap \mathcal{S}$ in $\mathcal{N}_1$ onto a neighborhood of $\mathcal{N}_2 \cap \mathcal{S}$ in $\mathcal{N}_2$. This automorphism induces an isomorphism of the induced Poisson structures in the neighborhoods.*

This Lemma provides meaning to the expression "transverse Poisson structure *to a symplectic leaf*", widely used in the literature. Again we remark that, if x is a regular point, then the transverse Poisson structure to $\mathcal{S}$ at x will be trivial (see Corollary 33), therefore we will restrict our attention to singular points.

The transverse Poisson structure just built uses a decomposition (and projection) different from the one considered by Cushman & Roberts [3]. We find this construction easier to work with and, in the particular case of a Lie-Poisson structure, the formula we arrive at is simpler to use.

2.2 A Simple Formula for the Transverse Poisson Structure to a Coadjoint Orbit

Given a Lie algebra $\mathfrak{g}$, we have already established in Section 1.3 that $\mathfrak{g}^*$ is endowed with a natural Poisson structure, which can be characterized by the bundle map $\mathcal{P}^{\#}$ defined by

$$\begin{array}{rccc} \mathcal{P}^{\#}_{\mu}: & \mathfrak{g} & \to & \mathfrak{g}^* \\ & X & \mapsto & \mathrm{ad}^*_X \mu \end{array},$$

where $\mu \in \mathfrak{g}^*$ and ad^* stands for the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

The tangent space (at μ) to the symplectic leaf $[\mu]$ is the image of $\mathcal{P}^{\#}_{\mu}$, i.e., the orbit of μ for ad^*. Furthermore, Corollary 27 establishes that $[\mu]$ is precisely the orbit $\mathcal{O}_\mu$ for the coadjoint action

$$\mathrm{Ad}^* : \mathbf{G} \to \mathfrak{g}^*,$$

(here $\mathbf{G}$ is the connected and simply-connected Lie group with Lie algebra $\mathfrak{g}$)

We will now apply the construction given in the last section in order to build a transverse Poisson structure to the coadjoint orbit $\mathcal{O}_\mu$. The first step is to find a suitable submanifold of $\mathfrak{g}^*$, i.e., one which intersects $\mathcal{O}_\mu$ transversally. We remark that the isotropy subalgebra

$$\mathfrak{g}_\mu = \{X \in \mathfrak{g} : \mathrm{ad}^*_X \mu = 0\}$$

is nothing but the kernel of $\mathcal{P}^\#_\mu$. In the next Lemma, the direct sum $\oplus$ refers to $\mathfrak{g}_\mu$ and $\mathfrak{h}$ as vector spaces, not as Lie algebras ($\mathfrak{h}$ is not necessarily a Lie algebra).

Lemma 41 (Weinstein [18], Cushman & Roberts [3]) *Let* $(\mathfrak{g}^*, \{,\}_L)$ *be the dual of a Lie algebra equipped with the Lie-Poisson bracket and* μ *in* $\mathfrak{g}^*$. *Let* $\mathfrak{h}$ *a supplement of* $\mathfrak{g}_\mu$ *in* $\mathfrak{g}$, *i.e.,*

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{h}. \tag{2.7}$$

Then the affine subspace

$$\mathcal{N} = \mu + \mathfrak{h}^\circ$$

is a submanifold which is transversal to the coadjoint orbit $\mathcal{O}_\mu$ *at* μ.

Proof. In this case, we have $\mathcal{M} = \mathfrak{g}^*$ and $\mathrm{T}_\mu\mathcal{N} = \mathfrak{h}^\circ$. On the other hand,

$$\mathrm{T}_\mu\mathcal{S} = \mathrm{T}_\mu\mathcal{O}_\mu = \mathrm{ad}^*_\mathfrak{g}(\mu).$$

By Lemma 35,

$$\mathrm{T}^\circ_\mu\mathcal{S} = \ker(\mathcal{P}^\#_\mu) = \mathfrak{g}_\mu.$$

If we dualize (2.7), we obtain

$$\mathfrak{g}^* = \mathfrak{g}^\circ_\mu \oplus \mathfrak{h}^\circ,$$

which is just decomposition 2.1 of Definition 34,

$$\mathrm{T}_x\mathcal{M} = \mathrm{T}_x\mathcal{N} \oplus \mathrm{T}_x\mathcal{S},$$

meaning that the submanifold $\mu+\mathfrak{h}^\circ$ and the coadjoing orbit intersect transversally at μ. ■

Before we go any further, we take a closer look at some useful identifications. First, we have that $\mathfrak{h}^\circ$ is naturally identified with $\mathfrak{g}^*_\mu$. Second, if $\mathcal{N}$ is the transverse submanifold $\mu + \mathfrak{h}^\circ$ and ν belongs to a neighborhood of 0 in $\mathfrak{h}^\circ$, then

$$\mathrm{T}_{\mu+\nu}\mathcal{N} \cong \mathfrak{h}^\circ \left(\cong \mathfrak{g}^*_\mu\right).$$

Finally,

$$\begin{aligned} \mathcal{P}^{\#}_{\mu+\nu}(\mathrm{T}^{\circ}_{\mu+\nu}\mathcal{N}) &\cong \mathcal{P}^{\#}_{\mu+\nu}(\mathfrak{h}) \\ &= \mathrm{ad}^{*}_{\mathfrak{h}}(\mu+\nu). \end{aligned}$$

Thus, decomposition 2.3 defined earlier may be written as

$$\mathfrak{g}^{*} = \mathfrak{h}^{\circ} \oplus \mathrm{ad}^{*}_{\mathfrak{h}}(\mu+\nu) \tag{2.8}$$

and the projection π is, in this specific case,

$$\pi_{\nu} : \mathfrak{g}^{*} \to \mathfrak{h}^{\circ}(\cong \mathfrak{g}^{*}_{\mu}),$$

with kernel $\mathrm{ad}^{*}_{\mathfrak{h}}(\mu+\nu)$.

The following result provides a new formula for the transverse Poisson structure to $\mathcal{O}_{\mu}$, which is simpler to use than the one in [3].

Theorem 42 (Formula for the transverse Poisson structure) *Let $\mu \in \mathfrak{g}^{*}$ be such that $\mathfrak{h}$ is a supplement of $\mathfrak{g}_{\mu}$ in $\mathfrak{g}$, i.e.,*

$$\mathfrak{g} = \mathfrak{g}_{\mu} \oplus \mathfrak{h}.$$

Then the transverse Poisson structure to the coadjoint orbit $\mathcal{O}_{\mu}$, defined in the transverse *submanifold*

$$\mathcal{N} = \mu + \mathfrak{h}^{\circ},$$

is given by the bundle map $\mathcal{P}^{\#}_{\mathcal{N}} : \mathfrak{g}_{\mu} \to \mathfrak{g}^{*}_{\mu}$ defined by

$$\left(\mathcal{P}^{\#}_{\mathcal{N}}\right)_{\mu+\nu}(X) = \pi_{\nu} \circ \mathrm{ad}^{*}_{X}\nu, \tag{2.9}$$

where $X \in \mathfrak{g}_{\mu}$, $\nu \in \mathfrak{h}^{\circ}$ and $\pi_{\nu} : \mathfrak{g}^{*} \to \mathfrak{h}^{\circ}$ is the projection such that

$$\ker(\pi_{\nu}) = \mathrm{ad}^{*}_{\mathfrak{h}}(\mu+\nu).$$

Proof. Let $X \in \mathfrak{g}_{\mu}$. Then, according to what is stated in section 2.1, we may express $\mathcal{P}^{\#}_{\mathcal{N}}$ as follows:

$$\left(\mathcal{P}^{\#}_{\mathcal{N}}\right)_{\mu+\nu}(X) = \pi_{\nu} \circ \mathcal{P}^{\#}_{\mu+\nu} \circ \pi^{*}_{\nu}(X),$$

where $\pi_{\nu} : \mathfrak{g}^{*} \to \mathfrak{h}^{\circ}$ is the projection associated to decomposition 2.8. But noticing that

$$\pi^{*}_{\nu}(X) \in \mathfrak{g},$$

we know from decomposition 2.7 (see Lemma 41) that $\pi_\nu^*(X)$ can be written in an unique manner as

$$\pi_\nu^*(X) = Y + Z,$$

with $Y \in \mathfrak{g}_\mu,\ Z \in \mathfrak{h}$. Also, we know that

$$\mathcal{P}^{\#}_{\mu+\nu}(\pi_\nu^*(X)) = \mathrm{ad}^*_{\pi_\nu^*(X)}(\mu+\nu),$$

and therefore we are able to compute $\left(\mathcal{P}^{\#}_{\mathcal{N}}\right)_{\mu+\nu}(X)$:

$$\begin{aligned}
\left(\mathcal{P}^{\#}_{\mathcal{N}}\right)_{\mu+\nu}(X) &= \pi_\nu\left(\mathcal{P}^{\#}_{\mu+\nu}(\pi_\nu^*(X))\right) \\
&= \pi_\nu\left(\mathrm{ad}^*_{Y+Z}(\mu+\nu)\right) \\
&= \pi_\nu\left(\mathrm{ad}^*_{Y}(\mu+\nu)\right) + \pi_\nu\left(\mathrm{ad}^*_{Z}(\mu+\nu)\right) \\
&\overset{Z\in\mathfrak{h}}{=} \pi_\nu\left(\mathrm{ad}^*_{Y}(\mu+\nu)\right) \\
&= \pi_\nu\left(\mathrm{ad}^*_{Y}(\mu) + \mathrm{ad}^*_{Y}(\nu)\right) \\
&\overset{Y\in\mathfrak{g}_\mu}{=} \pi_\nu\left(\mathrm{ad}^*_{Y}(\nu)\right).
\end{aligned}$$

We will now prove that $Y = X$, which completes the proof. By definition,

$$\pi_\nu^*(X) \in \mathfrak{g}$$

is the only vector satisfying

$$\forall \mu' \in \mathfrak{g}^*, \langle \mu',\, \pi_\nu^*(X)\rangle = \langle \pi_\nu(\mu'),\, X\rangle\,,$$

i.e.,

$$\forall \mu' \in \mathfrak{g}^*\ \langle \mu',\ Y+Z\rangle = \langle \pi_\nu(\mu'),\, X\rangle\,.$$

In particular, if $\mu' \in \mathfrak{h}^\circ$, we obtain

$$\begin{aligned}
\langle \mu',\ Y\rangle &= \langle \pi_\nu(\mu'),\, X\rangle\,. \\
&= \langle \mu',\, X\rangle\,.
\end{aligned}$$

We have just concluded that $Y - X$ belongs to $(\mathfrak{h}^\circ)^\circ$, i.e., belongs to $\mathfrak{h}$. On the other hand, $Y - X$ is also in $\mathfrak{g}_\mu$. Since

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{h},$$

we have $Y = X$. ■

The most useful feature of this result is that, given μ and once a complement $\mathfrak{h}$ to $\mathfrak{g}_\mu$ has been chosen, the computation of the transverse Poisson structure to the coadjoint orbit $\mathcal{O}_\mu$ reduces to the simple computation of the projection π_ν.

As a corollary, we obtain the following result of P. Molino [12].

Corollary 43 *If* $\mathfrak{h}$ *is a supplement of* $\mathfrak{g}_\mu$ *in* $\mathfrak{g}$ *such that*

$$[\mathfrak{g}_\mu,\ \mathfrak{h}] \subseteq \mathfrak{h},$$

then $\mathcal{P}_{\mathcal{N}}^{\#}$ *is the Lie-Poisson structure on* $\mathfrak{g}_\mu^*$ *and is thus linear in* ν.

Remark 44 We remark that *linear* Poisson structures are defined only on vector spaces. That is not the case for $\mathcal{N} = \mu + \mathfrak{h}^\circ$ (unless μ equals zero). Nevertheless, it is common to define as *linear* any Poisson structure on $\mathcal{N}$ whose expression is linear on a system of affine coordinates on $\mathcal{N}$.

Proof. Suppose that X belongs to $\mathfrak{g}_\mu$ and $\mu + \nu$ is a point in the transverse submanifold $\mu + \mathfrak{h}^\circ$. Then the following equivalences take place:

$$\begin{aligned} \mathrm{ad}_X^*\nu \in \mathfrak{h}^\circ \quad &\Leftrightarrow \quad \langle \mathrm{ad}_X^*\nu,\ Y\rangle = 0, \quad \forall Y \in \mathfrak{h} \\ &\Leftrightarrow \quad \langle \nu,\ [X,\ Y]\rangle = 0, \quad \forall Y \in \mathfrak{h}. \end{aligned}$$

Now suppose that $[\mathfrak{g}_\mu,\ \mathfrak{h}] \subseteq \mathfrak{h}$. Then $[X,\ Y]$ belongs to $\mathfrak{h}$. In addition, ν belongs to $\mathfrak{h}^\circ$, so

$$\langle \nu,\ [X,\ Y]\rangle = 0, \quad \forall Y \in \mathfrak{h},$$

i.e.

$$\mathrm{ad}_X^*\nu \in \mathfrak{h}^\circ.$$

Now we recall that π_ν projects vectors of $\mathfrak{g}^*$ onto $\mathfrak{h}^\circ$. Thus

$$\pi_\nu\left(\mathrm{ad}_X^*\nu\right) = \mathrm{ad}_X^*\nu.$$

The linearity of the Poisson bracket follows from the properties of the coadjoint action. ■

Remark 45 It is clear from the proof that, whenever the condition

$$[\mathfrak{g}_\mu, \mathfrak{h}] \subset \mathfrak{h}$$

holds, there is no need to compute the projection π, since it will be the identity. Nevertheless we have included the computation of π in the appendices, for control.

2.3 Some Historical Notes on the Transverse Poisson Structure to a Coadjoint Orbit

One may ask if the transverse Poisson structure to a coadjoint orbit is always the Lie-Poisson structure on $\mathfrak{g}_\mu^*$, and thus a linear Poisson structure as described in Section 1.3. In fact, Weinstein stated it as a theorem when he first introduced the concept of transverse Poisson manifold to the symplectic leaf at a point (see [18]). As a consequence of that theorem we would have, for every element X in $\mathfrak{g}_\mu$,

$$\left(\mathcal{P}_{\mathcal{N}}^{\#}\right)_{\mu+\nu}(X) = \mathrm{ad}_X^*\nu, \tag{2.10}$$

where $\nu \in \mathfrak{g}_\mu^* \cong \mathfrak{h}^\circ$.

That is not the case in general. However, as we have seen, the additional hypothesis of Corollary 43, known as Molino Condition ([12], [19]), guarantees that the transverse Poisson structure will indeed be given by 2.10.

There were further attempts to establish necessary and/or sufficient conditions for linearity (or at least "polynomiality") of the transverse Poisson structure to a coadjoint orbit. In 1986, Oh proved [13] that the transverse Poisson structure to a coadjoint orbit $\mathcal{O}_\mu$ is at most quadratic if there is a supplement $\mathfrak{h}$ of $\mathfrak{g}_\mu$ in $\mathfrak{g}$ which is a subalgebra. In order to do that, he used Dirac's constraint bracket formula, following a result due to Courant and Montgomery:

Proposition 46 (Dirac Constraint Bracket Formula) *Let* $(\mathcal{M};\{,\})$ *be a Poisson manifold,* x *a point in* $\mathcal{M}$*,* $\mathcal{S}$ *the symplectic leaf through* x*,* $\mathcal{N}$ *a transverse submanifold to* $\mathcal{S}$ *at* x*,* U *neighborhood of* x *as in the setting of Theorem 30* (Weinstein's Splitting Theorem) *and* $y \in U$*. Consider functions* $\psi_1, \ldots, \psi_{2l}$ *such that*

$$\mathcal{N} = \{y \in U : \quad \psi_1(y) = 0, \ldots, \psi_{2l}(y) = 0\}$$

and denote by C *the (non-singular) matrix such that*

$$\mathrm{C}_{ij}(y) = \{\psi_i,\, \psi_j\}(y).$$

Then the transverse Poisson structure to $\mathcal{S}$ *is given by*

$$\{f,\, h\}_{\mathcal{N}} = \{\widetilde{F},\, \widetilde{H}\} - \sum_{i,j=1}^{2l} \{\widetilde{F},\, \psi_i\}(\mathrm{C}^{-1})_{ij}\{\psi_j,\, \widetilde{H}\}, \tag{2.11}$$

where $f, h \in C^\infty(\mathcal{N})$ *and* $\widetilde{F}, \widetilde{H}$ *are arbitrary extensions to* U *of* f, h.

Marsden & Ratiu [9] obtained the same formula (2.11) in a special case of Poisson reduction. They considered the Poisson structure on the reduced manifold associated with the triplet $(\mathcal{M}, \mathcal{N}, E)$, where the vector subundle $E \subset \mathrm{T}\mathcal{M}|_{\mathcal{N}}$ is given by

$$E = \mathcal{P}^{\#}\left(\mathrm{T}^{\circ}\mathcal{N}\right)$$

and

$$\mathcal{P}^{\#}\left(\mathrm{T}^{\circ}\mathcal{N}\right) \cap \mathrm{T}\mathcal{N} = \{0\}.$$

In such case, the leaves of the induced foliation are just points and the reduced submanifold is $\mathcal{N}$ itself. We observe that the fact that the same formula applies to both situations (transverse and reduction), implies that the transverse Poisson structure to the symplectic leaf $\mathcal{S}$ can be regarded as the result of Poisson reduction. Since Marsden and Ratiu have not proved the formula above in the reduction situation, we will do so below.

Consider

$$\psi_i : \mathcal{M} \to \mathbb{R},\ i = 1, \ldots, 2l$$

so that

$$\mathcal{N} = \{y \in U : \quad \psi_1(y) = 0, \ldots, \psi_{2l}(y) = 0\}.$$

We need the following two lemmas:

Lemma 47 *Let f be a real-valued function on $\mathcal{N}$. Consider F and $\widetilde{F}$ extensions of f to U such that:*

- *$\widetilde{F}$ is arbitrary;*
- $\mathrm{d}F(E) = 0$.

Then, we have

$$\mathrm{d}(\widetilde{F} - F) \in \mathrm{T}^{\circ}\mathcal{N},$$

i.e.,

$$\mathrm{d}\widetilde{F} = \mathrm{d}F + \sum_{i=1}^{2l} a_i \mathrm{d}\psi_i,$$

where each a_i is a real function on U.

Proof. Consider $y \in \mathcal{N}$. We have

$$(\widetilde{F} - F)(y) = 0, \quad y \in \mathcal{N},$$

therefore the restriction of its differential to $\mathrm{T}\mathcal{N}$ is zero. Hence,

$$\mathrm{d}(\widetilde{F} - F) \in \mathrm{T}^{\circ}\mathcal{N}.$$

Since

$$\mathrm{T}^{\circ}\mathcal{N} = \mathrm{span}\{\mathrm{d}\psi_1, \ldots, \mathrm{d}\psi_{2l}\},$$

we have

$$\mathrm{d}(\widetilde{F} - F) = \sum_{i=1}^{2l} a_i \mathrm{d}\psi_i,$$

for some real valued functions a_i. ■

Lemma 48 *If F is an extension of $f \in C^{\infty}(\mathcal{N})$ to U such that $\mathrm{d}F(E) = 0$, then*

$$\{\psi_i, F\} = 0, \quad \forall i,$$

where $\{,\}$ is the Poisson bracket on $\mathcal{M}$.

Proof. We have that

$$\begin{aligned} \mathrm{d}F(E) = 0 \ &\Leftrightarrow \ \mathrm{d}F\left(\mathcal{P}^{\#}(\mathrm{d}\psi_i)\right) = 0 \quad \forall i \\ &\Leftrightarrow \ \{\psi_i, F\} = 0 \quad \forall i, \end{aligned}$$

and the lemma is proved. ■

Now let F and H be, as before, extensions to U of real-valued functions f, h on $\mathcal{N}$ such that

$$\mathrm{d}F(E) = \mathrm{d}H(E) = 0.$$

If we apply Poisson reduction to $(\mathcal{M}, \mathcal{N}, E)$, we obtain

$$\{f, h\}_{\mathcal{N}} = \{F, H\} \circ i, \tag{2.12}$$

where $i : \mathcal{N} \to \mathcal{M}$ is the inclusion. But in order to take $\widetilde{F}$ and $\widetilde{H}$ arbitrary extensions of f and h, one cannot use the reduction formula 2.12 directly. We have to find out what is the relation between $\{\widetilde{F}, \widetilde{H}\}(y)$ and $\{F, H\}(y)$, at each point y of $\mathcal{N}$. This turns out to be a corrective term to be added to the reduction formula 2.12. By Lemma 47,

$$\mathrm{d}(\widetilde{F} - F) = \sum_{i=1}^{2l} a_i \mathrm{d}\psi_i, \quad \mathrm{d}(\widetilde{H} - H) = \sum_{i=1}^{2l} b_i \mathrm{d}\psi_i,$$

(where a_i, $b_i \in C^\infty(\mathcal{N})$). Using Lemma 48 and Lemma 47 consecutively, we have

$$\begin{aligned}
\{\psi_i,\, \widetilde{F}\} &= \{\psi_i,\, \widetilde{F} - F\} \\
&= \left\langle \mathrm{d}(\widetilde{F} - F),\, \mathcal{P}^\#(\mathrm{d}\psi_i) \right\rangle \\
&= \sum_j a_j \left\langle \mathrm{d}\psi_j,\, \mathcal{P}^\#(\mathrm{d}\psi_i) \right\rangle \\
&= \sum_j a_j \{\psi_i,\, \psi_j\}, \\
&= \sum_j a_j \mathrm{C}_{ij}, \\
&= [\mathrm{C}\, a]_i\,,
\end{aligned}$$

or, in vector notation,

$$\{\psi,\, \widetilde{F}\} = \mathrm{C}\, a.$$

After left multiplication by C^{-1} on both sides, we obtain

$$a_i = \sum_j \left[\mathrm{C}^{-1}\right]_{ij} \{\psi_j,\, \widetilde{F}\} \quad \forall i \in \{1, \ldots, 2l\}.$$

Analogousy,

$$b_i = \sum_j \left[\mathrm{C}^{-1}\right]_{ij} \{\psi_j,\, \widetilde{H}\} \quad \forall i \in \{1, \ldots, 2l\}.$$

Now using Lemma 47 and Lemma 48 successively, we have

$$\begin{aligned}
\{\widetilde{F},\, \widetilde{H}\} &= \left\langle \mathrm{d}\widetilde{H}, \mathcal{P}^\# \left(\mathrm{d}\widetilde{F}\right) \right\rangle \\
&= \left\langle \mathrm{d}H, \mathcal{P}^\#(\mathrm{d}F) \right\rangle + \sum_j b_j \{F,\, \psi_j\} \\
&+ \sum_i a_i \{\psi_i,\, H\} + \sum_{i,j} a_i b_j \{\psi_i, \psi_j\} \\
&= \{F,\, H\} + \sum_{i,j} a_i b_j \mathrm{C}_{ij},
\end{aligned}$$

and a straightforward computation - based on substituting a_i and b_j by the expressions deduced above - proves formula 2.11. Please note that the previous computations make sense only for points of $\mathcal{N}$.

We remark that Falceto and Zambon [6] have later argued that the assumptions of the Marsden-Ratiu theorem for Poisson reduction are too strong, and they have weakened these hypothesis in order to be able to recover a greater amount of Poisson structures (on quotients of M). However, to perform reduction in the case we described above, there was no need to apply this extended result.

Later, Saint-Germain [15] derived a formula for the computation of the transverse Poisson structure to a coadjoint orbit from Dirac's constraint bracket formula. Saint Germain used natural identifications typical of the Lie-Poisson case, as described below (we use the notations and definitions of Sections 1.3 and 2.2). Let

$$\{X_1, \dots, X_{m-2l}\}$$

be a basis for $\mathfrak{g}_\mu$ and

$$\{Y_1, \dots, Y_{2l}\}$$

be a basis for $\mathfrak{h}$. Together they form a basis for $\mathfrak{g}$. Consider μ in $\mathfrak{g}^*$, ν in $\mathfrak{h}^\circ$ and the matrices

$$\mathrm{A}(\nu) = ([X_k, X_n](\nu)), \quad \mathrm{B}(\nu) = ([X_k, Y_i](\nu)), \quad \mathrm{C}(\mu+\nu) = ([Y_i, Y_j](\mu+\nu)),$$

where

$$i, j \in \{1, \dots, 2l\}, \quad k, n \in \{1, \dots, m-2l\}.$$

Then Dirac's constraint bracket formula translates to Saint Germain's Formula,

$$\mathcal{P}_{\mathcal{N}}(\mu+\nu) = \mathrm{A}(\nu) + \mathrm{B}(\nu)\mathrm{C}^{-1}(\mu+\nu)\mathrm{B}^T(\nu). \tag{2.13}$$

In 1996, Damianou [4] formulated the following conjecture:

Conjecture 49 *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then the transverse Poisson structure to any coadjoint orbit in $\mathfrak{g}^*$ is polynomial.*

In 2002, Cushman & Roberts [3] proved Damianou's conjecture using their own formula for the transverse Poisson structure to a coadjoint orbit. Cushman and Roberts started from a decomposition of the cotangent space of the Poisson manifold,

$$\mathrm{T}^*_x\mathcal{M} = \mathrm{T}^\circ_x\mathcal{N} \oplus (\mathcal{P}^\#_x(\mathrm{T}^\circ_x\mathcal{N}))^\circ,$$

which is the dual decomposition of 2.3. When taking $\mathcal{M} = \mathfrak{g}^*$, and $x = \mu+\nu$ they considered the projection associated to the dual decomposition of 2.7:

$$\pi_{\mathfrak{g}^\circ_\mu} : \mathfrak{g}^* \to \mathfrak{g}^\circ_\mu,$$

with kernel $\mathfrak{h}^\circ$ and verified that

$$(\mathcal{P}_x^{\#}(\mathrm{T}_x^\circ\mathcal{N}))^\circ = \left\{X \in \mathfrak{g} : \pi_{\mathfrak{g}_\mu^\circ}(\mathrm{ad}_X^*(\mu+\nu)) = 0\right\}.$$

We remark that this projection, $\pi_{\mathfrak{g}_\mu^\circ}$, does not vary with the point $\mu+\nu$ in the transverse submanifold $\mu+\mathfrak{h}^\circ$. Then, they considered U a neighborhood of the origin in $\mathfrak{h}^\circ$ and proceeded to define a map

$$Y : (\mu+U)\times\mathfrak{g}_\mu \to \mathfrak{h}$$

in the following way: Y is the solution (which is proven unique) of the equation

$$\pi_{\mathfrak{g}_\mu^\circ}(\mathrm{ad}_{X+Y}^*(\mu+\nu)) = 0. \tag{2.14}$$

Therefore, Y depends on $X \in \mathfrak{g}_\mu$ and $\nu \in U \subset \mathfrak{h}^\circ$. Roberts and Cushman eventually obtained the following formula for $\{,\}_\mathcal{N}$, the transverse Poisson structure to the coadjoint orbit at μ :

$$\{X_1,\, X_2\}_\mathcal{N}(\mu+\nu) = \langle\mu+\nu,\, [X_1+\, Y_\nu(X_1),\, X_2+\, Y_\nu(X_2)]_\mathfrak{g}\rangle$$

or equivalently

$$\{X_1,\, X_2\}_\mathcal{N}(\mu+\nu) = \left\langle\nu,\, [X_1,\, X_2]_{\mathfrak{g}_\mu}\right\rangle - \langle\mu+\nu,\, [Y_\nu(X_1),\, Y_\nu(X_2)]_\mathfrak{h}\rangle\,.$$

Thus, computing the transverse Poisson structure is basically to determine $Y_\nu(X)$, i.e., to solve equation 2.14 for Y.

In 2005, Sabourin [14] studied the *semisimple complex* case. In the case of nilpotent adjoint orbits which are subregular or have dimension 2, a wide class of supplements $\mathfrak{h}$ was given where the transverse Poisson structure is quadratic.

In 2006, Damianou, Sabourin and Vanhaecke [5] introduced the notion of quasi-degree of a Poisson structure on quasi-homogeneous coordinates. Considering the Lie-Poisson structure on *complex semisimple Lie algebras*, they proved that, in suitable coordinates, the quasi-degree of the tranverse Poisson structures is -2.

Remark 50 None of the results we have referred to above implies that, *for arbitrary* $\mathfrak{h}$, the transverse Poisson structure on $\mathcal{N} = \mu+\mathfrak{h}^\circ$ will be linear (affine) nor quadratic/polynomial. When said that a transverse Poisson structure is *linear* (resp. *polynomial*), it is meant that there is one of such choices of $\mathfrak{h}$ which results in a linear (resp. polynomial) transverse Poisson structure.

2.4 Examples

In order to simplify the consultation of the appendices, in this first example we specify the notations of the Maple files, even though they sometimes seem to be too heavy.

Example 51 We take the semisimple Lie algebra

$$\mathfrak{g} = \mathfrak{so}(4) = \{A \in \mathcal{M}_{4\times 4} : A = -A^t\}.$$

A basis for $\mathfrak{g}$ is, for example

$$\begin{aligned} X_1 &= E_{1,2} - E_{2,1}, \\ X_2 &= E_{1,3} - E_{3,1}, \\ X_3 &= E_{1,4} - E_{4,1}, \\ X_4 &= E_{2,3} - E_{3,2}, \\ X_5 &= E_{2,4} - E_{4,2}, \\ X_6 &= E_{3,4} - E_{4,3}, \end{aligned}$$

where $E_{i,j}$ is an elementary matrix whose unit entry $(i,\, j)$ is the only non-zero entry. We denote its dual basis in $\mathfrak{g}^*$ by $(X^1, \ldots, X^6)$ and consider, the Lie-Poisson structure on $\mathfrak{g}^*$.

As described in Section 1.3, we identify X_i in $\mathfrak{g}$ with x_i in $\mathfrak{g}^{**}$ when computing the Lie-Poisson bracket. We remark that if $\mu \in \mathfrak{g}^*$, we have

$$\{x_i,\, x_j\}_L\,(\mu) \cong \sum_k c_{ij}^k x_k(\mu)$$

where the c_{ij}^k are the structure constants of the Lie algebra $\mathfrak{so}(4)$ in the basis above. Hence, the Poisson matrix is, in these coordinates,

$$\mathrm{P} = \begin{pmatrix} . & -x_4 & -x_5 & x_2 & x_3 & . \\ x_4 & . & -x_6 & -x_1 & . & x_3 \\ x_5 & x_6 & . & . & -x_1 & -x_2 \\ -x_2 & x_1 & . & . & -x_6 & x_5 \\ -x_3 & . & x_1 & x_6 & . & -x_4 \\ . & -x_3 & x_2 & -x_5 & x_4 & . \end{pmatrix}$$

(dots stand for zeros). Its determinant is zero, so there are no 6-rank points in $\mathfrak{g}^*$. Therefore, points of rank 4 (which do exist) are of maximum rank. Again we point out that linear Poisson structures are analytical and hence, by Lemma 9, these points of maximum rank are exactly the regular ones. The transverse Poisson structure on these points will be trivial and the remaining points of $\mathcal{M}$ will be singular. The eigenvalues of the Poisson matrix are

$$\begin{aligned}
\lambda_1 &= 0, \\
\lambda_2 &= 0, \\
\lambda_3 &= -\lambda_4 = i\sqrt{(x_5 - x_2)^2 + (x_1 + x_6)^2 + (x_4 + x_3)^2}, \\
\lambda_5 &= -\lambda_6 = i\sqrt{(x_5 + x_2)^2 + (x_1 - x_6)^2 + (x_4 - x_3)^2}.
\end{aligned}$$

Hence, the points of rank 2 have the following form:

$$(a,\, b,\, c,\, -c,\, b,\, -a) \;\text{ or }\; (a,\, b,\, c,\, c,\, -b,\, a),$$

with $(a,\, b,\, c) \neq (0,\, 0,\, 0)$. We choose

$$\mu = (a,\, b,\, c,\, -c,\, b,\, -a).$$

Then

$$\begin{aligned}
\mathfrak{g}_\mu &= \{X \in \mathfrak{g} : \mathrm{ad}^*_X \mu \equiv 0\} \\
&= \ker \mathrm{P}(\mu) \\
&= \mathrm{span}\,\{F_1,\, F_2,\, F_3,\, F_4\}\,,
\end{aligned}$$

where

$$\begin{aligned}
F_1 &= X_1 + X_6, \\
F_2 &= X_2 - X_5, \\
F_3 &= X_3 + X_4, \\
F_4 &= cX_4 - bX_5 + aX_6.
\end{aligned}$$

Now we must choose a supplement $\mathfrak{h}$ of $\mathfrak{g}_\mu$ in $\mathfrak{g}$. Consider

$$\mathfrak{h} = \mathrm{span}\,\{G_1 = X_1,\, G_2 = X_2\}\,.$$

Then

$$\begin{aligned}
\mathfrak{h}^\circ &= \mathrm{span}\{H^1 = X^3,\, H^2 = X^4,\, H^3 = X^5,\, H^4 = X^6\}. \\
&= \{(0,\, 0,\, y_1,\, y_2,\, y_3,\, y_4) : y_i \in \mathbb{R}\}
\end{aligned}$$

and

$$\mathcal{N} = \{(a,\, b,\, c + y_1,\, -c + y_2,\, b + y_3,\, -a + y_4) : \quad y_i \in \mathbb{R}\}.$$

Taking into account formula 2.9 of Theorem 42, we want to find an expression for the projection

$$\pi_\nu : \mathfrak{g}^* \to \mathfrak{h}^\circ,$$

with kernel $\mathrm{ad}^*_{\mathfrak{h}}(\mu+\nu)$. We start by finding a basis for the kernel. It is necessary to compute

$$\mathrm{ad}^*_{G_i}(\mu+\nu),\ i=1,2,$$

and for that purpose we use the Lie-Poisson matrix at point $\mu+\nu$. We remark that the i-th row of the matrix $P(\mu+\nu)$ is exactly the vector

$$\mathrm{ad}^*_{X_i}(\mu+\nu).$$

Then, the following two vectors define a basis for $\mathrm{ad}^*_{\mathfrak{h}}(\mu+\nu)$:

$$\begin{array}{rcl} W^1 &=& \mathrm{ad}^*_{G_1}(\mu+\nu) \\ &=& (0,\, c-y_2,\, -b-y_3,\, b,\, c-y_1,\, 0); \\ W^2 &=& \mathrm{ad}^*_{G_2}(\mu+\nu) \\ &=& (-c+y_2,\, 0,\, a-y_4,\, -a,\, 0,\, c+y_1). \end{array}$$

We consider the matrix

$$M = ([H^1][H^2][H^3][H^4][W^1][W^2]),$$

obtained by concatenation of the vectors. Finding $\pi_\nu(u)$ is the same as solving the system of linear equations

$$M\lambda = u, \tag{2.15}$$

for λ and choosing

$$\pi_\nu(u) = \lambda_1 H^1 + \lambda_2 H^2 + \lambda_3 H^3 + \lambda_4 H^4.$$

Consider the basis for $\mathfrak{g}_\mu$ defined earlier. In order to find the Poisson matrix for the transverse structure, we now compute $\pi_\nu(\mathrm{ad}^*_{F_1}\nu)$. Through the close connection between ad^* and the Poisson matrix, we know that

$$\mathrm{ad}^*_{F_1}\nu = (0,\, y_1+y_2,\, y_3,\, y_3,\, -y_1-y_2,\, 0).$$

We solve equation 2.15 for $u = \mathrm{ad}^*_{F_1}\nu$, i.e.,

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ y_1+y_2 \\ y_3 \\ y_3 \\ -y_1-y_2 \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned}\pi_\nu(\mathrm{ad}^*_{F_1}\nu) &= \sum_{i=1}^{4}\lambda_i F_i \\ &= (0,\,0,\,y_3+(b+y_3)z,\,y_3-bz,\,-y_1-y_2-(c+y_1)z,\,0),\end{aligned}$$

where

$$z=\frac{(y_1+y_2)}{c-y_2}.$$

Finally, we can compute the first row of the transverse Poisson matrix:

$$\begin{aligned}(\mathrm{P}_{\mathcal{N}})_{11}(\mu+\nu) &= \left\langle \pi_\nu(\mathrm{ad}^*_{F_1}\nu),\,F_1\right\rangle = 0\\ (\mathrm{P}_{\mathcal{N}})_{12}(\mu+\nu) &= \left\langle \pi_\nu(\mathrm{ad}^*_{F_1}\nu),\,F_2\right\rangle = -\tfrac{(2c+y_1-y_2)(y_1+y_2)}{c-y_2}\\ (\mathrm{P}_{\mathcal{N}})_{13}(\mu+\nu) &= \left\langle \pi_\nu(\mathrm{ad}^*_{F_1}\nu),\,F_3\right\rangle = -\tfrac{y_3(2c+y_1-y_2)}{c-y_2}\\ (\mathrm{P}_{\mathcal{N}})_{14}(\mu+\nu) &= \left\langle \pi_\nu(\mathrm{ad}^*_{F_1}\nu),\,F_4\right\rangle = -\tfrac{c(by_2+cy_3+by_1-y_2y_3)+b(y_1^2-y_2^2)}{c-y_2}\end{aligned}$$

In the same way, starting with

$$\pi_\nu(\mathrm{ad}^*_{F_i}\nu),\; i=2,\,3,\,4$$

we obtain the remaining entries of the Poisson matrix:

$$\begin{aligned}(\mathrm{P}_{\mathcal{N}})_{23}(\mu+\nu) &= -\tfrac{y_4(2c+y_1-y_2)}{c-y_2}\\ (\mathrm{P}_{\mathcal{N}})_{24}(\mu+\nu) &= \tfrac{c(ay_2-cy_4+y_2y_4+ay_1)+a(y_1^2-y_2^2)}{c-y_2}\\ (\mathrm{P}_{\mathcal{N}})_{34}(\mu+\nu) &= \tfrac{a(y_3y_1-y_3y_2+cy_3)+b(y_4y_1-y_4y_2+cy_4)}{c-y_2}.\end{aligned}$$

We have obtained non-polynomial entries after computing a transverse Poisson structure to the dual of a semisimple Lie algebra. And yet, as we have seen, this example does not contradict Damianou's conjecture. There may be another affine subspace $\mu+\mathfrak{h}^\circ$, transversal to the coadjoint orbit, in which the transverse Poisson structure is polynomial. We refer the reader to Appendix A for further details on the computation of this transverse Poisson structure.

Example 52 We will now consider the Lie algebra

$$\mathfrak{g}=\mathfrak{se}(3)=\mathfrak{so}(3)\ltimes\mathbb{R}^3$$

of the euclidean group

$$\mathbf{SE}(3)=\mathbf{SO}(3)\ltimes\mathbb{R}^3,$$

considering the usual action of $\mathbf{SO}(3)$ in $\mathbb{R}^3$.

We must determine which points μ in $\mathfrak{g}^*$ are singular with respect to the Lie-Poisson structure. We consider the basis of $\mathfrak{so}(3) \ltimes \mathbb{R}^3$ formed by the following elements:

$$\begin{aligned}
X_1 &\cong \left(\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 \\ \cdot & 1 & \cdot \end{pmatrix}, (0,\, 0,\, 0) \right), \\
X_2 &\cong \left(\begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot \end{pmatrix}, (0,\, 0,\, 0) \right), \\
X_3 &\cong \left(\begin{pmatrix} \cdot & -1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, (0,\, 0,\, 0) \right), \\
X_4 &\cong ((0), (1,\, 0,\, 0)), \\
X_5 &\cong ((0), (0,\, 1,\, 0)), \\
X_6 &\cong ((0), (0,\, 0,\, 1)) \cdot
\end{aligned}$$

In this basis, the Poisson matrix is

$$\mathrm{P} = \begin{pmatrix} \cdot & x_3 & -x_2 & \cdot & x_6 & -x_5 \\ -x_3 & \cdot & x_1 & -x_6 & \cdot & x_4 \\ x_2 & -x_1 & \cdot & x_5 & -x_4 & \cdot \\ \cdot & x_6 & -x_5 & \cdot & \cdot & \cdot \\ -x_6 & \cdot & x_4 & \cdot & \cdot & \cdot \\ x_5 & -x_4 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

We have

$$\det \mathrm{P}(\mu) \equiv 0, \quad \forall \mu \in \mathfrak{g}^*,$$

and again there are no 6-rank points. As in the previous example, we are in presence of an analytical Poisson structure and therefore the points of maximal rank are the regular ones. All points such that $x_i \neq 0$, for some $i = 4, 5, 6$ have rank 4, which is maximal. At these points the transverse Poisson structure is trivial, and again we turn our attention to the points of rank 2. They are all points of the form

$$\mu = (a,\, b,\, c,\, 0,\, 0,\, 0),$$

with $a^2 + b^2 + c^2 \neq 0$.

We will now compute the transverse Poisson structure to the coadjoint orbit of any of these points. We have

$$\begin{aligned}
\mathfrak{g}_\mu &= \ker \mathrm{P}(\mu) \\
&= \mathrm{span}\{aX_1 + bX_2 + cX_3,\, X_4,\, X_5,\, X_6\},
\end{aligned}$$

For instance, a possible choice for the supplement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$ is

$$\mathfrak{h} = \operatorname{span}\{X_1,\ X_2\},$$

and we obtain (as in the previous example, $\{X^1, X^2, X^3, X^4, X^5, X^6\}$ is the dual basis of the basis of $\mathfrak{g}$ above)

$$\mathfrak{h}^\circ = \operatorname{span}\{X^3,\ X^4,\ X^5,\ X^6\}$$

Considering an arbitrary element in

$$\mathcal{N} = \{(a,\ b,\ c + y_1,\ y_2,\ y_3,\ y_4), \quad y_i \in \mathbb{R}\}$$

and doing all the computations as in the previous example, we obtain the following matrix for the transverse Poisson structure (due to lack of space, we present only the upper-diagonal part of the matrix):

$$\frac{1}{c+y_1}\begin{pmatrix} . & -c(by_4 - cy_3 - y_3y_1) & c(ay_4 - cy_2 - y_2y_1) & c(by_2 - ay_3) \\ * & . & -y_4^2 & y_3y_4 \\ * & * & . & -y_2y_4 \\ * & * & * & . \end{pmatrix},$$

The entries of this Poisson matrix are rational functions in the variables y_i, as in Example 51. However, this does not imply that all other transverse Poisson structures to the same coadjoint orbits will also be non-polynomial. For all we know, there could even be a linear one amongst them.

In fact, the transverse Poisson structures of examples 51 and 52 are substantially different in nature. Those differences will become more evident in Chapter 4.

Chapter 3

On Sufficient Conditions for Linearity of the Transverse Poisson Structure

Examples 51 and 52 illustrate the fundamental question:

Given a certain singular point μ of a Lie-Poisson structure, how simple can a transverse Poisson structure to its orbit be? For example, under which conditions can we guarantee the existence of a linear transverse Poisson structure to the coadjoint orbit $\mathcal{O}_\mu$?

First we point out that, in order to adress the second question properly, we must prove that the notion of linear transverse Poisson structure *to a coadjoint orbit* is well defined. There is an infinity of transverse manifolds to $\mathcal{O}_\mu$, each one endowed with its own transverse Poisson structure. Although we know that they are all Poisson-diffeomorphic (see Lemma 40), the linearity of a specific transverse Poisson structure to $\mathcal{O}_\mu$ does not guarantee the linearity of all others. We present an example of this (Example 62) in Section 3.2 below.

By "linear transverse Poisson structure to $\mathcal{O}_\mu$ at μ" we mean that there is at least one transverse Poisson structure *through* μ which is linear. We will now verify that this implies the existence, *at any other point in* $\mathcal{O}_\mu$, of at least one transverse Poisson structure which is linear, giving meaning to the expression "*linear* transverse Poisson structure to a coadjoint orbit". The following lemma will be useful for that purpose.

Lemma 53 *Let V be a vector space. Consider U_1, U_2, V_1, V_2 vector subspaces of V such that*

$$V = U_i \oplus V_i, \quad i = 1, 2,$$

the projections

$$\pi_i : V \to U_i, \quad i = 1, 2,$$

with $\ker \pi_i = V_i$*, and an isomorphism* I *such that*

$$\mathrm{I}(U_1) = U_2, \quad \mathrm{I}(V_1) = V_2.$$

Then

$$\pi_2 \circ \mathrm{I} = \mathrm{I} \circ \pi_1.$$

Proof. Let v be a vector in V. Then there are vectors u_1 in U_1 and v_1 in V_1 such that

$$v = v_1 + u_1.$$

We have

$$\begin{aligned} \pi_2 \circ \mathrm{I}(v) &= \pi_2 \left(\mathrm{I}(u_1) + \mathrm{I}(v_1)\right) \\ &= \pi_2 \left(\mathrm{I}(u_1)\right) \\ &= \mathrm{I}(u_1). \end{aligned}$$

(notice that $\mathrm{I}(v_1) \in \ker \pi_2$). On the other hand, we have

$$\begin{aligned} \mathrm{I} \circ \pi_1(v) &= \mathrm{I} \circ \pi_1(u_1 + v_1) \\ &= \mathrm{I}(u_1), \end{aligned}$$

because $v_1 \in \ker \pi_1$. ■

Theorem 54 *Let* μ_1 *be a singular point of a Lie-Poisson manifold* $(\mathfrak{g}^*, \{,\}_L)$ *and let* $\mathcal{O}$ *denote the coadjoint orbit through* μ_1*. Suppose that there is a linear transverse Poisson structure to* $\mathcal{O}$ *at* μ_1*. Then, given* μ_2 *in* $\mathcal{O}$*, there is a linear transverse Poisson structure to* $\mathcal{O}$ *at* μ_2*.*

Proof. First we show how the isotropy subalgebras of μ_1 and μ_2 are related. Let g be an element of $\mathbf{G}$ such that $\mu_2 = \mathrm{Ad}^*_g \mu_1$. Let X be an element of $\mathfrak{g}_{\mu_1}$. For all Y in $\mathfrak{g}$, we have

$$\begin{aligned} \langle \mathrm{ad}^*_X \mu_2,\ Y \rangle &= \left\langle \mathrm{Ad}^*_g \mu_1,\ \mathrm{ad}_X Y \right\rangle \\ &= \langle \mu_1,\ \mathrm{Ad}_{g^{-1}}[X,\ Y] \rangle \\ &= \langle \mu_1,\ [\mathrm{Ad}_{g^{-1}} X,\ \mathrm{Ad}_{g^{-1}} Y] \rangle \\ &= \left\langle \mu_1,\ \mathrm{ad}_{\mathrm{Ad}_{g^{-1}} X}(\mathrm{Ad}_{g^{-1}} Y) \right\rangle \\ &= \left\langle \mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}} X}(\mu_1),\ \mathrm{Ad}_{g^{-1}} Y \right\rangle. \end{aligned}$$

Since $\mathrm{Ad}_{g^{-1}}$ is an automorphism of $\mathfrak{g}$, having

$$\langle \mathrm{ad}^*_X \mu_2,\ Y \rangle = 0, \quad \forall Y \in \mathfrak{g}$$

is the same as having

$$\left\langle \mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}} X}(\mu_1),\ Y \right\rangle = 0, \quad \forall Y \in \mathfrak{g},$$

i.e., $\mathrm{Ad}_{g^{-1}} X \in \mathfrak{g}_{\mu_1}$. We have hence proved that

$$\mathfrak{g}_{\mu_2} = \mathrm{Ad}_g(\mathfrak{g}_{\mu_1}). \tag{3.1}$$

Now, suppose that the linear transverse Poisson structure to $\mathcal{O}$ through μ_1 (which exists by hypothesis) is defined on the affine subspace

$$\mathcal{N}_1 = \mu_1 + \mathfrak{h}_1^\circ,$$

where $\mathfrak{h}_1$ is a supplement of $\mathfrak{g}_{\mu_1}$ in $\mathfrak{g}$. Such linear Poisson structure shall be denoted by $\{,\}_1$. Then, we have that

$$\mathfrak{h}_2 = \mathrm{Ad}_g \mathfrak{h}_1 \tag{3.2}$$

is a supplement of $\mathfrak{g}_{\mu_2}$ in $\mathfrak{g}$ (again, because Ad_g is an automorphism of $\mathfrak{g}$) and we may define the transverse Poisson structure to $\mathcal{O}$ through μ_2 on the submanifold

$$\mathcal{N}_2 = \mu_2 + \mathfrak{h}_2^\circ.$$

We will show that this Poisson structure, which we will denote by $\{,\}_2$, is linear. Straightforward computations using (3.2) and the properties of Ad_g used above show that

$$\mathfrak{h}_2^\circ = \mathrm{Ad}^*_g \mathfrak{h}_1^\circ, \tag{3.3}$$

and that

$$\mathrm{ad}^*_{\mathrm{Ad}_g Y}\left(\mathrm{Ad}^*_g \nu\right) = \mathrm{Ad}^*_g\left(\mathrm{ad}^*_Y \nu\right), \quad \forall Y \in \mathfrak{g}, \forall \nu \in \mathfrak{g}^*. \tag{3.4}$$

Let $X_2,\ Y_2$ be in $\mathfrak{g}_{\mu_2}$, ν_2, τ_2 be in $\mathfrak{h}_2^\circ$ and α, β be real numbers. We will now show that

$$\{X_2,\, Y_2\}_2\,(\mu_2 + \alpha\nu_2 + \beta\tau_2) = \alpha\,\{X_2,\, Y_2\}_2\,(\mu_2 + \nu_2) + \beta\,\{X_2,\, Y_2\}_2\,(\mu_2 + \tau_2).$$

Because of relations 3.1, 3.2 and 3.3, there are $X_1,\ Y_1,\ \nu_1,\ \tau_1$ such that

$$X_2 = \mathrm{Ad}_g X_1, \quad Y_2 = \mathrm{Ad}_g Y_1$$

and

$$\nu_2 = \mathrm{Ad}_g^* \nu_1 \quad \tau_2 = \mathrm{Ad}_g^* \tau_1.$$

Using Theorem 42, equation 3.4 and Lemma 53 consecutively, we have that

$$\begin{aligned}
\{X_2, Y_2\}_2 (\mu_2 + \alpha\nu_2 + \beta\tau_2) &= \left\langle \pi_{\alpha\nu_2+\beta\tau_2}(\mathrm{ad}^*_{Y_2}(\alpha\nu_2 + \beta\tau_2)), X_2 \right\rangle \\
&= \left\langle \pi_{\alpha\nu_2+\beta\tau_2}\left(\mathrm{Ad}^*_g \mathrm{ad}^*_{Y_1}(\alpha\nu_1 + \beta\tau_1)\right), X_2 \right\rangle \\
&= \left\langle \mathrm{Ad}^*_g(\pi_{\alpha\nu_1+\beta\tau_1}(\mathrm{ad}^*_{Y_1}(\alpha\nu_1 + \beta\tau_1))), \mathrm{Ad}_g X_1 \right\rangle \\
&= \left\langle \pi_{\alpha\nu_1+\beta\tau_1}(\mathrm{ad}^*_{Y_1}(\alpha\nu_1 + \beta\tau_1)), X_1 \right\rangle \\
&= \{X_1, Y_1\}_1 (\mu_1 + \alpha\nu_1 + \beta\tau_1).
\end{aligned}$$

By hypothesis, the Poisson bracket $\{,\}_1$ is linear, i.e.,

$$\{X_1, Y_1\}_1 (\mu_1 + \alpha\nu_1 + \beta\tau_1) = \alpha \{X_1, Y_1\}_1 (\mu_1 + \nu_1) + \beta \{X_1, Y_1\}_1 (\mu_1 + \tau_1).$$

A straightforward computation (basically, reversing the previous process) shows that also $\{,\}_2$ is linear. ■

We already know a sufficient condition (on the supplement $\mathfrak{h}$) for linearity of the transverse Poisson structure, which is Molino condition (Corollary 43). Still, Molino condition gives us no indication about how to find such an $\mathfrak{h}$, if it exists.

In this chapter we will establish a new sufficient condition for the existence of such an $\mathfrak{h}$, which in addition explicites which $\mathfrak{h}$ to consider in order to obtain a linear transverse Poisson structure. Some properties of symmetric bilinear forms in Lie algebras will be useful for that purpose.

3.1 Symmetric Bilinear Forms on a Lie algebra

Definition 55 Let $\mathbf{B}$ be a symmetric bilinear form on $\mathfrak{g}$ and $\mathfrak{g}_0$ a subalgebra of $\mathfrak{g}$. $\mathbf{B}$ is said to be:

1. *Non-degenerate* if

$$\mathbf{B}(X, Y) = 0, \ \forall Y \in \mathfrak{g} \Leftrightarrow X = \vec{0};$$

2. Ad-*invariant* if

$$\mathbf{B}(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)) = \mathbf{B}(X, Y) \quad \forall X, Y \in \mathfrak{g}, \forall g \in \mathbf{G},$$

   where $\mathbf{G}$ is the connected and simply connected Lie group whose Lie algebra is $\mathfrak{g}$;

3. ad-*invariant* if

$$\mathbf{B}(X,[Y,Z])+\mathbf{B}([X,Z],Y)=0, \quad \forall X,\ Y,\ Z\in\mathfrak{g};$$

4. $\mathrm{ad}_{\mathfrak{g}_0}$-*invariant* if

$$\mathbf{B}(X,[Y,Z])+\mathbf{B}([X,Z],Y)=0, \quad \forall X,\ Y\ \in\mathfrak{g},\ Z\in\mathfrak{g}_0.$$

The third condition can be derived from the second by taking time derivatives, and is usually referred to as its infinitesimal version. Both obviously imply condition 4, for any Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$. For instance, the Killing form of a Lie algebra $\mathfrak{g}$,

$$\begin{array}{rccl}\mathbf{K}: & \mathfrak{g}\times\mathfrak{g} & \to & \mathbb{R}\\ & (X,\,Y) & \mapsto & \mathrm{tr}(\mathrm{ad}_X\circ\mathrm{ad}_Y)\end{array}$$

is an Ad-invariant symmetric bilinear form. Also, $\mathfrak{g}$ is semisimple if and only if $\mathbf{K}$ is non-degenerate.

Lemma 56 *Let $\mathfrak{g}_0\subset\mathfrak{g}$ be a subalgebra and let $\mathbf{B}$ be an $\mathrm{ad}_{\mathfrak{g}_0}$-invariant, symmetric and bilinear form. Let*

$$\mathfrak{g}_0^{\perp}=\{X\in\mathfrak{g}:\mathbf{B}(X,\,Y)=0,\forall Y\in\mathfrak{g}_0\}$$

be the orthogonal of $\mathfrak{g}_0$ with respect to $\mathbf{B}$. Then

$$[\mathfrak{g}_0,\mathfrak{g}_0^{\perp}]\subset\mathfrak{g}_0^{\perp}.$$

If, furthermore, $\mathbf{B}|_{\mathfrak{g}_0\times\mathfrak{g}_0}$ is non-degenerate, then

$$\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_0^{\perp}.$$

Proof. First, we prove that $[\mathfrak{g}_0,\mathfrak{g}_0^{\perp}]\subset\mathfrak{g}_0^{\perp}$. Let $X\in\mathfrak{g}_0,\ Y\in\mathfrak{g}_0^{\perp}$. We have

$$\mathbf{B}([Z,\,X],\,Y)=0, \quad \forall\, Z\in\mathfrak{g}_0,$$

because

$$[X,Z]\in\mathfrak{g}_0,\ \ Y\in\mathfrak{g}_0^{\perp}.$$

Because $\mathbf{B}$ is an $\mathrm{ad}_{\mathfrak{g}_0}$-invariant bilinear form (Definition 55), we have

$$\mathbf{B}(Z,[X,\,Y])=0, \quad \forall\, Z\in\mathfrak{g}_0,$$

i.e.

$$[X,\, Y] \in \mathfrak{g}_0^{\perp}.$$

Now we prove that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0^{\perp}$. Because the restriction of $\mathbf{B}$ to $\mathfrak{g}_0$ is non-degenerate, $\mathfrak{g}_0$ and $\mathfrak{g}_0^{\perp}$ intersect only at zero, so we only need to prove that $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_0^{\perp}$. This is a consequence of the Orthogonal Splitting Lemma [11], which we will now prove.

The non-degeneracy of $\mathbf{B}|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ implies that the following map is a linear isomorphism:

$$\begin{array}{rccc} \overline{\mathbf{B}}: & \mathfrak{g}_0 & \to & \mathfrak{g}_0^* \\ & X & \mapsto & \overline{\mathbf{B}}(X), \end{array}$$

where

$$\langle \overline{\mathbf{B}}(X),\, Y \rangle = \mathbf{B}(X,\, Y), \quad \forall\, X \in \mathfrak{g}_0. \tag{3.5}$$

For each $X \in \mathfrak{g}$, we consider

$$\begin{array}{rccc} \mathbf{B}_X: & \mathfrak{g}_0 & \to & \mathbb{R} \\ & Y & \mapsto & \mathbf{B}(X,\, Y). \end{array}$$

$\mathbf{B}_X$ is an element of $\mathfrak{g}_0^*$, so there is one and only one X_0 in $\mathfrak{g}_0$ such that

$$\mathbf{B}_X = \overline{\mathbf{B}}(X_0).$$

Hence,

$$\mathbf{B}(X,\, Y) = \mathbf{B}(X_0,\, Y), \quad \forall\, Y \in \mathfrak{g}_0,$$

or, in other words, $X - X_0$ is an element of $\mathfrak{g}_0^{\perp}$. Therefore we may write

$$X = X_0 + Z,$$

where $X_0 \in \mathfrak{g}_0$ and $Z \in \mathfrak{g}_0^{\perp}$. ■

3.2 Our Sufficient Condition for Linearity of the Transverse Poisson Structure

If $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ in the conditions of Lemma 56 then $\mathfrak{g}_0^{\perp}$ satisfies both

$$[\mathfrak{g}_0, \mathfrak{g}_0^{\perp}] \subset \mathfrak{g}_0^{\perp}$$

and

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0^{\perp}.$$

Suppose that the two conditions of Lemma 56 are valid for an isotropy subalgebra $\mathfrak{g}_\mu$. Then

$$\mathfrak{h} = \mathfrak{g}_\mu^\perp$$

satisfies Molino condition and Corollary 43 shows that the transverse Poisson structure to $\mathcal{O}_\mu$, on

$$\mathcal{N} = \mu + \left(\mathfrak{g}_\mu^\perp\right)^\circ,$$

is linear. This argument proves the following theorem.

Theorem 57 *Let $\mathfrak{g}$ be a Lie algebra and consider μ in $\mathfrak{g}^*$. Let $\mathbf{B}$ be a symmetric bilinear form in $\mathfrak{g}$ such that*

1. *$\mathbf{B}$ is $\mathrm{ad}_{\mathfrak{g}_\mu}$-invariant;*

2. *$\mathbf{B}|_{\mathfrak{g}_\mu \times \mathfrak{g}_\mu}$ is non-degenerate.*

Then the transverse Poisson structure on

$$\mathcal{N} = \mu + \left(\mathfrak{g}_\mu^\perp\right)^\circ$$

is linear ($\mathfrak{g}_\mu^\perp$ being the orthogonal with respect to $\mathbf{B}$).

This sufficient condition has a fundamental advantage in relation with Molino condition. Having found the bilinear form $\mathbf{B}$, the transverse manifold on which the transverse Poisson structure is linear is already determined:

$$\mathcal{N} = \mu + \left(\mathfrak{g}_\mu^\perp\right)^\circ.$$

Based on some known bilinear forms in suitable Lie algebras, several cases of linear transverse Poisson structures to coadjoint orbits immediately emerge. For example, the Killing form $\mathbf{K}$ always satisfies condition 1 of Theorem 57. One only has to verify the second condition. If $\mathfrak{g}$ is a semisimple Lie algebra of compact type, the Killing form in $\mathfrak{g}$ is negative definite and, consequently, its restriction to any subalgebra is non-degenerate (see Corollary 58), satisfying condition 2 of Theorem 57. Hence, in this case there will be a linear transverse Poisson structure to every coadjoint orbit. However, a more general result can be derived:

Corollary 58 *Let $\mathfrak{g}$ be a compact type Lie algebra (i.e., there is a compact Lie group whose Lie algebra is $\mathfrak{g}$). Then, for any μ in $\mathfrak{g}$, there is a linear transverse Poisson structure to the coadjoint orbit $\mathcal{O}_\mu$.*

Proof. If $\mathfrak{g}$ is a compact type Lie algebra, we may define a symmetric bilinear form $\mathbf{B}$ which is also positive definite and ad-invariant (see [11]). In particular, $\mathbf{B}$ is $\mathrm{ad}_{\mathfrak{g}_\mu}$invariant for every μ in $\mathfrak{g}^*$. We will establish that its restriction to any subalgebra $\mathfrak{g}_\mu$ is non-degenerate. Indeed, considering Y in $\mathfrak{g}_\mu$ and supposing that

$$\mathbf{B}|_{\mathfrak{g}_\mu \times \mathfrak{g}_\mu}(X, Y) = 0, \quad \forall X \in \mathfrak{g}_\mu,$$

we get, in particular,

$$\mathbf{B}(Y, Y) = 0.$$

But then $Y = 0$ (because $\mathbf{B}$ is positive definite). Theorem 57 completes the proof. ■

Now suppose $\mathfrak{g}$ is a semisimple Lie algebra. The Killing form $\mathbf{K}$, being non-degenerate in semisimple Lie algebras, provides an isomorphism, $\overline{\mathbf{K}}$, between $\mathfrak{g}$ and $\mathfrak{g}^*$ (see 3.5 in the proof of Lemma 56). Then it makes sense to say that an element of $\mathfrak{g}^*$ is semisimple or nilpotent. We recall that an element X in $\mathfrak{g}$ is a *semisimple (resp. nilpotent) element of* $\mathfrak{g}$ if

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$$

is a semisimple (resp. nilpotent) linear operator. We refer the reader to [7] and [17] for further details.

Corollary 59 *Let $\mathfrak{g}$ be a semisimple Lie algebra and μ a semisimple element of $\mathfrak{g}^*$ (meaning that μ is identified through* $\mathbf{K}$ *with a semisimple element X in $\mathfrak{g}$). Then there is a linear transverse Poisson structure to $\mathcal{O}_\mu$.*

Proof. Consider the semisimple element X in $\mathfrak{g}$ which corresponds to μ. First we point out that

$$\mathfrak{g}_\mu = \mathfrak{z}(X),$$

where $\mathfrak{z}(X)$ is the centralizer of X in $\mathfrak{g}$. Indeed, considering Y in $\mathfrak{g}_\mu$, we have

$$\begin{aligned} \mathrm{ad}^*_Y \mu = 0 \;&\Leftrightarrow\; \langle \mathrm{ad}^*_Y \mu,\, Z\rangle = 0, \quad \forall Z \in \mathfrak{g} \\ &\Leftrightarrow\; \langle \mu,\, [Z, Y]\rangle = 0, \quad \forall Z \in \mathfrak{g} \\ &\Leftrightarrow\; \mathbf{K}(X, [Z, Y]) = 0, \quad \forall Z \in \mathfrak{g} \\ &\Leftrightarrow\; \mathbf{K}([X, Y], Z) = 0, \quad \forall Z \in \mathfrak{g} \qquad (3.6) \\ &\Leftrightarrow\; [X, Y] = 0, \end{aligned}$$

where in 3.6 we have used the symmetry and ad-invariance of $\mathbf{K}$ and the last identity follows from the non-degeneracy of $\mathbf{K}$.

Together with Theorem 4.1.6 of [17], this implies that $\mathbf{K}|_{\mathfrak{g}_\mu \times \mathfrak{g}_\mu}$ is non-degenerate and that

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^\perp.$$

Theorem 57 completes the proof. ■

In the next corollary, $\mathbf{G}$ will denote any Lie group with Lie algebra $\mathfrak{g}$ and $\mathbf{G}_\mu$ will stand for the isotropy subgroup of $\mu \in \mathfrak{g}^*$, i.e.,

$$\mathbf{G}_\mu = \{g \in \mathbf{G} : \mathrm{Ad}_g^* \mu = \mu\}.$$

Corollary 60 *If $\mu \in \mathfrak{g}^*$ is such that $\mathfrak{g}_\mu$ is semisimple or $\mathbf{G}_\mu$ is compact, then there is a linear transverse Poisson structure to the coadjoint orbit $\mathcal{O}_\mu$.*

Proof. In the first situation, the adjoint representation of $\mathfrak{g}_\mu$ on $\mathfrak{g}$ is faithful. Then $\mathbf{K}$, the Killing form of $\mathfrak{g}$, is non-degenerate when restricted to $\mathfrak{g}_\mu$ and Theorem 57 can again be used (with $\mathbf{B} = \mathbf{K}$).

In the second case, every representation of $\mathbf{G}_\mu$ on a finite-dimensional vector space V is completely reducible. Consider the adjoint representation of $\mathbf{G}_\mu$ on $\mathfrak{g}$:

$$\begin{array}{rccl} \mathrm{Ad}^\mu : & \mathbf{G}_\mu & \to & Hom(\mathfrak{g}) \\ & g & \mapsto & \mathrm{d}(L_g \circ R_{g^{-1}})_e \end{array}$$

where L_g and R_g are left product and right product by g, respectively. Clearly $\mathfrak{g}_\mu$ is an Ad^μ-invariant subspace. By complete reducibility of Ad^μ, there is an Ad^μ-invariant supplement, say $\mathfrak{h}$, to $\mathfrak{g}_\mu$. This means that

$$\mathrm{Ad}^\mu(g)(\mathfrak{h}) \subset \mathfrak{h}, \quad \forall g \in \mathbf{G}_\mu,$$

which in turn implies that

$$[\mathfrak{g}_\mu, \mathfrak{h}] \subset \mathfrak{h}$$

and Molino condition may be used. ■

Remark 61 In the last theorem, the condition "$\mathbf{G}_\mu$ compact" cannot be weakened to "$\mathfrak{g}_\mu$ of compact type." The tools to give a counter-example will be given in Chapter 4 (see Example 77). The $\mathbf{G}_\mu$ compact situation, with the necessary adaptations, was suggested to us by R. Loja Fernandes for Poisson manifolds in general.

Example 62 Again we consider the Lie algebra

$$\mathfrak{so}(4) = \{A \in \mathcal{M}_{4\times 4} : A = -A^T\}.$$

This is a compact type Lie algebra, hence Corollary 58 assures that there will be a linear transverse Poisson structure to any coadjoint orbit. We will find that specific transverse submanifold $\mathcal{N}$. First, we compute its Killing form $\mathbf{K}$. We already know it is negative definite, because $\mathfrak{so}(4)$ is semisimple.

The matrix of $\mathbf{K}$ on the basis $\{X_1, \ldots, X_6\}$ (see Example 51) is

$$\begin{pmatrix} -4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ . & -4 & . & . & . & . \\ . & . & -4 & . & . & . \\ . & . & . & -4 & . & . \\ . & . & . & . & -4 & . \\ . & . & . & . & . & -4 \end{pmatrix}.$$

Therefore, Theorem 57 guarantees that taking μ in $\mathfrak{so}(4)^*$ and choosing

$$\mathcal{N} = \mu + \left(\mathfrak{g}_\mu^\perp\right)^\circ$$

(ortogonal with respect to $\mathbf{K}$), we obtain a linear transverse Poisson structure to $\mathcal{O}_\mu$. For example, taking the same point as in Example 51,

$$\mu = (a,\, b,\, c,\, -c,\, b,\, -a),$$

then

$$\mathfrak{g}_\mu^\perp = \operatorname{span}\{c(X_1 - X_6) - a(X_3 - X_4),\ -b(X_1 - X_6) + a(X_2 + X_5)\}.$$

Then the formula of Theorem 42 on the submanifold

$$\{(a + y_1, b + y_2, c + y_3, (y_4 - 1)c + y_3, (1 - y_4)b - y_2, (y_4 - 1)a + y_1);\ y_i \in \mathbb{R}\}$$

produces the following Poisson matrix for the transverse Poisson structure:

$$\mathrm{P}_{\mathcal{N}} = \begin{pmatrix} . & -4y_3 - 2cy_4 & 4y_2 + 2by_4 & 2cy_2 - 2by_3 \\ 4y_3 + 2cy_4 & . & -4y_1 - 2ay_4 & 2ay_3 - 2cy_1 \\ -4y_2 - 2by_4 & 4y_1 + 2ay_4 & . & 2by_1 - 2ay_2 \\ 2by_3 - 2cy_2 & 2cy_1 - 2ay_3 & 2ay_2 - 2by_1 & . \end{pmatrix}.$$

The entries are linear in the y_i's (coordinates of the points in the transverse manifold).

Again we remark that, if another subspace $\mathfrak{h}$ was chosen (and consequently another transverse manifold), we could obtain non-linear transverse Poisson structures, like in Example 51.

Example 63 Now we consider the Lie algebra

$$\mathfrak{sp}(4)=\left\{\left(\begin{array}{cc} A_1 & A_2 \\ A_3 & -A_1^T \end{array}\right) : A_i \in \mathcal{M}_{2\times 2}, A_2 = A_2^T, A_3 = A_3^T\right\},$$

which is semisimple but not of compact type. We choose the basis of $\mathfrak{sp}(4)$ as follows:

$$\begin{array}{rcl} X_1 & = & E_{1,1}-E_{3,3} \\ X_2 & = & E_{1,2}-E_{4,3} \\ X_3 & = & E_{2,1}-E_{3,4} \\ X_4 & = & E_{2,2}-E_{4,4} \\ X_5 & = & E_{1,3} \\ X_6 & = & E_{1,4}+E_{2,3} \\ X_7 & = & E_{2,4} \\ X_8 & = & E_{3,1} \\ X_9 & = & E_{3,2}+E_{4,1} \\ X_{10} & = & E_{4,2} \end{array}$$

The Lie-Poisson matrix is

$$\left(\begin{array}{cccccccccc} . & x_2 & -x_3 & . & 2x_5 & x_6 & . & -2x_8 & -x_9 & . \\ * & . & x_1-x_4 & x_2 & . & 2x_5 & x_6 & -x_9 & -2x_{10} & . \\ * & * & . & -x_3 & x_6 & 2x_7 & . & . & -2x_8 & -x_9 \\ * & * & * & . & . & x_6 & 2x_7 & . & -x_9 & -2x_{10} \\ * & * & * & * & . & . & . & x_1 & x_2 & . \\ * & * & * & * & * & . & . & x_3 & x_1+x_4 & x_2 \\ * & * & * & * & * & * & . & . & x_3 & x_4 \\ * & * & * & * & * & * & * & . & . & . \\ * & * & * & * & * & * & * & * & . & . \\ * & * & * & * & * & * & * & * & * & . \end{array}\right).$$

There are no points of rank 10, therefore points of rank 8 (which do exist) are regular. We choose a 6-rank point, for example

$$\begin{array}{rcl} \mu & = & X^1+X^4 \\ & = & (1,0,0,1,0,\ldots,0). \end{array}$$

The isotropy subalgebra is

$$\mathfrak{g}_\mu = \operatorname{span}\{X_1,X_2,X_3,X_4\}.$$

Taking the same basis $X_1,\ldots,X_{10}$, the Killing form of $\mathfrak{sp}(4)$ is represented by the matrix

$$
\mathbf{K} = \begin{pmatrix}
12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 \\
\cdot & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 12 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot
\end{pmatrix}.
$$

The non-degeneracy of $\mathbf{K}|_{\mathfrak{g}_\mu \times \mathfrak{g}_\mu}$ is obvious, so choosing

$$
\mathfrak{h} = \mathfrak{g}_\mu^\perp = \operatorname{span}\{X_5,\, X_6,\, X_7,\, X_8,\, X_9,\, X_{10}\},
$$

Theorem 57 assures that on the submanifold

$$
\mathcal{N} = \mu + \mathfrak{h}^\circ = \{(1 + y_1, y_2, y_3, 1 + y_4, 0, \ldots, 0) : y_1, \ldots, y_4 \in \mathbb{R}\},
$$

the transverse Poisson structure is linear. Indeed, the computations in Appendix D produce:

$$
\mathrm{P}_{\mathcal{N}} = \begin{pmatrix}
\cdot & y_2 & -y_3 & \cdot \\
-y_2 & \cdot & y_1 - y_4 & y_2 \\
y_3 & y_4 - y_1 & \cdot & -y_3 \\
\cdot & -y_2 & y_3 & \cdot
\end{pmatrix}.
$$

In the previous example, we could have used Corollary 59 to establish the linearity of the transverse Poisson structure instead of using Theorem 57 directly. The element X corresponding to μ through the Killing form is

$$
X = \frac{1}{12} \begin{pmatrix}
1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & \cdot & -1
\end{pmatrix},
$$

which is a semisimple element of $\mathfrak{sp}(4)$.

We proceed to give an example that illustrates Corollary 60.

Example 64 Let $\mathfrak{g}$ be generated by the vectors $X_1, \dots, X_5$, with the Lie bracket given by:

$$[X_1,\, X_2] = -2X_2 + X_4 + 2X_5 \quad [X_1,\, X_3] = 2X_3 \quad [X_2,\, X_3] = -X_1 + X_4,$$

$$[X_1,\, X_5] = [X_4,\, X_5] = -[X_2,\, X_4] = X_4,$$

all other being zero. The matrix for the Lie-Poisson structure on $\mathfrak{g}^*$ is

$$\mathrm{P} = \begin{pmatrix} \cdot & -2x_2 + x_4 + 2x_5 & 2x_3 & \cdot & x_4 \\ 2x_2 - x_4 - 2x_5 & \cdot & x_4 - x_1 & -x_4 & \cdot \\ -2x_3 & x_1 - x_4 & \cdot & \cdot & \cdot \\ \cdot & x_4 & \cdot & \cdot & x_4 \\ -x_4 & \cdot & \cdot & -x_4 & \cdot \end{pmatrix}.$$

If we take $\mu = (1, 0, 0, 1, 0)$, then

$$\mathfrak{g}_\mu = \operatorname{span}\{X_3,\, X_2 - X_5,\, X_1 - X_4\}.$$

We have

$$[X_3,\, X_2 - X_5] = X_1 - X_4, \quad [X_3,\, X_1 - X_4] = -2X_3$$

and

$$[X_2 - X_5,\, X_1 - X_4] = 2X_2 - 2X_5,$$

so this is $\mathfrak{sl}(2)$, a simple Lie algebra. Hence Corollary 60 guarantees that there is a transverse Poisson structure which is linear. Indeed, choosing

$$\mathfrak{h} = \operatorname{span}\{X_4,\, X_5\},$$

we obtain the transverse Poisson structure given by

$$\mathrm{P} = \begin{pmatrix} \cdot & y_1 & -2y_3 \\ -y_1 & \cdot & 2y_2 \\ -2y_3 & -2y_2 & \cdot \end{pmatrix}.$$

We remark that the $\mathfrak{h}$ above is the orthogonal of $\mathfrak{g}_\mu$ with respect to the Killing form of $\mathfrak{g}$ (see Theorem 57), which is given by

$$\begin{pmatrix} 8 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 4 & \cdot & 1 \\ \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 \end{pmatrix}.$$

3.3 On the Relation between Two Sufficient Conditions for Linearity

We now know two sufficient conditions for the existence of a linear transverse Poisson structure to a coadjoint orbit. The one provided by Molino (Corollary 43) and the conditions of Theorem 57. We start our comparison between the two by noticing that if the conditions of Theorem 57 are satisfied, then Molino condition holds with $\mathfrak{h}$ the $\mathbf{B}$-orthogonal of $\mathfrak{g}_\mu$. The first assertion of the theorem below is then proved. Moreover, there are important classes of Lie algebras where the two conditions are equivalent.

Theorem 65 *Let $\mathfrak{g}$ be a Lie algebra and consider the Lie-Poisson structure on $\mathfrak{g}^*$.*

1. *If there is a point $\mu \in \mathfrak{g}^*$ and a symmetric bilinear form $\mathbf{B}$ satisfying conditions 1 and 2 of Theorem 57, then $\mathfrak{h} = \mathfrak{g}_\mu^\perp$ satisfies Molino condition.*

2. *If an element μ of $\mathfrak{g}^*$ is such that $\mathfrak{g}_\mu$ is semisimple or of compact type, then any supplement $\mathfrak{h}$ satisfying Molino condition may be regarded as $\mathfrak{g}_\mu^\perp$, with respect to an $\mathrm{ad}_{\mathfrak{g}_\mu}$-invariant symmetric bilinear form $\mathbf{B}$ which satisfies the conditions of Theorem 57.*

Proof. To check the second assertion, we will construct a symmetric bilinear form

$$\mathbf{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

that satisfies the conditions of Theorem 57. First we note that, if $\mathfrak{g}_\mu$ is semisimple or of compact type, there is a symmetric bilinear form on $\mathfrak{g}_\mu$,

$$\mathbf{B}_\mu : \mathfrak{g}_\mu \times \mathfrak{g}_\mu \to \mathbb{R}$$

which is also ad-invariant and non-degenerate. If $\mathfrak{g}_\mu$ is semisimple, take $\mathbf{B}_\mu$ to be its Killing form. If $\mathfrak{g}_\mu$ is of compact type take a positive definite ad-invariant symmetric bilinear form. We use the splitting

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{h} \tag{3.7}$$

to define $\mathbf{B}$:

$$\mathbf{B}\,(X,\ Y) = \mathbf{B}_\mu\,(\pi(X),\, \pi(Y))$$

where $\pi : \mathfrak{g} \to \mathfrak{g}_\mu$ is the projection with kernel $\ker(\pi) = \mathfrak{h}$. Note that

$$\mathbf{B}(X,\, Y_2) = 0, \quad \forall\, Y_2 \in \mathfrak{h}. \tag{3.8}$$

Since $\mathbf{B}_\mu$ is non-degenerate, then so is $\mathbf{B}|_{\mathfrak{g}_\mu \times \mathfrak{g}_\mu}$. We will show that $\mathbf{B}$ is also $\mathrm{ad}_{\mathfrak{g}_\mu}$-invariant, i.e.,

$$\mathbf{B}\left(X, [Y, Z]\right) + \mathbf{B}\left([X, Z], Y\right) = 0 \quad \forall\, X,\ Y \in \mathfrak{g}, \quad \forall\, Z \in \mathfrak{g}_\mu.$$

Indeed, with $X_1, Y_1 \in \mathfrak{g}_\mu$, $X_2, Y_2 \in \mathfrak{h}$ coming from decomposition 3.7,

$$\begin{aligned}
\mathbf{B}\left(X, [Y, Z]\right) + \mathbf{B}\left([X, Z], Y\right) &= \mathbf{B}\left(X_1 + X_2, [Y_1 + Y_2, Z]\right) \\
&+ \mathbf{B}\left([X_1 + X_2, Z], Y_1 + Y_2\right) \\
&= \mathbf{B}\left(X_1, [Y_1 + Y_2, Z]\right) + \mathbf{B}\left([X_1 + X_2, Z], Y_1\right) \\
&= \mathbf{B}\left(X_1, [Y_1, Z]\right) + \mathbf{B}\left(X_1, [Y_2, Z]\right) \\
&+ \mathbf{B}\left([X_1, Z], Y_1\right) + \mathbf{B}\left([X_2, Z], Y_2\right) \\
&= \mathbf{B}\left(X_1, [Y_2, Z]\right) + \mathbf{B}\left([X_2, Z], Y_1\right) \\
&= 0
\end{aligned}$$

(we have used 3.8, ad-invariance of $\mathbf{B}_\mu = \mathbf{B}|_{\mathfrak{g}_\mu \times \mathfrak{g}_\mu}$ and Molino condition). ■

The next example will make clear that neither of these two conditions are necessary for linearity of the transverse Poisson structure. We will find a Lie algebra $\mathfrak{g}$ and an element μ of $\mathfrak{g}^*$ such that there is no $\mathfrak{h}$ satisfying Molino condition but still there is an $\mathfrak{h}$ such that the transverse Poisson structure on

$$\mathcal{N} = \mu + \mathfrak{h}^\circ$$

is linear. In addition, this shows that the converse of Damianou's conjecture is not valid. Semisimplicity is not necessary for "polynomiality" of the transverse Poisson structure.

Example 66 Take $\mathfrak{g}$ to be the direct sum of the open book algebra ($\mathbb{R}$ acting on $\mathbb{R}^3$ by the identity matrix) with the 2-dimensional non-abelian Lie algebra A_2. On the natural basis

$$\{T,\ X_1,\ X_2,\ X_3,\ Y_1,\ Y_2\}$$

for $\mathfrak{g}$, the non-zero brackets are

$$[T,\ X_i] = X_i, \quad [Y_1,\ Y_2] = Y_1.$$

As usually, we identify the above basis with linear coordinates $\{t,\ x_1,\ x_2,\ x_3,\ y_1,\ y_2\}$ in $\mathfrak{g}^*$ and we obtain the following Poisson matrix for the Lie-Poisson structure:

$$\mathrm{P} = \begin{pmatrix} \cdot & x_1 & x_2 & x_3 & \cdot & \cdot \\ -x_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -x_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -x_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & y_1 \\ \cdot & \cdot & \cdot & \cdot & -y_1 & \cdot \end{pmatrix}$$

Its determinant is zero, therefore the 4-rank points are of maximum rank. We consider the following point of rank 2 (and hence singular) in $\mathfrak{g}^*$:

$$\mu = (0,\ 1,\ 0,\ 0,\ 0,\ 0).$$

Its isotropy subalgebra $\mathfrak{g}_\mu$ is generated by the vectors

$$X_2,\ X_3,\ Y_1,\ Y_2.$$

For any choice of the supplement $\mathfrak{h}$ of $\mathfrak{g}_\mu$ in $\mathfrak{g}$ one has

$$\mathrm{span}\{X_2,\ X_3\} \subset [\mathfrak{h},\ \mathfrak{g}_\mu]$$

so $\mathfrak{h}$ does not satisfy Molino condition. Let us compute the transverse Poisson structure to $\mathcal{O}_\mu$ for the following choice for the supplement of $\mathfrak{g}_\mu$:

$$\mathfrak{h}_0 = \mathrm{span}\{T, X_1\}.$$

We have

$$\mathfrak{h}_0^\circ = \mathrm{span}\{X^2,\ X^3,\ Y^1,\ Y^2\},$$

so the transverse submanifold is given by

$$\mathcal{N} = \{(0,\ 1,\ z_1,\ z_2,\ z_3, z_4) :\ z_1, \ldots, z_4 \in \mathbb{R}\}.$$

The usual computations produce the following Poisson matrix for the transverse Poisson structure:

$$\mathrm{P}_{\mathcal{N}} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & z_3 \\ \cdot & \cdot & -z_3 & \cdot \end{pmatrix}$$

This Poisson structure is linear in the affine z-coordinates for $\mathcal{N}_{\mathfrak{h}_0}$. We therefore conclude that Molino condition is not necessary for linearity (and consequently, neither are the conditions of Theorem 57). Moreover, this is also an example of a linear transverse Poisson structure to a coadjoint orbit in a Lie algebra which is not semisimple.

We have said above that the condition of Theorem 57 is stronger than Molino condition, because Molino condition comes as its direct consequence for all Lie algebras. Nevertheless, for all we know until now, the two conditions could even be equivalent. The following example will clarify this point, showing that this is not the case.

Example 67 Consider the Lie algebra

$$\mathfrak{g} = \mathrm{A}_2 \oplus \mathrm{A}_2.$$

We choose the basis $\{X,\ Y\}$ of A_2 such that

$$[X,\ Y] = X.$$

As before, we consider $\{x_1,\ y_1,\ x_2,\ y_2\}$, "natural" linear coordinates in $\mathfrak{g}^*$. The Lie-Poisson structure on $\mathfrak{g}^*$ is given by the following block-diagonal matrix:

$$\mathrm{P} = \begin{pmatrix} \begin{pmatrix} \cdot & x_1 \\ -x_1 & \cdot \end{pmatrix} & \\ & \begin{pmatrix} \cdot & x_2 \\ -x_2 & \cdot \end{pmatrix} \end{pmatrix}$$

Choose a singular point in $\mathfrak{g}^*$, for example one such that $x_1 = 0$ and $x_2 \neq 0$. Then its isotropy subalgebra is

$$\mathfrak{g}_\mu = \mathrm{span}\{X_1,\ Y_1\}.$$

Taking

$$\mathfrak{h} = \mathrm{span}\{X_2,\ Y_2\} = \mathrm{A}_2,$$

we have a supplement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$ such that Molino condition is satisfied, and thus there is a transverse Poisson structure to $\mathcal{O}_\mu$ which is linear. Nevertheless, Theorem 57 is not applicable, because there is no invariant symmetric bilinear form on A_2 which is also non-degenerate.

Chapter 4

A Necessary Condition for Linearity of the Transverse Poisson Structure

In the previous chapter, we were able to use the two sufficient conditions at our disposal to establish the linearity of the transverse Poisson structure to the coadjoint orbits in several classes of Lie-Poisson manifolds. However, it would be desirable to assess the existence of a linear transverse Poisson structure to a coadjoint orbit without having to exhibit it. If such a transverse submanifold is not found, one remains in doubt. Is it or is it not possible to find another supplement $\mathfrak{h}$ such that the Poisson structure on $\mu + \mathfrak{h}^\circ$ is linear?

Moreover, at this point we still haven't clarified the situation in $\mathfrak{se}(3)^*$(see Example 52). That is why we will now turn our attention to find a necessary condition for linearity of the transverse Poisson structure to a coadjoint orbit. Later on, this necessary condition will be used to rule out linearity in specific cases.

We begin by introducing the notion of *linear approximation* to a Poisson structure at a zero rank point.

4.1 The Linear Approximation at a Zero Rank Point

In this section we follow Weinstein's approach in [18].

Let $(\mathcal{M}, \mathcal{P})$ be a Poisson manifold. Consider the following subsets of

$C^\infty(\mathcal{M})$:

$$\begin{aligned}
\mathbf{m}_{x_0} &= \{f \in C^\infty(\mathcal{M});\ f(x_0) = 0\},\\
\mathbf{m}^2_{x_0} &= \{f \in C^\infty(\mathcal{M});\ f(x_0) = 0,\ \mathrm{d}f(x_0) = 0\}.
\end{aligned}$$

Lemma 68 *Let x_0 be a zero rank point in $\mathcal{M}$. Then $\mathbf{m}_{x_0}$ and $\mathbf{m}^2_{x_0}$ are subalgebras of $(C^\infty(\mathcal{M});\ \{,\})$. In addition, $\mathbf{m}^2_{x_0}$ is an ideal.*

Proof. We want to prove that

$$\{\mathbf{m}_{x_0},\ \mathbf{m}_{x_0}\} \subseteq \mathbf{m}_{x_0}$$

and that

$$\{\mathbf{m}^2_{x_0},\ C^\infty(\mathcal{M})\} \subseteq \mathbf{m}^2_{x_0}.$$

Because x_0 is a zero rank point, we have $\{f,\ g\}(x_0) = 0$ for any f and g, i.e.,

$$\{C^\infty(M),\ C^\infty(M)\} \subset \mathbf{m}_{x_0}.$$

In particular, $\mathbf{m}_{x_0}$ is a subalgebra in $C^\infty(\mathcal{M})$. Now we will prove that $\mathbf{m}^2_{x_0}$ is an ideal in $C^\infty(\mathcal{M})$. Let f in $\mathbf{m}^2_{x_0}$ and g in $C^\infty(\mathcal{M})$. Then

$$\mathrm{d}f_{x_0} = 0 \Rightarrow \frac{\partial f}{\partial x_i}(x_0) = 0, \quad \forall i \tag{4.1}$$

Now we compute $\mathrm{d}\left(\{f,\ g\}\right)_{x_0}$ to check that $\{f,\ g\} \in \mathbf{m}^2_{x_0}$:

$$\begin{aligned}
\mathrm{d}\left(\{f,\ g\}\right)_{x_0} &= \mathrm{d}\left[\sum_{i<j}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)\mathcal{P}_{ij}\right]_{x_0}\\
&= \sum_{i<j}\mathrm{d}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)_{x_0}\mathcal{P}_{ij}(x_0)\\
&+ \sum_{i<j}\left(\frac{\partial f}{\partial x_i}(x_0)\frac{\partial g}{\partial x_j}(x_0) - \frac{\partial f}{\partial x_j}(x_0)\frac{\partial g}{\partial x_i}(x_0)\right)\mathrm{d}\left(\mathcal{P}_{ij}\right)_{x_0}\\
&= 0.
\end{aligned}$$

In the last identity we used condition 4.1 and the fact that $\mathcal{P}(x_0) = 0$. We conclude that $\{f,\ g\}$ belongs to $\mathbf{m}^2_{x_0}$, and consequently $\mathbf{m}^2_{x_0}$ is an ideal in $C^\infty(\mathcal{M})$. ■

Theorem 69 (Weinstein [18]) *Let x_0 be a zero rank point in $(\mathcal{M},\, \mathcal{P})$. Then the tangent space $\mathrm{T}_{x_0}\mathcal{M}$ inherits a linear Poisson structure from $(\mathcal{M},\, \mathcal{P})$, denoted $(\mathrm{T}_{x_0}\mathcal{M},\, \mathcal{P}^0)$. In local coordinates, that structure is the first order Taylor expansion of the original Poisson structure (in M) at x_0.*

Proof. Consider the isomorphism

$$\begin{array}{rccc} \mathrm{I}: & \mathrm{T}^*_{x_0}\mathcal{M} & \longrightarrow & \mathbf{m}_{x_0}/\mathbf{m}^2_{x_0} \\ & \alpha & \mapsto & [f] \end{array},$$

where f is any function such that $f(x_0) = 0$, $df_{x_0} = \alpha$.

We will define a Lie algebra structure in $\mathbf{m}_{x_0}/\mathbf{m}^2_{x_0}$, starting with the Poisson bracket on $\mathbf{m}_{x_0}\,(\subset C^\infty(\mathcal{M}))$:

$$\{[f],\, [g]\}' \overset{def}{=} [\{f,\, g\}]$$

We must verify that this bracket is well-defined. By hypothesis,

$$\begin{array}{rcl} f \sim f' & \Leftrightarrow & \mathrm{d}f_{x_0} = \mathrm{d}f'_{x_0}, \\ g \sim g' & \Leftrightarrow & \mathrm{d}g_{x_0} = \mathrm{d}g'_{x_0}. \end{array}$$

and, on the other hand,

$$\begin{aligned} \mathrm{d}\left(\{f,\, g\}\right)(x_0) &= \mathrm{d}\left[\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\mathcal{P}_{ij}(x)\right]_{x_0} \\ &= \sum_{i<j}\mathrm{d}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\right)_{x_0}\underbrace{\mathcal{P}_{ij}(x_0)}_{=0} \\ &+ \sum_{i<j}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\mathrm{d}(\mathcal{P}_{ij})_{x_0} \\ &= \sum_{i<j}\left(\frac{\partial f'}{\partial x_i}\frac{\partial g'}{\partial x_j} - \frac{\partial g'}{\partial x_i}\frac{\partial f'}{\partial x_j}\right)\mathrm{d}(\mathcal{P}_{ij})_{x_0} \\ &= \mathrm{d}\left(\{f',\, g'\}\right)(x_0). \end{aligned} \tag{4.2}$$

Hence,

$$[\{f,\, g\}] = [\{f',\, g'\}]\,,$$

and this bracket on $\mathbf{m}_{x_0}/\mathbf{m}^2_{x_0}$ is well defined. Skew-symmetry follows from the skew-symmetry of the original Poisson structure. As for the Jacobi identity, we observe that

$$\{\{[f],\, [g]\}',\, [h]\}' = \{[\{f,\, g\}],\, [h]\}' = [\{\{f,\, g\}\}',\, h\}]$$

and it also follows from the properties of the Poisson structure on $\mathcal{M}$.

We have thus defined a Lie algebra structure in $\mathbf{m}_{x_0} \big/ \mathbf{m}_{x_0}^2$ given by

$$\{[f],\, [g]\}' = [\{f,\, g\}].$$

The isomorphism I allows us to define a Lie algebra structure in $\mathrm{T}^*_{x_0}(\mathcal{M})$, given by

$$[\alpha,\, \beta] = \mathrm{I}^{-1}\left(\{\mathrm{I}(\alpha), \mathrm{I}(\beta)\}'\right).$$

Consequently, there is a Lie-Poisson structure on

$$(\mathrm{T}^*_{x_0}\mathcal{M})^* \cong \mathrm{T}_{x_0}\mathcal{M}, \tag{4.3}$$

which we will call $\mathcal{P}^0$.

Now we take $x_1, \ldots, x_n$ local coordinates in the manifold $\mathcal{M}$, with

$$x_i(x_0) = 0 \quad \forall i,$$

and $X_1, \ldots, X_n$, (linear) coordinates in $\mathrm{T}_{x_0}\mathcal{M}$, given by

$$X_i = (\mathrm{d}x_i)_{x_0}.$$

Having identification 4.3 in mind, we conclude that

$$\begin{aligned}
\mathcal{P}^0_{ij} &= \{X_i,\, X_j\}^0 \\
&= \mathrm{I}^{-1}\left(\{\mathrm{I}(X_i),\, \mathrm{I}(X_j)\}'\right) \\
&= \mathrm{I}^{-1}\left(\left\{\mathrm{I}\left(\mathrm{d}x_i\right)_{x_0}, \mathrm{I}\left(\mathrm{d}x_j\right)_{x_0}\right\}'\right) \\
&= \mathrm{I}^{-1}\left(\{[x_i],\, [x_j]\}'\right) \\
&= \mathrm{I}^{-1}\left([\{x_i,\, x_j\}]\right) \\
&= \mathrm{I}^{-1}\left([\mathcal{P}_{ij}(x)]\right)
\end{aligned}$$

A possible representative for the class $[\mathcal{P}_{ij}(x)]$ is

$$f_{ij} = \underbrace{\mathcal{P}_{ij}(x_0)}_{=0} + \sum_{k=1}^{m} \frac{\partial \mathcal{P}_{ij}}{\partial x_k}(x_0) x_k,$$

the linear component of the Taylor series of $\mathcal{P}_{ij}(x)$. Indeed, we have

$$\begin{aligned}
\mathrm{d}f_{x_0} &= c^1_{ij}(\mathrm{d}x_1)_{x_0} + \ldots + c^m_{ij}(\mathrm{d}x_m)_{x_0} \\
&= c^1_{ij}X_1 + \ldots + c^m_{ij}X_m
\end{aligned}$$

where

$$c^k_{ij} = \frac{\partial \mathcal{P}_{ij}}{\partial x_k}(x_0).$$

Therefore,

$$\{X_i,\, X_j\}^0 = \sum_{k=1}^{m} c_{ij}^k X_k$$

and the proof is concluded. ■

Definition 70 $(\mathrm{T}_{x_0}\mathcal{M},\, \mathcal{P}^0)$ *is known as* the linear approximation to $(\mathcal{M},\, \mathcal{P})$ at x_0.

Definition 71 Let $(\mathcal{M},\, \mathcal{P})$ be a Poisson manifold and x_0 a zero rank point. $(\mathcal{M},\, \mathcal{P})$ is said to be *linearizable at* x_0 if it is locally Poisson-diffeomorphic to $(\mathrm{T}_{x_0}\mathcal{M},\, \mathcal{P}^0)$ (around x_0).

4.2 A Necessary Condition for Linearity

Now we adress the following problem: Let $(\mathcal{M},\, \mathcal{P})$ be a Poisson manifold and x_0 a zero rank point. Suppose that $\mathcal{P}$ is Poisson-diffeomorphic to a linear Poisson structure $\mathcal{Q}$. What can be said about the linear approximation $\mathcal{P}^0$?

We start with the following Lemma:

Lemma 72 *Let* $(\mathcal{M};\, \{,\})$ *be a Poisson manifold and* x_0 *a zero rank point. Then*

$$\{\mathrm{d}f_{x_0},\, \mathrm{d}g_{x_0}\}^0 = \mathrm{d}\left(\{f,\, g\}\right)_{x_0}, \quad \forall f,\, g \in C^\infty(\mathcal{M}).$$

Proof. As in the previous section, we define $\{X_1, \ldots, X_n\}$ linear coordinates on $\mathrm{T}_{x_0}\mathcal{M}$ such that

$$X_k = (\mathrm{d}x_k)_{x_0}.$$

We have that

$$
\begin{aligned}
(\mathrm{d}\{f,\, g\})_{x_0} &= \sum_k \left.\frac{\partial\{f,\, g\}}{\partial x_k}\right|_{x_0} \cdot X_k \\
&= \sum_k \frac{\partial}{\partial x_k}\left.\left(\sum_{i,j} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\{x_i,\, x_j\}\right)\right|_{x_0} \cdot X_k \\
&= \sum_k \left(\sum_{i,j} \frac{\partial f}{\partial x_i}(x_0)\frac{\partial g}{\partial x_j}(x_0)\, \left.\frac{\partial\{x_i,\, x_j\}}{\partial x_k}\right|_{x_0}\right) \cdot X_k \\
&= \sum_{i,j} \frac{\partial f}{\partial x_i}(x_0)\frac{\partial g}{\partial x_j}(x_0) \sum_k \left.\frac{\partial\{x_i,\, x_j\}}{\partial x_k}\right|_{x_0} \cdot X_k \\
&= \sum_{i,j} \frac{\partial f}{\partial x_i}(x_0)\frac{\partial g}{\partial x_j}(x_0)\, \{X_i,\, X_j\}^0 \\
&= \left\{\sum_i \frac{\partial f}{\partial x_i}(x_0)\cdot X_i,\, \sum_j \frac{\partial g}{\partial x_j}(x_0)\cdot X_j\right\}^0 \\
&= \{\mathrm{d}f_{x_0},\, \mathrm{d}g_{x_0}\}^0 ,
\end{aligned}
$$

concluding the proof. ■

Theorem 73 *Let $(\mathcal{M}_1,\, \mathcal{P}_1)$ and $(\mathcal{M}_2,\, \mathcal{P}_2)$ be Poisson manifolds which are locally Poisson-diffeomorphic, x_0 a zero rank point of $\mathcal{M}_1$ and y_0 the corresponding zero rank point of $\mathcal{M}_2$. Then $(\mathrm{T}_{x_0}\mathcal{M}_1,\, \mathcal{P}_1^0)$ and $(\mathrm{T}_{y_0}\mathcal{M}_2,\, \mathcal{P}_2^0)$ are also Poisson-diffeomorphic.*

Proof. Let $x_1, \ldots, x_n$ be coordinates in $\mathcal{M}_1$ and $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ the diffeomorphism mentioned above. Then ψ induces coordinates $y_1, \ldots, y_n$ in $\mathcal{M}_2$ such that

$$\{y_i \circ \psi,\, y_j \circ \psi\}_1 = \{y_i,\, y_j\}_2 \circ \psi.$$

Now let us define coordinates $X_1, \ldots, X_n$ in $\mathrm{T}_{x_0}\mathcal{M}_1$, coordinates $Y_1, \ldots, Y_n$ in $\mathrm{T}_{y_0}\mathcal{M}_2$ and

$$\Psi : \mathrm{T}_{x_0}\mathcal{M}_1 \to \mathrm{T}_{y_0}\mathcal{M}_2$$

in the following way:

$$X_i = (\mathrm{d}x_i)_{x_0}, \quad Y_i = (\mathrm{d}y_i)_{y_0}, \quad \Psi = (\mathrm{d}\psi)_{x_0}.$$

First we note that

$$
\begin{aligned}
Y_k \circ \Psi &= (\mathrm{d}y_k)_{y_0} \circ (\mathrm{d}\psi)_{x_0} \\
&= \mathrm{d}\,(y_k \circ \psi)_{x_0} \\
&= (\mathrm{d}\psi_k)_{x_0}.
\end{aligned}
$$

Also, Ψ is a linear isomorphism because ψ is a diffeomorphism. Now we prove that Ψ is a Poisson map, which completes the proof:

$$\begin{aligned}
\{Y_i \circ \Psi,\, Y_j \circ \Psi\}_1^0 &= \left\{(\mathrm{d}\psi_i)_{x_0},\, (\mathrm{d}\psi_j)_{x_0}\right\}_1^0 \\
&= \mathrm{d}\left(\{\psi_i,\, \psi_j\}_1\right)_{x_0} && (4.4) \\
&= \mathrm{d}\left(\{y_i,\, y_j\}_2\right)_{y_0} \circ (\mathrm{d}\psi)_{x_0} && (4.5) \\
&= \left\{(\mathrm{d}y_i)_{y_0},\, (\mathrm{d}y_j)_{y_0}\right\}_2^0 \circ (\mathrm{d}\psi)_{x_0} && (4.6) \\
&= \{Y_i,\, Y_j\}_2^0 \circ \Psi.
\end{aligned}$$

We have used Lemma 72 in 4.4 and 4.6. In 4.5 we used the hypothesis of ψ being a Poisson map and the chain rule. ■

Remark 74 Taking in account Theorem 73, we remark that $(\mathcal{M},\, \mathcal{P})$ is linearizable at x_0 if it is locally Poisson-diffeomorphic to some linear Poisson structure. The linear approximation $(\mathrm{T}_{x_0}\mathcal{M},\, \mathcal{P}^0)$ may be taken as a representative of the linear structures which are Poisson-diffeomorphic to $\mathcal{P}$, provided that they exist.

Then we can formulate the following necessary condition for linearity:

If $(\mathcal{M},\, \mathcal{P})$ is locally Poisson-diffeomorphic to a linear Poisson structure, then $(\mathcal{M},\, \mathcal{P})$ must be locally Poisson-diffeomorphic to $(\mathrm{T}_{x_0}\mathcal{M},\, \mathcal{P}^0)$.

4.3 Examples in Transverse Poisson Structures

In particular cases, namely in transverse Poisson structures, we can use the necessary condition for linearity above to infer the existence of a linear transverse Poisson structure to a coadjoint orbit $\mathcal{O}$. We know by Lemma 40 that any two transverse Poisson structures $\mathcal{P}_{\mathcal{N}_1}$ and $\mathcal{P}_{\mathcal{N}}$ to $\mathcal{O}$ are Poisson-diffeomorphic. Suppose $\mathcal{P}_{\mathcal{N}}$ is linear. Then, by the necessary condition above, $\mathcal{P}_{\mathcal{N}_1}$ is Poisson-diffeomorphic to its linear approximation $\mathcal{P}^0_{\mathcal{N}_1}$. As a consequence:

If a transverse Poisson structure to a coadjoint orbit is not Poisson-diffeomorphic to its first order Taylor expansion, then there is no linear transverse Poisson structure to that orbit.

In the following examples we will consider singular points in the dual of some Lie algebras which admit no linear transverse Poisson structure to its coadjoint orbit. In each case, we will use Theorem 73 to prove this.

Example 75 We consider again $\mathfrak{sp}(4)^*$, the dual of the same Lie algebra of example 63, but we study the transverse Poisson structure to the coadjoint orbit through a different point μ :

$$\mu = (0, 1, 0, \ldots, 0).$$

This is a 6-rank point. We note that μ is identified through $\mathbf{K}$ with

$$X = \frac{1}{12} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

which is a nilpotent element of $\mathfrak{sp}(4)$. Therefore, Corollary 59 cannot be used. In fact, none of the sufficient conditions for linearity of Chapter 3 holds at this coadjoint orbit, because the transverse Poisson structure to the coadjoint orbit through μ is non-linearizable. We proceed to prove this fact. We have

$$\mathfrak{g}_\mu = \operatorname{span}\{X_1 + X_4, X_3, X_7, X_8\}$$

and we choose a random $\mathfrak{h}$, for example

$$\mathfrak{h} = \operatorname{span}\{X_1, X_2, X_5, X_6, X_9, X_{10}\},$$

and we compute the transverse Poisson structure. A basis for $\mathfrak{h}^\circ$ is, for example,

$$\{X^3, X^4, X^7, X^8\}$$

and we have

$$\mu + \nu = (0, 1, y_1, y_2, 0, 0, y_3, y_4, 0, 0)\,.$$

Through the usual computations, we obtain the matrix of the transverse Poisson structure:

$$\mathrm{P}_{\mathcal{N}} = \begin{pmatrix} \cdot & \cdot & 2y_3 & -2y_4 \\ \cdot & \cdot & -2y_2y_3 & 2y_4y_2 \\ -2y_3 & 2y_2y_3 & \cdot & -y_2y_1 \\ 2y_4 & -2y_4y_2 & y_2y_1 & \cdot \end{pmatrix}.$$

Now we will prove that this transverse Poisson structure is non-linearizable. The first step is to notice that the set of all its zero rank points is the union of two one-dimensional manifolds:

$$\{(y_1, 0, 0, 0) : \, y_1 \in \mathbb{R}\} \cup \{(0, \, y_2, 0, 0, 0) : \, y_2 \in \mathbb{R}\}$$

However, its linear approximation $\mathcal{P}^0_{\mathcal{N}}$ at μ (i.e., at $y = 0$) has a different behaviour. We have that

$$\mathcal{P}^0_{\mathcal{N}} = \begin{pmatrix} \cdot & \cdot & 2y_3 & -2y_4 \\ \cdot & \cdot & \cdot & \cdot \\ -2y_3 & \cdot & \cdot & \cdot \\ 2y_4 & \cdot & \cdot & \cdot \end{pmatrix}.$$

The set of zero rank points of this Poisson structure is a 2-dimensional hyperplane:

$$\{(y_1, y_2, 0, 0) : y_1 \in \mathbb{R}, y_2 \in \mathbb{R}\}.$$

Therefore, $\mathrm{P}_{\mathcal{N}}$ and $\mathrm{P}^0_{\mathcal{N}}$ cannot be locally Poisson-diffeomorphic and hence $\mathrm{P}_{\mathcal{N}}$ is not Poisson-diffeomorphic to any linear Poisson structure.

We remark that any other choice of $\mathfrak{h}$ would produce a nonlinearizable transverse Poisson structure.

Remark 76 This example also shows that semisimplicity of the Lie algebra is not sufficient to guarantee linearity of the transverse Poisson structure (although it is sufficient to guarantee "polynomiality", as proved by Cushman & Roberts [3]).

Example 77 In order to give the counter-example referred to in Remark 61, we have constructed a Lie algebra $\mathfrak{g}$ and chosen a point $\mu \in \mathfrak{g}^*$ with a compact-type isotropy subalgebra. Let $\mathfrak{g}$ be the real 4-dimensional Lie algebra with basis $\{T_1, T_2, X_1, X_2\}$ and brackets given by:

$$\begin{aligned} [T_1, T_2] = 0, \quad [T_1, X_1] = T_2, \quad & [T_1, X_2] = kX_1, \\ [T_2, X_1] = 0, \quad & [T_2, X_2] = T_2, \\ & [X_1, X_2] = T_1 + X_1, \end{aligned}$$

where k is an arbitrary real number. Then, the Poisson matrix for the Lie-Poisson structure on $\mathfrak{g}^*$ is given by (dots stand for zeros)

$$\mathrm{P} = \begin{pmatrix} \cdot & \cdot & t_2 & kx_1 \\ \cdot & \cdot & \cdot & t_2 \\ -t_2 & \cdot & \cdot & t_1 + x_1 \\ -kx_1 & -t_2 & -t_1 - x_1 & \cdot \end{pmatrix}$$

We take $\mu = (1, 0, 0, 1) \in \mathfrak{g}^*$. Then:

$$\mathfrak{g}_\mu = \text{span}\{T_1, T_2\},$$

which is obviously of compact type (for example, this $\mathfrak{g}_\mu$ is the Lie algebra of the 2-torus).

Now take the following supplement of $\mathfrak{g}_\mu$:

$$\mathfrak{h} = \text{span}\{X_1, X_2\}.$$

Then

$$\mathcal{N} = \mu + \mathfrak{h}^\circ = \{(1 + y_1, y_2, 0, 1) : y_1, y_2 \in \mathbb{R}\}$$

and the usual computations produce:

$$\mathrm{P}_{\mathcal{N}} = \begin{pmatrix} 0 & -\frac{y_2^2}{1+y_1} \\ \frac{y_2^2}{1+y_1} & 0 \end{pmatrix}$$

Again there are obstructions to linearizability, since

$$\mathrm{P}^0_{\mathcal{N}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 78 Again we consider the Lie algebra

$$\mathfrak{g} = \mathfrak{se}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3,$$

described in example 52. In that example, we have computed the transverse Poisson structure to the coadjoint orbit through points of rank 2 of the Poisson structure. Such points have the form

$$\mu = (a,\, b,\, c,\, 0,\, 0,\, 0),$$

with $a^2 + b^2 + c^2 \neq 0$. The transverse Poisson structure, as computed in Example 52, is given by:

$$\frac{1}{c + y_1} \begin{pmatrix} . & -c(by_4 - cy_3 - y_3 y_1) & c(ay_4 - cy_2 - y_2 y_1) & c(by_2 - ay_3) \\ * & . & -y_4^2 & y_3 y_4 \\ * & * & . & -y_2 y_4 \\ * & * & * & . \end{pmatrix},$$

clearly non-linear. But until now, doubt remains. Is it linear for a different choice of $\mathfrak{h}$? To give an answer, we take a closer look at its linear approximation at μ :

$$\mathcal{P}_{\mathcal{N}}^{0} = -c \begin{pmatrix} . & by_4 - cy_3 & -ay_4 + cy_2 & ay_3 - by_2 \\ -by_4 + cy_3 & . & . & . \\ ay_4 - cy_2 & . & . & . \\ -ay_3 + by_2 & . & . & . \end{pmatrix}.$$

The set of zero rank points in $\mathcal{P}_{\mathcal{N}}$ is then the 1-dimensional submanifold

$$\{(y_1, 0, 0, 0) : y_1 \in \mathbb{R}\}.$$

On the other hand, the set of zero rank points in $\mathcal{P}_{\mathcal{N}}^{0}$ is a 2-dimensional submanifold:

$$\{(y_1, ay_2, by_2, cy_2) : y_1, y_2 \in \mathbb{R}\}.$$

Therefore, $\mathcal{P}_{\mathcal{N}}$ is not Poisson-diffeomorphic to its linear approximation and hence is not linearizable.

Chapter 5

On the Polynomiality of the Transverse Poisson Structure

Theorem 73 allowed us to conclude that no linear transverse Poisson structures could exist, in the examples of the last section of the previous chapter. At this point, a logical step would be to investigate if, in such cases, there were obstructions to the existence of quadratic, or even polynomial of higher degree transverse Poisson structures.

Unfortunately, it is not possible to obtain an analogous of Theorem 73 for quadratic Poisson structures using Taylor approximations of degree two. This is due to the fact that, in general, the quadratic Taylor approximation of a Poisson tensor $\mathcal{P}$ is not a Poisson tensor. We recall that $\mathcal{P}$ is Poisson if and only if

$$[\mathcal{P}, \mathcal{P}]_S = 0.$$

Using coordinates (see for example [16]), it can be proved that only the first non vanishing term of the Taylor series of $\mathcal{P}$ is guaranteed to satisfy this requirement (and also the last non vanishing term, if $\mathcal{P}$ is polynomial).

Therefore, a possible generalization of Theorem 73 could only produce obstructions to the existence of homogeneous polynomial Poisson structures of the particular degree of the referred first non vanishing term of the Taylor series of $\mathcal{P}$. In spite of this difficulty, we were able to draw conclusions in the case of $\mathfrak{se}(3)^*$.

5.1 Obstructions to Polynomiality in $\mathfrak{se}(3)^*$

Example 79 Consider again the Lie-Poisson structure on $\mathfrak{se}(3)^*$. In this specific Lie algebra, one can find obstructions to the polynomiality of the transverse Poisson structure. We will show that, on any transverse submanifold to a coadjoint orbit, a polynomial transverse Poisson structure would have to be linear and hence (see Example 78), does not exist.

In order to do this we will parametrize all possible supplements to $\mathfrak{g}_\mu$ and compute explicitly the transverse Poisson structure on an arbitrary transverse submanifold. We will then analyse the possible degree of an eventual polynomial structure.

We start with a change of basis that makes computations less cumbersome. Our new basis differs from the one we used in Examples 52 and 78 by replacing X_3 by $aX_1 + bX_2 + cX_3$ (we recall that $c \neq 0$ so that indeed this defines a new basis for $\mathfrak{se}(3)$). In this new basis, the Lie-Poisson structure for $\mathfrak{se}(3)^*$ is given by the skew-symmetric matrix with lower-triangular part as follows:

$$\begin{pmatrix} . & * & * & * & * & * \\ \frac{ax_1+bx_2-x_3}{c} & . & * & * & * & * \\ \frac{abx_1+(b^2+c^2)x_2-bx_3}{c} & \frac{ax_3-(a^2+c^2)x_1-abx_2}{c} & . & * & * & * \\ . & x_6 & bx_6-cx_5 & . & * & * \\ -x_6 & . & cx_4-ax_6 & . & . & * \\ x_5 & -x_4 & cx_5-bx_4 & . & . & . \end{pmatrix}.$$

The arbitrary singular point μ now writes as

$$\mu = (a, b, a^2 + b^2 + c^2, 0, 0, 0).$$

The advantage of the new basis is that now we have

$$\mathfrak{g}_\mu = \operatorname{span}\{X_3, X_4, X_5, X_6\}$$

and standard linear algebra can be used to show that any supplement $\mathfrak{h}$ to $\mathfrak{g}_\mu$ has the form:

$$\mathfrak{h}_{A,B} = \operatorname{span}\{X_1 + A_2X_2 + \ldots + A_6X_6, X_2 + B_3X_3 + \ldots + B_6X_6\},$$

for some real numbers A_i, $i = 2, \ldots, 6$ and B_j, $j = 3, \ldots, 6$.

Proceeding with the computations, we arrived at the transverse Poisson structure on $\mathcal{N}_{A,B} = \mu + \mathfrak{h}^\circ_{A,B}$, from which we highlight the following facts:

- Each entry (i, j) in the transverse Poisson structure is of the form:

$$\frac{Q_{i,j}}{D},$$

  where $Q_{i,j}$ is an homogeneous quadratic polynomial in the y-variables and D is the common denominator to all entries;

- D is of the form:

$$D = c^2 + \lambda_1 y_1 + \ldots + \lambda_4 y_4,$$

  where each λ_i depends on a complicated way on a, b, c and on the parameters A_i and B_j;

- the expression for λ_1 can be put in the form:

$$\lambda_1 = (1 + B_3 b + A_3 a - A_2 B_3 a)^2 + (A_3 c - A_2 B_3 c)^2 + B_3^2 b^2;$$

From these facts we conclude that:

1. any nonzero entry Q_{ij}/D is polynomial (of degree $n_{i,j}$) if and only if:

$$\lambda_1 = \ldots = \lambda_4 = 0 \ \vee \ n_{i,j} = 1;$$

2. for any A_i, B_j and μ_k we have $\lambda_1 \neq 0$ (recall that $c \neq 0$).

Summing up, the transverse Poisson structure on any affine transverse manifold $\mathcal{N}_{A,B}$ is polynomial if and only if it is trivial or of degree 1, but Example 78 rules out both possibilities. The transverse Poisson structure on $\mathcal{N}_{A,B}$ cannot be linear and also is not trivial (because its linear approximation is not trivial). We can therefore conclude that there is no polynomial transverse Poisson structure to any singular coadjoint orbit of $\mathfrak{se}(3)^*$.

5.2 Polynomializable Poisson Structures

We now know that there are no polynomial transverse Poisson structures in $\mathfrak{se}(3)^*$ apart from the trivial ones. However, the conclusions of Example 79 leave open the possibility of "polynomializing" any (non trivial) transverse Poisson structure to a coadjoint orbit of $\mathfrak{se}(3)^*$ through a non-affine Poisson diffeomorphism.

In fact, we confirm this in the next example. The referee of [2] mentioned in his report that, although he had not proved it, he felt that the transverse Poisson structure computed in Example 52 was diffeomorphic to a specific polynomial Poisson structure. We were not able to confirm nor deny his conjecture, but our efforts to prove it have lead us to a polynomial (although of higher degree) Poisson structure.

Example 80 Again we take the Lie-Poisson structure on $\mathfrak{se}(3)^*$. For simplicity, we consider

$$\mu = (0, 0, 1, 0, 0, 0),$$

in the same basis as in examples 52 and 78. We will find a polynomial Poisson structure which is diffeomorphic to

$$\mathrm{P}_{\mathcal{N}} = \begin{pmatrix} \cdot & y_3 & -y_2 & \cdot \\ * & \cdot & \frac{-y_4^2}{1+y_1} & \frac{y_3 y_4}{1+y_1} \\ * & * & \cdot & \frac{-y_2 y_4}{1+y_1} \\ * & * & * & \cdot \end{pmatrix},$$

the transverse Poisson structure to the coadjoint orbit through μ, where

$$\mathcal{N} = \{(0, 0, 1 + y_1, y_2, y_3, y_4) : \quad y_i \in \mathbb{R}\}.$$

According to the referee of [2], the Poisson structure given by

$$\mathrm{Q} = \begin{pmatrix} \cdot & z_3 & -z_2 & \cdot \\ * & \cdot & -z_4^2 & \cdot \\ * & * & \cdot & \cdot \\ * & * & * & \cdot \end{pmatrix},$$

is Poisson diffeomorphic to $\mathrm{P}_{\mathcal{N}}$. Since the maximum rank of both structures is two and Poisson diffeomorphisms send symplectic leaves to symplectic leaves, a possible way to check the referee's claim is to take Casimir functions of both Poisson structures and then try to relate them through diffeomorphisms. More precisely, if

$$\psi : (\mathcal{N}, \mathrm{P}_{\mathcal{N}}) \to (\mathcal{M}, \mathrm{Q})$$

is a Poisson diffeomorphism, then

$$c_i = k_i \circ \psi, \quad i \in \{1, 2\}, \tag{5.1}$$

where c_i and k_i are Casimir functions of $\mathrm{P}_{\mathcal{N}}$ and Q, respectively.

First we consider the case of $\mathrm{P}_{\mathcal{N}}$. Since its symplectic leaves are the intersections of $\mathcal{N}$ with the simplectic leaves of the Lie-Poisson structure P on $\mathfrak{se}(3)^*$, two independent Casimirs c_i of $\mathrm{P}_{\mathcal{N}}$ are easily obtained from two independent Casimir functions f_i of P. One can easily check, using the notations of examples 52 and 78, that

$$f_1 = x_1 x_4 + x_2 x_5 + x_3 x_6, \quad f_2 = x_4^2 + x_5^2 + x_6^2$$

are Casimirs of P, from which we obtain

$$c_1 = (1 + y_1) y_4, \quad c_2 = y_2^2 + y_3^2 + y_4^2.$$

On the other hand,

$$k_1 = z_4, \quad k_2 = z_2^2 + z_3^2 - 2z_1z_4^2$$

are two independent Casimirs of Q. Now suppose ψ is a diffeomorphism which satisfies conditions 5.1 above, for example

$$\begin{array}{rccl} \psi: & \mathcal{N} & \to & \mathcal{M} \\ & (y_1, y_2, y_3, y_4) & \mapsto & \left(-\frac{1}{2}(1+y_1)^{-2}, y_2, y_3, y_4(1+y_1)\right). \end{array}$$

It turns out that this diffeomorphism does not transform $\mathrm{P}_{\mathcal{N}}$ into Q. Moreover, it doesn't even polynomialize the transverse Poisson structure. Instead, we obtain

$$\mathrm{R} = \psi_*\mathrm{P}_{\mathcal{N}} = \begin{pmatrix} \cdot & (2z_1)^{\frac{3}{2}} z_3 & -(2z_1)^{\frac{3}{2}} z_2 & \cdot \\ * & \cdot & -(2z_1)^{\frac{3}{2}} z_4^2 & \cdot \\ * & * & \cdot & \\ * & * & * & \cdot \end{pmatrix}.$$

However, this Poisson structure is easily polynomializable. It suffices to consider

$$\begin{array}{rccl} \psi': & \mathcal{M} & \to & \mathcal{M}' \\ & (z_1, z_2, z_3, z_4) & \mapsto & (\sqrt{z_1}, z_2, z_3, z_4), \end{array}$$

obtaining

$$\mathrm{R}' = \psi'_*\mathrm{R} = \begin{pmatrix} \cdot & \sqrt{2}w_1^2w_3 & -\sqrt{2}w_1^2w_2 & \cdot \\ * & \cdot & -2\sqrt{2}w_1^3w_4^2 & \cdot \\ * & * & \cdot & \cdot \\ * & * & * & \cdot \end{pmatrix}.$$

We have hence found a transverse Poisson structure to a coadjoint orbit of $\mathfrak{se}(3)^*$ which is not linear nor linearizable (by Example 78) but, nevertheless, is polynomializable of degree 5. We were able to lower the degree to 4 by using the following diffeomorphism instead of $\psi' \circ \psi$:

$$\begin{array}{rccl} \varphi: & \mathcal{N} & \to & \mathcal{M} \\ & (y_1, y_2, y_3, y_4) & \mapsto & \left(\frac{1}{2}(1+y_1)^{-1}, y_2, y_3, y_4\sqrt{1+y_1}\right). \end{array}$$

The resulting Poisson structure is

$$\mathrm{T} = \varphi_*\mathrm{P}_{\mathcal{N}} = \begin{pmatrix} \cdot & -2z_1^2z_3 & 2z_1^2z_2 & \cdot \\ * & \cdot & 4z_1^2z_4^2 & z_1z_3z_4 \\ * & * & \cdot & -z_1z_2z_4 \\ * & * & * & \cdot \end{pmatrix}.$$

We have then showed that:

1. there is no polynomial transverse Poisson Structure to any coadjoint orbit of $\mathfrak{se}(3)^*$;

2. nevertheless, any transverse Poisson Structure to the coadjoint orbit of $\mu = (0, 0, 1, 0, 0, 0)$ is locally polynomializable.

This shows the difference between the concepts of "polynomial" and "polynomializable".

Unfortunately we could not succeed in finding a similar example to show that there is a difference between "linear" and "linearizable", i.e., an example such that:

1. there is no linear transverse Poisson Structure to a certain coadjoint orbit;

2. there is a linearizable transverse Poisson Structure to the same coadjoint orbit.

Conclusion

All in all, we were able to accomplish many of the objectives we had set ourselves. We proved a sufficient condition for linearity of the transverse Poisson structure to a coadjoint orbit which is, in a certain sense, complementary to the condition obtained by Molino in 1984. Our condition is equivalent to Molino's in important classes of Lie-Poisson manifolds, and in general it is stronger. Therefore, it applies to a stricter set of Lie-Poisson manifolds than Molino condition, but in spite of that it proved to have straightforward applications in a variety of cases, where the validity of the condition by Molino was not at all evident. In addition, we used our own formula for computing examples of transverse Poisson structures which illustrate the applications referred above.

We were also able to clarify the situation in $\mathfrak{se}(3)^*$. We proved that, in the $\mathfrak{se}(3)$ case:

1. there are no linearizable transverse Poisson structures to its orbits (and therefore there are no linear transverse Poisson structures);
2. there are also no polynomial transverse Poisson structures to its coadjoint orbits;
3. nevertheless, there are coadjoint orbits whose transverse Poisson structures are Poisson-diffeomorphic to polynomial Poisson structures.

Although 2 and 3 are particular results (meaning that they may only apply to $\mathfrak{se}(3)^*$), 1 involves a necessary condition for linearity which can be checked for any Lie-Poisson manifold.

The defiance of finding a geometric characterization of linear transverse Poisson structures or, more precisely, a condition for linearity which was both sufficient and necessary, was not achieved. That would be our ultimate goal regarding linearity on the transverse Poisson structure, but we were always aware that it was a very difficult task. In our opinion, it is improbable that there is such a condition to be found.

Appendix A

Maple file for Example 51 - $\mathfrak{so}(4)^*$

```
> restart: with(LinearAlgebra):
```

> #Dimension of the Lie algebra g;

```
> dimg := 6;
```

$$dimg := 6 \quad (1)$$

> #The Lie-Poisson matrix is the following:

```
> Poisson:=Matrix([[0,-x[4],-x[5],x[2],x[3],0], [x[4],0,-x[6],-x[1],0,x[3]], [x[5],x[6],0,0,-x[1],-x[2]], [-x[2],x[1],0,0,-x[6],x[5]], [-x[3],0,x[1],x[6],0,-x[4]], [0,-x[3],x[2],-x[5],x[4],0]]); Determinant(Poisson);
```

$$Poisson := \begin{bmatrix} 0 & -x_4 & -x_5 & x_2 & x_3 & 0 \\ x_4 & 0 & -x_6 & -x_1 & 0 & x_3 \\ x_5 & x_6 & 0 & 0 & -x_1 & -x_2 \\ -x_2 & x_1 & 0 & 0 & -x_6 & x_5 \\ -x_3 & 0 & x_1 & x_6 & 0 & -x_4 \\ 0 & -x_3 & x_2 & -x_5 & x_4 & 0 \end{bmatrix}$$

$$0$$

```
> mu:=<a,b,c,-c,b,-a>;
```

$$\mu := \begin{bmatrix} a \\ b \\ c \\ -c \\ b \\ -a \end{bmatrix}$$

> #Dimension and codimension of the isotropy subalgebra g_mu (which is also the dimension of the transverse Poisson structure);

```
> dimtrans := 4; codimtrans := 2;
```

$$dimtrans := 4$$

$$codimtrans := 2 \quad (2)$$

> #Computing **g_mu (= ker P(mu)):**

```
> Poisson_mu:=Poisson: for i from 1 to dimg do Poisson_mu:= subs(x[i]=mu[i],Poisson_mu) end do: Poisson_mu;
```

$$\begin{bmatrix} 0 & c & -b & b & c & 0 \\ -c & 0 & a & -a & 0 & c \\ b & -a & 0 & 0 & -a & -b \\ -b & a & 0 & 0 & a & b \\ -c & 0 & a & -a & 0 & c \\ 0 & -c & b & -b & -c & 0 \end{bmatrix} \quad (3)$$

```
> NullSpace(Poisson_mu);
```

$$\left\{ \begin{bmatrix} \frac{a}{c} \\ \frac{b}{c} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{a}{c} \\ -\frac{b}{c} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{4}$$

> #MAPLE generates a random basis for **g_mu** each time the worksheet is computed. For coherence, we choose our own set of generators:

```
> F[1] := ⟨1, 0, 0, 0, 0, 1⟩ : F[2] := ⟨0, 1, 0, 0, -1, 0⟩ : F[3] := ⟨0, 0, 1,
    1, 0, 0⟩ : F[4] := ⟨0, 0, 0, c, -b, a⟩ :
```

Basis for ***h***:

```
> G[1]:=<1,0,0,0,0,0>: G[2]:=<0,1,0,0,0,0>:
```

Basis for ***h***°:

```
> H[1]:=<0,0,1,0,0,0>: H[2]:=<0,0,0,1,0,0>: H[3]:=<0,0,0,0,1,
  0>: H[4]:=<0,0,0,0,0,1>:
```

Consider ***nu*** arbitrary element of ***h***°:

```
> for i from 1 to dimtrans do k[i]:=VectorScalarMultiply(H[i],y
  [i]) end do: nu:=sum(k[j],j=1..dimtrans); munu:=VectorAdd(mu,
  nu);
```

$$\nu := \begin{bmatrix} 0 \\ 0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$munu := \begin{bmatrix} a \\ b \\ c + y_1 \\ -c + y_2 \\ b + y_3 \\ -a + y_4 \end{bmatrix}$$

```
> Poisson_munu:=Poisson: for i from 1 to dimg do Poisson_munu:=
  subs(x[i]=munu[i],Poisson_munu) end do: Poisson_munu;
```

$$\begin{bmatrix} 0 & c-y_2 & -b-y_3 & b & c+y_1 & 0 \\ -c+y_2 & 0 & a-y_4 & -a & 0 & c+y_1 \\ b+y_3 & -a+y_4 & 0 & 0 & -a & -b \\ -b & a & 0 & 0 & a-y_4 & b+y_3 \\ -c-y_1 & 0 & a & -a+y_4 & 0 & c-y_2 \\ 0 & -c-y_1 & b & -b-y_3 & -c+y_2 & 0 \end{bmatrix}$$

Computation of ad*_G1(mu + nu) and ad*_G2(mu + nu):

```
> for i from 1 to dimg do for j from 1 to codimtrans do k[j,i]
  := ScalarMultiply(Row(Poisson_munu,i),G[j][i]) end do end do:
  for j from 1 to codimtrans do W[j]:= eval(sum(k[j,l],l=1..
  dimg)) end do;
```

$$W_1 := \begin{bmatrix} 0 & c-y_2 & -b-y_3 & b & c+y_1 & 0 \end{bmatrix}$$

$$W_2 := \begin{bmatrix} -c+y_2 & 0 & a-y_4 & -a & 0 & c+y_1 \end{bmatrix}$$

> Computation of Pi(ad*_Fi(nu)):

```
> M:= Matrix([H[1], H[2], H[3], H[4], Transpose(W[1]),
  Transpose(W[2])]); Mi:=MatrixInverse(M):
```

$$M := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -c+y_2 \\ 0 & 0 & 0 & 0 & c-y_2 & 0 \\ 1 & 0 & 0 & 0 & -b-y_3 & a-y_4 \\ 0 & 1 & 0 & 0 & b & -a \\ 0 & 0 & 1 & 0 & c+y_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & c+y_1 \end{bmatrix}$$

Computing Z[1]:=ad*_F1(nu), Z[2]:=ad*_F2(nu), Z[3]:=ad*_F3(nu), Z[4]:=ad*_F4(nu):

```
> Poisson_nu:=Poisson: for i from 1 to dimg do Poisson_nu:=
  subs(x[i]=nu[i],Poisson_nu) end do: Poisson_nu;
```

$$\begin{bmatrix} 0 & -y_2 & -y_3 & 0 & y_1 & 0 \\ y_2 & 0 & -y_4 & 0 & 0 & y_1 \\ y_3 & y_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y_4 & y_3 \\ -y_1 & 0 & 0 & y_4 & 0 & -y_2 \\ 0 & -y_1 & 0 & -y_3 & y_2 & 0 \end{bmatrix} \tag{5}$$

```
> for i from 1 to dimg do for j from 1 to dimtrans do k[j,i]:=
  ScalarMultiply(Row(Poisson_nu,i), F[j][i]) end do end do: for
  j from 1 to dimtrans do Z[j]:= eval(sum(k[j,l], l=1..dimg))
  end do;
```

$$Z_1 := \begin{bmatrix} 0 & -y_2-y_1 & -y_3 & -y_3 & y_1+y_2 & 0 \end{bmatrix}$$

$$Z_2 := \left[\begin{array}{cccccc} y_1 + y_2 & 0 & -y_4 & -y_4 & 0 & y_1 + y_2 \end{array}\right]$$

$$Z_3 := \left[\begin{array}{cccccc} y_3 & y_4 & 0 & 0 & -y_4 & y_3 \end{array}\right]$$

$$Z_4 := \left[\begin{array}{cccccc} b\,y_1 & -a\,y_1 & 0 & -b\,y_4 - a\,y_3 & -c\,y_4 + a\,y_2 & c\,y_3 + b\,y_2 \end{array}\right]$$

Computing the projections of the previous vectors:

```
> for j from 1 to dimtrans do v[j]:=MatrixVectorMultiply(Mi,
  Transpose(Z[j])) end do:
```

```
> for i from 1 to dimtrans do for l from 1 to dimtrans do k[i,
  l]:= ScalarMultiply(H[l],v[i][l]) end do end do: for i from 1
  to dimtrans do Pi_nu[i]:= eval(sum(k[i,o], o=1..dimtrans))
  end do;
```

$$\pi nu_1 := \left[\begin{array}{c} 0 \\ 0 \\ \dfrac{(b + y_3)\,(-y_2 - y_1)}{c - y_2} - y_3 \\ -\dfrac{b\,(-y_2 - y_1)}{c - y_2} - y_3 \\ -\dfrac{(c + y_1)\,(-y_2 - y_1)}{c - y_2} + y_1 + y_2 \\ 0 \end{array}\right]$$

$$\pi nu_2 := \left[\begin{array}{c} 0 \\ 0 \\ \dfrac{(a - y_4)\,(y_1 + y_2)}{c - y_2} - y_4 \\ -\dfrac{a\,(y_1 + y_2)}{c - y_2} - y_4 \\ 0 \\ \dfrac{(c + y_1)\,(y_1 + y_2)}{c - y_2} + y_1 + y_2 \end{array}\right]$$

$$\pi nu_3 := \begin{bmatrix} 0 \\ 0 \\ \frac{(a - y_4)\, y_3}{c - y_2} + \frac{(b + y_3)\, y_4}{c - y_2} \\ -\frac{a\, y_3}{c - y_2} - \frac{b\, y_4}{c - y_2} \\ -\frac{(c + y_1)\, y_4}{c - y_2} - y_4 \\ \frac{(c + y_1)\, y_3}{c - y_2} + y_3 \end{bmatrix}$$

$$\pi nu_4 := \begin{bmatrix} 0 \\ 0 \\ \frac{(a - y_4)\, b\, y_1}{c - y_2} - \frac{(b + y_3)\, a\, y_1}{c - y_2} \\ -b\, y_4 - a\, y_3 \\ \frac{(c + y_1)\, a\, y_1}{c - y_2} - c\, y_4 + a\, y_2 \\ \frac{(c + y_1)\, b\, y_1}{c - y_2} + c\, y_3 + b\, y_2 \end{bmatrix}$$

Finally, the entries of the Poisson matrix:

```
> printlevel:=2; for i from 1 to dimtrans do for j from i to
  dimtrans do trans[i,j]:= simplify(DotProduct(Transpose(Pi_nu
  [i]), F[j],conjugate=false)) od od;
```

$$printlevel := 2$$

$$trans_{1,1} := 0$$

$$trans_{1,2} := -\frac{2\, c\, y_1 + 2\, c\, y_2 + y_1^2 - y_2^2}{c - y_2}$$

$$trans_{1,3} := -\frac{y_3\, (2\, c + y_1 - y_2)}{c - y_2}$$

$$trans_{1,4} := -\frac{c\, b\, y_1 + c\, b\, y_2 + c^2\, y_3 - c\, y_3\, y_2 + b\, y_1^2 - b\, y_2^2}{c - y_2}$$

$$trans_{2,2} := 0$$

$$trans_{2,3} := -\frac{y_4\, (2\, c + y_1 - y_2)}{c - y_2}$$

$$trans_{2, 4} := \frac{c\,a\,y_1 + c\,a\,y_2 - c^2\,y_4 + c\,y_4\,y_2 + a\,y_1^2 - a\,y_2^2}{c - y_2}$$

$$trans_{3, 3} := 0$$

$$trans_{3, 4} := \frac{c\,b\,y_4 + c\,a\,y_3 + b\,y_4\,y_1 - b\,y_4\,y_2 + a\,y_3\,y_1 - a\,y_3\,y_2}{c - y_2}$$

$$trans_{4, 4} := 0$$

>

Appendix B

Maple file for Example 52 - $\mathfrak{se}(3)^*$

```
> restart:with(LinearAlgebra):
> #Dimension of the Lie algebra g;
> dimg := 6;
```

$$dimg := 6 \quad (1)$$

```
> #The Lie-Poisson matrix is the following:
> Poisson:=Matrix([[0,x[3],-x[2],0,x[6],-x[5]],[-x[3],0,x[1],-x[6],0,x[4]],[x[2],-x[1],0,x[5],-x[4],0],[0,x[6],-x[5],0,0,0],[-x[6],0,x[4],0,0,0],[x[5],-x[4],0,0,0,0]]); Determinant(Poisson);
```

$$Poisson := \begin{bmatrix} 0 & x_3 & -x_2 & 0 & x_6 & -x_5 \\ -x_3 & 0 & x_1 & -x_6 & 0 & x_4 \\ x_2 & -x_1 & 0 & x_5 & -x_4 & 0 \\ 0 & x_6 & -x_5 & 0 & 0 & 0 \\ -x_6 & 0 & x_4 & 0 & 0 & 0 \\ x_5 & -x_4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$0$$

```
> mu:=<a,b,c,0,0,0>;
```

$$\mu := \begin{bmatrix} a \\ b \\ c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
> #Dimension and codimension of the isotropy subalgebra g_mu (which is also the dimension of the transverse Poisson structure);
> dimtrans := 4; codimtrans := 2;
```

$$dimtrans := 4$$

$$codimtrans := 2 \quad (2)$$

> #Basis for **g_mu (= ker P(mu)):**

```
> Poisson_mu:=Poisson: for i from 1 to dimg do Poisson_mu:=subs(x[i]=mu[i],Poisson_mu) end do: Poisson_mu;
```

$$\begin{bmatrix} 0 & c & -b & 0 & 0 & 0 \\ -c & 0 & a & 0 & 0 & 0 \\ b & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3)$$

```
> NullSpace(Poisson_mu);
```

$$\left\{\begin{bmatrix}\frac{a}{c}\\\frac{b}{c}\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}\right\} \tag{4}$$

> #MAPLE generates a random basis for **g_mu** each time the worksheet is computed. For coherence, we choose our own set of generators:

```
> F[1] := ⟨a, b, c, 0, 0, 0⟩ : F[2] := ⟨0, 0, 0, 1, 0, 0⟩ : F[3] := ⟨0, 0, 0,
    0, 1, 0⟩ : F[4] := ⟨0, 0, 0, 0, 0, 1⟩ :
```

Basis for ***h***:

```
> G[1]:=<1,0,0,0,0,0>: G[2]:=<0,1,0,0,0,0>:
```

Basis for ***h***^{q}:

```
> H[1]:=<0,0,1,0,0,0>: H[2]:=<0,0,0,1,0,0>: H[3]:=<0,0,0,0,1,
  0>: H[4]:=<0,0,0,0,0,1>:
```

Consider ***nu*** arbitrary element of ***h***^{q}:

```
> for i from 1 to dimtrans do k[i]:=VectorScalarMultiply(H[i],y
  [i]) end do: nu:=sum(k[j],j=1..dimtrans); munu:=VectorAdd(mu,
  nu);
```

$$\nu := \begin{bmatrix}0\\0\\y_1\\y_2\\y_3\\y_4\end{bmatrix}$$

$$munu := \begin{bmatrix}a\\b\\c+y_1\\y_2\\y_3\\y_4\end{bmatrix}$$

```
> Poisson_munu:=Poisson: for i from 1 to dimg do Poisson_munu:=
  subs(x[i]=munu[i],Poisson_munu) end do: Poisson_munu;
```

$$\begin{bmatrix} 0 & c+y_1 & -b & 0 & y_4 & -y_3 \\ -c-y_1 & 0 & a & -y_4 & 0 & y_2 \\ b & -a & 0 & y_3 & -y_2 & 0 \\ 0 & y_4 & -y_3 & 0 & 0 & 0 \\ -y_4 & 0 & y_2 & 0 & 0 & 0 \\ y_3 & -y_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Computation of ad*_G1(mu + nu) and ad*_G2(mu + nu):

```
> for i from 1 to dimg do for j from 1 to codimtrans do k[j,i]
:=ScalarMultiply(Row(Poisson_munu,i),G[j][i]) end do end do:
for j from 1 to codimtrans do W[j]:=eval(sum(k[j,l],l=1..
dimg)) end do;
```

$$W_1 := \begin{bmatrix} 0 & c+y_1 & -b & 0 & y_4 & -y_3 \end{bmatrix}$$

$$W_2 := \begin{bmatrix} -c-y_1 & 0 & a & -y_4 & 0 & y_2 \end{bmatrix}$$

> Computation of Pi(ad*_Fi(nu)):

```
> M:=Matrix([H[1],H[2],H[3],H[4],Transpose(W[1]),Transpose(W[2]
)]); Mi:=MatrixInverse(M):
```

$$M := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -c-y_1 \\ 0 & 0 & 0 & 0 & c+y_1 & 0 \\ 1 & 0 & 0 & 0 & -b & a \\ 0 & 1 & 0 & 0 & 0 & -y_4 \\ 0 & 0 & 1 & 0 & y_4 & 0 \\ 0 & 0 & 0 & 1 & -y_3 & y_2 \end{bmatrix}$$

Computing Z[1]:=ad*_F1(nu), Z[2]:=ad*_F2(nu), Z[3]:=ad*_F3(nu), Z[4]:=ad*_F4(nu):

```
> Poisson_nu:=Poisson: for i from 1 to dimg do Poisson_nu:=subs
(x[i]=nu[i],Poisson_nu) end do: Poisson_nu;
```

$$\begin{bmatrix} 0 & y_1 & 0 & 0 & y_4 & -y_3 \\ -y_1 & 0 & 0 & -y_4 & 0 & y_2 \\ 0 & 0 & 0 & y_3 & -y_2 & 0 \\ 0 & y_4 & -y_3 & 0 & 0 & 0 \\ -y_4 & 0 & y_2 & 0 & 0 & 0 \\ y_3 & -y_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> for i from 1 to dimg do for j from 1 to dimtrans do k[j,i]:=
ScalarMultiply(Row(Poisson_nu,i),F[j][i]) end do end do: for
j from 1 to dimtrans do Z[j]:=eval(sum(k[j,l],l=1..dimg)) end
do;
```

$$Z_1 := \begin{bmatrix} -b\,y_1 & a\,y_1 & 0 & -b\,y_4+c\,y_3 & a\,y_4-c\,y_2 & -a\,y_3+b\,y_2 \end{bmatrix}$$

$$Z_2 := \begin{bmatrix} 0 & y_4 & -y_3 & 0 & 0 & 0 \end{bmatrix}$$

$$Z_3 := \begin{bmatrix} -y_4 & 0 & y_2 & 0 & 0 & 0 \end{bmatrix}$$

$$Z_4 := \begin{bmatrix} y_3 & -y_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Computing the projections of the previous vectors:

```
> for j from 1 to dimtrans do v[j]:=MatrixVectorMultiply(Mi,
  Transpose(Z[j])) end do:
> for i from 1 to dimtrans do for l from 1 to dimtrans do k[i,
  l]:=ScalarMultiply(H[l],v[i][l]) end do end do: for i from 1
  to dimtrans do Pi_nu[i]:=eval(sum(k[i,o],o=1..dimtrans)) end
  do;
```

$$\pi nu_1 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{y_4 b y_1}{c + y_1} - b y_4 + c y_3 \\ -\frac{y_4 a y_1}{c + y_1} + a y_4 - c y_2 \\ -\frac{y_2 b y_1}{c + y_1} + \frac{y_3 a y_1}{c + y_1} - a y_3 + b y_2 \end{bmatrix}$$

$$\pi nu_2 := \begin{bmatrix} 0 \\ 0 \\ \frac{b y_4}{c + y_1} - y_3 \\ 0 \\ -\frac{y_4^2}{c + y_1} \\ \frac{y_3 y_4}{c + y_1} \end{bmatrix}$$

$$\pi nu_3 := \begin{bmatrix} 0 \\ 0 \\ -\frac{a\,y_4}{c+y_1} + y_2 \\ \frac{y_4^2}{c+y_1} \\ 0 \\ -\frac{y_2\,y_4}{c+y_1} \end{bmatrix}$$

$$\pi nu_4 := \begin{bmatrix} 0 \\ 0 \\ \frac{a\,y_3}{c+y_1} - \frac{b\,y_2}{c+y_1} \\ -\frac{y_3\,y_4}{c+y_1} \\ \frac{y_2\,y_4}{c+y_1} \\ 0 \end{bmatrix}$$

Finally, the entries of the Poisson matrix:

```
> printlevel:=2;for i from 1 to dimtrans do for j from i to
dimtrans do trans[i,j]:=simplify(DotProduct(Transpose(Pi_nu
[i]),F[j],conjugate=false)) od od;
```

$$printlevel := 2$$

$$trans_{1,1} := 0$$

$$trans_{1,2} := \frac{c\,(-b\,y_4 + c\,y_3 + y_3\,y_1)}{c+y_1}$$

$$trans_{1,3} := \frac{c\,(a\,y_4 - c\,y_2 - y_2\,y_1)}{c+y_1}$$

$$trans_{1,4} := -\frac{c\,(a\,y_3 - b\,y_2)}{c+y_1}$$

$$trans_{2,2} := 0$$

$$trans_{2,3} := -\frac{y_4^2}{c+y_1}$$

$$trans_{2,4} := \frac{y_3 y_4}{c + y_1}$$

$$trans_{3,3} := 0$$

$$trans_{3,4} := -\frac{y_2 y_4}{c + y_1}$$

$$trans_{4,4} := 0$$

```
> TRANS:=Matrix(dimtrans,dimtrans,symbol=alpha,shape=
antisymmetric): for i from 1 to dimtrans do for j from i+1 to
dimtrans do TRANS:=subs(alpha[i,j]=trans[i,j],TRANS) od od:
TRANS;
```

$$\left[\left[0, \frac{c\left(-b\,y_4 + c\,y_3 + y_3 y_1\right)}{c + y_1}, \frac{c\left(a\,y_4 - c\,y_2 - y_2 y_1\right)}{c + y_1}, -\frac{c\left(a\,y_3 - b\,y_2\right)}{c + y_1}\right],\right.$$

$$\left[-\frac{c\left(-b\,y_4 + c\,y_3 + y_3 y_1\right)}{c + y_1}, 0, -\frac{y_4^2}{c + y_1}, \frac{y_3 y_4}{c + y_1}\right],$$

$$\left[-\frac{c\left(a\,y_4 - c\,y_2 - y_2 y_1\right)}{c + y_1}, \frac{y_4^2}{c + y_1}, 0, -\frac{y_2 y_4}{c + y_1}\right],$$

$$\left.\left[\frac{c\left(a\,y_3 - b\,y_2\right)}{c + y_1}, -\frac{y_3 y_4}{c + y_1}, \frac{y_2 y_4}{c + y_1}, 0\right]\right]$$

>

Appendix C

Maple file for Example 62 - $\mathfrak{so}(4)^*$

```
> restart:with(LinearAlgebra):
```

\> #Dimension of the Lie algebra g;

```
> dimg := 6;
```

$$dimg := 6 \qquad (1)$$

\> #The Lie-Poisson matrix is the following:

```
> Poisson:=Matrix([[0,-x[4],-x[5],x[2],x[3],0],[x[4],0,-x[6],-x[1],0,x[3]],[x[5],x[6],0,0,-x[1],-x[2]],[-x[2],x[1],0,0,-x[6],x[5]],[-x[3],0,x[1],x[6],0,-x[4]],[0,-x[3],x[2],-x[5],x[4],0]]); Determinant(Poisson);
```

$$Poisson := \begin{bmatrix} 0 & -x_4 & -x_5 & x_2 & x_3 & 0 \\ x_4 & 0 & -x_6 & -x_1 & 0 & x_3 \\ x_5 & x_6 & 0 & 0 & -x_1 & -x_2 \\ -x_2 & x_1 & 0 & 0 & -x_6 & x_5 \\ -x_3 & 0 & x_1 & x_6 & 0 & -x_4 \\ 0 & -x_3 & x_2 & -x_5 & x_4 & 0 \end{bmatrix}$$

$$0$$

```
> mu:=<a,b,c,-c,b,-a>;
```

$$\mu := \begin{bmatrix} a \\ b \\ c \\ -c \\ b \\ -a \end{bmatrix}$$

\> #Dimension and codimension of the isotropy subalgebra g_mu (which is also the dimension of the transverse Poisson structure);

```
> dimtrans := 4; codimtrans := 2;
```

$$dimtrans := 4$$

$$codimtrans := 2 \qquad (2)$$

\> #Basis for **g_mu (= ker P(mu))**:

```
> Poisson_mu:=Poisson: for i from 1 to dimg do Poisson_mu:=subs(x[i]=mu[i],Poisson_mu) end do: Poisson_mu;
```

$$\begin{bmatrix} 0 & c & -b & b & c & 0 \\ -c & 0 & a & -a & 0 & c \\ b & -a & 0 & 0 & -a & -b \\ -b & a & 0 & 0 & a & b \\ -c & 0 & a & -a & 0 & c \\ 0 & -c & b & -b & -c & 0 \end{bmatrix} \qquad (3)$$

```
> NullSpace(Poisson_mu);
```

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{a}{c} \\ -\frac{b}{c} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{a}{c} \\ \frac{b}{c} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{4}$$

> #MAPLE generates a random basis for **g_mu** each time the worksheet is computed. For coherence, we choose our own set of generators:

```
> F[1] := ⟨1, 0, 0, 0, 0, 1⟩ : F[2] := ⟨0, 1, 0, 0, -1, 0⟩ : F[3] := ⟨0, 0, 1,
    1, 0, 0⟩ : F[4] := ⟨0, 0, 0, c, -b, a⟩ :
```

Basis for ***h***:

```
> G[1]:=<c,0,-a,a,0,-c>:G[2]:=<-b,a,0,0,a,b>:
```

Basis for ***h***^{q}:

```
> H[1]:=<1,0,0,0,0,1>: H[2]:=<0,1,0,0,-1,0>: H[3]:=<0,0,1,1,0,
  0>: H[4]:=<0,0,0,c,-b,a>:
```

Consider ***nu*** arbitrary element of ***h***^{q}:

```
> for i from 1 to dimtrans do k[i]:=VectorScalarMultiply(H[i],y
  [i]) end do: nu:=sum(k[j],j=1..dimtrans); munu:=VectorAdd(mu,
  nu);
```

$$\nu := \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_3 + y_4 c \\ -y_2 - y_4 b \\ y_1 + y_4 a \end{bmatrix}$$

$$munu := \begin{bmatrix} a + y_1 \\ b + y_2 \\ c + y_3 \\ -c + y_3 + y_4 c \\ b - y_2 - y_4 b \\ -a + y_1 + y_4 a \end{bmatrix}$$

```
> Poisson_munu:=Poisson: for i from 1 to dimg do Poisson_munu:=
  subs(x[i]=munu[i],Poisson_munu) end do: Poisson_munu;
```

$$[[0, c - y_3 - y_4 c, -b + y_2 + y_4 b, b + y_2, c + y_3, 0],$$

$$[-c + y_3 + y_4 c, 0, a - y_1 - y_4 a, -a - y_1, 0, c + y_3],$$

$$[b - y_2 - y_4 b, -a + y_1 + y_4 a, 0, 0, -a - y_1, -b - y_2],$$

$[-b-y_2, a+y_1, 0, 0, a-y_1-y_4 a, b-y_2-y_4 b],$

$[-c-y_3, 0, a+y_1, -a+y_1+y_4 a, 0, c-y_3-y_4 c],$

$[0, -c-y_3, b+y_2, -b+y_2+y_4 b, -c+y_3+y_4 c, 0]]$

Computation of ad*_G1(mu + nu) and ad*_G2(mu + nu):

```
> for i from 1 to dimg do for j from 1 to codimtrans do k[j,i]
  :=ScalarMultiply(Row(Poisson_munu,i),G[j][i]) end do end do:
  for j from 1 to codimtrans do W[j]:=eval(sum(k[j,l],l=1..
  dimg)) end do;
```

$$W_1 := [-a(b-y_2-y_4 b)+a(-b-y_2), c(c-y_3-y_4 c)-a(-a+y_1+y_4 a)+a(a+y_1)-c(-c-y_3), c(-b+y_2+y_4 b)-c(b+y_2), c(b+y_2)-c(-b+y_2+y_4 b), c(c+y_3)-a(-a-y_1)+a(a-y_1-y_4 a)-c(-c+y_3+y_4 c), -a(-b-y_2)+a(b-y_2-y_4 b)]$$

$$W_2 := [a(-c+y_3+y_4 c)+a(-c-y_3), -b(c-y_3-y_4 c)+b(-c-y_3), -b(-b+y_2+y_4 b)+a(a-y_1-y_4 a)+a(a+y_1)+b(b+y_2), -b(b+y_2)+a(-a-y_1)+a(-a+y_1+y_4 a)+b(-b+y_2+y_4 b), -b(c+y_3)+b(-c+y_3+y_4 c), a(c+y_3)+a(c-y_3-y_4 c)]$$

> Computation of Pi(ad*_Fi(nu)):

```
> M:=Matrix([H[1],H[2],H[3],H[4],Transpose(W[1]),Transpose(W[2]
  )]); Mi:=MatrixInverse(M):
```

$$M := [[1, 0, 0, 0, -a(b-y_2-y_4 b)+a(-b-y_2), a(-c+y_3+y_4 c)+a(-c-y_3)],$$
$$[0, 1, 0, 0, c(c-y_3-y_4 c)-a(-a+y_1+y_4 a)+a(a+y_1)-c(-c-y_3), -b(c-y_3-y_4 c)+b(-c-y_3)],$$
$$[0, 0, 1, 0, c(-b+y_2+y_4 b)-c(b+y_2), -b(-b+y_2+y_4 b)+a(a-y_1-y_4 a)+a(a+y_1)+b(b+y_2)],$$
$$[0, 0, 1, c, c(b+y_2)-c(-b+y_2+y_4 b), -b(b+y_2)+a(-a-y_1)+a(-a+y_1+y_4 a)+b(-b+y_2+y_4 b)],$$
$$[0, -1, 0, -b, c(c+y_3)-a(-a-y_1)+a(a-y_1-y_4 a)-c(-c+y_3+y_4 c), -b(c+y_3)+b(-c+y_3+y_4 c)],$$
$$[1, 0, 0, a, -a(-b-y_2)+a(b-y_2-y_4 b), a(c+y_3)+a(c-y_3-y_4 c)]]$$

Computing Z[1]:=ad*_F1(nu), Z[2]:=ad*_F2(nu), Z[3]:=ad*_F3(nu), Z[4]:=ad*_F4(nu):

```
> Poisson_nu:=Poisson: for i from 1 to dimg do Poisson_nu:=subs
  (x[i]=nu[i],Poisson_nu) end do: Poisson_nu;
```

$$\begin{bmatrix} 0 & -y_3 - y_4 c & y_2 + y_4 b & y_2 & y_3 & 0 \\ y_3 + y_4 c & 0 & -y_1 - y_4 a & -y_1 & 0 & y_3 \\ -y_2 - y_4 b & y_1 + y_4 a & 0 & 0 & -y_1 & -y_2 \\ -y_2 & y_1 & 0 & 0 & -y_1 - y_4 a & -y_2 - y_4 b \\ -y_3 & 0 & y_1 & y_1 + y_4 a & 0 & -y_3 - y_4 c \\ 0 & -y_3 & y_2 & y_2 + y_4 b & y_3 + y_4 c & 0 \end{bmatrix}$$

```
> for i from 1 to dimg do for j from 1 to dimtrans do k[j,i]:=
  ScalarMultiply(Row(Poisson_nu,i),F[j][i]) end do end do: for
  j from 1 to dimtrans do Z[j]:=eval(sum(k[j,l],l=1..dimg)) end
  do;
```

$$Z_1 := \left[\ 0 \quad -2 y_3 - y_4 c \quad 2 y_2 + y_4 b \quad 2 y_2 + y_4 b \quad 2 y_3 + y_4 c \quad 0\ \right]$$

$$Z_2 := \left[\ 2 y_3 + y_4 c \quad 0 \quad -2 y_1 - y_4 a \quad -2 y_1 - y_4 a \quad 0 \quad 2 y_3 + y_4 c\ \right]$$

$$Z_3 := \left[\ -2 y_2 - y_4 b \quad 2 y_1 + y_4 a \quad 0 \quad 0 \quad -2 y_1 - y_4 a \quad -2 y_2 - y_4 b\ \right]$$

$$Z_4 := \left[-c y_2 + b y_3, c y_1 - a y_3, -b y_1 + a y_2, -b\left(y_1 + y_4 a\right) + a\left(y_2 + y_4 b\right), c\left(-y_1 - y_4 a\right) + a\left(y_3 + y_4 c\right), c\left(-y_2 - y_4 b\right) - b\left(-y_3 - y_4 c\right)\right]$$

Computing the projections of the previous vectors:

```
> for j from 1 to dimtrans do v[j]:=MatrixVectorMultiply(Mi,
  Transpose(Z[j])) end do:
> for i from 1 to dimtrans do for l from 1 to dimtrans do k[i,
  l]:=ScalarMultiply(H[l],v[i][l]) end do end do: for i from 1
  to dimtrans do Pi_nu[i]:=eval(sum(k[i,o],o=1..dimtrans)) end
  do;
```

$$\pi nu_1 := \left[\left[\frac{1}{2} \frac{a b\left(-2 y_3 - y_4 c\right)}{c^2 + a^2 + b^2} + \frac{1}{2} \frac{a b\left(2 y_3 + y_4 c\right)}{c^2 + a^2 + b^2}\right],\right.$$

$$\left[\frac{1}{2} \frac{\left(c^2 + a^2 + 2 b^2\right)\left(-2 y_3 - y_4 c\right)}{c^2 + a^2 + b^2} - \frac{1}{2} \frac{\left(c^2 + a^2\right)\left(2 y_3 + y_4 c\right)}{c^2 + a^2 + b^2}\right],$$

$$\left[\frac{1}{2} \frac{b c\left(-2 y_3 - y_4 c\right)}{c^2 + a^2 + b^2} + \frac{1}{2} \frac{\left(2 c^2 + a^2 + b^2\right)\left(2 y_2 + y_4 b\right)}{c^2 + a^2 + b^2} + \frac{1}{2} \frac{\left(a^2 + b^2\right)\left(2 y_2 + y_4 b\right)}{c^2 + a^2 + b^2} + \frac{1}{2} \frac{b c\left(2 y_3 + y_4 c\right)}{c^2 + a^2 + b^2}\right],$$

$$\left[\frac{1}{2} \frac{b c\left(-2 y_3 - y_4 c\right)}{c^2 + a^2 + b^2} + \frac{1}{2} \frac{\left(2 c^2 + a^2 + b^2\right)\left(2 y_2 + y_4 b\right)}{c^2 + a^2 + b^2} + \frac{1}{2} \frac{\left(a^2 + b^2\right)\left(2 y_2 + y_4 b\right)}{c^2 + a^2 + b^2} + \frac{1}{2} \frac{b c\left(2 y_3 + y_4 c\right)}{c^2 + a^2 + b^2} + \left(-\frac{b\left(-2 y_3 - y_4 c\right)}{c^2 + a^2 + b^2}\right.\right.$$

$$-\frac{b\left(2y_3+y_4c\right)}{c^2+a^2+b^2}\Bigg)c\Bigg],$$

$$\left[-\frac{1}{2}\,\frac{\left(c^2+a^2+2b^2\right)\left(-2y_3-y_4c\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(c^2+a^2\right)\left(2y_3+y_4c\right)}{c^2+a^2+b^2}-\Bigg(\right.$$

$$\left.-\frac{b\left(-2y_3-y_4c\right)}{c^2+a^2+b^2}-\frac{b\left(2y_3+y_4c\right)}{c^2+a^2+b^2}\Bigg)b\right],$$

$$\left[\frac{1}{2}\,\frac{a\,b\left(-2y_3-y_4c\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{a\,b\left(2y_3+y_4c\right)}{c^2+a^2+b^2}+\Bigg(-\frac{b\left(-2y_3-y_4c\right)}{c^2+a^2+b^2}\right.$$

$$\left.-\frac{b\left(2y_3+y_4c\right)}{c^2+a^2+b^2}\Bigg)a\right]\Bigg]$$

$$\pi nu_2 := \begin{bmatrix} \frac{1}{2}\,\frac{\left(c^2+2a^2+b^2\right)\left(2y_3+y_4c\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(b^2+c^2\right)\left(2y_3+y_4c\right)}{c^2+a^2+b^2} \\ 0 \\ \frac{1}{2}\,\frac{\left(2c^2+a^2+b^2\right)\left(-2y_1-y_4a\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(a^2+b^2\right)\left(-2y_1-y_4a\right)}{c^2+a^2+b^2} \\ \frac{1}{2}\,\frac{\left(2c^2+a^2+b^2\right)\left(-2y_1-y_4a\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(a^2+b^2\right)\left(-2y_1-y_4a\right)}{c^2+a^2+b^2} \\ 0 \\ \frac{1}{2}\,\frac{\left(c^2+2a^2+b^2\right)\left(2y_3+y_4c\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(b^2+c^2\right)\left(2y_3+y_4c\right)}{c^2+a^2+b^2} \end{bmatrix}$$

$$\pi nu_3 := \Bigg[\Bigg[\frac{1}{2}\,\frac{\left(c^2+2a^2+b^2\right)\left(-2y_2-y_4b\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{a\,b\left(2y_1+y_4a\right)}{c^2+a^2+b^2}$$

$$+\frac{1}{2}\,\frac{a\,b\left(-2y_1-y_4a\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(b^2+c^2\right)\left(-2y_2-y_4b\right)}{c^2+a^2+b^2}\Bigg],$$

$$\left[\frac{1}{2}\,\frac{\left(c^2+a^2+2b^2\right)\left(2y_1+y_4a\right)}{c^2+a^2+b^2}-\frac{1}{2}\,\frac{\left(c^2+a^2\right)\left(-2y_1-y_4a\right)}{c^2+a^2+b^2}\right],$$

$$\left[\frac{1}{2}\,\frac{b\,c\left(2y_1+y_4a\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{b\,c\left(-2y_1-y_4a\right)}{c^2+a^2+b^2}\right],$$

$$\left[\frac{1}{2}\,\frac{b\,c\left(2y_1+y_4a\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{b\,c\left(-2y_1-y_4a\right)}{c^2+a^2+b^2}+\Bigg(-\frac{b\left(2y_1+y_4a\right)}{c^2+a^2+b^2}\right.$$

$$\left.-\frac{b\left(-2y_1-y_4a\right)}{c^2+a^2+b^2}\Bigg)c\right],$$

$$\left[-\frac{1}{2}\,\frac{\left(c^2+a^2+2\,b^2\right)\,\left(2\,y_1+y_4\,a\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(c^2+a^2\right)\,\left(-2\,y_1-y_4\,a\right)}{c^2+a^2+b^2}-\left(-\frac{b\,\left(2\,y_1+y_4\,a\right)}{c^2+a^2+b^2}-\frac{b\,\left(-2\,y_1-y_4\,a\right)}{c^2+a^2+b^2}\right)\,b\right],$$

$$\left[\frac{1}{2}\,\frac{\left(c^2+2\,a^2+b^2\right)\,\left(-2\,y_2-y_4\,b\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{a\,b\,\left(2\,y_1+y_4\,a\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{a\,b\,\left(-2\,y_1-y_4\,a\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(b^2+c^2\right)\,\left(-2\,y_2-y_4\,b\right)}{c^2+a^2+b^2}+\left(-\frac{b\,\left(2\,y_1+y_4\,a\right)}{c^2+a^2+b^2}-\frac{b\,\left(-2\,y_1-y_4\,a\right)}{c^2+a^2+b^2}\right)\,a\right]\Bigg]$$

$$\pi nu_4:=\Bigg[\Bigg[\frac{1}{2}\,\frac{\left(c^2+2\,a^2+b^2\right)\,\left(-c\,y_2+b\,y_3\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{a\,b\,\left(c\,y_1-a\,y_3\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{a\,c\,\left(-b\,y_1+a\,y_2\right)}{c^2+a^2+b^2}-\frac{1}{2}\,\frac{a\,c\,\left(-b\,\left(y_1+y_4\,a\right)+a\,\left(y_2+y_4\,b\right)\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{a\,b\,\left(c\,\left(-y_1-y_4\,a\right)+a\,\left(y_3+y_4\,c\right)\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(b^2+c^2\right)\,\left(c\,\left(-y_2-y_4\,b\right)-b\,\left(-y_3-y_4\,c\right)\right)}{c^2+a^2+b^2}\Bigg],$$

$$\left[\frac{1}{2}\,\frac{a\,b\,\left(-c\,y_2+b\,y_3\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(c^2+a^2+2\,b^2\right)\,\left(c\,y_1-a\,y_3\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{b\,c\,\left(-b\,y_1+a\,y_2\right)}{c^2+a^2+b^2}-\frac{1}{2}\,\frac{b\,c\,\left(-b\,\left(y_1+y_4\,a\right)+a\,\left(y_2+y_4\,b\right)\right)}{c^2+a^2+b^2}-\frac{1}{2}\,\frac{\left(c^2+a^2\right)\,\left(c\,\left(-y_1-y_4\,a\right)+a\,\left(y_3+y_4\,c\right)\right)}{c^2+a^2+b^2}-\frac{1}{2}\,\frac{a\,b\,\left(c\,\left(-y_2-y_4\,b\right)-b\,\left(-y_3-y_4\,c\right)\right)}{c^2+a^2+b^2}\right],$$

$$\left[\frac{1}{2}\,\frac{a\,c\,\left(-c\,y_2+b\,y_3\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{b\,c\,\left(c\,y_1-a\,y_3\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{\left(2\,c^2+a^2+b^2\right)\,\left(-b\,y_1+a\,y_2\right)}{c^2+a^2+b^2}\right.$$

$$+\frac{1}{2}\,\frac{\left(a^2+b^2\right)\left(-b\left(y_1+y_4\,a\right)+a\left(y_2+y_4\,b\right)\right)}{c^2+a^2+b^2}$$

$$+\frac{1}{2}\,\frac{b\,c\left(c\left(-y_1-y_4\,a\right)+a\left(y_3+y_4\,c\right)\right)}{c^2+a^2+b^2}$$

$$-\frac{1}{2}\,\frac{a\,c\left(c\left(-y_2-y_4\,b\right)-b\left(-y_3-y_4\,c\right)\right)}{c^2+a^2+b^2}\Bigg],$$

$$\Bigg[\frac{1}{2}\,\frac{a\,c\left(-c\,y_2+b\,y_3\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{b\,c\left(c\,y_1-a\,y_3\right)}{c^2+a^2+b^2}$$

$$+\frac{1}{2}\,\frac{\left(2\,c^2+a^2+b^2\right)\left(-b\,y_1+a\,y_2\right)}{c^2+a^2+b^2}$$

$$+\frac{1}{2}\,\frac{\left(a^2+b^2\right)\left(-b\left(y_1+y_4\,a\right)+a\left(y_2+y_4\,b\right)\right)}{c^2+a^2+b^2}$$

$$+\frac{1}{2}\,\frac{b\,c\left(c\left(-y_1-y_4\,a\right)+a\left(y_3+y_4\,c\right)\right)}{c^2+a^2+b^2}$$

$$-\frac{1}{2}\,\frac{a\,c\left(c\left(-y_2-y_4\,b\right)-b\left(-y_3-y_4\,c\right)\right)}{c^2+a^2+b^2}+\Bigg(-\frac{a\left(-c\,y_2+b\,y_3\right)}{c^2+a^2+b^2}$$

$$-\frac{b\left(c\,y_1-a\,y_3\right)}{c^2+a^2+b^2}-\frac{c\left(-b\,y_1+a\,y_2\right)}{c^2+a^2+b^2}+\frac{c\left(-b\left(y_1+y_4\,a\right)+a\left(y_2+y_4\,b\right)\right)}{c^2+a^2+b^2}$$

$$-\frac{b\left(c\left(-y_1-y_4\,a\right)+a\left(y_3+y_4\,c\right)\right)}{c^2+a^2+b^2}+\frac{a\left(c\left(-y_2-y_4\,b\right)-b\left(-y_3-y_4\,c\right)\right)}{c^2+a^2+b^2}\Bigg)$$

$$c\Bigg],$$

$$\Bigg[-\frac{1}{2}\,\frac{a\,b\left(-c\,y_2+b\,y_3\right)}{c^2+a^2+b^2}-\frac{1}{2}\,\frac{\left(c^2+a^2+2\,b^2\right)\left(c\,y_1-a\,y_3\right)}{c^2+a^2+b^2}$$

$$-\frac{1}{2}\,\frac{b\,c\left(-b\,y_1+a\,y_2\right)}{c^2+a^2+b^2}+\frac{1}{2}\,\frac{b\,c\left(-b\left(y_1+y_4\,a\right)+a\left(y_2+y_4\,b\right)\right)}{c^2+a^2+b^2}$$

$$+\frac{1}{2}\,\frac{\left(c^2+a^2\right)\left(c\left(-y_1-y_4\,a\right)+a\left(y_3+y_4\,c\right)\right)}{c^2+a^2+b^2}$$

$$+\frac{1}{2}\,\frac{a\,b\left(c\left(-y_2-y_4\,b\right)-b\left(-y_3-y_4\,c\right)\right)}{c^2+a^2+b^2}-\Bigg(-\frac{a\left(-c\,y_2+b\,y_3\right)}{c^2+a^2+b^2}$$

$$-\frac{b\left(c\,y_1-a\,y_3\right)}{c^2+a^2+b^2}-\frac{c\left(-b\,y_1+a\,y_2\right)}{c^2+a^2+b^2}+\frac{c\left(-b\left(y_1+y_4\,a\right)+a\left(y_2+y_4\,b\right)\right)}{c^2+a^2+b^2}$$

$$- \frac{b\left(c\left(-y_1 - y_4 a\right) + a\left(y_3 + y_4 c\right)\right)}{c^2 + a^2 + b^2} + \frac{a\left(c\left(-y_2 - y_4 b\right) - b\left(-y_3 - y_4 c\right)\right)}{c^2 + a^2 + b^2}\Bigg)$$

$$b\Bigg],$$

$$\Bigg[\frac{1}{2}\frac{\left(c^2 + 2a^2 + b^2\right)\left(-c y_2 + b y_3\right)}{c^2 + a^2 + b^2} + \frac{1}{2}\frac{a b\left(c y_1 - a y_3\right)}{c^2 + a^2 + b^2}$$

$$+ \frac{1}{2}\frac{a c\left(-b y_1 + a y_2\right)}{c^2 + a^2 + b^2} - \frac{1}{2}\frac{a c\left(-b\left(y_1 + y_4 a\right) + a\left(y_2 + y_4 b\right)\right)}{c^2 + a^2 + b^2}$$

$$+ \frac{1}{2}\frac{a b\left(c\left(-y_1 - y_4 a\right) + a\left(y_3 + y_4 c\right)\right)}{c^2 + a^2 + b^2}$$

$$+ \frac{1}{2}\frac{\left(b^2 + c^2\right)\left(c\left(-y_2 - y_4 b\right) - b\left(-y_3 - y_4 c\right)\right)}{c^2 + a^2 + b^2} + \Bigg(-\frac{a\left(-c y_2 + b y_3\right)}{c^2 + a^2 + b^2}$$

$$- \frac{b\left(c y_1 - a y_3\right)}{c^2 + a^2 + b^2} - \frac{c\left(-b y_1 + a y_2\right)}{c^2 + a^2 + b^2} + \frac{c\left(-b\left(y_1 + y_4 a\right) + a\left(y_2 + y_4 b\right)\right)}{c^2 + a^2 + b^2}$$

$$- \frac{b\left(c\left(-y_1 - y_4 a\right) + a\left(y_3 + y_4 c\right)\right)}{c^2 + a^2 + b^2} + \frac{a\left(c\left(-y_2 - y_4 b\right) - b\left(-y_3 - y_4 c\right)\right)}{c^2 + a^2 + b^2}\Bigg)$$

$$a\Bigg]\Bigg]$$

Finally, the entries of the Poisson matrix:

```
> printlevel:=2;for i from 1 to dimtrans do for j from i to
  dimtrans do trans[i,j]:=simplify(DotProduct(Transpose(Pi_nu
  [i]),F[j],conjugate=false)) od od;
```

$$printlevel := 2$$

$$trans_{1,1} := 0$$

$$trans_{1,2} := -4 y_3 - 2 y_4 c$$

$$trans_{1,3} := 4 y_2 + 2 y_4 b$$

$$trans_{1,4} := 2 c y_2 - 2 b y_3$$

$$trans_{2,2} := 0$$

$$trans_{2,3} := -4 y_1 - 2 y_4 a$$

$$trans_{2,4} := -2 c y_1 + 2 a y_3$$

$$trans_{3,3} := 0$$

$$trans_{3,4} := 2 b y_1 - 2 a y_2$$

$$trans_{4,4} := 0$$

```
> TRANS:=Matrix(dimtrans,dimtrans,symbol=alpha,shape=
antisymmetric): for i from 1 to dimtrans do for j from i+1 to
dimtrans do TRANS:=subs(alpha[i,j]=trans[i,j],TRANS) od od:
TRANS;
```

$$\begin{bmatrix} 0 & -4\,y_3 - 2\,y_4\,c & 4\,y_2 + 2\,y_4\,b & 2\,c\,y_2 - 2\,b\,y_3 \\ 4\,y_3 + 2\,y_4\,c & 0 & -4\,y_1 - 2\,y_4\,a & -2\,c\,y_1 + 2\,a\,y_3 \\ -4\,y_2 - 2\,y_4\,b & 4\,y_1 + 2\,y_4\,a & 0 & 2\,b\,y_1 - 2\,a\,y_2 \\ -2\,c\,y_2 + 2\,b\,y_3 & 2\,c\,y_1 - 2\,a\,y_3 & -2\,b\,y_1 + 2\,a\,y_2 & 0 \end{bmatrix}$$

```
>
```

Appendix D

Maple file for Example 63 - $\mathfrak{sp}(4)^*$

```
> restart:with(LinearAlgebra):
> #Dimension of the Lie algebra g;
> dimg := 10;
```

$$dimg := 10 \qquad (1)$$

```
> #The Lie-Poisson matrix is the following:
> Poisson:=Matrix([[0,x[2],-x[3],0,2*x[5],x[6],0,-2*x[8],-x[9],0],[-x[2],0,x[1]-x[4],x[2],0,2*x[5],x[6],-x[9],-2*x[10],0],[x[3],x[4]-x[1],0,-x[3],x[6],2*x[7],0,0,-2*x[8],-x[9]],[0,-x[2],x[3],0,0,x[6],2*x[7],0,-x[9],-2*x[10]],[-2*x[5],0,-x[6],0,0,0,0,x[1],x[2],0],[-x[6],-2*x[5],-2*x[7],-x[6],0,0,0,x[3],x[1]+x[4],x[2]],[0,-x[6],0,-2*x[7],0,0,0,0,x[3],x[4]],[2*x[8],x[9],0,0,-x[1],-x[3],0,0,0,0],[x[9],2*x[10],2*x[8],x[9],-x[2],-x[1]-x[4],-x[3],0,0,0],[0,0,x[9],2*x[10],0,-x[2],-x[4],0,0,0]]); Determinant(Poisson);
```

$$Poisson := \begin{bmatrix} 0 & x_2 & -x_3 & 0 & 2\,x_5 & x_6 & 0 & -2\,x_8 & -x_9 & 0 \\ -x_2 & 0 & x_1 - x_4 & x_2 & 0 & 2\,x_5 & x_6 & -x_9 & -2\,x_{10} & 0 \\ x_3 & x_4 - x_1 & 0 & -x_3 & x_6 & 2\,x_7 & 0 & 0 & -2\,x_8 & -x_9 \\ 0 & -x_2 & x_3 & 0 & 0 & x_6 & 2\,x_7 & 0 & -x_9 & -2\,x_{10} \\ -2\,x_5 & 0 & -x_6 & 0 & 0 & 0 & 0 & x_1 & x_2 & 0 \\ -x_6 & -2\,x_5 & -2\,x_7 & -x_6 & 0 & 0 & 0 & x_3 & x_1 + x_4 & x_2 \\ 0 & -x_6 & 0 & -2\,x_7 & 0 & 0 & 0 & 0 & x_3 & x_4 \\ 2\,x_8 & x_9 & 0 & 0 & -x_1 & -x_3 & 0 & 0 & 0 & 0 \\ x_9 & 2\,x_{10} & 2\,x_8 & x_9 & -x_2 & -x_1 - x_4 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & x_9 & 2\,x_{10} & 0 & -x_2 & -x_4 & 0 & 0 & 0 \end{bmatrix}$$

$$0$$

```
> mu:=<1,0,0,1,0,0,0,0,0,0>;
```

$$\mu := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
> #Dimension and codimension of the isotropy subalgebra g_mu (which is also the dimension of the transverse Poisson structure);
> dimtrans := 4; codimtrans := 6;
```

$$dimtrans := 4$$

$$codimtrans := 6 \tag{2}$$

> #Basis for **g_mu (= ker P(mu))**:

```
> Poisson_mu:=Poisson: for i from 1 to dimg do Poisson_mu:=
subs(x[i]=mu[i],Poisson_mu) end do: Poisson_mu;
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

```
> F[1] := <1, 0, 0, 0, 0, 0, 0, 0, 0, 0> : F[2] := <0, 1, 0, 0, 0, 0, 0, 0, 0,
0> : F[3] := <0, 0, 1, 0, 0, 0, 0, 0, 0, 0> : F[4] := <0, 0, 0, 1, 0, 0,
0, 0, 0, 0> :
```

Basis for ***h***:

```
> G[1]:=<0,0,0,0,1,0,0,0,0,0>: G[2]:=<0,0,0,0,0,1,0,0,0,0>: G
[3]:=<0,0,0,0,0,0,1,0,0,0>: G[4]:=<0,0,0,0,0,0,0,1,0,0>: G[5]
:=<0,0,0,0,0,0,0,0,1,0>: G[6]:=<0,0,0,0,0,0,0,0,0,1>:
```

Basis for $\boldsymbol{h}^q$:

```
> H[1]:=<1,0,0,0,0,0,0,0,0,0,0>: H[2]:=<0,1,0,0,0,0,0,0,0,0,0>: H
[3]:=<0,0,1,0,0,0,0,0,0,0,0>: H[4]:=<0,0,0,1,0,0,0,0,0,0,0>:
```

Consider ***nu*** arbitrary element of $\boldsymbol{h}^q$:

```
> for i from 1 to dimtrans do k[i]:=VectorScalarMultiply(H[i],y
[i]) end do: nu:=sum(k[j],j=1..dimtrans); munu:=VectorAdd(mu,
nu);
```

$$\nu := \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$munu := \begin{bmatrix} 1+y_1 \\ y_2 \\ y_3 \\ 1+y_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
> Poisson_munu:=Poisson: for i from 1 to dimg do Poisson_munu:=
  subs(x[i]=munu[i],Poisson_munu) end do: Poisson_munu;
```

$$\big[[0, y_2, -y_3, 0, 0, 0, 0, 0, 0, 0],$$
$$[-y_2, 0, y_1 - y_4, y_2, 0, 0, 0, 0, 0, 0],$$
$$[y_3, y_4 - y_1, 0, -y_3, 0, 0, 0, 0, 0, 0],$$
$$[0, -y_2, y_3, 0, 0, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 0, 1 + y_1, y_2, 0],$$
$$[0, 0, 0, 0, 0, 0, 0, y_3, 2 + y_1 + y_4, y_2],$$
$$[0, 0, 0, 0, 0, 0, 0, 0, y_3, 1 + y_4],$$
$$[0, 0, 0, 0, -1 - y_1, -y_3, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, -y_2, -2 - y_1 - y_4, -y_3, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, -y_2, -1 - y_4, 0, 0, 0]\big]$$

Computation of ad*_G1(mu + nu) and ad*_G2(mu + nu):

```
> for i from 1 to dimg do for j from 1 to codimtrans do k[j,i]
  :=ScalarMultiply(Row(Poisson_munu,i),G[j][i]) end do end do:
  for j from 1 to codimtrans do W[j]:=eval(sum(k[j,l],l=1..
  dimg)) end do;
```

$$W_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+y_1 & y_2 & 0 \end{bmatrix}$$

$$W_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_3 & 2+y_1+y_4 & y_2 \end{bmatrix}$$

$$W_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_3 & 1+y_4 \end{bmatrix}$$

$$W_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & -1-y_1 & -y_3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W_5 := \begin{bmatrix} 0 & 0 & 0 & 0 & -y_2 & -2-y_1-y_4 & -y_3 & 0 & 0 & 0 \end{bmatrix}$$

$$W_6 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -y_2 & -1-y_4 & 0 & 0 & 0 \end{bmatrix}$$

> Computation of Pi(ad*_Fi(nu)):

```
> for i from 1 to 6 do W[i] := Transpose(W[i]) end do:
> M:=Matrix([H[1],H[2],H[3],H[4],W[1],W[2],W[3],W[4],W[5],W[6]]); Mi:=MatrixInverse(M):
```

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1-y_1 & -y_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_3 & -2-y_1-y_4 & -y_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_3 & -1-y_4 \\ 0 & 0 & 0 & 0 & 1+y_1 & y_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_2 & 2+y_1+y_4 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_2 & 1+y_4 & 0 & 0 & 0 \end{bmatrix}$$

Computing Z[1]:=ad*_F1(nu), Z[2]:=ad*_F2(nu), Z[3]:=ad*_F3(nu), Z[4]:=ad*_F4(nu):

```
> Poisson_nu:=Poisson: for i from 1 to dimg do Poisson_nu:=subs(x[i]=nu[i],Poisson_nu) end do: Poisson_nu;
```

$$\begin{bmatrix} 0 & y_2 & -y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -y_2 & 0 & y_1-y_4 & y_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_3 & y_4-y_1 & 0 & -y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y_2 & y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_1 & y_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_3 & y_1+y_4 & y_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_3 & y_4 \\ 0 & 0 & 0 & 0 & -y_1 & -y_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y_2 & -y_1-y_4 & -y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -y_2 & -y_4 & 0 & 0 & 0 \end{bmatrix}$$

```
> for i from 1 to dimg do for j from 1 to dimtrans do k[j,i]:= ScalarMultiply(Row(Poisson_nu,i),F[j][i]) end do end do: for j from 1 to dimtrans do Z[j]:=eval(sum(k[j,l],l=1..dimg)) end do;
```

$$Z_1 := \begin{bmatrix} 0 & y_2 & -y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Z_2 := \begin{bmatrix} -y_2 & 0 & y_1-y_4 & y_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Z_3 := \left[\begin{array}{cccccccccc} y_3 & y_4 - y_1 & 0 & -y_3 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$Z_4 := \left[\begin{array}{cccccccccc} 0 & -y_2 & y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Computing the projections of the previous vectors:

```
> for j from 1 to dimtrans do v[j]:=MatrixVectorMultiply(Mi,
  Transpose(Z[j])) end do:
> for i from 1 to dimtrans do for l from 1 to dimtrans do k[i,
  l]:=ScalarMultiply(H[l],v[i][l]) end do end do: for i from 1
  to dimtrans do Pi_nu[i]:=eval(sum(k[i,o],o=1..dimtrans)) end
  do:
```

Finally, the entries of the Poisson matrix:

```
> printlevel:=2;for i from 1 to dimtrans do for j from i to
  dimtrans do trans[i,j]:=simplify(DotProduct(Transpose(Pi_nu
  [i]),F[j],conjugate=false)) od od;
```

$$printlevel := 2$$

$$trans_{1,1} := 0$$

$$trans_{1,2} := y_2$$

$$trans_{1,3} := -y_3$$

$$trans_{1,4} := 0$$

$$trans_{2,2} := 0$$

$$trans_{2,3} := y_1 - y_4$$

$$trans_{2,4} := y_2$$

$$trans_{3,3} := 0$$

$$trans_{3,4} := -y_3$$

$$trans_{4,4} := 0$$

```
> TRANS:=Matrix(dimtrans,dimtrans,symbol=alpha,shape=
  antisymmetric): for i from 1 to dimtrans do for j from i+1 to
  dimtrans do TRANS:=subs(alpha[i,j]=trans[i,j],TRANS) od od:
  TRANS;
```

$$\left[\begin{array}{cccc} 0 & y_2 & -y_3 & 0 \\ -y_2 & 0 & y_1 - y_4 & y_2 \\ y_3 & y_4 - y_1 & 0 & -y_3 \\ 0 & -y_2 & y_3 & 0 \end{array}\right]$$

```
>
```

Appendix E

Maple file for Example 64

```
> restart: with(LinearAlgebra):
> #Dimension of the Lie algebra g;
> dimg := 5;
```

$$dimg := 5 \quad (1)$$

```
> #The Lie-Poisson matrix is the following:
> Poisson:=Matrix([[0,-2*x[2]+x[4]+2*x[5],2*x[3],0,x[4]],[2*x[2]-x[4]-2*x[5],0,x[4]-x[1],-x[4],0],[-2*x[3],x[1]-x[4],0,0,0],[0,x[4],0,0,x[4]],[-x[4],0,0,-x[4],0]]); Determinant(Poisson);
```

$$Poisson := \begin{bmatrix} 0 & -2x_2+x_4+2x_5 & 2x_3 & 0 & x_4 \\ 2x_2-x_4-2x_5 & 0 & x_4-x_1 & -x_4 & 0 \\ -2x_3 & x_1-x_4 & 0 & 0 & 0 \\ 0 & x_4 & 0 & 0 & x_4 \\ -x_4 & 0 & 0 & -x_4 & 0 \end{bmatrix}$$

$$0$$

```
> mu:=<1,0,0,1,0>;
```

$$\mu := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

```
> #Dimension and codimension of the isotropy subalgebra g_mu (which is also the dimension of the transverse Poisson structure);
> dimtrans := 3; codimtrans := 2;
```

$$dimtrans := 3$$

$$codimtrans := 2 \quad (2)$$

```
> #Computing g_mu (= ker P(mu)):
> Poisson_mu:=Poisson: for i from 1 to dimg do Poisson_mu:=subs(x[i]=mu[i],Poisson_mu) end do: Poisson_mu;
```

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 \end{bmatrix} \quad (3)$$

```
> NullSpace(Poisson_mu);
```

(4)

$$\left\{\begin{bmatrix}0\\-1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}\right\} \quad (4)$$

> #MAPLE generates a random basis for **g_mu** each time the worksheet is computed. For coherence, we choose our own set of generators:

```
> F[1] := <0, 0, 1, 0, 0> : F[2] := <0, 1, 0, 0,-1> : F[3] := <1, 0, 0,-1,
    0> :
```

Basis for $\boldsymbol{h}$:

```
> G[1]:=<0,0,0,1,0>: G[2]:=<0,0,0,0,1>:
```

Basis for $\boldsymbol{h}^a$:

```
> H[1]:=<1,0,0,0,0>: H[2]:=<0,1,0,0,0>: H[3]:=<0,0,1,0,0>:
```

Consider ***nu*** arbitrary element of $\boldsymbol{h}^a$:

```
> for i from 1 to dimtrans do k[i]:=VectorScalarMultiply(H[i],y
  [i]) end do: nu:=sum(k[j],j=1..dimtrans); munu:=VectorAdd(mu,
  nu);
```

$$\nu := \begin{bmatrix}y_1\\y_2\\y_3\\0\\0\end{bmatrix}$$

$$munu := \begin{bmatrix}1+y_1\\y_2\\y_3\\1\\0\end{bmatrix}$$

```
> Poisson_munu:=Poisson: for i from 1 to dimg do Poisson_munu:=
  subs(x[i]=munu[i],Poisson_munu) end do: Poisson_munu;
```

$$\begin{bmatrix}0 & -2y_2+1 & 2y_3 & 0 & 1\\ 2y_2-1 & 0 & -y_1 & -1 & 0\\ -2y_3 & y_1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1\\ -1 & 0 & 0 & -1 & 0\end{bmatrix}$$

Computation of ad*_G1(mu + nu) and ad*_G2(mu + nu):

```
> for i from 1 to dimg do for j from 1 to codimtrans do k[j,i]
  := ScalarMultiply(Row(Poisson_munu,i),G[j][i]) end do end do:
  for j from 1 to codimtrans do W[j]:= eval(sum(k[j,l],l=1..
  dimg)) end do;
```

$$W_1 := \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$W_2 := \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

> Computation of Pi(ad*_Fi(nu)):

```
> M:= Matrix([H[1], H[2], H[3], Transpose(W[1]), Transpose(W[2]
)]); Mi:=MatrixInverse(M):
```

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Computing Z[1]:=ad*_F1(nu), Z[2]:=ad*_F2(nu), Z[3]:=ad*_F3(nu), Z[4]:=ad*_F4(nu):

```
> Poisson_nu:=Poisson: for i from 1 to dimg do Poisson_nu:=
subs(x[i]=nu[i],Poisson_nu) end do: Poisson_nu;
```

$$\begin{bmatrix} 0 & -2\,y_2 & 2\,y_3 & 0 & 0 \\ 2\,y_2 & 0 & -y_1 & 0 & 0 \\ -2\,y_3 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \textbf{(5)}$$

```
> for i from 1 to dimg do for j from 1 to dimtrans do k[j,i]:=
ScalarMultiply(Row(Poisson_nu,i), F[j][i]) end do end do: for
j from 1 to dimtrans do Z[j]:= eval(sum(k[j,l], l=1..dimg))
end do;
```

$$Z_1 := \begin{bmatrix} -2\,y_3 & y_1 & 0 & 0 & 0 \end{bmatrix}$$

$$Z_2 := \begin{bmatrix} 2\,y_2 & 0 & -y_1 & 0 & 0 \end{bmatrix}$$

$$Z_3 := \begin{bmatrix} 0 & -2\,y_2 & 2\,y_3 & 0 & 0 \end{bmatrix}$$

Computing the projections of the previous vectors:

```
> for j from 1 to dimtrans do v[j]:=MatrixVectorMultiply(Mi,
Transpose(Z[j])) end do:
> for i from 1 to dimtrans do for l from 1 to dimtrans do k[i,
l]:= ScalarMultiply(H[l],v[i][l]) end do end do: for i from 1
to dimtrans do Pi_nu[i]:= eval(sum(k[i,o], o=1..dimtrans))
end do;
```

$$\pi nu_1 := \begin{bmatrix} -2\,y_3 \\ y_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\pi nu_2 := \begin{bmatrix} 2\,y_2 \\ 0 \\ -y_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\pi nu_3 := \begin{bmatrix} 0 \\ -2\,y_2 \\ 2\,y_3 \\ 0 \\ 0 \end{bmatrix}$$

Finally, the entries of the Poisson matrix:

```
> printlevel:=2; for i from 1 to dimtrans do for j from i to
  dimtrans do trans[i,j]:= simplify(DotProduct(Transpose(Pi_nu
  [i]), F[j],conjugate=false)) od od;
```

$$printlevel := 2$$

$$trans_{1,\,1} := 0$$

$$trans_{1,\,2} := y_1$$

$$trans_{1,\,3} := -2\,y_3$$

$$trans_{2,\,2} := 0$$

$$trans_{2,\,3} := 2\,y_2$$

$$trans_{3,\,3} := 0$$

```
> TRANS:=Matrix(dimtrans,dimtrans,symbol=alpha,shape=
  antisymmetric): for i from 1 to dimtrans do for j from i+1 to
  dimtrans do TRANS:=subs(alpha[i,j]=trans[i,j],TRANS) od od:
  TRANS;
```

$$\begin{bmatrix} 0 & y_1 & -2\,y_3 \\ -y_1 & 0 & 2\,y_2 \\ 2\,y_3 & -2\,y_2 & 0 \end{bmatrix}$$

```
>
```

Appendix F

Maple file for Example 66 - $(\mathfrak{od}(4) \oplus \mathrm{A}_2)^*$

```
> restart:with(LinearAlgebra):
```

> #Dimension of the Lie algebra g;

```
> dimg := 6;
```

$$dimg := 6 \quad (1)$$

> #For the sake of simplicity of this worksheet, the names of the coordinates on g* were changed. The Lie-Poisson matrix in the "new" coordinates is the following:

```
> Poisson:=Matrix([[0,x[2],x[3],x[4],0,0],[-x[2],0,0,0,0,0],[-x
[3],0,0,0,0,0],[-x[4],0,0,0,0,0],[0,0,0,0,0,x[5]],[0,0,0,0,-x
[5],0]]); Determinant(Poisson);
```

$$Poisson := \begin{bmatrix} 0 & x_2 & x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & 0 & 0 & 0 & 0 \\ -x_3 & 0 & 0 & 0 & 0 & 0 \\ -x_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & -x_5 & 0 \end{bmatrix}$$

$$0$$

```
> mu := ⟨0, 1, 0, 0, 0, 0⟩;
```

$$\mu := \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

> #Dimension and codimension of the isotropy subalgebra g_mu (which is also the dimension of the transverse Poisson structure);

```
> dimtrans := 4; codimtrans := 2;
```

$$dimtrans := 4$$

$$codimtrans := 2 \quad (3)$$

> #Basis for **g_mu (= ker P(mu)):**

```
> Poisson_mu:=Poisson: for i from 1 to dimg do Poisson_mu:=
subs(x[i]=mu[i],Poisson_mu) end do: Poisson_mu;
```

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (4)$$

```
> F[1] := ⟨0, 0, 1, 0, 0, 0⟩ : F[2] := ⟨0, 0, 0, 1, 0, 0⟩ : F[3] := ⟨0, 0, 0,
    0, 1, 0⟩ : F[4] := ⟨0, 0, 0, 0, 0, 1⟩ :
```

Basis for $\boldsymbol{h}$:

```
> G[1]:=<1,0,0,0,0,0>: G[2]:=<0,1,0,0,0,0>:
```

Basis for $\boldsymbol{h}^a$:

```
> H[1]:=<0,0,1,0,0,0>: H[2]:=<0,0,0,1,0,0>: H[3]:=<0,0,0,0,1,
  0>: H[4]:=<0,0,0,0,0,1>:
```

Consider ***nu*** arbitrary element of $\boldsymbol{h}^a$:

```
> for i from 1 to dimtrans do k[i]:=VectorScalarMultiply(H[i],z
  [i]) end do: nu:=sum(k[j],j=1..dimtrans); munu:=VectorAdd(mu,
  nu);
```

$$\nu := \begin{bmatrix} 0 \\ 0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

$$munu := \begin{bmatrix} 0 \\ 1 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

```
> Poisson_munu:=Poisson: for i from 1 to dimg do Poisson_munu:=
  subs(x[i]=munu[i],Poisson_munu) end do: Poisson_munu;
```

$$\begin{bmatrix} 0 & 1 & z_1 & z_2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -z_1 & 0 & 0 & 0 & 0 & 0 \\ -z_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_3 \\ 0 & 0 & 0 & 0 & -z_3 & 0 \end{bmatrix}$$

Computation of ad*_G1(mu + nu) and ad*_G2(mu + nu):

```
> for i from 1 to dimg do for j from 1 to codimtrans do k[j,i]
  :=ScalarMultiply(Row(Poisson_munu,i),G[j][i]) end do end do:
  for j from 1 to codimtrans do W[j]:=eval(sum(k[j,l],l=1..
  dimg)) end do;
```

$$W_1 := \begin{bmatrix} 0 & 1 & z_1 & z_2 & 0 & 0 \end{bmatrix}$$

$$W_2 := \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

> Computation of Pi(ad*_Fi(nu)):

```
> M:=Matrix([H[1],H[2],H[3],H[4],Transpose(W[1]),Transpose(W[2]
```

```
)]); Mi:=MatrixInverse(M):
```

$$M := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & z_1 & 0 \\ 0 & 1 & 0 & 0 & z_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Computing Z[1]:=ad*_F1(nu), Z[2]:=ad*_F2(nu), Z[3]:=ad*_F3(nu), Z[4]:=ad*_F4(nu):

```
> Poisson_nu:=Poisson: for i from 1 to dimg do Poisson_nu:=subs
(x[i]=nu[i],Poisson_nu) end do: Poisson_nu;
```

$$\begin{bmatrix} 0 & 0 & z_1 & z_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -z_1 & 0 & 0 & 0 & 0 & 0 \\ -z_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_3 \\ 0 & 0 & 0 & 0 & -z_3 & 0 \end{bmatrix}$$

```
> for i from 1 to dimg do for j from 1 to dimtrans do k[j,i]:=
ScalarMultiply(Row(Poisson_nu,i),F[j][i]) end do end do: for
j from 1 to dimtrans do Z[j]:=eval(sum(k[j,l],l=1..dimg)) end
do;
```

$$Z_1 := \begin{bmatrix} -z_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Z_2 := \begin{bmatrix} -z_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Z_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & z_3 \end{bmatrix}$$

$$Z_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & -z_3 & 0 \end{bmatrix}$$

Computing the projections of the previous vectors:

```
> for j from 1 to dimtrans do v[j]:=MatrixVectorMultiply(Mi,
Transpose(Z[j])) end do:
> for i from 1 to dimtrans do for l from 1 to dimtrans do k[i,
l]:=ScalarMultiply(H[l],v[i][l]) end do end do: for i from 1
to dimtrans do Pi_nu[i]:=eval(sum(k[i,o],o=1..dimtrans)) end
do;
```

$$\pi nu_1 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\pi nu_2 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\pi nu_3 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ z_3 \end{bmatrix}$$

$$\pi nu_4 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -z_3 \\ 0 \end{bmatrix}$$

Finally, the entries of the Poisson matrix:

```
> printlevel:=2;for i from 1 to dimtrans do for j from i to
  dimtrans do trans[i,j]:=simplify(DotProduct(Transpose(Pi_nu
  [i]),F[j],conjugate=false)) od od;
```

$$printlevel := 2$$

$$trans_{1,1} := 0$$

$$trans_{1,2} := 0$$

$$trans_{1,3} := 0$$

$$trans_{1,4} := 0$$

$$trans_{2,2} := 0$$

$$trans_{2,3} := 0$$

$$trans_{2,4} := 0$$

$$trans_{3,3} := 0$$

$$trans_{3,4} := z_3$$

$$trans_{4,4} := 0$$

```
> TRANS:=Matrix(dimtrans,dimtrans,symbol=alpha,shape=
```

```
antisymmetric): for i from 1 to dimtrans do for j from i+1 to
dimtrans do TRANS:=subs(alpha[i,j]=trans[i,j],TRANS) od od:
TRANS;
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_3 \\ 0 & 0 & -z_3 & 0 \end{bmatrix}$$

```
>
```

Appendix G

Maple file for Example 75 - $\mathfrak{sp}(4)^*$

```
> restart:with(LinearAlgebra):
```

> #Dimension of the Lie algebra g;

```
> dimg := 10;
```

$$dimg := 10 \qquad (1)$$

> #The Lie-Poisson matrix is the following:

```
> Poisson:=Matrix([[0,x[2],-x[3],0,2*x[5],x[6],0,-2*x[8],-x[9],
0],[-x[2],0,x[1]-x[4],x[2],0,2*x[5],x[6],-x[9],-2*x[10],0],[x
[3],x[4]-x[1],0,-x[3],x[6],2*x[7],0,0,-2*x[8],-x[9]],[0,-x[2]
,x[3],0,0,x[6],2*x[7],0,-x[9],-2*x[10]],[-2*x[5],0,-x[6],0,0,
0,0,x[1],x[2],0],[-x[6],-2*x[5],-2*x[7],-x[6],0,0,0,x[3],x[1]
+x[4],x[2]],[0,-x[6],0,-2*x[7],0,0,0,0,x[3],x[4]],[2*x[8],x
[9],0,0,-x[1],-x[3],0,0,0,0],[x[9],2*x[10],2*x[8],x[9],-x[2],
-x[1]-x[4],-x[3],0,0,0],[0,0,x[9],2*x[10],0,-x[2],-x[4],0,0,
0]]); Determinant(Poisson);
```

$$Poisson := \begin{bmatrix} 0 & x_2 & -x_3 & 0 & 2\,x_5 & x_6 & 0 & -2\,x_8 & -x_9 & 0 \\ -x_2 & 0 & x_1 - x_4 & x_2 & 0 & 2\,x_5 & x_6 & -x_9 & -2\,x_{10} & 0 \\ x_3 & x_4 - x_1 & 0 & -x_3 & x_6 & 2\,x_7 & 0 & 0 & -2\,x_8 & -x_9 \\ 0 & -x_2 & x_3 & 0 & 0 & x_6 & 2\,x_7 & 0 & -x_9 & -2\,x_{10} \\ -2\,x_5 & 0 & -x_6 & 0 & 0 & 0 & 0 & x_1 & x_2 & 0 \\ -x_6 & -2\,x_5 & -2\,x_7 & -x_6 & 0 & 0 & 0 & x_3 & x_1 + x_4 & x_2 \\ 0 & -x_6 & 0 & -2\,x_7 & 0 & 0 & 0 & 0 & x_3 & x_4 \\ 2\,x_8 & x_9 & 0 & 0 & -x_1 & -x_3 & 0 & 0 & 0 & 0 \\ x_9 & 2\,x_{10} & 2\,x_8 & x_9 & -x_2 & -x_1 - x_4 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & x_9 & 2\,x_{10} & 0 & -x_2 & -x_4 & 0 & 0 & 0 \end{bmatrix}$$

$$0$$

```
> mu:=<0,1,0,0,0,0,0,0,0,0>;
```

$$\mu := \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

> #Dimension and codimension of the isotropy subalgebra g_mu (which is also the dimension of the transverse Poisson structure);

```
> dimtrans := 4; codimtrans := 6;
```

$$dimtrans := 4$$

$$codimtrans := 6 \qquad (2)$$

> #Basis for **g_mu (= ker P(mu))**:

```
> Poisson_mu:=Poisson: for i from 1 to dimg do Poisson_mu:=
  subs(x[i]=mu[i],Poisson_mu) end do: Poisson_mu;
```

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3)$$

```
> NullSpace(Poisson_mu);
```

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad (4)$$

> #MAPLE generates a random basis for **g_mu** each time the worksheet is computed. For coherence, we choose our own set of generators:

```
> F[1] := ⟨1, 0, 0, 1, 0, 0, 0, 0, 0, 0⟩ : F[2] := ⟨0, 0, 1, 0, 0, 0, 0, 0, 0,
  0⟩ : F[3] := ⟨0, 0, 0, 0, 0, 0, 1, 0, 0, 0⟩ : F[4] := ⟨0, 0, 0, 0, 0, 0,
  0, 1, 0, 0⟩ :
```

Basis for ***h***:

```
> G[1]:=<1,0,0,0,0,0,0,0,0,0>: G[2]:=<0,1,0,0,0,0,0,0,0,0>: G
  [3]:=<0,0,0,0,1,0,0,0,0,0>: G[4]:=<0,0,0,0,0,1,0,0,0,0>: G[5]
  :=<0,0,0,0,0,0,0,0,1,0>: G[6]:=<0,0,0,0,0,0,0,0,0,1>:
```

Basis for $\boldsymbol{h}^q$:

```
> H[1]:=<0,0,1,0,0,0,0,0,0,0>: H[2]:=<0,0,0,1,0,0,0,0,0,0>: H
  [3]:=<0,0,0,0,0,0,1,0,0,0>: H[4]:=<0,0,0,0,0,0,0,1,0,0>:
```

Consider ***nu*** arbitrary element of $\boldsymbol{h}^q$:

```
> for i from 1 to dimtrans do k[i]:=VectorScalarMultiply(H[i],y
  [i]) end do: nu:=sum(k[j],j=1..dimtrans); munu:=VectorAdd(mu,
  nu);
```

$$\nu := \begin{bmatrix} 0 \\ 0 \\ y_1 \\ y_2 \\ 0 \\ 0 \\ y_3 \\ y_4 \\ 0 \\ 0 \end{bmatrix}$$

$$munu := \begin{bmatrix} 0 \\ 1 \\ y_1 \\ y_2 \\ 0 \\ 0 \\ y_3 \\ y_4 \\ 0 \\ 0 \end{bmatrix}$$

```
> Poisson_munu:=Poisson: for i from 1 to dimg do Poisson_munu:=
  subs(x[i]=munu[i],Poisson_munu) end do: Poisson_munu;
```

$$\begin{bmatrix} 0 & 1 & -y_1 & 0 & 0 & 0 & 0 & -2\,y_4 & 0 & 0 \\ -1 & 0 & -y_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1 & y_2 & 0 & -y_1 & 0 & 2\,y_3 & 0 & 0 & -2\,y_4 & 0 \\ 0 & -1 & y_1 & 0 & 0 & 0 & 2\,y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2\,y_3 & 0 & 0 & 0 & 0 & y_1 & y_2 & 1 \\ 0 & 0 & 0 & -2\,y_3 & 0 & 0 & 0 & 0 & y_1 & y_2 \\ 2\,y_4 & 0 & 0 & 0 & 0 & -y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,y_4 & 0 & -1 & -y_2 & -y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -y_2 & 0 & 0 & 0 \end{bmatrix}$$

Computation of ad*_G1(mu + nu) and ad*_G2(mu + nu):

```
> for i from 1 to dimg do for j from 1 to codimtrans do k[j,i]
  :=ScalarMultiply(Row(Poisson_munu,i),G[j][i]) end do end do:
  for j from 1 to codimtrans do W[j]:=eval(sum(k[j,l],l=1..
  dimg)) end do;
```

$$W_1 := \begin{bmatrix} 0 & 1 & -y_1 & 0 & 0 & 0 & 0 & -2\,y_4 & 0 & 0 \end{bmatrix}$$

$$W_2 := \begin{bmatrix} -1 & 0 & -y_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_4 := \begin{bmatrix} 0 & 0 & -2\,y_3 & 0 & 0 & 0 & 0 & y_1 & y_2 & 1 \end{bmatrix}$$

$$W_5 := \begin{bmatrix} 0 & 0 & 2\,y_4 & 0 & -1 & -y_2 & -y_1 & 0 & 0 & 0 \end{bmatrix}$$

$$W_6 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -y_2 & 0 & 0 & 0 \end{bmatrix}$$

> Computation of Pi(ad*_Fi(nu)):

```
> for i from 1 to 6 do W[i] := Transpose(W[i]) end do:
> M:=Matrix([H[1],H[2],H[3],H[4],W[1],W[2],W[3],W[4],W[5],W[6]]
  ); Mi:=MatrixInverse(M):
```

$$M := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -y_1 & -y_2 & 0 & -2\,y_3 & 2\,y_4 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \\ 0 & 0 & 0 & 1 & -2\,y_4 & 0 & 0 & y_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Computing Z[1]:=ad*_F1(nu), Z[2]:=ad*_F2(nu), Z[3]:=ad*_F3(nu), Z[4]:=ad*_F4(nu):

```
> Poisson_nu:=Poisson: for i from 1 to dimg do Poisson_nu:=subs
  (x[i]=nu[i],Poisson_nu) end do: Poisson_nu;
```

$$\begin{bmatrix} 0 & 0 & -y_1 & 0 & 0 & 0 & 0 & -2\,y_4 & 0 & 0 \\ 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1 & y_2 & 0 & -y_1 & 0 & 2\,y_3 & 0 & 0 & -2\,y_4 & 0 \\ 0 & 0 & y_1 & 0 & 0 & 0 & 2\,y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\,y_3 & 0 & 0 & 0 & 0 & y_1 & y_2 & 0 \\ 0 & 0 & 0 & -2\,y_3 & 0 & 0 & 0 & 0 & y_1 & y_2 \\ 2\,y_4 & 0 & 0 & 0 & 0 & -y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,y_4 & 0 & 0 & -y_2 & -y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -y_2 & 0 & 0 & 0 \end{bmatrix}$$

```
> for i from 1 to dimg do for j from 1 to dimtrans do k[j,i]:=
  ScalarMultiply(Row(Poisson_nu,i),F[j][i]) end do end do: for
  j from 1 to dimtrans do Z[j]:=eval(sum(k[j,l],l=1..dimg)) end
  do;
```

$$Z_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2\,y_3 & -2\,y_4 & 0 & 0 \end{bmatrix}$$

$$Z_2 := \begin{bmatrix} y_1 & y_2 & 0 & -y_1 & 0 & 2\,y_3 & 0 & 0 & -2\,y_4 & 0 \end{bmatrix}$$

$$Z_3 := \begin{bmatrix} 0 & 0 & 0 & -2\,y_3 & 0 & 0 & 0 & 0 & y_1 & y_2 \end{bmatrix}$$

$$Z_4 := \begin{bmatrix} 2\,y_4 & 0 & 0 & 0 & 0 & -y_1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Computing the projections of the previous vectors:

```
> for j from 1 to dimtrans do v[j]:=MatrixVectorMultiply(Mi,
  Transpose(Z[j])) end do:
> for i from 1 to dimtrans do for l from 1 to dimtrans do k[i,
  l]:=ScalarMultiply(H[l],v[i][l]) end do end do: for i from 1
  to dimtrans do Pi_nu[i]:=eval(sum(k[i,o],o=1..dimtrans)) end
  do:
```

Finally, the entries of the Poisson matrix:

```
> printlevel:=2;for i from 1 to dimtrans do for j from i to
  dimtrans do trans[i,j]:=simplify(DotProduct(Transpose(Pi_nu
  [i]),F[j],conjugate=false)) od od;
```

$$printlevel := 2$$

$$trans_{1,\,1} := 0$$

$$trans_{1,\,2} := 0$$

$$trans_{1,\,3} := 2\,y_3$$

$$trans_{1,\,4} := -2\,y_4$$

$$trans_{2,\,2} := 0$$

$$trans_{2,\,3} := -2\,y_2\,y_3$$

$$trans_{2,4} := 2\,y_4\,y_2$$

$$trans_{3,3} := 0$$

$$trans_{3,4} := -y_2\,y_1$$

$$trans_{4,4} := 0$$

```
> TRANS:=Matrix(dimtrans,dimtrans,symbol=alpha,shape=
antisymmetric): for i from 1 to dimtrans do for j from i+1 to
dimtrans do TRANS:=subs(alpha[i,j]=trans[i,j],TRANS) od od:
TRANS;
```

$$\begin{bmatrix} 0 & 0 & 2\,y_3 & -2\,y_4 \\ 0 & 0 & -2\,y_2\,y_3 & 2\,y_4\,y_2 \\ -2\,y_3 & 2\,y_2\,y_3 & 0 & -y_2\,y_1 \\ 2\,y_4 & -2\,y_4\,y_2 & y_2\,y_1 & 0 \end{bmatrix}$$

```
>
```

Appendix H

Maple file for Example 77

```
> restart:with(LinearAlgebra):
> #Dimension of the Lie algebra g;
> dimg := 4;
```

$$dimg := 4 \quad (1)$$

```
> #For the sake of simplicity of this worksheet, the names of the coordinates on g* were changed (x[1] -> t[3], x[2] -> t[4]). The Lie-Poisson matrix in the "new" coordinates is the following:
> Poisson:=Matrix([[0,0,t[2],k*t[3]],[0,0,0,t[2]],[-t[2],0,0,t[1]+t[3]],[-k*t[3],-t[2],-t[1]-t[3],0]]); Determinant(Poisson);
```

$$Poisson := \begin{bmatrix} 0 & 0 & t_2 & k\,t_3 \\ 0 & 0 & 0 & t_2 \\ -t_2 & 0 & 0 & t_1 + t_3 \\ -k\,t_3 & -t_2 & -t_1 - t_3 & 0 \end{bmatrix}$$

$$t_2^4$$

```
> mu := ⟨1, 0, 0, 1⟩;
```

$$\mu := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (2)$$

```
> #Dimension and codimension of the isotropy subalgebra g_mu (which is also the dimension of the transverse Poisson structure);
> dimtrans := 2; codimtrans := 2;
```

$$dimtrans := 2$$

$$codimtrans := 2 \quad (3)$$

> #Basis for **g_mu (= ker P(mu))**:

```
> Poisson_mu:=Poisson: for i from 1 to dimg do Poisson_mu:=subs(t[i]=mu[i],Poisson_mu) end do: Poisson_mu;
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (4)$$

```
> F[1] := ⟨1, 0, 0, 0⟩ : F[2] := ⟨0, 1, 0, 0⟩ :
```

Basis for ***h***:

```
> G[1]:=<0,0,1,0>: G[2]:=<0,0,0,1>:
```

Basis for $\boldsymbol{h}^{a}$:

```
> H[1]:=<1,0,0,0>: H[2]:=<0,1,0,0>:
```

Consider ***nu*** arbitrary element of $\boldsymbol{h}^{a}$:

```
> for i from 1 to dimtrans do k[i]:=VectorScalarMultiply(H[i],y[i]) end do: nu:=sum(k[j],j=1..dimtrans); munu:=VectorAdd(mu,nu);
```

$$\nu := \begin{bmatrix} y_1 \\ y_2 \\ 0 \\ 0 \end{bmatrix}$$

$$munu := \begin{bmatrix} 1 + y_1 \\ y_2 \\ 0 \\ 1 \end{bmatrix}$$

```
> Poisson_munu:=Poisson: for i from 1 to dimg do Poisson_munu:=
subs(t[i]=munu[i],Poisson_munu) end do: Poisson_munu;
```

$$\begin{bmatrix} 0 & 0 & y_2 & 0 \\ 0 & 0 & 0 & y_2 \\ -y_2 & 0 & 0 & 1 + y_1 \\ 0 & -y_2 & -1 - y_1 & 0 \end{bmatrix}$$

Computation of ad*_G1(mu + nu) and ad*_G2(mu + nu):

```
> for i from 1 to dimg do for j from 1 to codimtrans do k[j,i]
:=ScalarMultiply(Row(Poisson_munu,i),G[j][i]) end do end do:
for j from 1 to codimtrans do W[j]:=eval(sum(k[j,l],l=1..
dimg)) end do;
```

$$W_1 := \begin{bmatrix} -y_2 & 0 & 0 & 1 + y_1 \end{bmatrix}$$

$$W_2 := \begin{bmatrix} 0 & -y_2 & -1 - y_1 & 0 \end{bmatrix}$$

> Computation of Pi(ad*_Fi(nu)):

```
> M:=Matrix([H[1],H[2],Transpose(W[1]),Transpose(W[2])]); Mi:=
MatrixInverse(M):
```

$$M := \begin{bmatrix} 1 & 0 & -y_2 & 0 \\ 0 & 1 & 0 & -y_2 \\ 0 & 0 & 0 & -1 - y_1 \\ 0 & 0 & 1 + y_1 & 0 \end{bmatrix}$$

Computing Z[1]:=ad*_F1(nu), Z[2]:=ad*_F2(nu), Z[3]:=ad*_F3(nu), Z[4]:=ad*_F4(nu):

```
> Poisson_nu:=Poisson: for i from 1 to dimg do Poisson_nu:=subs
(t[i]=nu[i],Poisson_nu) end do: Poisson_nu;
```

$$\begin{bmatrix} 0 & 0 & y_2 & 0 \\ 0 & 0 & 0 & y_2 \\ -y_2 & 0 & 0 & y_1 \\ 0 & -y_2 & -y_1 & 0 \end{bmatrix}$$

```
> for i from 1 to dimg do for j from 1 to dimtrans do k[j,i]:=
  ScalarMultiply(Row(Poisson_nu,i),F[j][i]) end do end do: for
  j from 1 to dimtrans do Z[j]:=eval(sum(k[j,l],l=1..dimg)) end
  do;
```

$$Z_1 := \begin{bmatrix} 0 & 0 & y_2 & 0 \end{bmatrix}$$

$$Z_2 := \begin{bmatrix} 0 & 0 & 0 & y_2 \end{bmatrix}$$

Computing the projections of the previous vectors:

```
> for j from 1 to dimtrans do v[j]:=MatrixVectorMultiply(Mi,
  Transpose(Z[j])) end do:
> for i from 1 to dimtrans do for l from 1 to dimtrans do k[i,
  l]:=ScalarMultiply(H[l],v[i][l]) end do end do: for i from 1
  to dimtrans do Pi_nu[i]:=eval(sum(k[i,o],o=1..dimtrans)) end
  do;
```

$$\pi nu_1 := \begin{bmatrix} 0 \\ -\frac{y_2^2}{1+y_1} \\ 0 \\ 0 \end{bmatrix}$$

$$\pi nu_2 := \begin{bmatrix} \frac{y_2^2}{1+y_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Finally, the entries of the Poisson matrix:

```
> printlevel:=2;for i from 1 to dimtrans do for j from i to
  dimtrans do trans[i,j]:=simplify(DotProduct(Transpose(Pi_nu
  [i]),F[j],conjugate=false)) od od;
```

$$printlevel := 2$$

$$trans_{1,\,1} := 0$$

$$trans_{1,\,2} := -\frac{y_2^2}{1+y_1}$$

$$trans_{2,\,2} := 0$$

```
> TRANS:=Matrix(dimtrans,dimtrans,symbol=alpha,shape=
  antisymmetric): for i from 1 to dimtrans do for j from i+1 to
  dimtrans do TRANS:=subs(alpha[i,j]=trans[i,j],TRANS) od od:
  TRANS;
```

$$\begin{bmatrix} 0 & -\frac{y_2^2}{1+y_1} \\ \frac{y_2^2}{1+y_1} & 0 \end{bmatrix}$$

Appendix I

Maple file for Example 79 - $\mathfrak{se}(3)^*$

```
> restart:with(LinearAlgebra):
> #Dimension of the Lie algebra g;
> dimg := 6;
```

$$dimg := 6 \tag{1}$$

```
> #The Lie-Poisson matrix is the following:
> Poisson:=Matrix(6,6,shape=antisymmetric): Poisson[1,2]:=-(a/c)*x[1]-(b/c)*x[2]+(1/c)*x[3]: Poisson[1,3]:=-(a*b/c)*x[1]-((b^2+c^2)/c)*x[2]+(b/c)*x[3]: Poisson[1,5]:=x[6]: Poisson[1,6]:=-x[5]: Poisson[2,3]:=((a^2+c^2)/c)*x[1]+(a*b/c)*x[2]-(a/c)*x[3]: Poisson[2,4]:=-x[6]: Poisson[2,6]:=x[4]: Poisson[3,4]:=-b*x[6]+c*x[5]: Poisson[3,5]:=a*x[6]-c*x[4]: Poisson[3,6]:=-a*x[5]+b*x[4]: Poisson; Determinant(Poisson);
```

$$\left[\left[0, -\frac{a x_1}{c} - \frac{b x_2}{c} + \frac{x_3}{c}, -\frac{a b x_1}{c} - \frac{(b^2+c^2) x_2}{c} + \frac{b x_3}{c}, 0, x_6, -x_5\right],\right.$$
$$\left[\frac{a x_1}{c} + \frac{b x_2}{c} - \frac{x_3}{c}, 0, \frac{(a^2+c^2) x_1}{c} + \frac{a b x_2}{c} - \frac{a x_3}{c}, -x_6, 0, x_4\right],$$
$$\left[\frac{a b x_1}{c} + \frac{(b^2+c^2) x_2}{c} - \frac{b x_3}{c}, -\frac{(a^2+c^2) x_1}{c} - \frac{a b x_2}{c} + \frac{a x_3}{c}, 0, -b x_6 + c x_5, a x_6 - c x_4, -a x_5 + b x_4\right],$$
$$\left[0, x_6, b x_6 - c x_5, 0, 0, 0\right],$$
$$\left[-x_6, 0, -a x_6 + c x_4, 0, 0, 0\right],$$
$$\left.\left[x_5, -x_4, a x_5 - b x_4, 0, 0, 0\right]\right]$$

$$0$$

```
> mu:=<a,b,a^2+b^2+c^2,0,0,0>;
```

$$\mu := \begin{bmatrix} a \\ b \\ a^2+b^2+c^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
> #Dimension and codimension of the isotropy subalgebra g_mu (which is also the dimension of the transverse Poisson structure);
> dimtrans := 4; codimtrans := 2;
```

$$dimtrans := 4$$

$$codimtrans := 2 \tag{2}$$

> #Basis for **g_mu (= ker P(mu))**:

```
> Poisson_mu:=Poisson: for i from 1 to dimg do Poisson_mu:=
  simplify((subs(x[i]=mu[i],Poisson_mu))) end do: Poisson_mu;
```

$$\begin{bmatrix} 0 & c & 0 & 0 & 0 & 0 \\ -c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

```
> F[1] := ⟨0, 0, 1, 0, 0, 0⟩ : F[2] := ⟨0, 0, 0, 1, 0, 0⟩ : F[3] := ⟨0, 0, 0,
    0, 1, 0⟩ : F[4] := ⟨0, 0, 0, 0, 0, 1⟩ :
```

Basis for ***h***:

```
> G[1]:=<1,A[2],A[3],A[4],A[5],A[6]>: G[2]:=<0,1,B[3],B[4],B[5]
  ,B[6]>:
```

Basis for ***h***^{a}:

```
> H[1]:=<0,0,1,0,0,0>: H[2]:=<0,0,0,1,0,0>: H[3]:=<0,0,0,0,1,
  0>: H[4]:=<0,0,0,0,0,1>:
```

Consider ***nu*** arbitrary element of ***h***^{a}:

```
> for i from 1 to dimtrans do k[i]:=VectorScalarMultiply(H[i],y
  [i]) end do: nu:=sum(k[j],j=1..dimtrans); munu:=VectorAdd(mu,
  nu);
```

$$\nu := \begin{bmatrix} 0 \\ 0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$munu := \begin{bmatrix} a \\ b \\ a^2 + b^2 + c^2 + y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

```
> Poisson_munu:=Poisson: for i from 1 to dimg do Poisson_munu:=
  subs(x[i]=munu[i],Poisson_munu) end do: Poisson_munu;
```

$$\Bigg[\Bigg[0,\ -\frac{a^2}{c} - \frac{b^2}{c} + \frac{a^2+b^2+c^2+y_1}{c},\ -\frac{a^2\,b}{c} - \frac{(b^2+c^2)\,b}{c} + \frac{b\,(a^2+b^2+c^2+y_1)}{c},\ 0,\ y_4,\ -y_3\Bigg],$$

$$\left[\frac{a^2}{c}+\frac{b^2}{c}-\frac{a^2+b^2+c^2+y_1}{c}, 0, \frac{(a^2+c^2)\,a}{c}+\frac{a\,b^2}{c}\right.$$

$$\left.-\frac{a\,(a^2+b^2+c^2+y_1)}{c}, -y_4, 0, y_2\right],$$

$$\left[\frac{a^2\,b}{c}+\frac{(b^2+c^2)\,b}{c}-\frac{b\,(a^2+b^2+c^2+y_1)}{c}, -\frac{(a^2+c^2)\,a}{c}-\frac{a\,b^2}{c}\right.$$

$$\left.+\frac{a\,(a^2+b^2+c^2+y_1)}{c}, 0, -b\,y_4+c\,y_3, a\,y_4-c\,y_2, -a\,y_3+b\,y_2\right],$$

$$\left[0, y_4, b\,y_4-c\,y_3, 0, 0, 0\right],$$

$$\left[-y_4, 0, -a\,y_4+c\,y_2, 0, 0, 0\right],$$

$$\left[y_3, -y_2, a\,y_3-b\,y_2, 0, 0, 0\right]\Big]$$

Computation of ad*_G1(mu + nu) and ad*_G2(mu + nu):

```
> for i from 1 to dimg do for j from 1 to codimtrans do k[j,i]
  :=ScalarMultiply(Row(Poisson_munu,i),G[j][i]) end do end do:
  for j from 1 to codimtrans do W[j]:=eval(sum(k[j,l],l=1..
  dimg)) end do:
```

> Computation of Pi(ad*_Fi(nu)):

```
> M:=Matrix([H[1],H[2],H[3],H[4],Transpose(W[1]),Transpose(W[2]
  )]); Mi:=MatrixInverse(M):
```

$$M := \left[\left[0, 0, 0, 0, A_2\left(\frac{a^2}{c}+\frac{b^2}{c}-\frac{a^2+b^2+c^2+y_1}{c}\right)+A_3\left(\frac{a^2\,b}{c}+\frac{(b^2+c^2)\,b}{c}\right.\right.\right.$$

$$\left.-\frac{b\,(a^2+b^2+c^2+y_1)}{c}\right)-A_5\,y_4+A_6\,y_3, \frac{a^2}{c}+\frac{b^2}{c}-\frac{a^2+b^2+c^2+y_1}{c}$$

$$\left.+B_3\left(\frac{a^2\,b}{c}+\frac{(b^2+c^2)\,b}{c}-\frac{b\,(a^2+b^2+c^2+y_1)}{c}\right)-B_5\,y_4+B_6\,y_3\right],$$

$$\left[0, 0, 0, 0, -\frac{a^2}{c}-\frac{b^2}{c}+\frac{a^2+b^2+c^2+y_1}{c}+A_3\left(-\frac{(a^2+c^2)\,a}{c}-\frac{a\,b^2}{c}\right.\right.$$

$$\left.+\frac{a\,(a^2+b^2+c^2+y_1)}{c}\right)+A_4\,y_4-A_6\,y_2, B_3\left(-\frac{(a^2+c^2)\,a}{c}-\frac{a\,b^2}{c}\right.$$

$$\left.\left.+\frac{a\,(a^2+b^2+c^2+y_1)}{c}\right)+B_4\,y_4-B_6\,y_2\right],$$

$$\left[1, 0, 0, 0, -\frac{a^2\,b}{c}-\frac{(b^2+c^2)\,b}{c}+\frac{b\,(a^2+b^2+c^2+y_1)}{c}+A_2\left(\frac{(a^2+c^2)\,a}{c}\right.\right.$$

$$+\frac{a b^2}{c}-\frac{a\left(a^2+b^2+c^2+y_1\right)}{c}\Bigg)+A_4\left(b y_4-c y_3\right)+A_5\left(-a y_4+c y_2\right)$$

$$+A_6\left(a y_3-b y_2\right), \frac{\left(a^2+c^2\right) a}{c}+\frac{a b^2}{c}-\frac{a\left(a^2+b^2+c^2+y_1\right)}{c}+B_4\left(b y_4\right.$$

$$\left.-c y_3\right)+B_5\left(-a y_4+c y_2\right)+B_6\left(a y_3-b y_2\right)\Bigg],$$

$$\left[0, 1, 0, 0, -A_2 y_4+A_3\left(-b y_4+c y_3\right), -y_4+B_3\left(-b y_4+c y_3\right)\right],$$

$$\left[0, 0, 1, 0, y_4+A_3\left(a y_4-c y_2\right), B_3\left(a y_4-c y_2\right)\right],$$

$$\left[0, 0, 0, 1, -y_3+A_2 y_2+A_3\left(-a y_3+b y_2\right), y_2+B_3\left(-a y_3+b y_2\right)\right]\Bigg]$$

Computing Z[1]:=ad*_F1(nu), Z[2]:=ad*_F2(nu), Z[3]:=ad*_F3(nu), Z[4]:=ad*_F4(nu):

```
> Poisson_nu:=Poisson: for i from 1 to dimg do Poisson_nu:=subs
  (x[i]=nu[i],Poisson_nu) end do: Poisson_nu;
```

$$\begin{bmatrix} 0 & \frac{y_1}{c} & \frac{b y_1}{c} & 0 & y_4 & -y_3 \\ -\frac{y_1}{c} & 0 & -\frac{a y_1}{c} & -y_4 & 0 & y_2 \\ -\frac{b y_1}{c} & \frac{a y_1}{c} & 0 & -b y_4+c y_3 & a y_4-c y_2 & -a y_3+b y_2 \\ 0 & y_4 & b y_4-c y_3 & 0 & 0 & 0 \\ -y_4 & 0 & -a y_4+c y_2 & 0 & 0 & 0 \\ y_3 & -y_2 & a y_3-b y_2 & 0 & 0 & 0 \end{bmatrix}$$

```
> for i from 1 to dimg do for j from 1 to dimtrans do k[j,i]:=
  ScalarMultiply(Row(Poisson_nu,i),F[j][i]) end do end do: for
  j from 1 to dimtrans do Z[j]:=eval(sum(k[j,l],l=1..dimg)) end
  do;
```

$$Z_1 := \begin{bmatrix} -\frac{b y_1}{c} & \frac{a y_1}{c} & 0 & -b y_4+c y_3 & a y_4-c y_2 & -a y_3+b y_2 \end{bmatrix}$$

$$Z_2 := \begin{bmatrix} 0 & y_4 & b y_4-c y_3 & 0 & 0 & 0 \end{bmatrix}$$

$$Z_3 := \begin{bmatrix} -y_4 & 0 & -a y_4+c y_2 & 0 & 0 & 0 \end{bmatrix}$$

$$Z_4 := \begin{bmatrix} y_3 & -y_2 & a y_3-b y_2 & 0 & 0 & 0 \end{bmatrix}$$

Computing the projections of the previous vectors:

```
> for j from 1 to dimtrans do v[j]:=MatrixVectorMultiply(Mi,
  Transpose(Z[j])) end do:
```

```
> for i from 1 to dimtrans do for l from 1 to dimtrans do k[i,
  l]:=ScalarMultiply(H[l],v[i][l]) end do end do: for i from 1
  to dimtrans do Pi_nu[i]:=eval(sum(k[i,o],o=1..dimtrans)) end
  do:
```

Finally, the entries of the Poisson matrix:

```
> printlevel:=2;for i from 1 to dimtrans do for j from i to
  dimtrans do trans[i,j]:=simplify(DotProduct(Transpose(Pi_nu
  [i]),F[j],conjugate=false)) od od;
```

$$printlevel := 2$$

$$trans_{1,1} := \Big(y_1\, c\,\big(-a\,A_4\,y_3\,y_1 - a\,B_5\,y_2\,A_2\,y_1 + a\,A_5\,y_2\,y_1 + a\,B_5\,c\,y_2\,A_6\,y_3 - a\,A_5\,c\,y_2\,B_6\,y_3 - a\,A_6\,c\,b\,y_2 + a\,B_4\,c^2\,y_3\,A_2 + a\,B_6\,c\,b\,y_2\,A_2 + b\,B_5\,c\,y_2\,A_4\,y_4 + a\,A_4\,c\,y_3^2\,B_6 - B_6\,c\,a^2\,y_3\,A_2 + b\,A_5\,c\,y_2^2\,B_6 + a\,A_4\,c\,b\,y_4 - a\,B_4\,c\,y_3^2\,A_6 - a\,B_5\,c^2\,y_2\,A_2 - b\,B_5\,c\,a\,y_4 - b\,A_5\,c\,y_2\,B_4\,y_4 - b\,A_4\,c\,y_3\,B_6\,y_2 - b\,B_5\,c\,y_2^2\,A_6 + b\,B_6\,c\,a\,y_3 - b\,B_4\,y_3\,y_1 - a\,A_4\,c\,y_3\,B_5\,y_4 + a\,B_4\,c\,y_3\,A_5\,y_4 + B_5\,c\,a^2\,y_4\,A_2 + b\,B_5\,y_2\,y_1 - a\,B_4\,c\,b\,y_4\,A_2 + b\,B_4\,c\,y_3\,A_6\,y_2 + a\,B_4\,y_3\,A_2\,y_1 - A_5\,c\,a^2\,y_4 - b\,B_4\,c^2\,y_3 + B_4\,c\,b^2\,y_4 - B_6\,c\,b^2\,y_2 + A_6\,c\,a^2\,y_3 - a\,A_4\,c^2\,y_3 + a\,A_5\,c^2\,y_2 + b\,B_5\,c^2\,y_2\big)\Big) \Big/ \big(-A_3\,a\,y_1\,B_5\,y_4\,c + B_3\,a\,y_1\,A_2\,c^2 + A_3\,a\,y_1\,B_6\,y_3\,c - A_4\,y_4\,c\,B_3\,b\,y_1 + A_4\,y_4\,c^2\,B_6\,y_3 + B_3\,a\,y_1^2\,A_2 + B_4\,y_4\,c^3\,A_2 + B_4\,y_4^2\,c^2\,A_5 - B_6\,y_2\,c^3\,A_2 - c^2\,B_3\,b\,y_1 - A_3\,a\,y_1\,c^2 - A_4\,y_4^2\,c^2\,B_5 - y_1\,B_5\,y_4\,c + y_1\,B_6\,y_3\,c - A_4\,y_4\,c\,y_1 + A_6\,y_2\,c\,y_1 - c^3\,B_5\,y_4 + c^3\,B_6\,y_3 - B_3\,b\,y_1^2 - A_3\,a\,y_1^2 - A_4\,y_4\,c^3 + A_6\,y_2\,c^3 - 2\,c^2\,y_1 - y_1^2 - c^4 + B_3\,a\,y_1\,A_5\,y_4\,c - B_3\,a\,y_1\,A_6\,y_3\,c + B_4\,y_4\,c\,A_2\,y_1 + B_4\,y_4\,c\,A_3\,b\,y_1 - B_4\,y_4\,c^2\,A_6\,y_3 - B_6\,y_2\,c\,A_2\,y_1 - B_6\,y_2\,c\,A_3\,b\,y_1 - B_6\,y_2\,c^2\,A_5\,y_4 + A_6\,y_2\,c\,B_3\,b\,y_1 + A_6\,y_2\,c^2\,B_5\,y_4\big)$$

$$trans_{1,2} := -\Big(c\,\big(-c^3\,y_3^2\,B_6 + c^4\,y_3 - b\,y_4^2\,A_4\,c^2 - c\,y_3^2\,y_1\,B_6 + c^3\,y_3\,B_5\,y_4 + c^3\,y_3\,A_4\,y_4 - c^3\,y_3\,A_6\,y_2 - b\,y_1\,y_4\,c - b\,y_4^2\,c^2\,B_5 + y_3\,y_1^2 + 2\,c^2\,y_3\,y_1 - b\,y_4\,c^3 - b\,y_4\,B_6\,y_2\,c^2\,A_2 - b\,y_4^3\,A_4\,c\,B_5 - b\,y_4^2\,y_1\,B_5 + b\,y_4\,y_1\,B_6\,y_3 + b\,y_4\,c^2\,B_6\,y_3 + b\,y_4^2\,A_4\,c\,B_6\,y_3 + b\,y_4^2\,B_4\,c^2\,A_2 + b\,y_4^3\,B_4\,c\,A_5 + b\,y_4^2\,A_6\,y_2\,c\,B_5 + b\,y_4\,A_6\,y_2\,c^2 - b\,y_4^2\,B_4\,c\,A_6\,y_3 - b\,y_4^2\,B_6\,y_2\,c\,A_5 - c^2\,y_3^2\,A_4\,y_4\,B_6 - c^3\,y_3\,B_4\,y_4\,A_2 - c^2\,y_3\,B_4\,y_4^2\,A_5 + c^3\,y_3\,B_6\,y_2\,A_2 + c^2\,y_3\,A_4\,y_4^2\,B_5 + c\,y_3\,y_1\,B_5\,y_4 + c\,y_3\,A_4\,y_4\,y_1 - c\,y_3\,A_6\,y_2\,y_1 - c\,y_3\,B_4\,y_4\,A_2\,y_1 + c^2\,y_3^2\,B_4\,y_4\,A_6 + c\,y_3\,B_6\,y_2\,A_2\,y_1 + c^2\,y_3\,B_6\,y_2\,A_5\,y_4 - c^2\,y_3\,A_6\,y_2\,B_5\,y_4 - a\,y_1\,A_2\,y_4\,B_6\,y_3 + a\,y_1\,y_4\,A_6\,y_3 - a\,y_1\,A_5\,y_4^2 + a\,y_1\,A_2\,y_4^2\,B_5\big)\Big) \Big/ \big(-A_3\,a\,y_1\,B_5\,y_4\,c + B_3\,a\,y_1\,A_2\,c^2 + A_3\,a\,y_1\,B_6\,y_3\,c - A_4\,y_4\,c\,B_3\,b\,y_1 + A_4\,y_4\,c^2\,B_6\,y_3 + B_3\,a\,y_1^2\,A_2 + B_4\,y_4\,c^3\,A_2 + B_4\,y_4^2\,c^2\,A_5 - B_6\,y_2\,c^3\,A_2$$

$$- c^{2} B_{3} b y_{1} - A_{3} a y_{1} c^{2} - A_{4} y_{4}^{2} c^{2} B_{5} - y_{1} B_{5} y_{4} c + y_{1} B_{6} y_{3} c - A_{4} y_{4} c y_{1} + A_{6} y_{2} c y_{1}$$

$$- c^{3} B_{5} y_{4} + c^{3} B_{6} y_{3} - B_{3} b y_{1}^{2} - A_{3} a y_{1}^{2} - A_{4} y_{4} c^{3} + A_{6} y_{2} c^{3} - 2 c^{2} y_{1} - y_{1}^{2} - c^{4}$$

$$+ B_{3} a y_{1} A_{5} y_{4} c - B_{3} a y_{1} A_{6} y_{3} c + B_{4} y_{4} c A_{2} y_{1} + B_{4} y_{4} c A_{3} b y_{1} - B_{4} y_{4} c^{2} A_{6} y_{3}$$

$$- B_{6} y_{2} c A_{2} y_{1} - B_{6} y_{2} c A_{3} b y_{1} - B_{6} y_{2} c^{2} A_{5} y_{4} + A_{6} y_{2} c B_{3} b y_{1} + A_{6} y_{2} c^{2} B_{5} y_{4} \Big)$$

$$trans_{1,\,3} := -\Big(c \Big(-c^{4} y_{2} + c^{3} y_{3} B_{6} y_{2} + c y_{3} B_{6} y_{2} y_{1} - a y_{4}^{2} B_{4} c^{2} A_{2} - a y_{4}^{3} B_{4} c A_{5} + a$$

$$y_{4}^{3} A_{4} c B_{5} + a y_{4}^{2} A_{4} y_{1} - a y_{4} A_{6} y_{2} y_{1} - a y_{4} c^{2} B_{6} y_{3} - a y_{4} A_{6} y_{2} c^{2} - a y_{4}^{2} B_{4} A_{2} y_{1} + a$$

$$y_{4}^{2} B_{4} c A_{6} y_{3} + a y_{4} B_{6} y_{2} A_{2} y_{1} + c^{3} y_{2} B_{4} y_{4} A_{2} + a y_{4}^{2} B_{6} y_{2} c A_{5} - a y_{4}^{2} A_{6} y_{2} c B_{5}$$

$$+ c^{2} y_{2} A_{4} y_{4} B_{6} y_{3} + c^{2} y_{2} B_{4} y_{4}^{2} A_{5} + b y_{1} B_{4} y_{4}^{2} - b y_{1} y_{4} B_{6} y_{2} - c^{2} y_{2} A_{4} y_{4}^{2} B_{5} - a$$

$$y_{4}^{2} A_{4} c B_{6} y_{3} + a y_{4} B_{6} y_{2} c^{2} A_{2} + a y_{4} c^{3} + c^{3} y_{2}^{2} A_{6} - 2 c^{2} y_{2} y_{1} - c y_{2} y_{1} B_{5} y_{4}$$

$$- c y_{2} A_{4} y_{4} y_{1} + c y_{2} B_{4} y_{4} A_{2} y_{1} - c^{2} y_{2} B_{4} y_{4} A_{6} y_{3} - c y_{2}^{2} B_{6} A_{2} y_{1} - c^{2} y_{2}^{2} B_{6} A_{5} y_{4} + c^{2}$$

$$y_{2}^{2} A_{6} B_{5} y_{4} - y_{2} y_{1}^{2} - c^{3} y_{2}^{2} B_{6} A_{2} + c y_{2}^{2} A_{6} y_{1} - c^{3} y_{2} B_{5} y_{4} - c^{3} y_{2} A_{4} y_{4} + a y_{1} y_{4} c + a$$

$$y_{4}^{2} c^{2} B_{5} + a y_{4}^{2} A_{4} c^{2}\Big)\Big) \Big/ \Big(-A_{3} a y_{1} B_{5} y_{4} c + B_{3} a y_{1} A_{2} c^{2} + A_{3} a y_{1} B_{6} y_{3} c$$

$$- A_{4} y_{4} c B_{3} b y_{1} + A_{4} y_{4} c^{2} B_{6} y_{3} + B_{3} a y_{1}^{2} A_{2} + B_{4} y_{4} c^{3} A_{2} + B_{4} y_{4}^{2} c^{2} A_{5} - B_{6} y_{2} c^{3} A_{2}$$

$$- c^{2} B_{3} b y_{1} - A_{3} a y_{1} c^{2} - A_{4} y_{4}^{2} c^{2} B_{5} - y_{1} B_{5} y_{4} c + y_{1} B_{6} y_{3} c - A_{4} y_{4} c y_{1} + A_{6} y_{2} c y_{1}$$

$$- c^{3} B_{5} y_{4} + c^{3} B_{6} y_{3} - B_{3} b y_{1}^{2} - A_{3} a y_{1}^{2} - A_{4} y_{4} c^{3} + A_{6} y_{2} c^{3} - 2 c^{2} y_{1} - y_{1}^{2} - c^{4}$$

$$+ B_{3} a y_{1} A_{5} y_{4} c - B_{3} a y_{1} A_{6} y_{3} c + B_{4} y_{4} c A_{2} y_{1} + B_{4} y_{4} c A_{3} b y_{1} - B_{4} y_{4} c^{2} A_{6} y_{3}$$

$$- B_{6} y_{2} c A_{2} y_{1} - B_{6} y_{2} c A_{3} b y_{1} - B_{6} y_{2} c^{2} A_{5} y_{4} + A_{6} y_{2} c B_{3} b y_{1} + A_{6} y_{2} c^{2} B_{5} y_{4}\Big)$$

$$trans_{1,\,4} := \Big(c \Big(-b y_{1} y_{2} c + a y_{1} y_{3} c + b y_{2}^{2} A_{6} c^{2} - a y_{3}^{2} c^{2} B_{6} + a y_{3} c^{3} - b y_{2} c^{3}$$

$$+ b y_{1} y_{3} B_{4} y_{4} + a y_{1} A_{2} y_{2} B_{5} y_{4} - a y_{1} y_{2} A_{5} y_{4} - a y_{3}^{2} A_{4} y_{4} c B_{6} - a y_{3} B_{4} y_{4} c^{2} A_{2}$$

$$- a y_{3} B_{4} y_{4}^{2} c A_{5} + a y_{3} B_{6} y_{2} c^{2} A_{2} + a y_{3} A_{4} y_{4}^{2} c B_{5} + a y_{3} A_{4} y_{4} y_{1} + a y_{3} c^{2} B_{5} y_{4}$$

$$+ a y_{3} A_{4} y_{4} c^{2} - a y_{3} A_{6} y_{2} c^{2} - a y_{3} B_{4} y_{4} A_{2} y_{1} + a y_{3}^{2} B_{4} y_{4} c A_{6} + a y_{3} B_{6} y_{2} c A_{5} y_{4}$$

$$- a y_{3} A_{6} y_{2} c B_{5} y_{4} + b y_{2} A_{4} y_{4} c B_{6} y_{3} + b y_{2} B_{4} y_{4} c^{2} A_{2} + b y_{2} B_{4} y_{4}^{2} c A_{5} - b$$

$$y_{2}^{2} B_{6} c^{2} A_{2} - b y_{2} A_{4} y_{4}^{2} c B_{5} - b y_{2} y_{1} B_{5} y_{4} - b y_{2} c^{2} B_{5} y_{4} + b y_{2} c^{2} B_{6} y_{3} - b y_{2} A_{4} y_{4} c^{2}$$

$$- b y_{2} B_{4} y_{4} c A_{6} y_{3} - b y_{2}^{2} B_{6} c A_{5} y_{4} + b y_{2}^{2} A_{6} c B_{5} y_{4}\Big)\Big) \Big/ \Big(-A_{3} a y_{1} B_{5} y_{4} c$$

$$+ B_{3} a y_{1} A_{2} c^{2} + A_{3} a y_{1} B_{6} y_{3} c - A_{4} y_{4} c B_{3} b y_{1} + A_{4} y_{4} c^{2} B_{6} y_{3} + B_{3} a y_{1}^{2} A_{2}$$

$$+ B_{4} y_{4} c^{3} A_{2} + B_{4} y_{4}^{2} c^{2} A_{5} - B_{6} y_{2} c^{3} A_{2} - c^{2} B_{3} b y_{1} - A_{3} a y_{1} c^{2} - A_{4} y_{4}^{2} c^{2} B_{5}$$

$$-y_1 B_5 y_4 c+y_1 B_6 y_3 c-A_4 y_4 c y_1+A_6 y_2 c y_1-c^3 B_5 y_4+c^3 B_6 y_3-B_3 b y_1^2-A_3 a$$
$$y_1^2-A_4 y_4 c^3+A_6 y_2 c^3-2 c^2 y_1-y_1^2-c^4+B_3 a y_1 A_5 y_4 c-B_3 a y_1 A_6 y_3 c$$
$$+B_4 y_4 c A_2 y_1+B_4 y_4 c A_3 b y_1-B_4 y_4 c^2 A_6 y_3-B_6 y_2 c A_2 y_1-B_6 y_2 c A_3 b y_1$$
$$-B_6 y_2 c^2 A_5 y_4+A_6 y_2 c B_3 b y_1+A_6 y_2 c^2 B_5 y_4\Big)$$
$$trans_{2,2} := -\Big(c^2 \Big(-A_2 y_4 B_6 y_3+A_3 b y_4^2 B_5-A_3 b y_4 B_6 y_3-A_3 y_3 B_5 y_4 c-B_3 b y_4 A_2 c$$
$$-B_3 b y_4^2 A_5+B_3 b y_4 A_6 y_3+B_3 y_3 A_2 y_1+B_3 y_3 A_5 y_4 c-B_3 y_3^2 A_6 c+y_4 A_6 y_3$$
$$-A_3 y_3 c^2-A_5 y_4^2+A_2 y_4^2 B_5+A_3 b y_4 c+A_3 y_3^2 B_6 c-A_3 y_3 y_1+B_3 y_3 A_2 c^2\Big) y_4\Big) \Big/ \Big($$
$$-A_3 a y_1 B_5 y_4 c+B_3 a y_1 A_2 c^2+A_3 a y_1 B_6 y_3 c-A_4 y_4 c B_3 b y_1+A_4 y_4 c^2 B_6 y_3+B_3 a$$
$$y_1^2 A_2+B_4 y_4 c^3 A_2+B_4 y_4^2 c^2 A_5-B_6 y_2 c^3 A_2-c^2 B_3 b y_1-A_3 a y_1 c^2-A_4 y_4^2 c^2 B_5$$
$$-y_1 B_5 y_4 c+y_1 B_6 y_3 c-A_4 y_4 c y_1+A_6 y_2 c y_1-c^3 B_5 y_4+c^3 B_6 y_3-B_3 b y_1^2-A_3 a$$
$$y_1^2-A_4 y_4 c^3+A_6 y_2 c^3-2 c^2 y_1-y_1^2-c^4+B_3 a y_1 A_5 y_4 c-B_3 a y_1 A_6 y_3 c$$
$$+B_4 y_4 c A_2 y_1+B_4 y_4 c A_3 b y_1-B_4 y_4 c^2 A_6 y_3-B_6 y_2 c A_2 y_1-B_6 y_2 c A_3 b y_1$$
$$-B_6 y_2 c^2 A_5 y_4+A_6 y_2 c B_3 b y_1+A_6 y_2 c^2 B_5 y_4\Big)$$
$$trans_{2,3} := -\Big(c \Big(B_3 a y_4 A_2 c^2+B_3 a y_4 A_2 y_1+B_3 a y_4^2 A_5 c-B_3 a y_4 A_6 y_3 c-B_3 y_2 c A_2 y_1$$
$$-B_3 y_2 c^2 A_5 y_4+B_3 y_3 c^2 A_6 y_2-A_3 a y_4 c^2-y_4 B_3 b y_1+y_4 B_6 y_3 c-A_3 a y_4 y_1$$
$$+A_3 c y_2 y_1-A_3 a y_4^2 B_5 c+A_3 a y_4 B_6 y_3 c+A_3 c^2 y_2 B_5 y_4-A_3 c^2 y_2 B_6 y_3$$
$$-B_3 y_2 c^3 A_2-B_5 y_4^2 c+A_3 c^3 y_2-y_4 c^2-y_4 y_1\Big) y_4\Big) \Big/ \Big(-A_3 a y_1 B_5 y_4 c$$
$$+B_3 a y_1 A_2 c^2+A_3 a y_1 B_6 y_3 c-A_4 y_4 c B_3 b y_1+A_4 y_4 c^2 B_6 y_3+B_3 a y_1^2 A_2$$
$$+B_4 y_4 c^3 A_2+B_4 y_4^2 c^2 A_5-B_6 y_2 c^3 A_2-c^2 B_3 b y_1-A_3 a y_1 c^2-A_4 y_4^2 c^2 B_5$$
$$-y_1 B_5 y_4 c+y_1 B_6 y_3 c-A_4 y_4 c y_1+A_6 y_2 c y_1-c^3 B_5 y_4+c^3 B_6 y_3-B_3 b y_1^2-A_3 a$$
$$y_1^2-A_4 y_4 c^3+A_6 y_2 c^3-2 c^2 y_1-y_1^2-c^4+B_3 a y_1 A_5 y_4 c-B_3 a y_1 A_6 y_3 c$$
$$+B_4 y_4 c A_2 y_1+B_4 y_4 c A_3 b y_1-B_4 y_4 c^2 A_6 y_3-B_6 y_2 c A_2 y_1-B_6 y_2 c A_3 b y_1$$
$$-B_6 y_2 c^2 A_5 y_4+A_6 y_2 c B_3 b y_1+A_6 y_2 c^2 B_5 y_4\Big)$$
$$trans_{2,4} := \Big(c \Big(-A_3 a y_3 c^2+A_3 b y_2 c^2-y_3 B_3 b y_1+A_2 y_2 B_5 y_4 c-A_2 y_2 B_6 y_3 c$$
$$-A_3 a y_3 B_5 y_4 c+A_3 a y_3^2 B_6 c+A_3 b y_2 B_5 y_4 c-A_3 b y_2 B_6 y_3 c+B_3 a y_3 A_2 c^2$$
$$+B_3 a y_3 A_2 y_1-y_3 c^2-y_3 y_1-y_3 B_5 y_4 c-y_2 A_5 y_4 c+y_2 A_6 y_3 c-A_3 a y_3 y_1+B_6$$

$$y_3^2\, c + B_3\, a\, y_3\, A_5\, y_4\, c - B_3\, a\, y_3^2\, A_6\, c - B_3\, b\, y_2\, A_2\, c^2 - B_3\, b\, y_2\, A_5\, y_4\, c + B_3\, b\, y_2\, A_6\, y_3\, c)$$

$$y_4) \big/ (-A_3\, a\, y_1\, B_5\, y_4\, c + B_3\, a\, y_1\, A_2\, c^2 + A_3\, a\, y_1\, B_6\, y_3\, c - A_4\, y_4\, c\, B_3\, b\, y_1 + A_4\, y_4\, c^2\, B_6\, y_3$$

$$+ B_3\, a\, y_1^2\, A_2 + B_4\, y_4\, c^3\, A_2 + B_4\, y_4^2\, c^2\, A_5 - B_6\, y_2\, c^3\, A_2 - c^2\, B_3\, b\, y_1 - A_3\, a\, y_1\, c^2 - A_4$$

$$y_4^2\, c^2\, B_5 - y_1\, B_5\, y_4\, c + y_1\, B_6\, y_3\, c - A_4\, y_4\, c\, y_1 + A_6\, y_2\, c\, y_1 - c^3\, B_5\, y_4 + c^3\, B_6\, y_3 - B_3\, b\, y_1^2$$

$$- A_3\, a\, y_1^2 - A_4\, y_4\, c^3 + A_6\, y_2\, c^3 - 2\, c^2\, y_1 - y_1^2 - c^4 + B_3\, a\, y_1\, A_5\, y_4\, c - B_3\, a\, y_1\, A_6\, y_3\, c$$

$$+ B_4\, y_4\, c\, A_2\, y_1 + B_4\, y_4\, c\, A_3\, b\, y_1 - B_4\, y_4\, c^2\, A_6\, y_3 - B_6\, y_2\, c\, A_2\, y_1 - B_6\, y_2\, c\, A_3\, b\, y_1$$

$$- B_6\, y_2\, c^2\, A_5\, y_4 + A_6\, y_2\, c\, B_3\, b\, y_1 + A_6\, y_2\, c^2\, B_5\, y_4)$$

$$trans_{3,\,3} := (c^2\, (-B_4\, y_4^2 + y_4\, B_6\, y_2 - A_3\, a\, y_4^2\, B_4 + A_3\, a\, y_4\, B_6\, y_2 + A_3\, y_2\, c\, B_4\, y_4 - A_3\, y_2^2\, c\, B_6$$

$$+ B_3\, a\, y_4\, c + B_3\, a\, y_4^2\, A_4 - B_3\, a\, y_4\, A_6\, y_2 - B_3\, y_2\, c^2 - B_3\, y_2\, y_1 - B_3\, y_2\, A_4\, y_4\, c + B_3$$

$$y_2^2\, A_6\, c)\, y_4) \big/ (-A_3\, a\, y_1\, B_5\, y_4\, c + B_3\, a\, y_1\, A_2\, c^2 + A_3\, a\, y_1\, B_6\, y_3\, c - A_4\, y_4\, c\, B_3\, b\, y_1$$

$$+ A_4\, y_4\, c^2\, B_6\, y_3 + B_3\, a\, y_1^2\, A_2 + B_4\, y_4\, c^3\, A_2 + B_4\, y_4^2\, c^2\, A_5 - B_6\, y_2\, c^3\, A_2 - c^2\, B_3\, b\, y_1$$

$$- A_3\, a\, y_1\, c^2 - A_4\, y_4^2\, c^2\, B_5 - y_1\, B_5\, y_4\, c + y_1\, B_6\, y_3\, c - A_4\, y_4\, c\, y_1 + A_6\, y_2\, c\, y_1 - c^3\, B_5\, y_4$$

$$+ c^3\, B_6\, y_3 - B_3\, b\, y_1^2 - A_3\, a\, y_1^2 - A_4\, y_4\, c^3 + A_6\, y_2\, c^3 - 2\, c^2\, y_1 - y_1^2 - c^4$$

$$+ B_3\, a\, y_1\, A_5\, y_4\, c - B_3\, a\, y_1\, A_6\, y_3\, c + B_4\, y_4\, c\, A_2\, y_1 + B_4\, y_4\, c\, A_3\, b\, y_1 - B_4\, y_4\, c^2\, A_6\, y_3$$

$$- B_6\, y_2\, c\, A_2\, y_1 - B_6\, y_2\, c\, A_3\, b\, y_1 - B_6\, y_2\, c^2\, A_5\, y_4 + A_6\, y_2\, c\, B_3\, b\, y_1 + A_6\, y_2\, c^2\, B_5\, y_4)$$

$$trans_{3,\,4} := -(c\, (A_6\, y_2^2\, c + B_3\, a\, y_3\, A_4\, y_4\, c - B_3\, a\, y_3\, A_6\, y_2\, c - B_3\, b\, y_2\, A_4\, y_4\, c + B_3\, b\, y_2^2\, A_6\, c$$

$$- B_3\, b\, y_2\, y_1 + A_3\, b\, y_2\, c\, B_4\, y_4 - A_3\, b\, y_2^2\, c\, B_6 + B_3\, a\, y_3\, c^2 - B_3\, b\, y_2\, c^2 - y_2\, A_3\, a\, y_1$$

$$- y_2\, A_4\, y_4\, c - A_2\, y_2^2\, c\, B_6 + y_3\, c\, B_6\, y_2 + A_2\, y_2\, c\, B_4\, y_4 + A_2\, y_2\, B_3\, a\, y_1 - A_3\, a\, y_3\, c\, B_4\, y_4$$

$$+ A_3\, a\, y_3\, c\, B_6\, y_2 - y_3\, c\, B_4\, y_4 - y_2\, c^2 - y_2\, y_1)\, y_4) \big/ (-A_3\, a\, y_1\, B_5\, y_4\, c + B_3\, a\, y_1\, A_2\, c^2$$

$$+ A_3\, a\, y_1\, B_6\, y_3\, c - A_4\, y_4\, c\, B_3\, b\, y_1 + A_4\, y_4\, c^2\, B_6\, y_3 + B_3\, a\, y_1^2\, A_2 + B_4\, y_4\, c^3\, A_2 + B_4$$

$$y_4^2\, c^2\, A_5 - B_6\, y_2\, c^3\, A_2 - c^2\, B_3\, b\, y_1 - A_3\, a\, y_1\, c^2 - A_4\, y_4^2\, c^2\, B_5 - y_1\, B_5\, y_4\, c + y_1\, B_6\, y_3\, c$$

$$- A_4\, y_4\, c\, y_1 + A_6\, y_2\, c\, y_1 - c^3\, B_5\, y_4 + c^3\, B_6\, y_3 - B_3\, b\, y_1^2 - A_3\, a\, y_1^2 - A_4\, y_4\, c^3 + A_6\, y_2\, c^3$$

$$- 2\, c^2\, y_1 - y_1^2 - c^4 + B_3\, a\, y_1\, A_5\, y_4\, c - B_3\, a\, y_1\, A_6\, y_3\, c + B_4\, y_4\, c\, A_2\, y_1 + B_4\, y_4\, c\, A_3\, b\, y_1$$

$$- B_4\, y_4\, c^2\, A_6\, y_3 - B_6\, y_2\, c\, A_2\, y_1 - B_6\, y_2\, c\, A_3\, b\, y_1 - B_6\, y_2\, c^2\, A_5\, y_4 + A_6\, y_2\, c\, B_3\, b\, y_1$$

$$+ A_6\, y_2\, c^2\, B_5\, y_4)$$

$$trans_{4,\,4} := -(c^2\, (-B_3\, a\, y_3^2\, A_4\, y_4 + A_2\, y_2^2\, B_5\, y_4 + A_3\, a\, y_3^2\, B_4\, y_4 - y_2^2\, A_5\, y_4 + A_3\, b\, y_2^2\, c - B_3\, a$$

$$y_3^2\,c + y_3^2\,B_4\,y_4 + y_3\,B_3\,b\,y_2\,A_4\,y_4 - y_3\,A_3\,b\,y_2\,B_4\,y_4 + y_3\,B_3\,b\,y_2\,c + y_3\,y_2\,A_4\,y_4$$
$$- y_3\,A_2\,y_2\,B_4\,y_4 + A_3\,b\,y_2^2\,B_5\,y_4 - B_3\,b\,y_2^2\,A_2\,c - B_3\,b\,y_2^2\,A_5\,y_4 - y_2\,A_3\,a\,y_3\,c$$
$$- y_2\,A_3\,a\,y_3\,B_5\,y_4 + y_2\,B_3\,a\,y_3\,A_2\,c - y_2\,y_3\,B_5\,y_4 + y_2\,B_3\,a\,y_3\,A_5\,y_4\big)\big) \Big/ \big(-A_3\,a\,y_1\,B_5\,y_4\,c$$
$$+ B_3\,a\,y_1\,A_2\,c^2 + A_3\,a\,y_1\,B_6\,y_3\,c - A_4\,y_4\,c\,B_3\,b\,y_1 + A_4\,y_4\,c^2\,B_6\,y_3 + B_3\,a\,y_1^2\,A_2$$
$$+ B_4\,y_4\,c^3\,A_2 + B_4\,y_4^2\,c^2\,A_5 - B_6\,y_2\,c^3\,A_2 - c^2\,B_3\,b\,y_1 - A_3\,a\,y_1\,c^2 - A_4\,y_4^2\,c^2\,B_5$$
$$- y_1\,B_5\,y_4\,c + y_1\,B_6\,y_3\,c - A_4\,y_4\,c\,y_1 + A_6\,y_2\,c\,y_1 - c^3\,B_5\,y_4 + c^3\,B_6\,y_3 - B_3\,b\,y_1^2 - A_3\,a$$
$$y_1^2 - A_4\,y_4\,c^3 + A_6\,y_2\,c^3 - 2\,c^2\,y_1 - y_1^2 - c^4 + B_3\,a\,y_1\,A_5\,y_4\,c - B_3\,a\,y_1\,A_6\,y_3\,c$$
$$+ B_4\,y_4\,c\,A_2\,y_1 + B_4\,y_4\,c\,A_3\,b\,y_1 - B_4\,y_4\,c^2\,A_6\,y_3 - B_6\,y_2\,c\,A_2\,y_1 - B_6\,y_2\,c\,A_3\,b\,y_1$$
$$- B_6\,y_2\,c^2\,A_5\,y_4 + A_6\,y_2\,c\,B_3\,b\,y_1 + A_6\,y_2\,c^2\,B_5\,y_4\big)$$

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